Discontinuous Homomorphisms from Banach Algebras of Operators

Richard James Skillicorn, MSci

A thesis submitted for the degree of
Doctor of Philosophy

Department of Mathematics and Statistics
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Abstract

The relationship between a Banach space X and its Banach algebra of bounded operators $\mathscr{B}(X)$ is rich and complex; this is especially so for non-classical Banach spaces. In this thesis we consider questions of the following form: does there exist a Banach space X such that $\mathscr{B}(X)$ has a particular (Banach algebra) property? If not, is there a quotient of $\mathscr{B}(X)$ with the property?

The first of these is the *uniqueness-of-norm problem* for Calkin algebras: does there exist a Banach space whose Calkin algebra lacks a unique complete norm? We show that there does indeed exist such a space, answering a classical open question [101].

Secondly, we turn our attention to splittings of extensions of Banach algebras. Work of Bade, Dales and Lykova [12] inspired the problem of whether there exist a Banach space X and an extension of $\mathscr{B}(X)$ which splits algebraically but not strongly; this asks for a special type of discontinuous homomorphism from $\mathscr{B}(X)$. Using the categorical notion of a pullback we obtain, jointly with N. J. Laustsen [71], new general results about extensions and prove that such a space exists.

The same space is used to answer our third question, which goes back to Helemskii, in the positive: is there a Banach space X such that $\mathscr{B}(X)$ has homological bidimension at least two? The proof uses techniques developed (with N. J. Laustsen [71]) during the solution to the second question.

We use two main Banach spaces to answer our questions. One is due to Read [90], the other to Argyros and Motakis [8]; the former plays a much more prominent role. Together with Laustsen [72], we prove a major original result about Read's space which allows for the new applications.

The conclusion of the thesis examines a class of operators on Banach spaces which have previously received little attention; these are a weak analogue of inessential operators.

Dedicated to the memory of Professor Charles J. Read (1958-2015)

For as in Adam all die, so in Christ all will be made alive.
1 Corinthians 15:22

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Much of the work in this thesis has been influenced by that of the late Professor Charles Read, formerly of the University of Leeds. I have spent many hours poring over his work, appreciating its brilliance. Of course I have only actually examined a very small part of his significant body of research, but nevertheless, I feel I have got to know him mathematically. I also had the pleasure of meeting him on a couple of occasions. Although I cannot say I got to know him well, he demonstrated a real interest in what I had done and spoke kindly to me. I was greatly saddened to hear of his sudden death in August 2015. Only the week before I had been speaking to him at *Banach Algebras and Applications 2015* in Toronto, Canada, and I was shocked to hear of his passing so soon after. Given its content, it seems right to dedicate the thesis to his memory.

Declaration

The research presented in this thesis has not been submitted for a higher degree elsewhere and is, to the best of my knowledge and belief, original and my own work, except as acknowledged herein.

Chapter 2 contains published work [101], which is reproduced, in a modified form, with permission.

Chapter 3 and Chapter 4 contain published joint work with N. J. Laustsen [71].

Chapter 5 (specifically Section 5.4) contains joint work with N. J. Laustsen [72] (submitted).

Contents

Chapter 1

Introduction

1.1 Thesis overview

This introductory chapter has two parts. The first collects notation, definitions and standard facts which we shall use throughout. We also record some well-known theorems for ease of reference. The second is a brief explanation of the Banach space which shall occupy our attention for a significant portion of the thesis. This Banach space construction of Read appears not to be widely understood or appreciated, at least in the sense that it has not been used in much subsequent work until now. Part of our aim is to change this by demonstrating some of its interesting applications. The key to these new applications is one of our original theorems, stated at the end of the chapter, which builds on Read's fundamental work.

Chapter 2 begins to explore these new applications in the context of Calkin algebras. It is a classical question whether a given Banach algebra (or class of Banach algebras) has a unique complete norm. For an arbitrary Banach space, the question of whether its Calkin algebra must have a unique complete norm was a long-standing open question. We answer this question in the negative, not in fact using Read's space, but a markedly different Banach space due to Argyros and Motakis, which has certain properties resembling those of Read's space. For the case of the weak Calkin algebra, Read's space is relevant; we demonstrate a similar result in this case. The chapter relies heavily on published work of the author [101].

Chapter 3 is about splittings of extensions of Banach algebras, and in particular the Banach algebra of bounded operators on a Banach space. An open problem since the 1990s is whether extensions of this Banach algebra which split algebraically also split strongly (such a question is related to the subject of automatic continuity). For many Banach spaces this is known to be true; together with N. J. Laustsen we show that for Read's space it is false: there is an extension which splits algebraically but not strongly. We apply our new theorem, and along the way develop new general results about extensions of Banach algebras. Parts of the chapter have been published [71].

Our fourth chapter continues in the vein of the third. Splittings of extensions are related to the concept of the homological bidimension of a Banach algebra; roughly speaking this measures its homological defects. Of interest is to determine the values of the homological bidimension for particular Banach algebras. In the case of the Banach algebra of bounded operators on a Banach space little is known, and it is a question going back to Helemskii as to whether there is a Banach space X such that $\mathscr{B}(X)$ has homological bidimension at least two. We demonstrate that there are at least three such Banach spaces, one of which is Read's space. The general theorems about extensions from Chapter 3 transfer usefully to this setting, making the proofs quite simple. The main original results of the chapter are joint with N. J. Laustsen and also appear in [71].

Having shown various new applications of Read's space, in Chapter 5 we turn to its construction. A barrier to full appreciation of the space is that grasping the full details of its construction is at times a challenging task. We attempt to present Read's results in a more elementary way. We labour the technical details in order to give a transparent proof of our main original theorem about the space. The results claimed in Chapter 1 are then completely justified. The author also hopes that a clear, self-contained exposition may perhaps be useful to others. Some of the exposition, along with the original results of Section 5.4 (which are joint with Laustsen), can be found in [72].

Chapter 6 examines a class of operators on Banach spaces which we term weakly inessential. These form an operator ideal (in the sense of Pietsch) which has many properties in common with the classical operator ideal of inessential operators. Indeed, the definition of a weakly inessential operator is obtained by simply replacing 'compact' with 'weakly compact' in the definition of an inessential operator. Despite its natural definition, this class has had virtually no attention in the literature because in almost all cases it coincides with the weakly compact operators. We show that there are actually many examples where this is not the case. Our motivating example is Read's space where the gap between them is 'as large as possible'. Given this, it seemed useful to develop a general theory; here there seems plenty of scope for future work.

1.2 Preliminaries

We write $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ for the set of non-negative integers. We denote Banach spaces by E, F, W, X, Y, Z, and the scalar field by $\mathbb{K}$, which is either $\mathbb{R}$ or $\mathbb{C}$. Let X be a Banach space. We write X^* for the (continuous) *dual space* of X. A sequence (x_n) in X *converges weakly* to $x \in X$, written $x_n \xrightarrow{w} x$, if $f(x_n) \to f(x)$ for every $f \in X^*$. A sequence (f_n) in X^* *converges weak** to $f \in X^*$, written $f_n \xrightarrow{w^*} f$, if $f_n(x) \to f(x)$ for every $x \in X$. There is a natural isometry $\kappa : X \to X^{**}$ given by $\kappa(x)(f) = f(x)$ for each $x \in X$ and $f \in X^*$, so we identify X with its image in X^{**}, which allows us to consider the quotient X^{**}/X. If κ is surjective then X is *reflexive*, and if $\dim(X^{**}/X) = 1$ then X is *quasi-reflexive*. The *duality bracket* between X and X^* is denoted by $\langle \cdot, \cdot \rangle$ and by convention we write the functional on the right. For a Hilbert space, this should not be confused with the *inner product*, written $(\cdot|\cdot)$, which we consider to be linear in the first argument.

Bounded, linear maps between Banach spaces are *bounded operators*, commonly labelled R, S, T. Let X and Y be Banach spaces. We write $\mathscr{B}(X, Y)$ for the Banach space of bounded operators from X to Y. If $X = Y$ we prefer $\mathscr{B}(X)$ to $\mathscr{B}(X, X)$; in this case $\mathscr{B}(X)$ is a Banach algebra, with multiplication given by composition. The *image* (or range) of a map $T : X \to Y$ is written $\operatorname{im} T$, and its *kernel* is $\ker T$. Let $T : X \to Y$ be a bounded operator. Then $\ker T$ is a closed subspace of X, and its *adjoint* $T^* : Y^* \to X^*$, the linear map given by $\langle x, T^* y^* \rangle = \langle Tx, y^* \rangle$ for $y^* \in Y^*, x \in X$, is also bounded, with $||T|| = ||T^*||$. The map T is *bounded below* if there exists $c > 0$ such that $c||x|| \leqslant ||Tx||$ for every $x \in X$. If T is bijective and T^{-1} is bounded, then it is an *isomorphism* and X and Y are *isomorphic*, written $X \simeq Y$ (the boundedness of T^{-1} is actually automatic by the Banach Isomorphism Theorem 1.2.2, below). We say that X *contains a copy* of Y if there is a closed subspace of X isomorphic to Y. A linear map $P : X \to X$ is a *projection* if $P^2 = P$. A subspace V of X is *algebraically complemented* if there is a subspace W of X such that $V \cap W = \{0\}$ and $V + W = \{v + w : v \in V, w \in W\} = X$, or equivalently, if there is a projection on X with range V. We write $X = V \boxplus W$. It is standard that every subspace of X is algebraically complemented. The *codimension* of V is the dimension of the vector space X/V. The *closure* of V is denoted by $\overline{V}$. A closed subspace Y of X is *complemented* if there is a closed subspace Z of X such that $Z \cap Y = \{0\}$ and $Z + Y = X$, or equivalently, if there is a bounded projection on X having range equal to Y. We write $X = Y \oplus Z$ for this situation; occasionally we say that X is the (internal) *direct sum* of Y and Z. A closed subspace of X with finite dimension or codimension is complemented.

Let H be a Hilbert space. The *Cauchy–Schwarz inequality* states that for all $x, y \in H$, $|(x|y)| \leqslant ||x||\,||y||$. When H is separable, this is an easy consequence of *Hölder's inequality*, $\sum_{n=1}^{\infty}|\alpha_n\beta_n| \leqslant \big(\sum_{n=1}^{\infty}|\alpha_n|^2\big)^{\frac{1}{2}}\big(\sum_{n=1}^{\infty}|\beta_n|^2\big)^{\frac{1}{2}}$ for all square-summable sequences of scalars (α_n) and (β_n), although we shall sometimes blur the distinction between the two inequalities. Two elements $x, y \in H$ are *orthogonal* if $(x|y) = 0$; in this situation *Pythagoras' Theorem* states that $||x||^2 + ||y||^2 = ||x+y||^2$. Let K be a closed subspace of H. We write $K^{\perp} = \{x \in H : (x|y) = 0 \text{ for every } y \in K\}$ for the *orthogonal complement* of K in H. Importantly, we have $K \oplus K^{\perp} = H$. Let $x \in H$, and denote by P_K the projection with range K and kernel $K^{\perp}$, termed the *orthogonal projection* onto K. Then $||P_K(x)|| = \inf\{||x-y|| : y \in K^{\perp}\}$. Non-zero orthogonal projections have norm 1.

Let X and Y be vector spaces. The (external) *direct sum* of X and Y, written $X \oplus Y$, is the vector space $\{(x,y) : x \in X, y \in Y\}$ with pointwise operations. In the case where X and Y are Banach spaces we may equip $X \oplus Y$ with a continuum of complete norms. For $1 \leqslant p < \infty$ the *p-norm* is $||(x,y)||_p = (||x||_X^p + ||y||_Y^p)^{\frac{1}{p}}$ and the *max norm* is $||(x,y)||_\infty = \max\{||x||_X, ||y||_Y\}$. Since $(X \oplus Y, ||\cdot||_p)$ and $(X \oplus Y, ||\cdot||_q)$ are isomorphic Banach spaces for any $p, q \in [1,\infty]$, we only specify the value of p if clarity is required.

Let $T : X \to Y$ be a bounded operator between Banach spaces X and Y. Then T is:

(i) *finite rank* if the image of T is finite-dimensional;

(ii) *approximable* if there exists a sequence (T_n) of finite rank operators from X to Y such that $T_n \to T$ in the operator norm as $n \to \infty$;

(iii) *compact* if every bounded sequence (x_n) in X has a subsequence (x_{n_k}) such that (Tx_{n_k}) converges in Y;

(iv) *weakly compact* if every bounded sequence (x_n) in X has a subsequence (x_{n_k}) such that (Tx_{n_k}) converges weakly in Y.

Let $\mathscr{F}(X,Y), \mathscr{A}(X,Y), \mathscr{K}(X,Y)$ and $\mathscr{W}(X,Y)$ denote the sets of finite rank, approximable, compact, and weakly compact operators, respectively. These are subspaces of $\mathscr{B}(X,Y)$, of which the latter three are closed. In the case where $X = Y$, $\mathscr{F}(X) = \mathscr{F}(X,X)$ is an ideal, and $\mathscr{A}(X), \mathscr{K}(X)$ and $\mathscr{W}(X)$ are closed ideals of $\mathscr{B}(X)$. A Banach space X has the *approximation property* (AP) if for each compact set $K \subseteq X$ and each $\varepsilon > 0$, there exists $T \in \mathscr{F}(X)$ such that $||Tx - x|| < \varepsilon$ for every $x \in K$, or equivalently, if $\mathscr{A}(Z,X) = \mathscr{K}(Z,X)$ for every Banach space Z. If there exists $C > 0$ (independent of K and ε) such that T can be chosen with $||T|| \leqslant C$, then X has the *bounded approximation property.*

A sequence $(x_n)_{n=1}^{\infty}$ in a Banach space X is a (Schauder) *basis* for X if for any $x \in X$ there is a unique sequence of scalars $(\alpha_n)_{n=1}^{\infty}$ such that $x = \sum_{n=1}^{\infty}\alpha_n x_n$. A sequence $(y_n)_{n=1}^{\infty}$ in X is a (Schauder) *basic sequence* if it is a Schauder basis for $\overline{\operatorname{span}}\,\{y_n : n \in \mathbb{N}\}$. Let X have a Schauder basis $(x_n)_{n=1}^{\infty}$. Then X is infinite-dimensional and separable. The basis is *normalised* if $||x_n|| = 1$ for every $n \in \mathbb{N}$. The sequence $(x_n/||x_n||)_{n=1}^\infty$ is a normalised basis for X. By a theorem of Banach, for each $m \in \mathbb{N}$, the projection $P_m : \sum_{n=1}^\infty \alpha_n x_n \mapsto \sum_{n=1}^m \alpha_n x_n$, $X \to X$, is bounded, and moreover, $\sup_{m\in\mathbb{N}} ||P_m|| < \infty$. If $\sup_{m\in\mathbb{N}} ||P_m|| = 1$ the basis (x_n) is *monotone*. For each $n \in \mathbb{N}$, the map $x_n^* : \sum_{j=1}^\infty \alpha_j x_j \mapsto \alpha_n$ for each $\sum_{j=1}^\infty \alpha_j x_j \in X$ is in X^*; such maps are called the *coordinate functionals*. The sequence $(x_n^*)_{n=1}^\infty$ of coordinate functionals is a Schauder basic sequence in X^*; if it forms a basis for X^* we say that (x_n) is *shrinking*. Let (p_n) be a sequence of natural numbers such that $1 = p_1 < p_2 < p_3 < \cdots$. For each $n \in \mathbb{N}$, let $\alpha_{p_n}, \dots, \alpha_{p_{n+1}-1}$ be a sequence of scalars, not all zero, and let $y_n = \sum_{j=p_n}^{p_{n+1}-1} \alpha_j x_j$. Then $(y_n)_{n=1}^\infty$ is a *block basic sequence* taken from (x_n).

Banach algebras will be denoted by A, B, C, D. Again $\mathbb{K}$ will denote the scalar field. Ideals of Banach algebras will usually be I or J, and two-sided unless stated otherwise. For a subset U of a Banach algebra A, we define the square of U to be $U^2 = \operatorname{span}\{ab : a, b \in U\}$. If U is an ideal of A, then U^2 is also an ideal of A.

The *unitisation* $\widetilde{A}$ of a Banach algebra A is the vector space $A \oplus \mathbb{K}$ equipped with the norm $||(a, \alpha)|| = ||a||_A + |\alpha|$, and the product $(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta)$ for $a, b \in A, \alpha, \beta \in \mathbb{K}$. This turns $\widetilde{A}$ into a unital Banach algebra containing A isometrically as an ideal of codimension one. We sometimes use the additive notation $a + \alpha 1_{\widetilde{A}}$ for (a, α). Let $\pi : A \to B$ be a continuous algebra homomorphism between Banach algebras A and B. Then the map

$$\widetilde{\pi} : (a, \alpha) \mapsto (\pi(a), \alpha), \qquad \widetilde{A} \to \widetilde{B}$$

is a continuous algebra homomorphism.

Let A and B be Banach algebras. A continuous bijective algebra homomorphism from A to B is an *isomorphism* and A and B are *isomorphic*, $A \cong B$. Again, the inverse is continuous by the Banach Isomorphism Theorem 1.2.2. A linear map $\varphi : A \to B$ is an *anti-homomorphism* if $\varphi(ab) = \varphi(b)\varphi(a)$ for each $a, b \in A$. A *character* on A is a (continuous) algebra homomorphism $\varphi : A \to \mathbb{K}$.

The vector space tensor product of A and B is denoted $A \otimes B$. The *projective norm* $|| \cdot ||_\pi$ on $A \otimes B$ is

$$||z||_\pi = \inf \left\{ \sum_{j=1}^n ||a_j||\,||b_j|| : z = \sum_{j=1}^n a_j \otimes b_j \right\}$$

for $z \in A \otimes B$, where the infimum is taken over all possible representations of z. The completion of $A \otimes B$ with respect to the projective norm is a Banach space, denoted by $A \widehat{\otimes} B$. Define a multiplication on simple tensors by $(a \otimes b)(c \otimes d) = ac \otimes bd$, and extend to the whole space. Then $A \widehat{\otimes} B$ is a Banach algebra with respect to this product, called the *projective tensor product* of A and B.

Let A be an algebra. The *Jacobson radical* of A, denoted $\operatorname{rad} A$, is defined as the intersection of all the maximal modular left ideals of A (see [24, §1.5] for details). In the case where A is unital we have

$$\operatorname{rad} A = \{a \in A : 1_A + ba \in \text{inv } A \text{ for all } b \in A\}$$

where inv A refers to the invertible elements of A. In general $\operatorname{rad} A$ is an ideal of A; when A is a Banach algebra, $\operatorname{rad} A$ is a closed ideal. If $\operatorname{rad} A = \{0\}$ then A is *semisimple*, while A is *radical* if $\operatorname{rad} A = A$.

A vector space Y is an A-*bimodule* if there are bilinear maps $(a, y) \mapsto a \cdot y$ and $(a, y) \mapsto y \cdot a$, $A \times Y \to Y$ such that $a \cdot (b \cdot y) = ab \cdot y$, $(y \cdot a) \cdot b = y \cdot ab$ and $a \cdot (y \cdot b) = (a \cdot y) \cdot b$ for each $a, b \in A$ and $y \in Y$. Suppose that Y is an A-bimodule. A *derivation* $\delta : A \to Y$ is a linear map such that $\delta(ab) = a \cdot \delta(b) + \delta(a) \cdot b$ for every $a, b \in A$. Suppose now that A is a Banach algebra. A Banach space Y is a *Banach* A-*bimodule* if it is an A-bimodule and there exists $C > 0$ satisfying $||a \cdot y|| \leqslant C||a|| \, ||y||$ and $||y \cdot a|| \leqslant C||y|| \, ||a||$ for all $y \in Y$ and $a, b \in A$. By passing to an equivalent norm on Y we can suppose that $C = 1$. Let Y and Z be Banach A-bimodules. A linear map $\varphi : Y \to Z$ is a *bimodule homomorphism* if $a \cdot \varphi(y) = \varphi(a \cdot y)$ and $\varphi(y) \cdot a = \varphi(y \cdot a)$ for every $y \in Y$ and $a \in A$. If there is a continuous bijective bimodule homomorphism $\varphi : Y \to Z$, then Y and Z are *isomorphic* as Banach A-bimodules.

We record some classical results which we shall use frequently. We refer to standard textbooks such as [3] and [78] for their proofs.

Theorem 1.2.1 (Open Mapping Theorem)**.** *Let X and Y be Banach spaces. The following are equivalent for a bounded operator $T : X \to Y$:*

(a) *T is surjective;*

(b) *T is an open mapping;*

(c) *there exists a constant $C > 0$ such that, for each $y \in Y$, there exists $x \in X$ with $||x|| \leqslant C||y||$ and $Tx = y$.*

Theorem 1.2.2 (Banach Isomorphism Theorem)**.** *Let X and Y be Banach spaces and let $T : X \to Y$ be a bounded bijective operator. Then $T^{-1} : Y \to X$ is a bounded operator.*

Theorem 1.2.3. *Let X and Y be Banach spaces, and let $T : X \to Y$ be a bounded operator. Then:*

(i) *(Schauder) T is compact if and only if T^* is compact;*

(ii) *(Gantmacher) T is weakly compact if and only if T^* is weakly compact if and only if $T^{**}(X^{**}) \subseteq Y$.*

Theorem 1.2.4 (Fundamental Isomorphism Theorem)**.** *Let X and Y be Banach spaces [Banach algebras], and let $T : X \to Y$ be a bounded operator [continuous algebra homomorphism]. Suppose that Z is a closed subspace [closed ideal] of* $\ker T$*, and let $Q_Z : X \to X/Z$ be the quotient map. Then there is a unique bounded operator [continuous algebra homomorphism] S such that the diagram*

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "Q_Z"'] & & Y \\ & X/Z \arrow[ur, "S"'] & \end{tikzcd}$$

is commutative. Moreover, if $\operatorname{im} T$ *is closed in Y then $X/\ker T$ is isomorphic to* $\operatorname{im} T$.

1.3 Introduction to Read's Banach space

The aim of this section is to introduce a Banach space construction of Read [90]. The study of Banach algebra theory has been profoundly shaped by the concept of *automatic continuity*; a central question of this theory is the following. Given a Banach algebra A, is it true that every derivation from A into a Banach A-bimodule is continuous?

Back in 1987, a semester on the theme of Banach algebras and automatic continuity was held at the University of Leeds, between March and July. One of the topics of the meeting was the continuity of derivations from $\mathscr{B}(X)$ into a Banach $\mathscr{B}(X)$-bimodule, where X is a Banach space. Johnson [59] had proved twenty years previously that, if X has a so-called continued bisection, then all derivations from $\mathscr{B}(X)$ are continuous; this covers most classical Banach spaces (as we shall see in Chapter 3, where this is discussed in detail). At the meeting the question was raised as to the situation for other Banach spaces. Two of the attendees in Leeds were R. J. Loy and G. A. Willis, who soon made progress on the continuity of derivations from $\mathscr{B}(J_2)$, where J_2 is the James space, and $\mathscr{B}(C[0, \omega_1])$, $C[0, \omega_1]$ being the continuous functions on the ordinal interval $[0, \omega_1]$ [76]. These spaces lack a continued bisection, but Loy and Willis showed that in fact all derivations from $\mathscr{B}(X)$ into a Banach $\mathscr{B}(X)$-bimodule are continuous for both spaces. Another meeting participant was Charles Read, who also set about trying to tackle the question, but with a different approach. His aim was to construct a new Banach space with a discontinuous derivation, rather than study existing spaces. His paper [90] shows that he was successful in this endeavour: there is a Banach space $E_{\mathcal{R}}$ and a discontinuous derivation from $\mathscr{B}(E_{\mathcal{R}})$ into the scalar field $\mathbb{K}$, where $\mathbb{K}$ can be made into a Banach $\mathscr{B}(E_{\mathcal{R}})$-bimodule ($E_{\mathcal{R}}$ is the symbol we shall adopt throughout for the space, although this was not Read's original notation; of course the letter $\mathcal{R}$ refers to Read). This remarkable result shows that we cannot extend Johnson's theorem to every Banach space, and since $E_{\mathcal{R}}$ is somewhat related to the James space, we cannot hope to extend it to a much wider class either.

Read's Banach space, $E_{\mathcal{R}}$, is ubiquitous in this thesis. We use it to consider uniqueness-of-norm questions in Chapter 2, to answer questions about extensions of Banach algebras in Chapter 3, to prove properties of the homological bidimension of $\mathscr{B}(X)$ in Chapter 4, and to examine a certain operator ideal in Chapter 6. To do this we shall need to understand the space well, and to extend some of the technical results in Read's paper.

The purpose of the current section is to present Read's main results, and expand on them—thereby we lay the foundations for the rest of the thesis. The proofs, however, will be deferred to Chapter 5. The reason for this is their rather technical nature. We felt it better to expound the new applications of the space before digging into the details later. Those who are more interested in the applications will be able to understand them fully having grasped the results in this section, although of course one would need to take our main original theorem on faith.

The technicalities should not put the reader off Chapter 5. The construction of the space $E_{\mathcal{R}}$ is quite involved, delicate, and certainly ingenious, but uses mostly notions that would be familiar after a graduate course in Banach space theory; deep results are generally not needed. In Chapter 5 we give a comprehensive account, filling in details that were omitted in [90], and presenting the key results in a more elementary way. Therefore the diligent reader will be well-rewarded by examining the details.

Despite the surprising nature of the space $E_{\mathcal{R}}$, it seems to have had little attention since its publication. Citations of Read's paper (for example [31], [45] and [76]) usually make note of it as a clever counterexample, but say little more, and do not appear to use the properties of the space beyond the discontinuous derivation. We believe this thesis contains the first expository account and the first new results concerning the space itself since the original paper. The author hopes that a thorough presentation of the construction will lead to a little more recognition of the brilliance of Read's work.

Note that the space $E_{\mathcal{R}}$ should not be confused with any of Read's other Banach space constructions, usually answering similarly difficult questions via a clever counterexample. One thinks of his example of a space with exactly two symmetric bases [87], a solution to the invariant subspace problem for Banach spaces in [88], and his example of a strictly singular operator with no non-trivial invariant subspaces [92], amongst many others. The space from [92] is loosely related to $E_{\mathcal{R}}$, as hopefully will become clear in Chapter 5. Looking at Read's other examples does give a nice insight into his pattern of thought, but we shall not be concerned with most of them here.

The Main Properties of Read's Space

How does one go about finding a discontinuous derivation from a given Banach algebra? Handily, there is a general elementary result, which seems to be folklore, that gives conditions for a unital Banach algebra to admit a discontinuous derivation. This is Read's starting point [90, Theorem 1].

Theorem 1.3.1. *Let A be a unital Banach algebra. Suppose that A contains a closed ideal I of codimension one such that I^2 has infinite codimension in A. Then there is a character $\varphi : A \to \mathbb{K}$ such that $\mathbb{K}$ is a Banach A-bimodule with respect to the maps*

$$\lambda \cdot a = \lambda\varphi(a) = a \cdot \lambda \qquad (\lambda \in \mathbb{K}, a \in A),$$

and there is a discontinuous derivation $\delta : A \to \mathbb{K}$. □

The discontinuous derivation produced by Theorem 1.3.1 is actually a *point derivation* at φ, meaning that $\delta(ab) = \varphi(a)\delta(b) + \delta(a)\varphi(b)$ for all $a, b \in A$. Point derivations play an important role in the general theory of derivations (see, *e.g.*, [24, §1.8]).

After considerable work, as we shall see in Chapter 5, Read obtained the following result [90, p. 306]. Note that although Read states his results for complex scalars only, they carry over verbatim to the real case.

Theorem 1.3.2 (Read)**.** *There exists a Banach space $E_{\mathcal{R}}$ such that $\mathscr{B}(E_{\mathcal{R}})$ has a closed ideal I of codimension one, and*

(i) *$\mathscr{W}(E_{\mathcal{R}}) \subseteq I \subseteq \mathscr{B}(E_{\mathcal{R}})$;*

(ii) *$\mathscr{B}(E_{\mathcal{R}})/\mathscr{W}(E_{\mathcal{R}})$ is infinite-dimensional;*

(iii) *$\overline{I^2} \subseteq \mathscr{W}(E_{\mathcal{R}})$.*

In particular, $\mathscr{B}(E_{\mathcal{R}})$ satisfies the conditions of Theorem 1.3.1, and so there is a discontinuous derivation $\delta : \mathscr{B}(E_{\mathcal{R}}) \to \mathbb{K}$.

We can actually say a little more than this, as our next well-known lemma explains (*cf.* [90, p. 305]).

Lemma 1.3.3. *Let A be a Banach algebra. Suppose that there is a discontinuous derivation $\delta : A \to Y$ into a Banach A-bimodule Y. Then there is a discontinuous algebra homomorphism from A into a Banach algebra B.*

Proof. Consider the Banach space $B = A \oplus Y$ with the norm $||(a,y)|| = ||a||_A + ||y||_Y$ for $a \in A$ and $y \in Y$. Define a product on B by

$$(a,y)(b,z) = (ab, a \cdot z + y \cdot b) \qquad (a, b \in A,\ y, z \in Y).$$

Then B is a Banach algebra, and the map $\theta : A \to B$ given by $\theta(a) = (a, \delta(a))$ is a discontinuous algebra homomorphism. $\square$

The converse is not true in general. Contrapositively, the lemma tells us that if every algebra homomorphism from A into a Banach algebra is continuous, then every derivation from A into a Banach A-bimodule is also continuous. Lemma 1.3.3 implies that there is a discontinuous homomorphism from $\mathscr{B}(E_{\mathcal{R}})$ into a Banach algebra, an interesting result in its own right, as no Banach space with this property was known until $E_{\mathcal{R}}$.

Read's paper left open the question of whether there is a Banach space X such that there is a discontinuous homomorphism from $\mathscr{B}(X)$, but all derivations are continuous (as he notes on [90, p. 305]). Dales, Loy and Willis produced a clever answer to this question in [28], where they construct such a space. The key property of their space E_{DLW} is that $\mathscr{B}(E_{\mathrm{DLW}})$ admits a quotient isomorphic to ℓ_∞, and thus has a discontinuous homomorphism into a Banach algebra (assuming the Continuum Hypothesis, see [28]). We shall make use of the Dales–Loy–Willis space in Chapter 4.

What type of space is $E_{\mathcal{R}}$? The fundamental building blocks of the space are James-type spaces. The classical James space J_2 was constructed in [55] (a modern presentation can be found in [78]). Edelstein and Mityagin [34] were the first to observe that the weakly compact operators on J_2 have codimension 1 in the bounded operators, that is, $\mathscr{B}(J_2) = \mathscr{W}(J_2) \oplus \mathbb{K}I_{J_2}$, where I_{J_2} denotes the identity operator. Looking at Theorem 1.3.2(ii), we would like $\mathscr{B}(X)/\mathscr{W}(X)$ to be infinite-dimensional. So suppose we let X be an infinite direct sum of James spaces, then since $\mathscr{B}(J_2) = \mathscr{W}(J_2) \oplus \mathbb{K}I_{J_2}$, we observe that $\mathscr{B}(X)/\mathscr{W}(X)$ is indeed infinite-dimensional (this can be seen by considering operators as infinite matrices). This is the basic intuition behind the first steps in Read's construction: James-type spaces are good building blocks. However, to obtain the codimension-one ideal I we need more complicated spaces than J_2, and we also need the direct sum to consist of a sequence of *different* spaces chosen in the right way. Once the direct sum is obtained, the remaining step is to take a quotient which makes the coordinates wrap around in a complicated fashion. All will be explained in detail in Chapter 5. The reader whose appetite has been whetted may now safely look ahead to that chapter, as it does not depend on the intermediate ones.

Thus far we claim no originality. We conclude this introductory chapter with our first major result, proved jointly with N.J. Laustsen, although we defer the proof until p. 115. The statement requires the notion of an extension of a Banach algebra.

Let B be a Banach algebra. An *extension* of B is a short exact sequence of Banach algebras and continuous algebra homomorphisms:

$$\{0\} \longrightarrow \ker\pi \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow \{0\}\,. \tag{1.3.1}$$

An extension of the form (1.3.1) *splits strongly* if there is a continuous algebra homomorphism $\theta : B \to A$ such that $\pi \circ \theta = \mathrm{id}_B$.

Consider the separable Hilbert space $\ell_2(\mathbb{N})$. Equip it with the *trivial product*, that is, define the multiplication to be $xy = 0$ for every $x, y \in \ell_2(\mathbb{N})$. Then $\ell_2(\mathbb{N})$ is trivially a Banach algebra, and so we can form its unitisation $\ell_2(\mathbb{N})^\sim$. It turns out that this unusual Banach algebra can be realised as a quotient of $\mathscr{B}(E_{\mathcal{R}})$.

Theorem 1.3.4. *There exists a continuous, unital, surjective algebra homomorphism β from $\mathscr{B}(E_{\mathcal{R}})$ onto $\ell_2(\mathbb{N})^\sim$, with* $\ker\beta = \mathscr{W}(E_{\mathcal{R}})$*, such that the extension*

$$\{0\} \longrightarrow \mathscr{W}(E_{\mathcal{R}}) \xrightarrow{\iota} \mathscr{B}(E_{\mathcal{R}}) \xrightarrow{\beta} \ell_2(\mathbb{N})^\sim \longrightarrow \{0\}$$

splits strongly.

The importance of this result is that when trying to prove certain properties about $\mathscr{B}(E_{\mathcal{R}})$, it can be enough to prove them for $\ell_2(\mathbb{N})^\sim$. Since $\ell_2(\mathbb{N})^\sim$ has an unusual algebra structure, this gives some unexpected results for $\mathscr{B}(E_{\mathcal{R}})$. This is our approach when considering splittings of extensions and homological bidimension of $\mathscr{B}(E_{\mathcal{R}})$ in Chapters 3 and 4.

Chapter 2

Uniqueness-of-norm for Calkin Algebras

In this chapter we shall examine the question of whether the Calkin algebra of a Banach space must have a unique complete algebra norm. We present a survey of known results, and make the observation that a recent Banach space construction of Argyros and Motakis provides the first negative answer. The parallel question for the weak Calkin algebra also has a negative answer; we demonstrate this using Read's space $E_{\mathcal{R}}$.

The inspiration for these results came from Theorem 1.3.4, which says that the weak Calkin algebra of $E_{\mathcal{R}}$ is isomorphic to $\ell_2(\mathbb{N})^{\sim}$. It follows quickly from this that $\mathscr{B}(E_{\mathcal{R}})/\mathscr{W}(E_{\mathcal{R}})$ does not have a unique complete norm. In fact, as we shall see below, one does not need the power of Theorem 1.3.4, because it is also a direct consequence of Read's Theorem 1.3.2. This begged the question of whether the Calkin algebra of $E_{\mathcal{R}}$ also lacks a unique complete norm. If true, this would provide an interesting counterexample to a long-open question. Despite considerable effort, the author made little progress on the problem for $E_{\mathcal{R}}$, and it is still unknown whether $\mathscr{B}(E_{\mathcal{R}})/\mathscr{K}(E_{\mathcal{R}})$ has a unique complete norm. It was not until reading a 2015 preprint of Argyros and Motakis [8] that the answer to the general question became clear. In [8] the authors construct a Banach space X_{AM} such that $\mathscr{B}(X_{\mathrm{AM}})$ has many properties in common with $\mathscr{B}(E_{\mathcal{R}})$—so much so that there is a discontinuous derivation from $\mathscr{B}(X_{\mathrm{AM}})$ (see Corollary 2.2.6). The difference is that the compact operators on X_{AM} play the role that the weakly compact operators do for $E_{\mathcal{R}}$; this led to the proof that the Calkin algebra of X_{AM} lacks a unique complete norm in exactly the same way as for $\mathscr{B}(E_{\mathcal{R}})/\mathscr{W}(E_{\mathcal{R}})$. In a sense this was entirely coincidental; the author doubts that Argyros and Motakis had Read's example in mind. Indeed, the constructions of the two spaces could not be more different, as can be seen by a cursory glance at [8] and Chapter 5. For the sake of brevity we have chosen to omit details of Argyros and Motakis' work and simply state their main results.

This chapter is organised as follows: the first section provides a history of the problem, and the second gives an account of the claimed counterexamples. Most of the chapter has been published as the paper [101].

2.1 History of the problem

Uniqueness-of-norm questions for Banach algebras have been studied for almost as long as Banach algebras themselves. Eidelheit was the first to publish on the topic; his 1940 paper [35] showed that the Banach algebra of bounded operators on a Banach space has a unique complete norm, and prompted the natural question of which other Banach algebras share this property. Gel'fand [40] quickly followed in 1941 with a similar result in the commutative case: his famous proof that commutative, semisimple Banach algebras have a unique complete norm. It was Rickart [94] who sought to tie the two ideas together in the 1950s by focusing on the problem of whether a (not necessarily commutative) semisimple Banach algebra always has a unique complete norm; this was solved positively by Johnson in 1967 [56], and his result remains the major achievement in the area. An attractive short proof of Johnson's result was given by Ransford; full details and further history can be found in [24, §5.1]. Beyond the semisimple setting (that is, once the Jacobson radical is non-zero), things are much less clear. Dales and Loy [27] developed theory to handle certain cases when the Jacobson radical is finite-dimensional, however, they noted that even when the radical is one-dimensional, a Banach algebra may lack a unique complete norm. Despite the considerable amount of work done on the uniqueness-of-norm problem, much remains unknown. In this chapter we address the question for two particular classes of Banach algebras: the Calkin algebra and weak Calkin algebra of a Banach space. Yood observed that in general a Calkin algebra need not be semisimple [113], and the same holds true for weak Calkin algebras, so the question has no immediate answer. As we shall see, there is a good reason for this.

It is time to be precise about some definitions.

Definition 2.1.1. A Banach algebra $(A, ||\cdot||)$ has a *unique complete norm* if any other complete algebra norm on A is equivalent to $||\cdot||$.

Recall that two norms $||\cdot||$ and $|||\cdot|||$ on a vector space A are *equivalent* if there exist constants $c, C > 0$ satisfying $c||a|| \leqslant |||a||| \leqslant C||a||$ for every $a \in A$. In the case where $||\cdot||$ and $|||\cdot|||$ are complete, it is enough, by the Banach Isomorphism Theorem 1.2.2, that one of these inequalities holds. An obvious fact we use throughout is that if A and B are isomorphic Banach algebras and A has a unique complete norm, then so does B.

Definition 2.1.2. Let X be a Banach space. The *Calkin algebra* of X is the Banach algebra $\mathscr{B}(X)/\mathscr{K}(X)$, and the *weak Calkin algebra* of X is the Banach algebra $\mathscr{B}(X)/\mathscr{W}(X)$.

Let us be clear about the questions we are considering.

1. Given a Banach space, must its Calkin algebra have a unique complete norm?
2. Must its weak Calkin algebra have a unique complete norm?

We refer to this as the *uniqueness-of-norm problem* for Calkin algebras. The questions have their roots in Calkin's study [18] of the Banach algebra $\mathscr{B}(H)/\mathscr{K}(H)$, where H is a separable Hilbert space. He showed that there are no proper, non-trivial closed ideals in $\mathscr{B}(H)/\mathscr{K}(H)$; this implies that $\mathscr{B}(H)/\mathscr{K}(H)$ has a unique complete norm since it is (semi-)simple. Once Yood had observed that there are non-semisimple Calkin algebras, Kleinecke defined the ideal of *inessential operators* on a Banach space X, denoted $\mathscr{E}(X)$, as a 'measure of non-semisimplicity' [65]. More specifically,

$$\mathscr{E}(X) = \{T \in \mathscr{B}(X) : T + \mathscr{K}(X) \in \operatorname{rad} \mathscr{B}(X)/\mathscr{K}(X)\},$$

and so $\mathscr{E}(X) = \mathscr{K}(X)$ if and only if the Calkin algebra of X is semisimple. The questions have also been considered indirectly by Johnson [57], Tylli [106] and Ware [107], in the context of wider work.

In the remainder of this section we give an overview of some known results. The author is not aware of any other survey of the same material, and it seemed helpful to draw together the scattered literature. We begin with a large class of Banach spaces for which the problem is quickly solved, as our first proposition, which is due to Johnson [57], shows. We give a proof to demonstrate some of the ideas under consideration.

Proposition 2.1.3. *Let X be a Banach space such that $X \simeq X \oplus X$. Then $\mathscr{B}(X)/\mathscr{K}(X)$ and $\mathscr{B}(X)/\mathscr{W}(X)$ have a unique complete norm.*

Proof. Let I be a closed ideal of $\mathscr{B}(X)$ and consider a complete algebra norm $|||\cdot|||$ on $\mathscr{B}(X)/I$ different from the quotient norm $||\cdot||$. Since $X \simeq X \oplus X$, a theorem of Johnson [57, Theorem 3.3; see remarks after Theorem 3.5] shows that every algebra homomorphism from $(\mathscr{B}(X)/I, ||\cdot||)$ into a Banach algebra is continuous (see also Chapter 3 and [24, Corollary 5.4.12]). In particular, the identity map

$$\iota : (\mathscr{B}(X)/I, ||\cdot||) \to (\mathscr{B}(X)/I, |||\cdot|||)$$

is continuous. So there exists $C > 0$ such that $|||b + I||| \leqslant C||b + I||$ for every $b \in \mathscr{B}(X)$, and hence the two norms on $\mathscr{B}(X)/I$ are equivalent by the Banach Isomorphism Theorem 1.2.2. We conclude by setting $I = \mathscr{K}(X)$ or $I = \mathscr{W}(X)$. $\square$

This result shows that a Calkin algebra may have a unique complete norm even if it is not semisimple. An example is $X = L_1[0, 1]$, because $X \simeq X \oplus X$, but $\mathscr{K}(X) \subsetneq \mathscr{E}(X)$. In fact this was Yood's original example [113].

The following Banach spaces have a Calkin algebra with a unique complete norm. The purpose of the list is not to give a comprehensive account, but rather a flavour of the wide variety of spaces sharing the property. Examples (ii)–(viii) follow because $X \simeq X \oplus X$.

(i) Any finite-dimensional Banach space.

(ii) ℓ_p for $1 \leqslant p \leqslant \infty$ and c_0. This generalises to $\ell_p(\mathbb{I})$ for $1 \leqslant p \leqslant \infty$ and $c_0(\mathbb{I})$ for any index set $\mathbb{I}$.

(iii) $\ell_p \oplus \ell_q$ for $1 \leqslant p < q \leqslant \infty$, and $c_0 \oplus \ell_q$ for $1 \leqslant q \leqslant \infty$. We may generalise this to $X \oplus Y$ where X, Y are Banach spaces such that $X \simeq X \oplus X$ and $Y \simeq Y \oplus Y$.

(iv) $(\bigoplus_{n=1}^{\infty} \ell_r^n)_{\ell_p}$ and $(\bigoplus_{n=1}^{\infty} \ell_r^n)_{c_0}$ for $p, r \in [1, \infty]$, where ℓ_r^n means $\mathbb{K}^n$ endowed with the r-norm.

(v) $C(K)$ for K an infinite compact metric space.

(vi) $L_p[0, 1]$ for $1 \leqslant p \leqslant \infty$.

(vii) $C^{(n)}[0, 1]$, the n-times continuously differentiable functions on $[0, 1]$, for all $n \in \mathbb{N}$.

(viii) Tsirelson's space T, and its dual T^* (see *e.g.*, [3, Exercise 10.6]).

(ix) J_p for $1 < p < \infty$, the pth James space. For $p = 2$ this follows from a result of Loy and Willis [76, Theorem 2.7], combined with [98, Remark 3.9] of Saksman and Tylli, while Laustsen [68, Proposition 4.9] showed later that $\mathscr{E}(J_p) = \mathscr{K}(J_p)$ for general $1 < p < \infty$.

(x) $C[0, \omega_1]$, where ω_1 denotes the first uncountable ordinal, because $\mathscr{K}(C[0, \omega_1]) = \mathscr{E}(C[0, \omega_1])$ (see *e.g.*, [63, Proposition 5.3]).

(xi) X_{AH}, the Argyros–Haydon solution to the scalar-plus-compact problem [6]. In this remarkable case we have $\mathscr{B}(X_{\mathrm{AH}})/\mathscr{K}(X_{\mathrm{AH}}) \cong \mathbb{K}$.

(xii) X_k for $k \in \mathbb{N} \cup \{\infty\}, k \geqslant 2$, Tarbard's reworkings of the Argyros–Haydon construction [103], [104]. For $k \in \mathbb{N}, k \geqslant 2$ these spaces have finite-dimensional Calkin algebras. The space X_∞ has a Calkin algebra isometrically isomorphic to the Banach algebra $\ell_1(\mathbb{N}_0)$ with convolution product, which is semisimple [24, Theorem 4.6.9(i)].

(xiii) For every countable compact metric space M, Motakis, Puglisi and Zisimopoulou [80] have constructed a Banach space X_M with Calkin algebra isomorphic (as a Banach algebra) to $C(M)$. These Calkin algebras are semisimple and thus have a unique complete norm.

(xiv) X_{KL}, Kania and Laustsen's [64] variant of the Argyros–Haydon space whose Calkin algebra is 3-dimensional.

Remark 2.1.4. One can also ask whether a Banach algebra has a unique norm, dropping the completeness assumption (that is, all algebra norms are equivalent). This stronger property holds for the Calkin algebra on certain Banach spaces. Indeed, Meyer [79] has shown that the Calkin algebras of ℓ_p for $1 \leqslant p < \infty$ and c_0 have a unique norm, and his results were substantially extended by Ware [107] to cases (iii) (excluding $q = \infty$, and the generalisation), (iv) (excluding $p = \infty$), (viii), and (ix), listed above. Ware's well-written and carefully referenced thesis [107] is an excellent source to find out more about the topic; his perspective differs from ours in that he hardly considers uniqueness of *complete* norms, instead focusing on the more stringent unique-norm condition. Zemánek [114] noticed that a result of Astala and Tylli provided the first example of a Calkin algebra with a non-unique norm; later Tylli [105, Remark 1] gave a related result showing the same is possible for the weak Calkin algebra. Ware [107, Example 3.2.8] demonstrated two more examples of Calkin algebras with a non-unique norm: one is the Tarbard space X_∞ in (xii), and the other is Read's space $E_{\mathcal{R}}$. However, in each of these cases, the norm that is not equivalent to the usual quotient norm fails to be complete, so no light is shed on our problem.

Let us quickly see why the Calkin algebra of $E_{\mathcal{R}}$ has a non-unique norm. First, we state a general lemma (*cf.* [24, Proposition 2.1.7]).

Lemma 2.1.5. *Let $(A, ||\cdot||)$ be a Banach algebra such that there is a discontinuous algebra homomorphism from A into a Banach algebra. Then A does not have a unique norm.*

Proof. Let $\theta : A \to B$ be a discontinuous algebra homomorphism into a Banach algebra $(B, ||\cdot||_B)$. Define a new algebra norm on A by

$$|||a||| = \max\{||a||, ||\theta(a)||_B\} \qquad (a \in A).$$

Then, because θ is unbounded, $||\cdot||$ and $|||\cdot|||$ are non-equivalent algebra norms on A. $\square$

Corollary 2.1.6. *The Calkin algebra $\mathscr{B}(E_{\mathcal{R}})/\mathscr{K}(E_{\mathcal{R}})$ does not have a unique norm.*

Proof. The Calkin algebra $\mathscr{B}(E_{\mathcal{R}})/\mathscr{K}(E_{\mathcal{R}})$ satisfies the conditions of Theorem 1.3.1 because $\mathscr{B}(E_{\mathcal{R}})$ does. Indeed, let I denote the ideal of codimension one in $\mathscr{B}(E_{\mathcal{R}})$ from Theorem 1.3.1. Then $I/\mathscr{K}(E_{\mathcal{R}})$ is a closed ideal of codimension one in the Calkin algebra, and $(I/\mathscr{K}(E_{\mathcal{R}}))^2 = I^2/\mathscr{K}(E_{\mathcal{R}})$ has infinite codimension. Hence Theorem 1.3.1 and Lemma 1.3.3 guarantee the existence of a discontinuous homomorphism $\theta : \mathscr{B}(E_{\mathcal{R}})/\mathscr{K}(E_{\mathcal{R}}) \to B$ into a Banach algebra B. The result follows from Lemma 2.1.5. $\square$

Remark 2.1.7. We observe that the norm $|||\cdot|||$ from Lemma 2.1.5 is not complete. Indeed, assume contrapositively that $|||\cdot|||$ is complete. Then for every $a \in A$ we have $||a|| \leqslant |||a|||$, and so by the Banach Isomorphism Theorem 1.2.2, $||\cdot||$ and $|||\cdot|||$ are equivalent. We also remark that the proof of Corollary 2.1.6 shows that $\mathscr{B}(E_{\mathcal{R}})$ lacks a unique norm.

The following Banach spaces have a weak Calkin algebra with a unique complete norm.

(i) Any Banach space X such that $X \simeq X \oplus X$, and any finite-dimensional Banach space; in particular, (i)–(viii) from the previous list.

(ii) Any reflexive Banach space, since $\mathscr{B}(X)/\mathscr{W}(X) = \{0\}$.

(iii) J_p for $1 < p < \infty$, due to the fact that $\mathscr{B}(J_p)/\mathscr{W}(J_p) \cong \mathbb{K}$ [34, p. 225].

(iv) $J_p \oplus J_q$ for $1 < p \leqslant q < \infty$. This follows by considering the operators on $J_p \oplus J_q$ as operator-valued 2×2 matrices (see Chapter 6 for the precise correspondence), and noting Loy and Willis' result [76, Theorem 4.5] that for $1 < p < q < \infty$, $\mathscr{B}(J_q, J_p) = \mathscr{K}(J_q, J_p)$ and $\mathscr{B}(J_p, J_q) = \mathscr{W}(J_p, J_q) \oplus \mathbb{K}I_{p,q}$, where $I_{p,q}$ denotes the formal inclusion. More generally, this holds for finite direct sums of James spaces.

(v) $C[0, \omega_1]$, X_{AH}, X_k for $k \in \mathbb{N} \cup \{\infty\}, k \geqslant 2$, and X_{KL}. These Banach spaces are each preduals of $\ell_1(\mathbb{I})$ for some index set $\mathbb{I}$, which has the Schur property. Hence, by Theorem 1.2.3, the weakly compact operators coincide with the compacts.

2.2 Calkin algebras with non-equivalent complete norms

Our first lemma is well known, but, nevertheless, we give a proof. It shows that one condition for a Banach algebra to lack a unique complete norm is to lack any sort of algebra structure at all.

Lemma 2.2.1. *Let $(A, ||\cdot||_A)$ be an infinite-dimensional Banach algebra such that $ab = 0$ for every $a, b \in A$. Then $\widetilde{A}$ does not have a unique complete norm.*

Proof. Since A is infinite-dimensional, there is a discontinuous linear bijection $\gamma : A \to A$. To see this, take a Hamel basis $(b_j)_{j\in J}$ for A (J some uncountable index set). Since J is uncountable, we may write it as the disjoint union of an infinite sequence $(j_n)_{n\in\mathbb{N}}$ and an uncountable subset J_0. Define $\gamma(b_{j_n}) = nb_{j_n}$ for $n \in \mathbb{N}$ and $\gamma(b_j) = b_j$ for $j \in J_0$, and extend by linearity. Then γ is a bijection because it maps a Hamel basis onto a Hamel basis, and it is unbounded. Define a new norm on $\widetilde{A}$ by

$$|||(a, \alpha)||| = ||\gamma(a)||_A + |\alpha|$$

for each (a, α) in $\widetilde{A}$. This is easily checked to be an algebra norm, since γ is a linear bijection. Let $((a_n, \alpha_n))_{n=1}^{\infty}$ be Cauchy in $(\widetilde{A}, |||\cdot|||)$. Then for each $\varepsilon > 0$ there exists n_0 such that

$$|||(a_n, \alpha_n) - (a_m, \alpha_m)||| = ||\gamma(a_n - a_m)||_A + |\alpha_n - \alpha_m| < \varepsilon$$

for all $n, m \geqslant n_0$. So $(\alpha_n)_{n=1}^{\infty}$ is Cauchy in $\mathbb{K}$ and $(\gamma(a_n))_{n=1}^{\infty}$ is Cauchy in $(A, ||\cdot||_A)$. Since $(A, ||\cdot||_A)$ and $\mathbb{K}$ are complete there exist $\alpha \in \mathbb{K}$ and $a \in A$ which are the respective limits. Then because γ is a bijection we have

$$|||(a_n, \alpha_n) - (\gamma^{-1}(a), \alpha)||| = ||\gamma(a_n - \gamma^{-1}(a))||_A + |\alpha_n - \alpha| \to 0$$

as $n \to \infty$. Since $(\gamma^{-1}(a), \alpha)$ is in $\widetilde{A}$ we conclude that $\widetilde{A}$ is complete with respect to $|||\cdot|||$. Now suppose that there exists $C > 0$ such that $|||(a, \alpha)||| \leqslant C||(a, \alpha)||$ for every $(a, \alpha) \in \widetilde{A}$. Then $|||(a, 0)||| \leqslant C||(a, 0)||$ for every $a \in A$, and so $||\gamma(a)||_A \leqslant C||a||_A$ for each $a \in A$; but γ is unbounded, a contradiction. Therefore $|||\cdot|||$ is not equivalent to $||\cdot||$ on $\widetilde{A}$. □

Recall that we refer to a Banach algebra A such that $ab = 0$ for every $a, b \in A$ as having the *trivial product*. This lemma may seem a small observation, but it can be applied to certain quotient algebras quite effectively. Firstly, let us see that there is a Banach space with a Calkin algebra lacking a unique complete norm. Before we state the result we require one more piece of terminology. A bounded operator on a Banach space X is *strictly singular* if it is not bounded below on any infinite-dimensional subspace. We write $\mathscr{S}(X)$ for the closed ideal of $\mathscr{B}(X)$ consisting of strictly singular operators, and record the standard fact that $\mathscr{K}(X) \subseteq \mathscr{S}(X)$. Argyros and Motakis [8, Theorem A] have recently constructed a Banach space with the following remarkable properties (the result for complex scalars is noted in the comments after [8, Theorem B]).

Theorem 2.2.2 (Argyros–Motakis)**.** *There exists a Banach space X_{AM} such that:*

(i) *X_{AM} is reflexive and has a Schauder basis;*

(ii) *$\mathscr{B}(X_{\mathrm{AM}}) = \mathscr{S}(X_{\mathrm{AM}}) \oplus \mathbb{K}I_{X_{\mathrm{AM}}}$ ($I_{X_{\mathrm{AM}}}$ denotes the identity operator);*

(iii) *the composition of any two strictly singular operators on X_{AM} is compact;*

(iv) *$\mathscr{S}(X_{\mathrm{AM}})$ is non-separable.*

Theorem 2.2.3. *The Calkin algebra $\mathscr{B}(X_{\mathrm{AM}})/\mathscr{K}(X_{\mathrm{AM}})$ of the space of Argyros and Motakis has at least two non-equivalent complete algebra norms.*

Proof. Observe that by Theorem 2.2.2(ii) we have

$$\mathscr{B}(X_{\mathrm{AM}})/\mathscr{K}(X_{\mathrm{AM}}) = \mathscr{S}(X_{\mathrm{AM}})/\mathscr{K}(X_{\mathrm{AM}}) \oplus \mathbb{K}(I_{X_{\mathrm{AM}}} + \mathscr{K}(X_{\mathrm{AM}}))$$

and so we can identify the Calkin algebra of X_{AM} with the unitisation of the Banach algebra $\mathscr{S}(X_{\mathrm{AM}})/\mathscr{K}(X_{\mathrm{AM}})$. We shall show that $\mathscr{S}(X_{\mathrm{AM}})/\mathscr{K}(X_{\mathrm{AM}})$ is infinite-dimensional and has the trivial product, and then implement Lemma 2.2.1. Suppose towards a contradiction that $\mathscr{S}(X_{\mathrm{AM}})/\mathscr{K}(X_{\mathrm{AM}})$ is finite-dimensional; then it is separable. By Theorem 2.2.2(i), X_{AM} is separable (since it has a basis) and reflexive, so X_{AM}^* is separable because its dual is separable [78, Corollary 1.12.12]. Take countable dense subsets M and N of X_{AM} and X_{AM}^*, respectively. Then it is a general fact (and easy to check) that $\operatorname{span}\{x \otimes f : x \in M, f \in N\}$ is dense in $\mathscr{F}(X_{\mathrm{AM}})$. Since X_{AM} has a basis, $\mathscr{F}(X_{\mathrm{AM}})$ is dense in $\mathscr{K}(X_{\mathrm{AM}})$, and so in fact $\operatorname{span}\{x \otimes f : x \in M, f \in N\}$ is dense in $\mathscr{K}(X_{\mathrm{AM}})$. Therefore $\mathscr{K}(X_{\mathrm{AM}})$ is separable. But the fact that separability is a three-space property [78, Corollary 1.12.10] means that $\mathscr{S}(X_{\mathrm{AM}})$ must also be separable, contradicting Theorem 2.2.2(iv). Therefore $\mathscr{S}(X_{\mathrm{AM}})/\mathscr{K}(X_{\mathrm{AM}})$ is infinite-dimensional. The fact that it has the trivial product follows from Theorem 2.2.2(iii), and so we may apply Lemma 2.2.1 to obtain the result. □

Corollary 2.2.4. *The Calkin algebra $\mathscr{B}(X_{\mathrm{AM}}^*)/\mathscr{K}(X_{\mathrm{AM}}^*)$ has at least two non-equivalent complete algebra norms.*

Proof. (*cf.* [107, Theorem 1.2.2]) By Theorem 2.2.2(i), X_{AM} is reflexive, and so the map

$$T + \mathscr{K}(X_{\mathrm{AM}}) \mapsto T^* + \mathscr{K}(X_{\mathrm{AM}}^*), \quad \mathscr{B}(X_{\mathrm{AM}})/\mathscr{K}(X_{\mathrm{AM}}) \to \mathscr{B}(X_{\mathrm{AM}}^*)/\mathscr{K}(X_{\mathrm{AM}}^*)$$

is an isometric anti-isomorphism by Schauder's Theorem 1.2.3(i). Theorem 2.2.3 allows us to choose two non-equivalent complete algebra norms on $\mathscr{B}(X_{\mathrm{AM}})/\mathscr{K}(X_{\mathrm{AM}})$, and these pass to non-equivalent complete algebra norms on $\mathscr{B}(X_{\mathrm{AM}}^*)/\mathscr{K}(X_{\mathrm{AM}}^*)$ using the anti-isomorphism. $\square$

Remark 2.2.5. With a little more work, one can see that there are actually *uncountably many* non-equivalent complete algebra norms on these Calkin algebras. To justify this, we refine Lemma 2.2.1 as follows. Consider an infinite-dimensional Banach algebra $(A, ||\cdot||)$ with the trivial product, and normalised Hamel basis $(b_j)_{j\in J}$. Since A is infinite-dimensional, J is necessarily uncountable [78, Theorem 1.5.8]. We may write $J = \bigcup_{\alpha\in\Gamma} J_\alpha$, where Γ is an index set such that $|\Gamma| = |J|$, J_α is countable for each $\alpha \in \Gamma$, and, importantly, $J_\alpha \cap J_\beta = \varnothing$ for each β distinct from α.

Fix $\alpha \in \Gamma$ and write $J_\alpha = \{j_{\alpha,n} : n \in \mathbb{N}\}$. Define $\gamma_\alpha : A \to A$ by $\gamma_\alpha(b_j) = nb_{j_{\alpha,n}}$ if $j = j_{\alpha,n} \in J_\alpha$ and $\gamma_\alpha(b_j) = b_j$ if $j \notin J_\alpha$, and extend by linearity. Then the norm $||a||_\alpha = ||\gamma_\alpha(a)||$ is a complete algebra norm on A not equivalent to $||\cdot||$. Take distinct α and β in Γ, and assume that there exists $C > 0$ such that $||a||_\alpha \leqslant C||a||_\beta$ for each $a \in A$. Then for each $n \in \mathbb{N}$, $n = ||\gamma_\alpha(b_{j_{\alpha,n}})|| = ||b_{j_{\alpha,n}}||_\alpha \leqslant C||b_{j_{\alpha,n}}||_\beta = C||\gamma_\beta(b_{j_{\alpha,n}})|| = C$, a contradiction. Thus $||\cdot||_\alpha$ and $||\cdot||_\beta$ are non-equivalent. These norms pass to the unitisation $\widetilde{A}$, as in Lemma 2.2.1, and therefore apply to the Calkin algebras, above.

Argyros and Motakis' original motivation for constructing X_{AM} was not related to Calkin algebras, but to invariant subspaces. Let X be a Banach space. The *invariant subspace problem* for X asks whether, for every $T \in \mathscr{B}(X)$, there exists a non-trivial closed subspace Y of X such that $T[Y] \subseteq Y$ (such a subspace is called *T-invariant* or just *invariant*; non-trivial means that Y is not $\{0\}$ or X). This problem has a long and rich history, and has proved extremely difficult in general. Indeed, the problem is still open for separable Hilbert spaces (and even for separable reflexive spaces), and no solution appears forthcoming without radical new techniques. The problem has been solved in the negative for general Banach spaces. Enflo [36] was the first to find a counterexample in the late 1970s, that is, a Banach space X, and an operator on X which has no non-trivial invariant closed subspace. Enflo's operator is essentially a shift operator, but the Banach space on which it acts is of enormous complexity. Read was the first to actually publish a counterexample [88]. He soon adapted his profound new methods to find an operator on ℓ_1 with no non-trivial invariant closed subspace [89]. Here, in contrast to Enflo's example, the space is simple but the operator is complicated. Later work by Read [91] has shown that every separable Banach space containing either c_0 or a complemented copy of ℓ_1 admits an operator without a non-trivial invariant closed subspace.

The Argyros–Motakis space moves in the opposite direction to these counterexamples. Its predecessor, the Argyros–Haydon space X_{AH} [6], has the property that every operator on X_{AH} has a non-trivial invariant subspace, making it the first infinite-dimensional example with the *invariant subspace property* (that is, a Banach space with a positive answer to the invariant subspace problem). This follows because each operator on X_{AH} is scalar plus compact, and Aronszajn and Smith [9] have shown that compact operators always have invariant subspaces, which, of course, are also invariant subspaces for scalar multiples of the identity. The Argyros–Haydon space is not reflexive, which raised the question of whether a separable reflexive Banach space with the invariant subspace property could be constructed using similar techniques. This was done successfully by Argyros and Motakis in [7]; the space constructed there, which we denote by X_{ISP}, is not quite the same as X_{AM}. The key difference is that the composition of any *three* strictly singular operators on X_{ISP} is compact, whereas it is unknown if this is the case for the composition of two. The space X_{AM} is therefore a refinement of X_{ISP}; it is also separable, reflexive and has the invariant subspace property, but the composition of any two strictly singular operators is compact. The paper [8] also provides new general methods for constructing spaces of this type, showing more clearly why the methods of [7] worked.

From its construction, X_{AM} seems very different to $E_{\mathcal{R}}$. But in fact they share the surprising property of having a discontinuous derivation from their algebra of operators (recall that this was Read's motivation for the creation of $E_{\mathcal{R}}$).

Corollary 2.2.6. *There is a discontinuous point derivation $\delta : \mathscr{B}(X_{\mathrm{AM}}) \to \mathbb{K}$.*

Proof. We seek to satisfy the conditions of Theorem 1.3.1. Certainly $\mathscr{B}(X_{\mathrm{AM}})$ is unital, and $\mathscr{S}(X_{\mathrm{AM}})$ forms a closed ideal of codimension one by Theorem 2.2.2(ii). Part (iii) of the same theorem ensures that $\mathscr{S}(X_{\mathrm{AM}})^2 \subseteq \mathscr{K}(X_{\mathrm{AM}})$, and so it remains to show that $\mathscr{K}(X_{\mathrm{AM}})$ has infinite codimension in $\mathscr{B}(X_{\mathrm{AM}})$. However, we have already demonstrated this in the proof of Theorem 2.2.3. □

Since X_{AM} is reflexive the weak Calkin algebra of X_{AM} has a unique complete norm. What can we say about weak Calkin algebras in general? We have seen that for many Banach spaces X, $\mathscr{B}(X)/\mathscr{W}(X)$ has a unique complete norm, but let us now give an example where this is not true. In line with the theme of this thesis, we shall use $E_{\mathcal{R}}$. The theorem follows directly from Read's results, using Lemma 2.2.1.

Theorem 2.2.7. *The weak Calkin algebra $\mathscr{B}(E_{\mathcal{R}})/\mathscr{W}(E_{\mathcal{R}})$ of the space of Read has at least two non-equivalent complete algebra norms.*

Proof. By Theorem 1.3.2 there is an ideal I of codimension one in $\mathscr{B}(E_{\mathcal{R}})$ such that $I^2 \subseteq \mathscr{W}(E_{\mathcal{R}}) \subseteq I$, and $\mathscr{B}(E_{\mathcal{R}})/\mathscr{W}(E_{\mathcal{R}})$ is infinite-dimensional. From this, it follows that

$$\mathscr{B}(E_{\mathcal{R}})/\mathscr{W}(E_{\mathcal{R}}) = I/\mathscr{W}(E_{\mathcal{R}}) \oplus \mathbb{K}(I_{E_{\mathcal{R}}} + \mathscr{W}(E_{\mathcal{R}})),$$

where $I_{E_{\mathcal{R}}}$ is the identity operator. Since $I^2 \subseteq \mathscr{W}(E_{\mathcal{R}})$, we see immediately that $I/\mathscr{W}(E_{\mathcal{R}})$ has the trivial product, and the fact that $\mathscr{B}(E_{\mathcal{R}})/\mathscr{W}(E_{\mathcal{R}})$ is infinite-dimensional ensures that $I/\mathscr{W}(E_{\mathcal{R}})$ is too. Thus we may apply Lemma 2.2.1 to the Banach algebra $I/\mathscr{W}(E_{\mathcal{R}})$ to obtain the result. $\square$

Chapter 3

Splittings of Extensions of Banach Algebras

An extension of a Banach algebra B is a general structure: a short exact sequence in the category **Balg** of Banach algebras and continuous algebra homomorphisms, with B on the right, playing the role of the quotient. It is natural to ask about splittings—when can we find a lift from the quotient to the full algebra? Two types of splitting shall interest us particularly. Algebraic splittings involve only the algebra structure and not the topology; strong splittings include the topology.

Splittings of extensions of Banach algebras are closely related to *Wedderburn decompositions*, and this is the context in which they were first studied. An algebra A has a Wedderburn decomposition if there is a subalgebra C of A such that $A = C \boxplus \operatorname{rad} A$, and Wedderburn proved that all finite-dimensional algebras have such a decomposition [108]. Feldman [38] considered the case where A is an infinite-dimensional Banach algebra. In this case A has a *strong Wedderburn decomposition* if there is a closed subalgebra D of A such that $A = D \oplus \operatorname{rad} A$, and Feldman observed that A need not have a strong Wedderburn decomposition in general (see also [24, Example 5.4.6]). Bade and Curtis [10], [11] then built on his work in the case where $A/\operatorname{rad} A$ is commutative. A large step forward in the study of extensions occurred in the 1960s when Kamowitz [61] and Johnson [58] introduced cohomological methods into Banach algebra theory. This equipped the abstract theory of extensions with many powerful results and ensured its continued importance. The cohomological viewpoint is taken up in Chapter 4, where we also describe the important contribution of Helemskii.

Extensions in the category of C^*-algebras have also received lots of attention (that is, short exact sequences of C^*-algebras and continuous $*$-homomorphisms) [14], [32], although we shall not consider these.

In 1999 Bade, Dales and Lykova published a wide-ranging memoir [12], studying extensions of Banach algebras and their splittings. Their purpose was to gather together the (by now) substantial literature, develop the abstract theory in more generality, and prove new results about classical Banach algebras. The aim of the present chapter is to answer some questions which they left open.

In the first section we look at the questions that captured their interest, and carefully define all our notions. We also record some known results. Secondly, we tackle one of the open problems from [12] about the Banach algebra $\mathscr{B}(X)$ for a Banach space X, namely, does every extension of $\mathscr{B}(X)$ which splits algebraically also split strongly? This is true for many Banach spaces, but was open for an arbitrary X. We introduce a new general theorem giving conditions upon which a Banach algebra admits an extension which splits algebraically but not strongly. The proof requires the use of pullbacks in the category of Banach algebras. While it is well known that pullbacks exist in **Balg**, we are not aware of any previous applications of the theory. For the convenience of the reader we give a complete account of the pullback construction in **Balg**. Then in the third section we apply this theorem to $\mathscr{B}(E_{\mathcal{R}})$, where $E_{\mathcal{R}}$ is Read's space. Using our main theorem about Read's space (Theorem 1.3.4) we find that the results drop out in a pleasing way. Read's work shows that there is a discontinuous homomorphism from $\mathscr{B}(E_{\mathcal{R}})$ into a Banach algebra; this result says that we can choose it to be of a very special type, thus providing a counterexample to the above question. Next we present another theorem about extensions which split algebraically but not strongly, with different conditions. This has the disadvantage of being less intuitive than the pullback construction, but the advantage of applying to possibly different Banach algebras, and being easy to generalise, as we subsequently do.

The author would like to thank Professor H. G. Dales, now at Lancaster University, for suggesting the main question considered in this chapter. The original results in Sections 3.2.2, 3.3.1 and 3.3.2, most of which appear in [71], were proved jointly with N. J. Laustsen.

3.1 The general theory

We want to properly introduce the terminology above. Extensions and strong splittings were briefly defined in Chapter 1; for ease of reference we record them again.

Definition 3.1.1. Let B be a Banach algebra. An *extension* of B is a short exact sequence of the form:

$$\{0\} \longrightarrow \ker\pi \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow \{0\}\ , \tag{3.1.1}$$

where A is a Banach algebra, $\pi : A \to B$ is a continuous, surjective algebra homomorphism, and ι is the inclusion map.

Our definition takes a slightly different perspective to previous authors (for example, [12, Definition 1.2] and [24, Definition 2.8.10], which are more general). Our perspective is advantageous in that, given a Banach algebra B, we shall be concerned with building an extension of B with certain properties, without regard to the form of the kernel. Within the more general viewpoint it is common to require the kernel to be isomorphic to some fixed Banach algebra C, and speak of *extensions of B by C*, in order to restrict the possibilities for the middle algebra A to a useful class. We shall make use of this idea in Chapter 4 in the context of Hochschild cohomology, but the overriding emphasis will be on finding Banach algebras having a continuous surjection onto B.

Definition 3.1.2. Let B be a Banach algebra. An extension of the form (3.1.1):

(i) *splits algebraically* if there is an algebra homomorphism $\rho : B \to A$ such that $\pi \circ \rho = \mathrm{id}_B$;

(ii) *splits strongly* if there is a *continuous* algebra homomorphism $\theta : B \to A$ such that $\pi \circ \theta = \mathrm{id}_B$;

(iii) is *admissible* if there is a continuous linear map $Q : B \to A$ such that $\pi \circ Q = \mathrm{id}_B$;

(iv) is *singular* if $(\ker \pi)^2 = \{0\}$;

(v) is *radical* if $\ker \pi \subseteq \mathrm{rad}\, A$.

An algebra homomorphism making an extension split algebraically or strongly is called a *splitting homomorphism*, or occasionally a (strong) *splitting*, and a continuous linear map making an extension admissible is an *admissible map*.

Two of the questions that featured in Bade, Dales and Lykova's work [12] were the following.

Question 1. For which Banach algebras B is it true that every extension of B which splits algebraically also splits strongly?

Question 2. For which Banach algebras B is it true that every singular, admissible extension of B splits, either algebraically or strongly?

They note [12, p. 9] that there are not always positive answers to these questions, and in fact most of this chapter will be dedicated to finding other situations where a negative answer can be given for Question 1. Question 2 will be considered in Chapter 4 in connection with homological bidimension.

What is known about Question 1 for various Banach algebras? The next two theorems summarise a number of the known results. Any unexplained terminology can be found in [12], although we try to give a brief description for most of the examples. This is not intended to be a complete summary, and so some of the results are not stated in their maximum possible generality (thus we avoid introducing too many further definitions). The first theorem is itself a collection (by Bade, Dales and Lykova) of results from the automatic continuity literature.

Theorem 3.1.3 (Bade–Dales–Lykova)**.** *Let B be a Banach algebra from the following list. Then every extension of B which splits algebraically also splits strongly.*

(i) *B is a C^*-algebra;*

(ii) *$B = L^1(G)$ for a compact group G;*

(iii) *B is a strong Ditkin algebra;*

(iv) *$B = \mathscr{B}(X)$ for a Banach space X such that $X \simeq X \oplus X$;*

(v) *$B = \mathscr{K}(X)$ for a Banach space X with (BAP) and such that $X \simeq X \oplus X$.*

Proof. (i)–(v) are covered in [12, Theorem 3.19]. A strong Ditkin algebra is a special type of Banach function algebra [12, p. 41]. (iv) uses the fact that when $X \simeq X \oplus X$, every algebra homomorphism from $\mathscr{B}(X)$ into Banach algebra is continuous, as we explain in Section 3.1.1, below. □

Theorem 3.1.4 (Bade–Dales–Lykova)**.** *Let B be a Banach algebra from the following list. Then there is an extension of B which splits algebraically but not strongly.*

(i) *$B = \ell_p$ with the pointwise product, for $1 < p < \infty$;*

(ii) *$B = C^{(n)}[0,1]$ for $n \in \mathbb{N}$;*

(iii) *$B = \ell_1(w)$ with convolution product, and w a radical weight;*

(iv) *$B = \mathrm{lip}_\alpha M$ for a compact metric space M and $\alpha \in (0,1)$.*

Proof. (i) [12, Theorem 5.1].

(ii) [12, Theorem 5.3]; the norm on $C^{(n)}[0,1]$ is $||f||_n = \sum_{k=0}^{n} \frac{1}{k!}||f^{(k)}||_\infty$, and the product is defined pointwise.

(iii) [12, Theorem 5.11, Theorem 5.12]; a sequence of real numbers $w = (w_n)_{n\in\mathbb{N}_0}$ is a *weight* if $w_0 = 1$ and for all $m, n \in \mathbb{N}_0$, $w_n > 0$ and $w_{m+n} \leqslant w_m w_n$. A weight is *radical* if $\lim_{n\to\infty} w_n^{1/n} = 0$. Given a weight w, $\ell_1(w)$ is the vector space $\ell_1(\mathbb{N}_0)$ equipped with the norm $||(a_n)_{n=0}^\infty||_w = \sum_{n=0}^\infty |a_n|w_n$. If $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty$ are in $\ell_1(w)$, the product is defined to be the convolution, that is, $((a_n) * (b_n))(m) = \sum_{j=0}^m a_j b_{m-j}$ for $m \in \mathbb{N}_0$.

(iv) [12, Theorem 5.8]; let (M, d) be a compact metric space and $\alpha \in (0,1)$. Then

$$\mathrm{lip}_\alpha M = \{f : M \to \mathbb{K} : \ |f(x) - f(y)|/d(x,y)^\alpha \to 0 \text{ as } d(x,y) \to 0, \ (x, y \in M)\}$$

with norm $||f||_\alpha = ||f||_\infty + \sup\{|f(x) - f(y)|/d(x,y)^\alpha : x, y \in M, x \neq y\}$, and the pointwise product. $\square$

Some comments are in order. The idea of an algebraic splitting implying a strong splitting is tied up with the notion of automatic continuity. Some of the results in Theorem 3.1.3 follow because every algebra homomorphism from the given Banach algebra into an arbitrary Banach algebra is continuous. This is the case for (iv) and (v). Interestingly, (i), (ii) and (iii) have a different flavour. The idea is that every homomorphism into another Banach algebra is 'close to continuous', and so we can replace any discontinuous splitting homomorphism with a continuous one.

In contrast, the examples in Theorem 3.1.4 have the property that there is an extension with a discontinuous splitting homomorphism, and, moreover, there is no possible continuous splitting homomorphism. In fact, more can be said: Bade, Dales and Lykova show that the extensions can be chosen to be *finite-dimensional* (meaning that the kernel is finite-dimensional), and therefore also admissible.

A necessary condition for an extension which splits algebraically but not strongly is, therefore, a discontinuous algebra homomorphism into a Banach algebra, but this is not usually sufficient. For a certain extension, one needs to rule out all possible continuous splitting homomorphisms, as well as demonstrating a discontinuous one. This means that there are still a number of open problems.

Problem 3.1.5. Let B be one of the Banach algebras in the following list. Is it true that each extension of B which splits algebraically also splits strongly?

(i) $B = L^1(G)$ for an arbitrary locally compact group G;

(ii) $B = A(\overline{\mathbb{D}})$, the disc algebra [24, Example 2.1.13(ii)];

(iii) $B = \mathscr{B}(X)$ for an arbitrary Banach space X;

(iv) $B = \mathscr{K}(X)$ for an arbitrary Banach space X;

(v) $B = \mathscr{B}(X)/\mathscr{K}(X)$ for an arbitrary Banach space X.

Let us formally raise two questions which we shall answer from this list.

Question 3. Is it true that for each Banach space X, every extension of $\mathscr{B}(X)$ which splits algebraically also splits strongly?

Question 4. Is it true that for each Banach space X, every extension of the Calkin algebra $\mathscr{B}(X)/\mathscr{K}(X)$ which splits algebraically also splits strongly?

In the course of this chapter we shall solve both questions in the negative: there are a Banach space X and extensions of $\mathscr{B}(X)$ and $\mathscr{B}(X)/\mathscr{K}(X)$ which split algebraically but not strongly. As we shall see, it remains open whether there are finite-dimensional extensions with the same properties (see the comments after Theorem 3.1.4).

We conclude the section with two general results which are basic to the theory. The first gives some equivalent characterisations of splittings, and explains the terminology: an extension of the form (3.1.1) splits if and only if A splits into two parts. The second records some basic relations between properties of extensions.

Proposition 3.1.6. *Let B be a Banach algebra and* (3.1.1) *an extension of B. Then the extension:*

(i) *is admissible if and only if there is a closed subspace X of A such that $A = X \oplus \ker \pi$;*

(ii) *splits algebraically if and only if there is a subalgebra C of A such that $A = C \boxplus \ker \pi$;*

(iii) *splits strongly if and only if there is a closed subalgebra D of A such that $A = D \oplus \ker \pi$.*

Proof. (i) Suppose that (3.1.1) is admissible, so that there exists $Q : B \to A$ such that $\pi \circ Q = \mathrm{id}_B$. Then $P = Q \circ \pi : A \to A$ is a bounded projection and $\ker P = \ker \pi$ because Q is injective. So let $X = \mathrm{im}\, P$.

If X is a closed subspace of A such that $A = X \oplus \ker \pi$, then $\pi|_X : X \to B$ is an isomorphism of Banach spaces. Thus $Q = (\pi|_X)^{-1} : B \to A$ is a bounded, linear, right inverse of π (considering the codomain to be A). Hence (3.1.1) is admissible.

(ii) and (iii) are similar. □

Proposition 3.1.7. *Let B be a Banach algebra and* (3.1.1) *an extension of B.*

(i) *If* (3.1.1) *splits strongly then it splits algebraically and is admissible.*

(ii) *If* (3.1.1) *is singular then it is radical.*

(iii) *If* (3.1.1) *is singular and admissible, with admissible map Q, then* $\ker \pi$ *is a Banach B-bimodule with respect to the maps*

$$b \cdot i = Q(b)i, \quad i \cdot b = iQ(b) \qquad (b \in B, i \in \ker \pi).$$

Proof. (i) This is clear from the definitions.

(ii) Suppose that (3.1.1) is singular. Consider the case when A is unital and choose $a \in \ker \pi$ and $b \in A$. Then $ba \in \ker \pi$ since $\ker \pi$ is an ideal, and so

$$(1_A + ba)(1_A - ba) = 1_A = (1_A - ba)(1_A + ba).$$

Hence $1_A + ba \in \mathrm{inv}A$, and so $a \in \mathrm{rad}\, A$. In the non-unital case the proof is a little more technical. We shall not need this case, so we omit the proof and refer to [24, Proposition 1.5.6(ii)].

(iii) We check that $\ker \pi$ is a Banach B-bimodule with respect to the given maps. First of all, $\ker \pi$ is a Banach space. Choose $\alpha, \beta \in \mathbb{K}$, $a, b \in B$ and $i, j \in \ker \pi$. Since Q is linear, it is not hard to see that $b \cdot (\alpha i + \beta j) = \alpha b \cdot i + \beta b \cdot j$ and $(\alpha a + \beta b) \cdot i = \alpha a \cdot i + \beta b \cdot i$.

Next, the associativity: notice that since Q is a right inverse of π, we have $Q(ab) - Q(a)Q(b) \in \ker \pi$ so $(Q(ab) - Q(a)Q(b))i = 0$ by singularity. Hence $a \cdot (b \cdot i) = Q(a)Q(b)i = Q(ab)i = ab \cdot i$, and so the left action is associative.

Analogous calculations involving the right action, together with the fact that $a \cdot (i \cdot b) = Q(a)iQ(b) = (a \cdot i) \cdot b$, show that $\ker \pi$ is a B-bimodule. Finally, $||b \cdot i|| = ||Q(b)i|| \leqslant ||Q|| \, ||b|| \, ||i||$ and $||i \cdot b|| = ||iQ(b)|| \leqslant ||Q|| \, ||b|| \, ||i||$, so $\ker \pi$ is a Banach B-bimodule. $\square$

3.1.1 Splittings of extensions of $\mathscr{B}(X)$

Let us analyse what is known about Question 3 for various Banach spaces X. We have already mentioned the case where $X \simeq X \oplus X$ in Theorem 3.1.3, but we now see what we can say beyond this setting.

Definition 3.1.8. Let X be a Banach space. A *continued bisection* of X is a pair $\{(Y_n), (Z_n)\}$ of sequences of closed subspaces of X such that $X = Y_1 \oplus Z_1$ and for each $n \in \mathbb{N}$:

$$Y_n = Y_{n+1} \oplus Z_{n+1} \qquad \text{and} \qquad Y_n \simeq Z_n.$$

Our definition is an amalgamation of Johnson's original definition [57, Definition 3.1], and the modern version for an arbitrary unital Banach algebra (see, *e.g.*, [24, Definition 1.3.24]). It is taken from [107], and is easily seen to be equivalent to Johnson's definition.

It is clear that every Banach space X such that $X \simeq X \oplus X$ has a continued bisection. But the properties are not equivalent because Loy and Willis [76, p. 327] observed that the class of spaces with a continued bisection is strictly larger than those with $X \simeq X \oplus X$, using an example of Figiel [39].

Theorem 3.1.9 (Johnson)**.** *Let X be a Banach space with a continued bisection. Then every algebra homomorphism from $\mathscr{B}(X)$ into a Banach algebra is continuous.*

Proof. This is [57, Theorem 3.3]; see also [24, Theorem 5.4.11]. $\square$

Applying Lemma 1.3.3 we see that, if X has a continued bisection, then every derivation from $\mathscr{B}(X)$ is continuous, as mentioned in Chapter 1.

Corollary 3.1.10. *Let X be a Banach space with a continued bisection. Then every extension of $\mathscr{B}(X)$ which splits algebraically also splits strongly.*

Proof. Suppose that X has a continued bisection. Let

$$\{0\} \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} \mathscr{B}(X) \longrightarrow \{0\} \tag{3.1.2}$$

be an extension of $\mathscr{B}(X)$ which splits algebraically, with splitting homomorphism $\rho : \mathscr{B}(X) \to A$. Then ρ is an algebra homomorphism from $\mathscr{B}(X)$ into a Banach algebra, so it is continuous by Theorem 3.1.9. Therefore (3.1.2) splits strongly. □

Example 3.1.11. For the following Banach spaces X, every algebra homomorphism from $\mathscr{B}(X)$ into a Banach algebra is continuous, and so every extension of $\mathscr{B}(X)$ which splits algebraically also splits strongly (by the same reasoning as in Corollary 3.1.10).

(i) Any finite dimensional Banach space.

(ii) Any Banach space X such that $X \simeq X \oplus X$ (see Chapter 2 for a list of examples, including many classical spaces).

(iii) $\mathfrak{F}$, Figiel's space which has a continued bisection but is not isomorphic to its square [39].

(iv) J_p for $1 < p < \infty$, the p^{th} James space. Willis has shown that every algebra homomorphism from $\mathscr{B}(J_2)$ into a Banach algebra is continuous [110], and his argument extends to arbitrary $p \in (1, \infty)$.

(v) $J_p(\omega_1)$ for $1 < p < \infty$, Edgar's p^{th} long James space. Kania and Kochanek [62, Theorem 3.12] proved that every algebra homomorphism from $\mathscr{B}(J_p(\omega_1))$ into a Banach algebra is continuous, just as for J_p.

(vi) $C[0, \omega_\eta]$, where η is a non-zero ordinal and ω_η denotes the first ordinal of cardinality $\aleph_\eta$. Indeed, Ogden [81] has shown that all homomorphisms from $\mathscr{B}(C[0, \omega_\eta])$ into a Banach algebra are continuous.

(vii) $\mathfrak{G}$, Gowers' space which satisfies $\mathfrak{G} \simeq \mathfrak{G} \oplus \mathfrak{G} \oplus \mathfrak{G}$ but $\mathfrak{G} \not\simeq \mathfrak{G} \oplus \mathfrak{G}$ [46]. A result of Laustsen [69, Proposition 1.9, Proposition 2.3] shows that $\mathscr{B}(\mathfrak{G})$ satisfies [24, Definition 1.3.24] (*cf.* Definition 3.1.8) because it contains a complemented copy of $\mathfrak{G} \oplus \mathfrak{G}$. Thus all homomorphisms from $\mathscr{B}(\mathfrak{G})$ are continuous since Johnson's proof carries over to this setting [24, Theorem 5.4.11]. Despite this, it is not clear whether $\mathfrak{G}$ has a continued bisection in the sense of Definition 3.1.8. Gowers and Maurey subsequently found a second example which is isomorphic to its cube but not its square [47].

In order to find a counterexample to Question 3, we need a Banach space X such that:

- there is a discontinuous algebra homomorphism ρ from $\mathscr{B}(X)$ into a Banach algebra A;
- there is a continuous, surjective algebra homomorphism $\pi : A \to \mathscr{B}(X)$;
- the kernel of π is algebraically complemented by the image of ρ, but is not complemented by any closed subalgebra of A.

Satisfying the first condition is already difficult, but it seems a good place to start our search for a Banach space with the above properties.

Example 3.1.12. Banach spaces which are known to admit discontinuous homomorphisms from $\mathscr{B}(X)$ into a Banach algebra include the following:

(i) $E_{\mathcal{R}}$;

(ii) E_{DLW}, Dales, Loy, and Willis' Banach space, assuming the Continuum Hypothesis [28];

(iii) X_{AM}, Argyros and Motakis' space (see Corollary 2.2.6).

We conclude the section with an observation about E_{DLW}, for which we shall need a further piece of terminology.

Definition 3.1.13. Let B be a Banach algebra, and Y a Banach B-bimodule. A linear map $S : B \to Y$ is *intertwining* if the maps

$$b \mapsto S(ab) - a \cdot Sb, \qquad b \mapsto S(ba) - Sb \cdot a, \qquad B \to Y, \qquad (b \in B)$$

are both continuous for each $a \in B$.

Every derivation is an intertwining map, and in many situations results about derivations can be generalised naturally to the setting of intertwining maps. A link to extensions is provided by the following theorem.

Theorem 3.1.14. *Let B be a Banach algebra, let Y be a Banach B-bimodule, and suppose that every intertwining map $S : B \to Y$ is continuous. Then every singular, admissible extension of B of the form* (3.1.1) *which splits algebraically also splits strongly, provided I is isomorphic to Y as a Banach B-bimodule.*

Proof. This is explained in [12, Theorem 2.13]. □

The next proposition perhaps provides a reason not to begin looking for a counterexample using the space of Dales, Loy and Willis.

Proposition 3.1.15. *Every singular, admissible extension of $\mathscr{B}(E_{\mathrm{DLW}})$ which splits algebraically also splits strongly.*

Proof. The main reason for the construction of the space E_{DLW} was to find an example of a Banach space such that there is a discontinuous homomorphism from $\mathscr{B}(E_{\mathrm{DLW}})$ into a Banach algebra, but such that all derivations from $\mathscr{B}(E_{\mathrm{DLW}})$ are continuous. Since all derivations are continuous, a theorem of Dales and Villena [24, Corollary 2.7.7] implies that every intertwining map from $\mathscr{B}(E_{\mathrm{DLW}})$ is continuous too.

Take a singular, admissible extension of $\mathscr{B}(E_{\mathrm{DLW}})$, say

$$\{0\} \longrightarrow \ker\pi \xrightarrow{\iota} A \xrightarrow{\pi} \mathscr{B}(E_{\mathrm{DLW}}) \longrightarrow \{0\}. \tag{3.1.3}$$

Then $\ker\pi$ is a Banach $\mathscr{B}(E_{\mathrm{DLW}})$-bimodule by Proposition 3.1.7(iii). Choose an intertwining map $S : \mathscr{B}(E_{\mathrm{DLW}}) \to \ker\pi$. By the observation above, S is continuous. Hence Theorem 3.1.14 implies that, if (3.1.3) splits algebraically, then it also splits strongly. □

3.2 Extensions and pullbacks of Banach algebras

One of the key results of the thesis is contained in this section (Theorem 3.2.5). Initially, it applies when one wants to find extensions which split algebraically but not strongly; in Chapter 4, it will apply to other types of extensions. Its proof relies on the theory of pullbacks in the category of Banach algebras and continuous algebra homomorphisms, so it seems an appropriate point to outline this theory.

3.2.1 Pullbacks in the category of Banach algebras

The notion of a *pullback* is a common one in category theory. Here our aim is to show that pullbacks exist and are unique up to isomorphism in the category **Balg**, where the objects are Banach algebras and the morphisms are continuous algebra homomorphisms. Pullbacks have been studied in the 'neighbouring' categories **C*alg** of C^*-algebras and continuous $*$-homomorphisms [82, §2.2] and **Ban** of Banach spaces and bounded operators [17], [21, p. 126], [23, p. 13], but apparently never in detail in **Balg**.

Definition 3.2.1. Let A, B, C be Banach algebras, and let $\alpha : A \to C$ and $\beta : B \to C$ be continuous algebra homomorphisms. We think of them as being set up in the following way.

$$
\begin{tikzcd}
& A \arrow[d, "\alpha"] \\
B \arrow[r, "\beta"] & C
\end{tikzcd}
\tag{3.2.1}
$$

A *coherent pair* (with respect to the diagram (3.2.1)) is a pair of continuous algebra homomorphisms $\gamma\colon D \to A$ and $\delta\colon D \to B$ defined on a common Banach algebra D and such that the following diagram is commutative.

$$
\begin{tikzcd}
D \arrow[r, "\gamma"] \arrow[d, "\delta"'] & A \arrow[d, "\alpha"] \\
B \arrow[r, "\beta"] & C
\end{tikzcd}
$$

A *pullback* for (3.2.1) is a triple (D, γ, δ), where D is a Banach algebra, (γ, δ) is a coherent pair, and for every Banach algebra E with coherent pair $\psi : E \to B$, $\varphi : E \to A$, there exists a unique continuous algebra homomorphism $\sigma : E \to D$ making the following diagram commutative.

$$
\begin{tikzcd}
E \arrow[r, dashed, "\exists!\ \sigma"] \arrow[rr, bend left, "\varphi"] \arrow[dr, bend right, "\psi"'] & D \arrow[r, "\gamma"] \arrow[d, "\delta"'] & A \arrow[d, "\alpha"] \\
& B \arrow[r, "\beta"] & C
\end{tikzcd}
\tag{3.2.2}
$$

Construction 3.2.2. Given Banach algebras A, B, C and continuous algebra homomorphisms $\alpha : A \to C$ and $\beta : B \to C$, we can construct a pullback for the diagram (3.2.1) explicitly.

Equip the cartesian product $A \times B$ with the usual pointwise algebra operations and the norm

$$
||(a, b)||_\infty = \max\{||a||_A, ||b||_B\} \qquad (a \in A,\ b \in B).
$$

Then $A \times B$ is a Banach algebra. Set

$$
D = \{(a, b) \in A \times B : \alpha(a) = \beta(b)\}. \tag{3.2.3}
$$

Then D is a Banach algebra since it is a closed subalgebra of $A \times B$, as is easily checked. Let $\gamma : (a, b) \mapsto a,\ D \to A$ and $\delta : (a, b) \mapsto b,\ D \to B$ be the natural projections on $A \times B$, restricted to D.

We *claim* that (D, γ, δ) is a pullback for the diagram (3.2.1). Firstly, since γ and δ are continuous algebra homomorphisms satisfying $\alpha \circ \gamma(a, b) = \alpha(a) = \beta(b) = \beta \circ \delta(a, b)$ for $(a, b) \in D$, we see that (γ, δ) is a coherent pair. Secondly, choose a Banach algebra E and a coherent pair $\varphi : E \to A$ and $\psi : E \to B$. Define a map $\sigma : E \to D$ by $\sigma(e) = (\varphi(e), \psi(e))$ for $e \in E$, which makes sense because (φ, ψ) is a coherent pair. Then σ is a continuous algebra homomorphism, while

$$\gamma \circ \sigma(e) = \varphi(e) \qquad \text{and} \qquad \delta \circ \sigma(e) = \psi(e)$$

for each $e \in E$. Hence the diagram (3.2.2) is commutative.

Finally, we must check that σ is unique. For this, let $\tau : E \to D$ be a continuous algebra homomorphism making the diagram (3.2.2) commutative (with τ in place of σ), and let $e \in E$. Then $\tau(e) = (a, b)$ for some $a \in A$ and $b \in B$. By the commutativity assumption, $\varphi(e) = \gamma \circ \tau(e) = a$ and $\psi(e) = \delta \circ \tau(e) = b$, and so $\tau(e) = (\varphi(e), \psi(e)) = \sigma(e)$. Thus $\sigma = \tau$, which implies that σ is unique. We conclude that the triple (D, γ, δ) is a pullback for the diagram (3.2.1).

So given the diagram (3.2.1) we can always form a pullback; the next task is to show that this is unique (in a suitable sense).

Proposition 3.2.3. *Let A, B, C be Banach algebras and let $\alpha : A \to C$ and $\beta : B \to C$ be continuous algebra homomorphisms. Suppose that $(D_1, \gamma_1, \delta_1)$ and $(D_2, \gamma_2, \delta_2)$ are pullbacks for the diagram* (3.2.1)*. Then there is a continuous algebra isomorphism $\sigma \colon D_2 \to D_1$ such that $\gamma_1 \circ \sigma = \gamma_2$ and $\delta_1 \circ \sigma = \delta_2$.*

Proof. Since $(D_1, \gamma_1, \delta_1)$ is a pullback and (γ_2, δ_2) is a coherent pair for (3.2.1), there exists a unique continuous algebra homomorphism $\sigma : D_2 \to D_1$ such that $\gamma_1 \circ \sigma = \gamma_2$ and $\delta_1 \circ \sigma = \delta_2$. Similarly, $(D_2, \gamma_2, \delta_2)$ is a pullback and (γ_1, δ_1) a coherent pair for (3.2.1), so there exists a unique continuous algebra homomorphism $\tau : D_1 \to D_2$ such that $\gamma_2 \circ \tau = \gamma_1$ and $\delta_2 \circ \tau = \delta_1$. Now the identities $\gamma_2 \circ \xi = \gamma_2$ and $\delta_2 \circ \xi = \delta_2$ are satisfied by $\xi = \mathrm{id}_{D_2}$ and $\xi = \tau \circ \sigma$. But the assumption that $(D_2, \gamma_2, \delta_2)$ is a pullback guarantees the uniqueness of the map ξ, and so $\mathrm{id}_{D_2} = \tau \circ \sigma$. Therefore τ is a left inverse of σ. A similar argument given by interchanging $(D_1, \gamma_1, \delta_1)$ and $(D_2, \gamma_2, \delta_2)$ shows that τ is also a right inverse of σ. Thus σ is an isomorphism and the proof is complete. $\square$

Given this proposition it makes sense to speak about *the* pullback for the diagram (3.2.1), and whenever we do so we shall always mean the one defined in Construction 3.2.2.

3.2.2 The connection to extensions

Having explained the theory of pullbacks in **Balg** we want to put it to work. The connection to extensions of Banach algebras is not difficult to make.

Lemma 3.2.4. *Let A, B, C be Banach algebras, let $\alpha\colon A \to C$ and $\beta\colon B \to C$ be continuous, surjective algebra homomorphisms, and let (D, γ, δ) be a pullback for the diagram* (3.2.1)*. Then the homomorphisms $\gamma : D \to A$ and $\delta : D \to B$ are surjective.*

Proof. By Proposition 3.2.3 we may suppose that (D, γ, δ) is the pullback from Construction 3.2.2. Let $a \in A$. Since β is surjective, there is $b \in B$ such that $\beta(b) = \alpha(a)$. By the definition of D, $(a, b) \in D$ and $\gamma(a, b) = a$. Thus γ is surjective. The proof that δ is surjective is analogous. □

We now come to our central result in this chapter. It shows that the problem of finding an extension of a Banach algebra B with certain properties may in certain cases be reduced to finding an extension of a quotient of B with the same properties.

Theorem 3.2.5. *Let B be a Banach algebra. Suppose that there are Banach algebras A and C and extensions:*

$$\{0\} \longrightarrow \ker\beta \xrightarrow{\iota} B \xrightarrow{\beta} C \longrightarrow \{0\} \tag{3.2.4}$$

$$\{0\} \longrightarrow \ker\alpha \xrightarrow{\iota} A \xrightarrow{\alpha} C \longrightarrow \{0\}\,. \tag{3.2.5}$$

Let (D, γ, δ) be the pullback for the diagram (3.2.1)*. Then there is an extension:*

$$\{0\} \longrightarrow \ker\delta \xrightarrow{\iota} D \xrightarrow{\delta} B \longrightarrow \{0\} \tag{3.2.6}$$

and the following hold:

(i) (3.2.6) *is singular if and only if* (3.2.5) *is singular.*

(ii) *Suppose that* (3.2.4) *splits strongly. Then* (3.2.6) *splits strongly if and only if* (3.2.5) *splits strongly.*

(iii) *Suppose that* (3.2.4) *splits algebraically. Then* (3.2.6) *splits algebraically if and only if* (3.2.5) *splits algebraically.*

(iv) *Suppose that* (3.2.4) *is admissible. Then* (3.2.6) *is admissible if and only if* (3.2.5) *is admissible.*

Proof. Our assumptions give the following setup.

$$\begin{tikzcd}
 & & & \{0\} \arrow[d] & \\
 & & & \ker\alpha \arrow[d] & \\
 & & & A \arrow[d, "\alpha"] & \\
\{0\} \arrow[r] & \ker\beta \arrow[r] & B \arrow[r, "\beta"] & C \arrow[r] \arrow[d] & \{0\} \\
 & & & \{0\} &
\end{tikzcd}$$

In the light of Lemma 3.2.4, and the fact that we may always construct the pullback (D, γ, δ) of (3.2.1), we obtain a commutative diagram:

$$\begin{tikzcd}
 & & \{0\} \arrow[d] & \{0\} \arrow[d] & \\
 & & \ker\delta \arrow[d] & \ker\alpha \arrow[d] & \\
\{0\} \arrow[r] & \ker\gamma \arrow[r] & D \arrow[r, "\gamma"] \arrow[d, "\delta"'] & A \arrow[r] \arrow[d, "\alpha"] & \{0\} \\
\{0\} \arrow[r] & \ker\beta \arrow[r] & B \arrow[r, "\beta"] \arrow[d] & C \arrow[r] \arrow[d] & \{0\} \\
 & & \{0\} & \{0\} &
\end{tikzcd} \tag{3.2.4}$$

(3.2.6) (3.2.5)

and (3.2.6) is an extension of B.

(i) We first claim that the map $\gamma|_{\ker\delta} : \ker\delta \to \ker\alpha$ is a bijection. If this is the case then (3.2.6) is singular if and only if (3.2.5) is singular. To establish the claim, note that since $\alpha\circ\gamma = \beta\circ\delta$, the range of $\gamma|_{\ker\delta}$ is contained in $\ker\alpha$. For injectivity, suppose that $\gamma(d) = \gamma(d')$ for some $d, d' \in \ker\delta$. Then $d - d' \in \ker\delta \cap \ker\gamma$. But it follows from the definition of γ and δ as corresponding projections that $\ker\delta\cap\ker\gamma = \{0\}$, so $d = d'$. Now choose $a \in \ker\alpha$. Then $(a, 0) \in \ker\delta$ because $\beta(0) = \alpha(a) = 0$, and $\gamma(a, 0) = a$. Thus $\gamma|_{\ker\delta}$ is a bijection.

We observe that in fact $\gamma|_{\ker\delta} : \ker\delta \to \ker\alpha$ is an isomorphism. Indeed, it is a continuous bijective algebra homomorphism between Banach algebras, and so by the Banach Isomorphism Theorem 1.2.2, its inverse is continuous.

(ii) Suppose that (3.2.4) splits strongly. Then there exists a continuous algebra homomorphism $\theta : C \to B$ such that $\beta\circ\theta = \mathrm{id}_C$.

Suppose that (3.2.6) also splits strongly; then there exists a continuous algebra homomorphism $\xi : B \to D$ such that $\delta\circ\xi = \mathrm{id}_B$. Define a map $\eta : C \to A$ by $\eta = \gamma\circ\xi\circ\theta$. This is the composition of continuous algebra homomorphisms, so it is certainly a continuous algebra homomorphism. Moreover,

$$\alpha\circ\eta = \alpha\circ\gamma\circ\xi\circ\theta = \beta\circ\delta\circ\xi\circ\theta = \beta\circ\theta = \mathrm{id}_C,$$

so (3.2.5) splits strongly, with splitting homomorphism η.

Conversely, suppose that (3.2.5) splits strongly, with splitting homomorphism $\eta : C \to A$. Then define $\xi : B \to D$ by $\xi(b) = (\eta\circ\beta(b), b)$. Note that this is an element of D because $\alpha(\eta\circ\beta(b)) = \beta(b)$. This is easily seen to be a continuous algebra homomorphism, and $\delta\circ\xi = \mathrm{id}_B$ because δ is the projection onto the second coordinate. Hence (3.2.6) splits strongly.

(iii) and (iv) are similar to (ii). $\square$

Given a Banach algebra B, we would like to construct an extension which splits algebraically but not strongly. The next result gives two conditions which, if satisfied, allow us to do this.

Corollary 3.2.6. *Let B be a Banach algebra. Suppose that there are a Banach algebra C and an extension:*

$$\{0\} \longrightarrow \ker\beta \xrightarrow{\iota} B \xrightarrow{\beta} C \longrightarrow \{0\} \tag{3.2.7}$$

which splits strongly. Suppose also that there are a Banach algebra A and an extension of C:

$$\{0\} \longrightarrow \ker\alpha \xrightarrow{\iota} A \xrightarrow{\alpha} C \longrightarrow \{0\} \tag{3.2.8}$$

which splits algebraically but not strongly. Then there is an extension of B which splits algebraically but not strongly.

Proof. Suppose that we have the two extensions (3.2.7) and (3.2.8). Then Theorem 3.2.5 implies there are a Banach algebra D and an extension of B:

$$\{0\} \longrightarrow \ker\delta \xrightarrow{\iota} D \xrightarrow{\delta} B \longrightarrow \{0\}. \tag{3.2.9}$$

By assumption, (3.2.7) splits strongly, and so Theorem 3.2.5 says that (3.2.9) splits algebraically if and only if (3.2.8) splits algebraically, and splits strongly if and only if (3.2.8) splits strongly. Conveniently, we have assumed that (3.2.8) splits algebraically but not strongly, and so (3.2.9) is an extension of B which splits algebraically but not strongly. $\square$

We conclude this section with a simple lemma about extensions and unitisations. Recall that the unitisation of a Banach algebra A is denoted by $\widetilde{A}$.

Lemma 3.2.7. *Let B be a Banach algebra, and let*

$$\{0\} \longrightarrow \ker\pi \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow \{0\} \tag{3.2.10}$$

be an extension of B. Then the following is an extension of $\widetilde{B}$:

$$\{0\} \longrightarrow \ker\widetilde{\pi} \xrightarrow{\iota} \widetilde{A} \xrightarrow{\widetilde{\pi}} \widetilde{B} \longrightarrow \{0\}\,. \tag{3.2.11}$$

Moreover:

(i) (3.2.11) *is singular if and only if* (3.2.10) *is singular;*

(ii) (3.2.11) *splits strongly if and only if* (3.2.10) *splits strongly;*

(iii) (3.2.11) *splits algebraically if and only if* (3.2.10) *splits algebraically;*

(iv) (3.2.11) *is admissible if and only if* (3.2.10) *is admissible.*

Proof. Since $\widetilde{\pi}$ is a continuous, surjective algebra homomorphism, (3.2.11) is an extension of $\widetilde{B}$.

(i) We have $\ker\widetilde{\pi} = \{(a,0) \in \widetilde{A} : a \in \ker\pi\}$, so choose $(a,0),(a',0) \in \ker\widetilde{\pi}$. Then $(a,0)(a',0) = (aa',0)$, so (3.2.11) is singular if and only if (3.2.10) is singular.

(ii) Suppose that (3.2.10) splits strongly, with continuous splitting homomorphism θ. Then $\widetilde{\theta} : (b,\lambda) \mapsto (\theta(b),\lambda)$, $\widetilde{B} \to \widetilde{A}$ (where $b \in B$, $\lambda \in \mathbb{K}$), is a continuous splitting homomorphism for (3.2.11).

Conversely, suppose that (3.2.11) splits strongly, with continuous splitting homomorphism $\eta : \widetilde{B} \to \widetilde{A}$. Denote by $j_1 : B \to \widetilde{B}$, the isometric algebra homomorphism $j_1(b) = (b,0)$, and by $P_1 : \widetilde{A} \to A$ the bounded operator $(a,\lambda) \mapsto a$. Then $P_1 \circ \eta \circ j_1$ is a continuous splitting homomorphism for (3.2.10).

(iii) and (iv) are proved analogously. $\square$

3.3 Extensions which split algebraically but not strongly

Corollary 3.2.6 gives conditions for finding extensions which split algebraically but not strongly. The task is now to apply this to solve Questions 3 and 4 in the negative, using Read's space $E_{\mathcal{R}}$. This is our first major application of Theorem 1.3.4. The second part of the section gives an alternative approach to proving the same result.

3.3.1 The pullback method

Theorem 3.3.1. *There is an extension of $\mathscr{B}(E_{\mathcal{R}})$ which splits algebraically, but it is not admissible, and so it does not split strongly.*

Proof. We want to satisfy the conditions of Corollary 3.2.6, with $B = \mathscr{B}(E_{\mathcal{R}})$. Set $C = \ell_2(\mathbb{N})^\sim$, where $\ell_2(\mathbb{N})$ has the trivial product. Then by Theorem 1.3.4 there is a continuous algebra homomorphism $\beta : \mathscr{B}(E_{\mathcal{R}}) \to C$ such that the extension

$$\{0\} \longrightarrow \ker\beta \xrightarrow{\iota} \mathscr{B}(E_{\mathcal{R}}) \xrightarrow{\beta} C \longrightarrow \{0\}$$

splits strongly. So the first condition is satisfied. We now need to find an extension of $C = \ell_2(\mathbb{N})^\sim$ which splits algebraically but not strongly. By Lemma 3.2.7, it is enough to find an extension of $\ell_2(\mathbb{N})$ which splits algebraically but not strongly.

To start, a theorem of Banach and Mazur [3, Theorem 2.3.1] gives a bounded linear surjection $\alpha_0 : \ell_1(\mathbb{N}) \to \ell_2(\mathbb{N})$. Now equip $\ell_1(\mathbb{N})$ with the trivial product to make it a Banach algebra. Then α_0 is a continuous algebra homomorphism since $\ell_2(\mathbb{N})$ has the trivial product. So

$$\{0\} \longrightarrow \ker\alpha_0 \xrightarrow{\iota} \ell_1(\mathbb{N}) \xrightarrow{\alpha_0} \ell_2(\mathbb{N}) \longrightarrow \{0\} \tag{3.3.1}$$

is an extension of $\ell_2(\mathbb{N})$. Now $\ker\alpha_0$ has an algebraic complement Z in $\ell_1(\mathbb{N})$ (as every subspace does), and in fact Z is a subalgebra because $\ell_1(\mathbb{N})$ has the trivial product. Hence (3.3.1) splits algebraically by Proposition 3.1.6(ii).

Assume that (3.3.1) is admissible. Then by Proposition 3.1.6(i) there is a closed subspace Y of $\ell_1(\mathbb{N})$ such that $\ker\alpha_0 \oplus Y = \ell_1(\mathbb{N})$. It follows that $\alpha_0|_Y : Y \to \ell_2(\mathbb{N})$ is an isomorphism of Banach spaces, using the Banach Isomorphism Theorem 1.2.2. But then $\ell_1(\mathbb{N})$ contains a complemented copy of $\ell_2(\mathbb{N})$, contradicting various theorems, including [3, Corollary 2.1.6]. Hence (3.3.1) is not admissible, and so it does not split strongly. $\square$

Example 3.3.2. Theorem 3.3.1 produces an extension of $\mathscr{B}(E_{\mathcal{R}})$ which splits algebraically but not strongly. But what does it look like? It has the form:

$$\{0\} \longrightarrow \ker\delta \xrightarrow{\iota} D \xrightarrow{\delta} \mathscr{B}(E_{\mathcal{R}}) \longrightarrow \{0\},$$

where $D = \{(a,b) \in \ell_1(\mathbb{N})^\sim \times \mathscr{B}(E_{\mathcal{R}}) : \alpha(a) = \beta(b)\}$, $\delta(a,b) = b$, β is the continuous algebra homomorphism onto $\ell_2(\mathbb{N})^\sim$ from Theorem 1.3.4, and α is the unitisation of $\alpha_0 : \ell_1(\mathbb{N}) \to \ell_2(\mathbb{N})$, defined above.

This provides an answer to Question 3: Read's space is a counterexample. We next consider splittings of extensions of Calkin algebras and Question 4. We begin with a positive result, and then show how Read's space is a counterexample in this case too.

Lemma 3.3.3. *Let A be a Banach algebra such that every algebra homomorphism from A into a Banach algebra is continuous, and let J be a closed ideal of A. Then every algebra homomorphism from A/J into a Banach algebra is continuous.*

Proof. Let B be a Banach algebra, and let $\rho : A/J \to B$ be an algebra homomorphism. Denote the quotient map by $Q_J : A \to A/J$. Then, by assumption, $\rho \circ Q_J : A \to B$ is a continuous algebra homomorphism. It is standard that quotient maps are open (see Proposition 5.1.1), meaning Q_J takes open sets to open sets. Also, ρ is continuous if and only if $\rho^{-1}(U)$ is open in A/J for every open set U in B.

Let U be open in B. Then $(\rho\circ Q_J)^{-1}(U)$ is open in A, and so $Q_J[(\rho\circ Q_J)^{-1}(U)]$ is open in A/J. Since Q_J is surjective, it easily follows that $Q_J[(\rho \circ Q_J)^{-1}(U)] = \rho^{-1}(U)$; thus ρ is continuous. $\square$

Proposition 3.3.4. *Let X be a Banach space with a continued bisection, and let J be a closed ideal of $\mathscr{B}(X)$. Then each extension of $\mathscr{B}(X)/J$ which splits algebraically also splits strongly.*

Proof. Let X have a continued bisection. Then every algebra homomorphism from $\mathscr{B}(X)$ into a Banach algebra is continuous by Theorem 3.1.9; hence every algebra homomorphism from $\mathscr{B}(X)/J$ into a Banach algebra is continuous by Lemma 3.3.3. It follows that each extension of $\mathscr{B}(X)/J$ which splits algebraically also splits strongly, as in Corollary 3.1.10. $\square$

Proposition 3.3.5. *For each closed ideal J of $\mathscr{B}(E_{\mathcal{R}})$ that is contained in $\mathscr{W}(E_{\mathcal{R}})$, there is an extension of $\mathscr{B}(E_{\mathcal{R}})/J$ which splits algebraically but not strongly. In* *particular, there are extensions of $\mathscr{B}(E_{\mathcal{R}})/\mathscr{K}(E_{\mathcal{R}})$ and $\mathscr{B}(E_{\mathcal{R}})/\mathscr{W}(E_{\mathcal{R}})$ which split algebraically but not strongly.*

Proof. We seek to fulfil the conditions of Corollary 3.2.6 with $B = \mathscr{B}(E_{\mathcal{R}})/J$. By Theorem 1.3.4 the extension

$$\{0\} \longrightarrow \mathscr{W}(E_{\mathcal{R}}) \longrightarrow \mathscr{B}(E_{\mathcal{R}}) \xrightarrow{\beta} \ell_2(\mathbb{N})^{\sim} \longrightarrow \{0\} \tag{3.3.2}$$

splits strongly, with continuous splitting homomorphism θ. It follows that there is an extension

$$\{0\} \longrightarrow \mathscr{W}(E_{\mathcal{R}})/J \longrightarrow \mathscr{B}(E_{\mathcal{R}})/J \xrightarrow{\widehat{\beta}} \ell_2(\mathbb{N})^{\sim} \longrightarrow \{0\} \tag{3.3.3}$$

where $\widehat{\beta}(T+J) = \beta(T)$ for every $T \in \mathscr{B}(E_{\mathcal{R}})$. Note that $\widehat{\beta}$ is well-defined because $J \subseteq \mathscr{W}(E_{\mathcal{R}})$. Write $Q_J : \mathscr{B}(E_{\mathcal{R}}) \to \mathscr{B}(E_{\mathcal{R}})/J$ for the quotient map. Now it is easy to see that $Q_J \circ \theta$ is a continuous splitting homomorphism for (3.3.3). So in the notation of Corollary 3.2.6, let $C = \ell_2(\mathbb{N})^{\sim}$. Then it suffices to find an extension of C which splits algebraically but not strongly. However, we have already found such an extension in the proof of Theorem 3.3.1, so the result follows. $\square$

3.3.2 An alternative method

Next we demonstrate a second method for finding extensions which split algebraically but not strongly. This leads to an interesting alternative proof of Theorem 3.3.1.

We first construct a useful Banach algebra.

Construction 3.3.6. Let C be a Banach algebra containing a proper closed ideal W such that $C^2 \subseteq W$, and let X be a Banach space with a linear contraction $q : X \to C$ such that $C = q(X) + W$. Define the Banach space

$$A_0 = X \oplus W$$

with pointwise vector space operations and the norm $||(x,w)||_1 = ||x||_X + ||w||_C$ for $x \in X$ and $w \in W$. Then endow A_0 with the product

$$(x,w)(y,v) = (0, q(x)q(y)+q(x)v+wq(y)+wv) \qquad (x,y \in X,\ w,v \in W). \tag{3.3.4}$$

Straightforward calculations show that A_0 is an algebra. Since $||q|| = 1$ we have

$$\begin{aligned}||(x,w)(y,v)|| &= ||(0, q(x)q(y) + q(x)v + wq(y) + wv)|| \\ &= ||q(x)q(y) + q(x)v + wq(y) + wv|| \\ &\leqslant ||q(x)||\,||q(y)|| + ||q(x)||\,||v|| + ||w||\,||q(y)|| + ||w||\,||v|| \\ &\leqslant \big(||x|| + ||w||\big)\big(||y|| + ||v||\big) = ||(x,w)||\,||(y,v)||.\end{aligned}$$

so that A_0 is a Banach algebra. Also define $\psi : A_0 \to C$ to be

$$\psi_0(x,w) = q(x) + w \qquad (x \in X, w \in W).$$

Then ψ_0 is a surjective algebra homomorphism of norm 1.

Theorem 3.3.7. *Let B be a unital Banach algebra containing a proper closed ideal W such that $B = D \oplus W \oplus \mathbb{K}1$ as a Banach space, where:*

(i) *D is a closed subspace of B;*

(ii) *$D^2 \subseteq W$;*

(iii) *$D \not\cong \ell_1(\Upsilon)$ for any index set Υ.*

Then there is a singular extension of B which splits algebraically, but is not admissible, and so it does not split strongly.

Proof. Choose a dense subset of the closed unit ball of D, say Γ, and recall that $\ell_1(\Gamma) = \{f : \Gamma \to \mathbb{K} \ : \sum_{\gamma\in\Gamma} |f(\gamma)| < \infty\}$. Now define

$$q : \ell_1(\Gamma) \to D, \qquad f \mapsto \sum_{\gamma\in\Gamma} f(\gamma)\gamma \qquad (f \in \ell_1(\Gamma)).$$

This is a linear surjection of norm 1 by the non-separable version of [3, Theorem 2.3.1].

Set $X = \ell_1(\Gamma)$ and $C = D \oplus W \subseteq B$, and consider q as a map into C. Then we can form the Banach algebra A_0 as in Construction 3.3.6. Now consider the Banach algebra $A = \widetilde{A_0}$ and the continuous, surjective algebra homomorphism $\psi = \widetilde{\psi}_0 : A \to B$ (viewing B as the unitisation of C). Hence the following is an extension of B:

$$\{0\} \longrightarrow \ker\psi \xrightarrow{\ \iota\ } A \xrightarrow{\ \psi\ } B \longrightarrow \{0\}. \tag{3.3.5}$$

Take an algebraic complement Z of $\ker q$ in $\ell_1(\Gamma)$ and consider the restriction

$$\psi|_{Z\oplus W\oplus\mathbb{K}} : Z \oplus W \oplus \mathbb{K} \subset A \to B.$$

This is a bijection because ψ is a surjection, and because $\ker\psi = \ker q \oplus \{0\} \oplus \{0\}$. Now define $\rho : B \to A$ to be the linear inverse of $\psi|_{Z\oplus W\oplus\mathbb{K}}$, with codomain considered to be A. Thus, by definition $\psi \circ \rho = \mathrm{id}_B$. Also, $Z \oplus W \oplus \mathbb{K}$ is a subalgebra of A by the definition of the product in (3.3.4), and so ρ is an algebra homomorphism. Therefore (3.3.5) splits algebraically.

To check that (3.3.5) is singular, choose $a, a' \in \ker\psi$, so that $a = (c, 0, 0)$ and $a' = (c', 0, 0)$ for some $c, c' \in \ker q$. Then by (3.3.4), $aa' = (c, 0, 0)(c', 0, 0) = (0, q(c)q(c'), 0) = (0, 0, 0)$.

We now *claim* that the following four conditions are equivalent:

(a) (3.3.5) splits strongly;

(b) (3.3.5) is admissible;

(c) $\ker q$ is complemented in $\ell_1(\Gamma)$;

(d) $D \simeq \ell_1(\Upsilon)$ for some index set Υ.

Suppose for a moment that we have proved the claim. We know by (iii) that (d) is not satisfied—hence (3.3.5) is not admissible, and therefore does not split strongly. Thus establishing the claim proves the result.

(a) $\Rightarrow$ (b) This follows by definition.

(b) $\Rightarrow$ (c) Suppose that there is a continuous linear map $Q : B \to A$ such that $\psi \circ Q = \mathrm{id}_B$. Then the map $P = Q \circ \psi$ is a bounded projection on A. Write $j_1 : \ell_1(\Gamma) \to A$ for the natural inclusion and $\pi_1 : A \to \ell_1(\Gamma)$ for the coordinate projection, and consider the map $P' = \pi_1 \circ P \circ j_1 : \ell_1(\Gamma) \to \ell_1(\Gamma)$. This is a bounded projection with $\ker P' = \ker q$. Hence $\ker q$ is complemented in $\ell_1(\Gamma)$ by $\mathrm{im}\, P'$.

(c) $\Rightarrow$ (a) Let $\ker q$ be complemented in $\ell_1(\Gamma)$, say by a closed subspace Z'. Then the map $\psi|_{Z'\oplus W\oplus\mathbb{K}} : Z' \oplus W \oplus \mathbb{K} \to B$ is a continuous bijective algebra homomorphism between Banach algebras, so its inverse is continuous by the Banach Isomorphism Theorem 1.2.2. Hence (3.3.5) splits strongly.

(c) $\Rightarrow$ (d) Suppose that $\ell_1(\Gamma) = Z' \oplus \ker q$, where Z' is closed in $\ell_1(\Gamma)$. Köthe proved in [67, Theorem 6] that every complemented subspace of $\ell_1(\Gamma)$ is isomorphic to $\ell_1(\Upsilon)$ for some index set Υ (the proof is similar to the case for $\Gamma = \mathbb{N}$, shown by Pełczyński in [83]; an alternative proof of Köthe's result, in English, is given in [95]). So $Z' \simeq \ell_1(\Upsilon)$ for some Υ. But now $q|_{Z'} : Z' \to D$ is an isomorphism of Z' onto D, so that $D \simeq \ell_1(\Upsilon)$.

(d) $\Rightarrow$ (c) Suppose that $D \simeq \ell_1(\Upsilon)$ for some index set Υ. Denote by $(e_j)_{j\in\Upsilon}$ the basis of $\ell_1(\Upsilon)$ (where $e_j(k) = 1$ if $k = j$ and zero otherwise). The map $q : \ell_1(\Gamma) \to D$ composed with the isomorphism $D \simeq \ell_1(\Upsilon)$ gives a bounded linear surjection $S : \ell_1(\Gamma) \to \ell_1(\Upsilon)$ between Banach spaces. By the Open Mapping Theorem 1.2.1, there exists $C > 0$ such that, for each $j \in \Upsilon$ there is $f_j \in \ell_1(\Gamma)$

satisfying $S(f_j) = e_j$ and $||f_j||_1 \leqslant C$. Now define $R : \ell_1(\Upsilon) \to \ell_1(\Gamma)$ by $R(\sum_{j\in\Upsilon} \lambda_j e_j) = \sum_{j\in\Upsilon} \lambda_j f_j$ for each $\sum_{j\in\Upsilon} \lambda_j e_j \in \ell_1(\Upsilon)$. It follows that R is a bounded linear operator and $SR = \mathrm{id}_{\ell_1(\Upsilon)}$. Therefore $RS : \ell_1(\Gamma) \to \ell_1(\Gamma)$ is a bounded projection with $\ker RS = \ker q$. Accordingly, the range of RS complements $\ker q$ in $\ell_1(\Gamma)$. $\square$

We can now give a different proof of Theorem 3.3.1.

Alternative proof of Theorem 3.3.1. We seek to satisfy the conditions of Theorem 3.3.7. Firstly, $B = \mathscr{B}(E_{\mathcal{R}})$ is a unital Banach algebra with identity $I_{E_{\mathcal{R}}}$. Theorem 1.3.4 assures us that $W = \mathscr{W}(E_{\mathcal{R}})$ is a proper closed ideal of $\mathscr{B}(E_{\mathcal{R}})$ and that there is a continuous algebra homomorphism $\theta : \ell_2(\mathbb{N})^\sim \to \mathscr{B}(E_{\mathcal{R}})$ which is a right inverse of the continuous surjection $\beta : \mathscr{B}(E_{\mathcal{R}}) \to \ell_2(\mathbb{N})^\sim$.

We next show that $\beta^{-1}(\ell_2(\mathbb{N})) = \theta(\ell_2(\mathbb{N})) + \mathscr{W}(E_{\mathcal{R}})$. Let $T \in \beta^{-1}(\ell_2(\mathbb{N}))$. Then $T - \theta\beta(T) \in \ker\beta = \mathscr{W}(E_{\mathcal{R}})$, so that

$$T = \theta\beta(T) + (T - \theta\beta(T)) \in \theta(\ell_2(\mathbb{N})) + \mathscr{W}(E_{\mathcal{R}}).$$

The reverse inclusion is clear by applying β, so $\beta^{-1}(\ell_2(\mathbb{N})) = \theta(\ell_2(\mathbb{N})) + \mathscr{W}(E_{\mathcal{R}})$.

Take $T \in \mathscr{B}(E_{\mathcal{R}})$. Then $\beta(T) \in \ell_2(\mathbb{N})^\sim = \ell_2(\mathbb{N}) + \mathbb{K}1_{\ell_2(\mathbb{N})^\sim}$. Since β is surjective there exist $S \in \mathscr{B}(E_{\mathcal{R}})$ and $\lambda \in \mathbb{K}$ such that $\beta(T) = \beta(S) - \lambda 1_{\ell_2(\mathbb{N})^\sim}$. Now, being surjective, β is unital, so $\beta(I_{E_{\mathcal{R}}}) = 1_{\ell_2(\mathbb{N})^\sim}$, which implies that $\beta(T - \lambda I_{E_{\mathcal{R}}}) = \beta(S) \in \ell_2(\mathbb{N})$. Hence $T = (T - \lambda I_{E_{\mathcal{R}}}) + \lambda I_{E_{\mathcal{R}}}$, so we deduce that $\mathscr{B}(E_{\mathcal{R}}) = \beta^{-1}(\ell_2(\mathbb{N})) + \mathbb{K}I_{E_{\mathcal{R}}}$. Also $I_{E_{\mathcal{R}}} \notin \beta^{-1}(\ell_2(\mathbb{N}))$, so $\mathscr{B}(E_{\mathcal{R}}) = \beta^{-1}(\ell_2(\mathbb{N})) \oplus \mathbb{K}I_{E_{\mathcal{R}}}$.

From these calculations and the fact that $\theta(\ell_2(\mathbb{N})) \cap \mathscr{W}(E_{\mathcal{R}}) = \{0\}$, we conclude that $\mathscr{B}(E_{\mathcal{R}}) = \theta(\ell_2(\mathbb{N})) \oplus \mathscr{W}(E_{\mathcal{R}}) \oplus \mathbb{K}I_{E_{\mathcal{R}}}$.

In the notation of Theorem 3.3.7, set $D = \theta(\ell_2(\mathbb{N}))$. Then conditions (i) and (ii) are satisfied because θ is multiplicative, injective and bounded below, so that D is a closed subalgebra of $\mathscr{B}(E_{\mathcal{R}})$ and $D^2 = \{0\} \subseteq W$. Condition (iii) is also satisfied because the restriction $\beta|_D : D \to \ell_2(\mathbb{N})$ is an isomorphism of Banach spaces, which implies that D cannot be isomorphic to $\ell_1(\Upsilon)$ for any index set Υ (for example, $\ell_2(\mathbb{N})$ is reflexive but $\ell_1(\Upsilon)$ is not). $\square$

Example 3.3.8. The extension of $\mathscr{B}(E_{\mathcal{R}})$ given by Theorem 3.3.7 is:

$$\{0\} \longrightarrow \ker\psi \longrightarrow \ell_1(\mathbb{N}) \oplus \mathscr{W}(E_{\mathcal{R}}) \oplus \mathbb{K} \xrightarrow{\psi} \mathscr{B}(E_{\mathcal{R}}) \longrightarrow \{0\} \qquad (3.3.6)$$

where $\psi : \ell_1(\mathbb{N}) \oplus \mathscr{W}(E_{\mathcal{R}}) \oplus \mathbb{K} \to \mathscr{B}(E_{\mathcal{R}})$ is given by $\psi(f, W, \lambda) = q(f) + W + \lambda I_{E_{\mathcal{R}}}$, for $f \in \ell_1(\mathbb{N}), W \in \mathscr{W}(E_{\mathcal{R}}), \lambda \in \mathbb{K}$. Note that this is (in general) different to the extension from Example 3.3.2.

In fact we can show that $\ker \psi = \operatorname{rad} A$, where $A = \ell_1(\Gamma) \oplus \mathscr{W}(E_{\mathcal{R}}) \oplus \mathbb{K}$. By Theorem 3.3.7, (3.3.6) is singular, and therefore radical by Proposition 3.1.7(ii). This means that $\ker \psi \subseteq \operatorname{rad} A$. Given an algebra A and an ideal $J \subseteq \operatorname{rad} A$, it is a standard result that $(\operatorname{rad} A)/J \subseteq \operatorname{rad}(A/J)$ [24, Theorem 1.5.4(ii)]. Combining this with the fact that $\mathscr{B}(X)$ is semisimple for every Banach space X [24, Proposition 1.5.6(ii)], we see that $\ker \psi = \operatorname{rad} A$.

Remark 3.3.9. The extension (3.3.6) is related to an interesting example of Yakovlev [112], which is also recorded in [24, Corollary 4.6.11]. He proved that there are a semisimple Banach algebra A and a singular extension of A which is not admissible, but which splits algebraically. Obtaining a Banach algebra and extension with these properties is not easy. We note that (3.3.6) has the same properties, yet $\mathscr{B}(E_{\mathcal{R}})$ is very different to the Banach algebra A, which is built from a direct sum of symmetric tensor products of Banach spaces.

The conditions for Theorem 3.3.7 are quite demanding—we need a distinct complemented structure inside our Banach algebra B. Can we weaken the assumptions? The next theorem and corollary do this to some extent. They are not a direct generalisation since we add an extra assumption about the density of W^2 in W.

Read claims without proof that $\mathscr{W}(E_{\mathcal{R}})^2$ is not dense in $\mathscr{W}(E_{\mathcal{R}})$ [90, p. 306], in contrast to the James space J_2, where $\mathscr{W}(J_2)^2$ is dense in $\mathscr{W}(J_2)$. The author has not been able to verify Read's claim, and it seems possible that it was a slip of the pen: certainly I^2 is not dense in I, where I is the codimension-one ideal of $\mathscr{B}(E_{\mathcal{R}})$, and the weakly compact operators on the James space form an ideal of codimension one, so perhaps this is what he had in mind. Therefore it is possible that Corollary 3.3.11 does apply to $\mathscr{B}(E_{\mathcal{R}})$, but so far no proof has been forthcoming.

Theorem 3.3.10. *Let C be a Banach algebra containing a proper closed ideal W such that $C^2 \subseteq W$, with W^2 norm dense in W. Suppose that we also have a Banach space X and a linear contraction $q : X \to C$ such that $C = q(X) + W$. Then there is an extension*

$$\{0\} \longrightarrow \ker \psi \xrightarrow{\iota} A_0 \xrightarrow{\psi_0} C \longrightarrow \{0\} \tag{3.3.7}$$

of C which splits algebraically. Moreover, (3.3.7) *splits strongly if and only if $q^{-1}(W)$ is complemented in X.*

Proof. Let A_0 be the Banach algebra and $\psi_0 : A_0 \to C$ the continuous surjection from Construction 3.3.6. Then by definition, (3.3.7) is an extension. Notice that $\ker \psi_0 = \{(x, -q(x)) : x \in q^{-1}(W)\}$.

Let Z be an algebraic complement of $q^{-1}(W)$ in X. Then $Z \oplus W$ is a subalgebra of A_0, and $\psi_0|_{Z\oplus W}$ is bijective. To see this it is enough to show that

$$(Z \oplus W) \oplus \ker \psi_0 = A_0.$$

If $(z, w) \in \ker \psi_0 \cap (Z \oplus W)$, then $\psi_0(z, w) = q(z) + w = 0$, so that $w = -q(z)$. Then $z \in Z \cap q^{-1}(W)$, so that $z = 0$ because their intersection is trivial by definition, and so $w = -q(0) = 0$. So $\ker \psi_0 \cap (Z \oplus W) = \{0\}$.

Also, if $a \in A_0$, then $a = (x, w)$ for some $x \in X, w \in W$. We have $x = z + y$ for some $z \in Z$ and $y \in q^{-1}(W)$. Then $a = (x, w) = (z, w + q(y)) + (y, -q(y))$ where $(z, w + q(y)) \in Z \oplus W$ and $(y, -q(y)) \in \ker \psi_0$. So $(Z \oplus W) + \ker \psi_0 \supseteq A_0$, and since the reverse is obvious we have $(Z \oplus W) \oplus \ker \psi_0 = A_0$.

So $\psi_0|_{Z\oplus W}$ is a bijection. Now let $\rho : C \to A$ be the algebraic inverse of $\psi_0|_{Z\oplus W}$. Then ρ is a splitting homomorphism for (3.3.7), so we have an algebraic splitting.

Suppose that $q^{-1}(W)$ is complemented in X, say by Z'. Then the map $\psi_0|_{Z'\oplus W}$ is a bijection, just as above, but it is now a bijective bounded operator between Banach spaces. By the Banach Isomorphism Theorem 1.2.2, its inverse is bounded, and the inverse is a strong splitting by construction.

For the converse, suppose that (3.3.7) splits strongly, with continuous splitting homomorphism $\varphi : C \to A_0$. Take $v, w \in W$ and write $\varphi(v) = (x, v')$ and $\varphi(y, w')$ for some $x, y \in X$ and $v', w' \in W$. Then using (3.3.4)

$$\begin{aligned} \varphi(vw) &= \varphi(v)\varphi(w) = (0, q(x)q(y) + q(x)w' + v'q(y) + v'w') \\ vw &= (\psi_0 \circ \varphi)(vw) = q(x)q(y) + q(x)w' + v'q(y) + v'w' \end{aligned}$$

so $\varphi(vw) = (0, vw)$. Therefore any element $t \in W^2$ has the property that $\varphi(t) = (0, t)$. By assumption W^2 is dense in W, so there exists a sequence (t_n) in W^2 such that $t_n \to w$ as $n \to \infty$. Then by the continuity of φ:

$$\varphi(w) = \varphi(\lim_{n\to\infty} t_n) = \lim_{n\to\infty} \varphi(t_n) = \lim_{n\to\infty} (0, t_n) = (0, w). \tag{3.3.8}$$

Now we *claim* that $Z' = \{x \in X : (x, 0) \in \operatorname{im} \varphi\}$ is a closed complement of $q^{-1}(W)$ in X. Since φ is a continuous right inverse of ψ_0, it has closed range. Therefore Z' is certainly closed.

Take $x \in X$. Then $\psi_0(x, 0) = q(x)$, and $\varphi(q(x)) = (y, w)$ for some $y \in X, w \in W$. Hence $q(x) = \psi_0\varphi(q(x)) = q(y) + w$ which implies that $q(x - y) \in W$. Write $x = (x - y) + y$. We have seen that $x - y \in q^{-1}(W)$, so it remains to show that $y \in Z'$. But this follows because $\varphi(q(x)) = (y, w) = (y, 0) + (0, w) = (y, 0) + \varphi(w)$ by (3.3.8), so $(y, 0) = \varphi(q(x) - w)$. Therefore $X = Z' + q^{-1}(W)$.

To see that Z' and $q^{-1}(W)$ have trivial intersection, take $x \in X$ such that

$q(x) \in W$ and $(x,0) \in \operatorname{im}\varphi$, say $(x,0) = \varphi(b)$ where $b \in C$. Then

$$q(x) = \psi_0(x,0) = \psi_0(\varphi(b)) = b$$

so that $b \in W$. By (3.3.8), $(x,0) = \varphi(b) = (0,b)$, so that $x = 0 = b$. Hence $Z' \cap q^{-1}(W) = \{0\}$, and so $X = Z' \oplus q^{-1}(W)$. This proves the claim and the result. □

Corollary 3.3.11. *Let B be a unital Banach algebra containing a closed ideal J of codimension 1 and a proper closed ideal W such that $J^2 \subseteq W$, with W^2 dense in W. Suppose that we also have a Banach space X and a linear contraction $q : X \to J$ such that $J = q(X) + W$. Then there is an extension*

$$\{0\} \longrightarrow \ker\psi \longrightarrow A \xrightarrow{\psi} B \longrightarrow \{0\} \tag{3.3.9}$$

of B which splits algebraically. Moreover, (3.3.9) *splits strongly if and only if $q^{-1}(W)$ is complemented in X.*

Proof. This follows from Theorem 3.3.10 and Lemma 3.2.7 (with $C = J$, $A = \widetilde{A_0}$ and $\psi = \widetilde{\psi_0}$) because we can unitise. Note that we are using the fact that $\widetilde{J} \cong B$, and so we can pass to an extension of B with the same properties. □

3.3.3 Extensions of algebras of operators

We give an application of Theorem 3.3.10. For a Banach space E, Bade, Dales and Lykova considered splittings of extensions of $\mathscr{B}(E)$ and $\mathscr{K}(E)$, but not other Banach algebras of operators, for example $\mathscr{W}(E)$ or $\mathscr{S}(E)$. We show that for these ideals of $\mathscr{B}(E)$, the situation is far from clear.

Proposition 3.3.12. *Let E be a Banach space, and let C be a proper closed subalgebra of $\mathscr{B}(E)$ such that*

(i) *$\mathscr{A}(E) \subseteq C$;*

(ii) *$\mathscr{A}(E)$ is not complemented in C;*

(iii) *$C^2 \subseteq \mathscr{A}(E)$.*

Then there is an extension of C which splits algebraically but not strongly.

Proof. Let C be a proper closed subalgebra of $\mathscr{B}(E)$. Then C is a Banach algebra in its own right. Looking at the statement of Theorem 3.3.10, we set $W = \mathscr{A}(E)$, which is contained in C by (i). By definition $\overline{\mathscr{A}(E)^2}$ is a closed ideal of $\mathscr{B}(E)$. But it is well known that $\mathscr{A}(E)$ is the smallest non-zero closed ideal of $\mathscr{B}(E)$; therefore $\mathscr{A}(E)^2$ is dense in $\mathscr{A}(E)$. Now set $X = C$ and choose $q : C \to C$ to be the identity map: this is certainly a linear contraction, and $q(C)+\mathscr{A}(E) = C$. Of course $q^{-1}(\mathscr{A}(E)) = \mathscr{A}(E)$, which, by (ii), is not complemented in C. The result now follows from Theorem 3.3.10. $\square$

Example 3.3.13. Consider $E = L_1[0,1]$. Set $C = \mathscr{W}(E)$, which is a closed subalgebra of $\mathscr{B}(E)$, and since E is not reflexive $\mathscr{W}(E) \subsetneq \mathscr{B}(E)$. Also, $L_1[0,1]$ has the approximation property so $\mathscr{A}(E) = \mathscr{K}(E)$, and of course $\mathscr{K}(E) \subseteq \mathscr{W}(E)$. Moreover, by the Dunford–Pettis Theorem [33, Corollary VI.8.13], $\mathscr{W}(E)^2 = \mathscr{K}(E)$ (see also Chapter 6).

Ghenciu and Lewis [41, Corollary 12] credit the following result to Emmanuele: Let F be a Banach space. If ℓ_1 is complemented in F, and F does not have the Schur property, then $\mathscr{K}(F)$ is not complemented in $\mathscr{W}(F)$. Now $L_1[0,1]$ certainly contains complemented copies of ℓ_1 [3, Lemma 5.1.1], and $L_1[0,1]$ lacks the Schur property [3, p. 102], so $\mathscr{K}(L_1[0,1])$ is not complemented in $\mathscr{W}(L_1[0,1])$. Proposition 3.3.12 now tells us that there is an extension of $\mathscr{W}(E)$ which splits algebraically but not strongly.

Example 3.3.14. One can show that $\mathscr{W}(L_1[0,1]) = \mathscr{S}(L_1[0,1])$, so that there is an extension of $\mathscr{S}(L_1[0,1])$ which splits algebraically but not strongly by the previous example. Let $E = L_1[0,1]$. We know that $\mathscr{W}(E) \subseteq \mathscr{S}(E)$ as a consequence of the Dunford–Pettis Theorem [3, Theorem 5.5.1]. For the converse, suppose that $T \in \mathscr{B}(E)$ is not weakly compact. Then by a theorem of Pełczyński [84, Part II, Theorem 1], there is a copy of ℓ_1 in E such that $T|_{\ell_1}$ is bounded below. Thus T is not strictly singular, and so $\mathscr{W}(E) = \mathscr{S}(E)$.

Subalgebras C satisfying Proposition 3.3.12 are 'very far from having an identity' because (ii) and (iii) say that C is somehow much larger than C^2. The next proposition is a result in the opposite direction. Recall that a *bounded left approximate identity* for a Banach algebra A is a bounded net (e_α) of elements in A such that for every $a \in A$, $\lim_\alpha e_\alpha a = a$.

Proposition 3.3.15. *Let E be a Banach space with a continued bisection, and let J be a closed ideal of $\mathscr{B}(E)$. If J has a bounded left approximate identity, then every extension of J which splits algebraically also splits strongly.*

Proof. Given the conditions, [24, Theorem 5.4.11] says that every algebra homomorphism from J into a Banach algebra is continuous. The result follows. $\square$

To consider a specific example, when does $\mathscr{W}(E)$ have a bounded left approximate identity (b.l.a.i.)? If E is reflexive then $\mathscr{B}(E) = \mathscr{W}(E)$ so $\mathscr{W}(E)$ even has an identity (provided E is non-zero). Suppose that E has (BAP) and $\mathscr{W}(E) = \mathscr{K}(E)$. Then $\mathscr{W}(E)$ has a b.l.a.i. by [24, Theorem 2.9.37], as noted by Bade, Dales and

Lykova [12, Theorem 3.19(ii)]; this includes spaces like ℓ_1 and c_0. For Banach spaces outside of this, it is mostly unclear whether or not $\mathscr{W}(E)$ has a b.l.a.i..

Chapter 4

Homological Bidimension

The concept of the homological bidimension of an algebra is standard in homology theory (see *e.g.*, [77, Chapter VII]). It associates a non-negative integer with an algebra, and captures something of the homological defects of the algebra. Roughly speaking, the higher the bidimension, the more defective the algebra. As a purely algebraic concept, the homological bidimension of a *Banach* algebra fails to take the topological structure into account, and thus the bidimension does not quite capture the information one may like. To rectify this, Helemskii transferred the notion to the topological setting. His concept of (topological) homological bidimension has an extra layer of structure which provides the right framework for studying the homology of Banach algebras. The topological version shall be our focus in this chapter; we shall not mention the algebraic version again, and so we drop the adjective 'topological' throughout. Thus the homological bidimension of a Banach algebra always refers to the topological notion.

Together with his students, Helemskii has calculated the homological bidimension of many Banach algebras, or given bounds on the possible values. See the fine book [52] for an exposition. Selivanov has shown that for every non-negative integer n there is a Banach algebra with homological bidimension equal to n, and that some important examples actually have infinite bidimension [99]. But for a general Banach algebra A calculating the bidimension is challenging, in particular because it requires knowledge about the cohomology groups with respect to an arbitrary Banach A-bimodule.

A significant open problem in the theory is to calculate the homological bidimension of $\mathscr{B}(H)$ for an (infinite-dimensional, separable) Hilbert space H. It is known to be at least one; little other progress seems to have been made. Helemskii expects it to be at least two [52, Chapter V, 2.5]. For an arbitrary Banach space X, the state of knowledge about the homological bidimension of $\mathscr{B}(X)$ is similar (as stated, for example, in [12, p. 27]). If X is finite-dimensional then the value is zero, and if X is infinite-dimensional and has the approximation property then it is at least one. No examples are known for which the bidimension of $\mathscr{B}(X)$ is at least two. If an example could be found this could potentially give some idea how to prove Helemskii's hunch about Hilbert space. In the present chapter we tackle this problem: is there a Banach space X such that the homological bidimension of $\mathscr{B}(X)$ is at least two? The author would like to thank Dr Z.A. Lykova (Newcastle, England) for suggesting this question, which seems to go back to Helemskii's seminar at Moscow State University. The question is connected to the topic of Chapter 3 by a result of Johnson (*cf.* Proposition 4.2.8), which relates singular admissible extensions of a Banach algebra to its second continuous cohomology groups. Thus, finding an example amounts to showing that there is a Banach space X and a singular, admissible extension of $\mathscr{B}(X)$ which does not split strongly. The machinery we have developed in Chapter 3 involving pullbacks can be applied to give several examples of such Banach spaces, as we explain below. In fact we give a useful approach for calculating the homological bidimension of a Banach algebra which satisfies certain conditions. Sadly, this approach sheds no light on the case of $\mathscr{B}(H)$. It perhaps gives some evidence to suggest Helemskii's intuition is correct, but since the examples we give are 'exotic' Banach spaces which are nothing like Hilbert space, it is not easy to tell.

The original results in this chapter, several of which feature in [71], are joint with N.J. Laustsen.

4.1 Cohomology groups

Fundamental to the whole chapter is the notion of cohomology groups and their continuous analogues. For each $n \in \mathbb{N}$ there is a corresponding n^{th} cohomology group. We begin by defining the first and second cohomology groups, before giving the general definition, to avoid overwhelming the reader with notation, and because they will be the most important cases for us. The algebraic theory of cohomology groups originates with Hochschild [54]; the 'Banach' versions were studied by Kamowitz [61], Guichardet [49], Helemskii [53] and Johnson [59]. First we introduce the algebraic versions.

Definition 4.1.1. Let B be an algebra and Y a B-bimodule. Recall that a linear map $D : B \to Y$ is a *derivation* if

$$D(ab) = a \cdot Db + Da \cdot b \qquad (a, b \in B).$$

The vector space of all derivations from B to Y is denoted $Z^1(B, Y)$. Let $x \in Y$.

Then the map $D_x : B \to Y$ given by

$$D_x(a) = a \cdot x - x \cdot a \qquad (a \in B)$$

is a derivation; such maps are called *inner derivations*. The set of all inner derivations from B to Y is a subspace of $Z^1(B, Y)$ which we denote by $N^1(B, Y)$. The *first cohomology group of B with coefficients in Y* is the quotient vector space

$$H^1(B, Y) = Z^1(B, Y)/N^1(B, Y).$$

A bilinear map $T : B \times B \to Y$ is a *2-cocycle* if

$$a \cdot T(b, c) - T(ab, c) + T(a, bc) - T(a, b) \cdot c = 0 \qquad (a, b, c \in B). \tag{4.1.1}$$

The vector space of 2-cocycles is $Z^2(B, Y)$. For each linear map $S : B \to Y$, define the bilinear map

$$(\delta^1 S)(a, b) = a \cdot Sb - S(ab) + Sa \cdot b \qquad (a, b \in B). \tag{4.1.2}$$

Then $\delta^1 S$ is a 2-cocycle, as is easily checked. A 2-cocycle T is a *2-coboundary* if there is a linear map S such that $T = \delta^1 S$. The subspace of $Z^2(B, Y)$ consisting of 2-coboundaries is $N^2(B, Y)$, and the vector space

$$H^2(B, Y) = Z^2(B, Y)/N^2(B, Y)$$

is the *second cohomology group of B with coefficients in Y*.

Next we form the continuous analogues. Let X and Y be Banach spaces. Recall that a bilinear map $T : X \times X \to Y$ is *bounded* if there exists $C > 0$ such that $||T(x, y)|| \leqslant C||x|| \, ||y||$ for each $(x, y) \in X \times X$; the norm of T is the infimum of all such constants. Similar definitions apply to n-linear maps for $n \in \mathbb{N}$.

Definition 4.1.2. Let B be a Banach algebra and Y a Banach B-bimodule. For each $n \in \mathbb{N}$, write $\mathscr{B}^n(B, Y)$ for the Banach space of all bounded n-linear maps from $B \times \cdots \times B$ to Y (n copies of B). For simplicity we prefer $\mathscr{B}(B, Y)$ to $\mathscr{B}^1(B, Y)$.

The Banach space of continuous derivations from B to Y is $\mathcal{Z}^1(B, Y) = Z^1(B, Y) \cap \mathscr{B}(B, Y)$. It is clear that each inner derivation from a Banach algebra B into a Banach B-bimodule is continuous, and so the *first Banach cohomology group of B with coefficients in Y* is the complete seminormed space

$$\mathcal{H}^1(B, Y) = \mathcal{Z}^1(B, Y)/N^1(B, Y).$$

Also, define

$$\mathscr{Z}^2(B,Y) = Z^2(B,Y) \cap \mathscr{B}^2(B,Y) \quad \text{and} \quad \mathscr{N}^2(B,Y) = \{\delta^1 S : S \in \mathscr{B}(B,Y)\}.$$

We refer to the Banach space $\mathscr{Z}^2(B,Y)$ as the *continuous 2-cocycles* and its (not necessarily closed) subspace $\mathscr{N}^2(B,Y)$ as the *continuous 2-coboundaries.* Then the *second Banach cohomology group of B with coefficients in Y* is the complete seminormed space

$$\mathscr{H}^2(B,Y) = \mathscr{Z}^2(B,Y)/\mathscr{N}^2(B,Y).$$

We now define a cohomology group corresponding to each natural number.

Definition 4.1.3. Let B be a Banach algebra and Y a Banach B-bimodule. Fix $n \in \mathbb{N}$ and write $L^n(B,Y)$ for the vector space of n-linear maps from $B \times \cdots \times B$ to Y (n copies of B). For convenience write $Y = L^0(B,Y)$. Then we can form the standard homology complex:

$$0 \to L^0(B,Y) \xrightarrow{\delta^0} L^1(B,Y) \xrightarrow{\delta^1} L^2(B,Y) \to \cdots \to L^n(B,Y) \xrightarrow{\delta^n} L^{n+1}(B,Y) \to \cdots,$$

so that $\delta^{m+1} \circ \delta^m = 0$ for each $m \in \mathbb{N}_0$, where $\delta^0 : y \mapsto D_y$, and the map δ^n is given by

$$\begin{aligned}(\delta^n T)(a_1, \ldots, a_{n+1}) &= a_1 \cdot T(a_2, \ldots, a_{n+1}) + \\ &\sum_{i=1}^{n} (-1)^i T(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1} T(a_1, \ldots, a_n) \cdot a_{n+1}\end{aligned}$$

for each $T \in L^n(B,Y)$ and $a_1, \ldots, a_{n+1} \in B$. The n^{th} *cohomology group of B with coefficients in Y* is the vector space

$$H^n(B,Y) = \ker \delta^n / \operatorname{im} \delta^{n-1}.$$

There is an analogous continuous complex:

$$0 \to \mathscr{B}^0(B,Y) \xrightarrow{\delta^0} \mathscr{B}^1(B,Y) \xrightarrow{\delta^1} \mathscr{B}^2(B,Y) \to \cdots \to \mathscr{B}^n(B,Y) \xrightarrow{\delta^n} \mathscr{B}^{n+1}(B,Y) \to \cdots,$$

and so the n^{th} *Banach cohomology group of B with coefficients in Y* is

$$\mathscr{H}^n(B,Y) = \ker \delta^n / \operatorname{im} \delta^{n-1}.$$

It is easily checked that these definitions coincide with our earlier ones for $n = 1, 2$. With the basic definitions in place we can define the homological bidimension. Helemskii's definition is different to ours, but he shows they are equivalent in [52, Chapter III, Theorem 5.15].

Definition 4.1.4. Let B be a Banach algebra. The *homological bidimension of* B is the non-negative integer

$$\operatorname{db} B = \min\{n \in \mathbb{N}_0 : \mathcal{H}^{n+1}(B, Y) = \{0\} \text{ for every Banach } B\text{-bimodule } Y\}$$

when this is finite. If there is no such non-negative integer, we set $\operatorname{db} B = \infty$.

A useful result when calculating the homological bidimension of a Banach algebra is Johnson's 'reduction of dimension' formula [59, p. 9]. For a Banach algebra B and a Banach B-bimodule Y, the Banach space $\mathscr{B}(B, Y)$ becomes a Banach B-bimodule when equipped with the module maps $(a, T) \mapsto a * T$ and $(a, T) \mapsto T * a$ from $B \times \mathscr{B}(B, Y)$ into $\mathscr{B}(B, Y)$, where

$$(a * T)(b) = a \cdot Tb, \quad (T * a)(b) = T(ab) - Ta \cdot b \qquad (b \in B).$$

Then for each $n \in \mathbb{N}$ there is a linear homeomorphism:

$$\mathcal{H}^n(B, \mathscr{B}(B, Y)) \simeq \mathcal{H}^{n+1}(B, Y). \tag{4.1.3}$$

In particular, if for every Banach B-bimodule Y we have $\mathcal{H}^1(B, Y) = \{0\}$, then $\mathcal{H}^2(B, Y) = \{0\}$ for every Banach B-bimodule Y.

4.2 Calculating second cohomology groups and homological bidimension

In this section we give an account of some of the known results on homological bidimension, especially concerning Banach algebras of the form $\mathscr{B}(X)$ for a Banach space X. Then, combining the results about pullbacks from Chapter 3 with standard techniques, we describe a method for showing that a Banach algebra (subject to certain conditions) has homological bidimension at least two.

Which Banach algebras have homological bidimension zero?

Proposition 4.2.1. *Let A be a semisimple, finite-dimensional Banach algebra. Then* $\operatorname{db} A = 0$.

Proof. We refer to [52, Chapter III, Theorem 5.17]; a more general result can be found in [24, Theorem 1.9.21]. □

So for a finite-dimensional Banach space X, $\operatorname{db} \mathscr{B}(X) = 0$. It is conjectured that the converse to Proposition 4.2.1 is true. Thus having homological bidimension equal to zero is a very strong condition; such algebras are termed *contractible*.

A partial result towards the conjecture was proved by Taylor (see [24, Corollary 2.8.49]).

Theorem 4.2.2 (Taylor)**.** *Let A be a Banach algebra such that* $\operatorname{db} A = 0$. *Suppose either that A is commutative, or that A has (AP) as a Banach space. Then A is semisimple and finite-dimensional.* □

Therefore a counterexample to the conjecture would at least have to be a non-commutative Banach algebra lacking the approximation property. The only natural example of a Banach space known to lack (AP) is $\mathscr{B}(H)$ for an infinite-dimensional separable Hilbert space H, a result of Szankowski [102]. In the next result we note that $\operatorname{db} \mathscr{B}(H) \geqslant 1$, so a counterexample to the conjecture would be a very strange Banach algebra indeed. Specialising to the case where $A = \mathscr{B}(X)$, the conjecture becomes that having $\operatorname{db} \mathscr{B}(X) = 0$ forces X to be finite-dimensional. A strong partial result in this direction was proved by Johnson [60, Proposition 5.1], generalising a theorem of Selivanov [100]; the approximation property again seems to be somehow important.

Theorem 4.2.3 (Johnson, Selivanov)**.** *Let X be an infinite-dimensional Banach space with (AP). Then* $\operatorname{db} \mathscr{B}(X) \geqslant 1$. □

Since a separable Hilbert space has the approximation property we obtain the following corollary.

Corollary 4.2.4. *Let H be an infinite-dimensional separable Hilbert space. Then* $\operatorname{db} \mathscr{B}(H) \geqslant 1$. □

What about $\operatorname{db} A = 1$? This is much harder to even attempt to characterise. Firstly, there are finite-dimensional Banach algebras A (not semisimple or commutative) that have $\operatorname{db} A = 1$ (see [52, p. 217]). Also, Selivanov [99] has proved that we may equip any complex Banach space $X \neq \mathbb{C}$ with an algebra structure making it into a Banach algebra with $\operatorname{db} X = 1$. Examples amongst the standard Banach algebras are much harder to come by. Accordingly, Helemskii asked the following general question [52, Chapter V, 2.5, Question 2].

Question 5. Let A be a semisimple, infinite-dimensional Banach algebra. Is it true that $\operatorname{db} A \geqslant 2$?

If true, this would mean that $\mathscr{B}(X)$ has homological bidimension at least two for every infinite-dimensional Banach space X. As explained in the introduction to the chapter, evidence for this is thin on the ground, as no examples are known. Let us formally state Lykova's more specific question, mentioned earlier.

Question 6. Does there exist a (necessarily infinite-dimensional) Banach space X such that $\operatorname{db}\mathscr{B}(X) \geqslant 2$?

Helemskii's question (Question 5) was partly motivated by his fundamental result in the commutative case, known as the *Global Dimension Theorem* [52, Chapter V, Assertion 2.21]. We denote the set of characters on a Banach algebra A by Φ_A.

Theorem 4.2.5 (Helemskii's Global Dimension Theorem)**.** *Let A be a commutative Banach algebra such that Φ_A is an infinite set. Then* $\operatorname{db} A \geqslant 2$. □

Even though this is a result about commutative Banach algebras, we shall use it in proving there are Banach spaces with $\operatorname{db}\mathscr{B}(X) \geqslant 2$.

There is a certain class of Banach algebras A for which proving that $\operatorname{db} A = 2$ is often possible: these are the *biprojective* Banach algebras. The definition goes as follows. The projective tensor product $A\widehat{\otimes}A$ is naturally a Banach A-bimodule, and there is a continuous product map $\pi_A : A\widehat{\otimes}A \to A$ such that $\pi_A(a \otimes b) = ab$ $(a, b \in A)$. We say that A is *biprojective* if there is a continuous bimodule homomorphism $\rho_A : A \to A\widehat{\otimes}A$ which is a right inverse of π_A. In the next result we see that biprojectivity provides an upper bound on the bidimension.

Theorem 4.2.6 (Helemskii)**.** *Let A be a biprojective Banach algebra. Then db $A \leqslant 2$. Suppose further that A is commutative and Φ_A is infinite or A is infinite-dimensional, semisimple and has (AP). Then* $\operatorname{db} A = 2$.

Proof. [52, Chapter V, Assertion 2.30] and [24, Theorem 2.8.56]. □

Example 4.2.7. The Banach sequence algebras c_0 and ℓ_1, equipped with the pointwise product, are both biprojective, commutative, and have an infinite set of characters. So they have homological bidimension equal to 2 [52, pp. 189-190].

This is a useful result, but not so much in our investigation of $\operatorname{db}\mathscr{B}(X)$ because a unital biprojective Banach algebra is contractible [24, Theorem 2.8.48]. New techniques are needed to study whether $\operatorname{db}\mathscr{B}(X) \geqslant 2$. Our first step is to make the link with extensions of $\mathscr{B}(X)$. The general principle goes back to pure algebra, and has been recognised by various authors (for example similar statements to ours are recorded by Johnson [59] and Helemskii [52, Chapter I, Theorem 1.10]), but in the case of $\mathscr{B}(X)$, no use has been found for it until now. Although this is a known result, we give a proof for completeness.

Proposition 4.2.8. *Let B be a Banach algebra. Then the following are equivalent:*

(a) *B has homological bidimension at least two;*

(b) *there is a Banach B-bimodule Y such that $\mathcal{H}^2(B, Y) \neq \{0\}$;*

(c) *there is a singular, admissible extension of B which does not split strongly.*

Proof. (a) $\Rightarrow$ (b) This follows from the definition.

(b) $\Rightarrow$ (a) Suppose that there is a Banach B-bimodule Y such that $\mathcal{H}^2(B, Y) \neq \{0\}$. Then $\operatorname{db} B \neq 1$. By Johnson's formula (4.1.3), $\mathcal{H}^1(B, \mathscr{B}(B, Y)) \neq \{0\}$. Thus $\operatorname{db} B$ is not zero either, and hence B has homological bidimension at least two.

(b) $\Rightarrow$ (c) Suppose that there is a Banach B-bimodule Y such that $\mathcal{H}^2(B, Y) \neq \{0\}$, that is, there exists a continuous 2-cocycle $T : B \times B \to Y$ which is not a continuous 2-coboundary. Consider the Banach space $B \oplus Y$ with the usual pointwise operations and the norm $||(b, y)||_1 = ||b||_B + ||y||_Y$ for $b \in B$ and $y \in Y$. Then equip $B \oplus Y$ with the product

$$(a, x)(b, y) = (ab, a \cdot y + x \cdot b + T(a, b)) \qquad (a, b \in B, x, y \in Y). \tag{4.2.1}$$

It is quickly checked, using the cocycle identity (4.1.1), that $B \oplus Y$ is a Banach algebra with this product (with respect to an equivalent norm). Let $\pi_1 : B \oplus Y \to B$ be the algebra homomorphism $\pi_1(b, y) = b$. Then we have an extension

$$\{0\} \longrightarrow \{0\} \oplus Y \xrightarrow{\iota} B \oplus Y \xrightarrow{\pi_1} B \longrightarrow \{0\} \tag{4.2.2}$$

which is singular, and admissible via the map $Q : b \mapsto (b, 0)$.

Assume that (4.2.2) splits strongly, with splitting homomorphism $\theta : B \to B \oplus Y$ and denote by $\pi_2 : B \oplus Y \to Y$, $\pi_2(b, y) = y$ the projection onto Y. Then $S = \pi_2 \circ \theta : B \to Y$ is a bounded linear map. For each $b \in B$ we have $(\pi_1 \circ \theta)(b) = b$ and $\theta(b) = (b, \pi_2 \circ \theta(b)) = (b, Sb)$. Choose $a, b \in B$. Now θ is multiplicative, so by (4.2.1) we obtain

$$(ab, S(ab)) = \theta(ab) = \theta(a)\theta(b) = (a, Sa)(b, Sb) = (ab, a \cdot Sb + Sa \cdot b + T(a, b)).$$

It follows that $S(ab) = a \cdot Sb + Sa \cdot b + T(a, b)$, and so

$$T(a, b) = -[S(ab) - S(a) \cdot b - a \cdot S(b)] = (\delta^1(-S))(a, b),$$

using (4.1.2). Hence $T = \delta^1(-S)$. This implies that T is a continuous 2-coboundary, contrary to assumption. Thus (4.2.2) cannot split strongly.

(c) $\Rightarrow$ (b) Suppose that

$$\{0\} \longrightarrow \ker \pi \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow \{0\} \tag{4.2.3}$$

is a singular, admissible extension of B which does not split strongly. Choose an admissible map $Q : B \to A$. Since (4.2.3) is a singular extension, Proposition

3.1.7(iii) implies that $\ker\pi$ is a Banach B-bimodule for the module maps $d \cdot a = dQ(a)$ and $a \cdot d = Q(a)d$ for each $a \in B$ and $d \in \ker\pi$.

We *claim* that the bilinear map $T : B \times B \to \ker\pi$ given by

$$T(a,b) = Q(a)Q(b) - Q(ab) \qquad (a, b \in B)$$

belongs to $\mathcal{Z}^2(B, \ker\pi)\backslash\mathcal{N}^2(B, \ker\pi)$. Firstly, note that $Q(a)Q(b) - Q(ab) \in \ker\pi$ for each $a, b \in B$ because π is multiplicative. We also have

$$||T(a,b)|| = ||Q(a)Q(b) - Q(ab)|| \leqslant (||Q||^2 + ||Q||)||a||\,||b||$$

for all $a, b \in B$ so that $T \in \mathscr{B}^2(B, \ker\pi)$. Now we check the cocycle identity (4.1.1); for every $a, b, c \in B$:

$$\begin{aligned}
&a \cdot T(b,c) - T(ab,c) + T(a,bc) - T(a,b) \cdot c \\
&= Q(a)\big(Q(b)Q(c) - Q(bc)\big) - \big(Q(ab)Q(c) - Q(abc)\big) \\
&+ \big(Q(a)Q(bc) - Q(abc)\big) - \big(Q(a)Q(b) - Q(ab)\big)Q(c) = 0
\end{aligned}$$

so $T \in \mathcal{Z}^2(B, \ker\pi)$.

Now suppose that $T \in \mathcal{N}^2(B, \ker\pi)$. Then by definition there exists $S \in \mathscr{B}(B, \ker\pi)$ such that $\delta^1 S = T$, that is, for every $a, b \in B$:

$$Q(a)Q(b) - Q(ab) = T(a,b) = a \cdot Sb - S(ab) + Sa \cdot b = Q(a)S(b) - S(ab) + S(a)Q(b). \tag{4.2.4}$$

Using (4.2.4) and the fact that $(\ker\pi)^2 = \{0\}$, it is now easy to see that the continuous linear map $Q - S : B \to A$ is a splitting homomorphism for (4.2.3). Therefore (4.2.3) splits strongly. But this contradicts our assumption. So T cannot be a continuous 2-cocycle, which implies that $\mathcal{H}^2(B, \ker\pi) \neq \{0\}$. $\square$

How can we make use of this result? Our next result is another that involves a slicing-type simplification. If we want to prove that a certain Banach algebra has homological bidimension at least two, it is enough to find a closed subalgebra with the property which is complemented by a closed ideal.

Corollary 4.2.9. *Let B be a Banach algebra. Suppose that there are a Banach algebra C and an extension:*

$$\{0\} \longrightarrow \ker\beta \xrightarrow{\iota} B \xrightarrow{\beta} C \longrightarrow \{0\} \tag{4.2.5}$$

which splits strongly. Suppose also that there are a Banach algebra A and an *extension of C:*

$$\{0\} \longrightarrow \ker\alpha \xrightarrow{\iota} A \xrightarrow{\alpha} C \longrightarrow \{0\} \tag{4.2.6}$$

which is singular and admissible but does not split strongly. Then there is a singular admissible extension of B which does not split strongly. Hence $\operatorname{db} B \geqslant 2$.

Proof. Given our assumptions, Theorem 3.2.5 implies that there is a singular admissible extension of B which does not split strongly. Hence, by Proposition 4.2.8, $\operatorname{db} B \geqslant 2$. □

We are now well-equipped to answer Question 6: there are at least three examples of Banach spaces X such that $\operatorname{db} \mathscr{B}(X) \geqslant 2$. One of these is Read's space $E_{\mathcal{R}}$, and another is the Dales–Loy–Willis space E_{DLW}. Before we make a precise statement let us introduce the final relevant space.

Definition 4.2.10. Let K be a compact Hausdorff space. A bounded operator $T \in \mathscr{B}(C(K))$ is a *weak multiplication* if there exist $g \in C(K)$ and $S \in \mathscr{W}(C(K))$ such that $Tf = gf + Sf$ for every $f \in C(K)$.

Koszmider [66], assuming the Continuum Hypothesis (CH), and Plebanek [86], with no assumptions beyond ZFC, have constructed an infinite compact Hausdorff space K_0 with no isolated points such that every $T \in \mathscr{B}(C(K_0))$ is a weak multiplication. We use this in the next theorem.

Theorem 4.2.11. *Let X be one of the following Banach spaces:*

(i) $X = E_{\mathcal{R}}$*;*

(ii) $X = C(K_0)$*, where K_0 has no isolated points and is such that every bounded operator $T \in \mathscr{B}(C(K_0))$ is a weak multiplication;*

(iii) $X = E_{\mathrm{DLW}}$*, the Dales–Loy–Willis space.*

Then $\operatorname{db} \mathscr{B}(X) \geqslant 2$.

Proof. We seek to apply Corollary 4.2.9 with $B = \mathscr{B}(X)$.

(i) Let $X = E_{\mathcal{R}}$ and take $C = \ell_2(\mathbb{N})^{\sim}$, where $\ell_2(\mathbb{N})$ has the trivial product. Theorem 1.3.4 says that the extension

$$\{0\} \longrightarrow \mathscr{W}(E_{\mathcal{R}}) \longrightarrow \mathscr{B}(E_{\mathcal{R}}) \xrightarrow{\beta} \ell_2(\mathbb{N})^{\sim} \longrightarrow \{0\} \tag{4.2.7}$$

splits strongly. Therefore by Lemma 3.2.7 and Corollary 4.2.9 it is enough to find an extension of $\ell_2(\mathbb{N})$ which is singular and admissible, but does not split strongly.

Take two unit vectors $\eta, \xi \in \ell_2(\mathbb{N})$ and let $\mu : \ell_2(\mathbb{N}) \times \ell_2(\mathbb{N}) \to \mathbb{K}$ be the nonzero bilinear contraction given by $\mu(\lambda, \zeta) = (\lambda|\eta)(\zeta|\xi)$ for $\lambda, \zeta \in \ell_2(\mathbb{N})$. Form the

Banach space $\ell_2(\mathbb{N})\oplus\mathbb{K}$ with the 1-norm and pointwise vector space operations, and equip it with the product $(\lambda, r)(\zeta, s) = (0, \mu(\lambda, \zeta))$ for $\lambda, \zeta \in \ell_2(\mathbb{N})$ and $r, s \in \mathbb{K}$. Then $\ell_2(\mathbb{N}) \oplus \mathbb{K}$ is a Banach algebra, and we have an extension

$$\{0\} \longrightarrow \{0\} \oplus \mathbb{K} \longrightarrow \ell_2(\mathbb{N}) \oplus \mathbb{K} \xrightarrow{\alpha} \ell_2(\mathbb{N}) \longrightarrow \{0\} \tag{4.2.8}$$

where $\alpha(\lambda, r) = \lambda$ for each $\lambda \in \ell_2(\mathbb{N}), r \in \mathbb{K}$. The extension is clearly singular, and $Q : \lambda \mapsto (\lambda, 0)$ is an admissible map.

Assume that $\theta : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N}) \oplus \mathbb{K}$ is a continuous splitting homomorphism for (4.2.8). Then because $\ell_2(\mathbb{N})$ has the trivial product

$$(0, \mu(\lambda, \zeta)) = \theta(\lambda)\theta(\zeta) = \theta(\lambda\zeta) = \theta(0) = (0, 0) \qquad (\lambda, \zeta \in \ell_2(\mathbb{N})).$$

Thus $\mu = 0$, but this is false. So (4.2.8) does not split strongly.

(ii) Let K_0 have no isolated points and be such that every $T \in \mathscr{B}(C(K_0))$ is a weak multiplication. It is shown in [26, Theorem 6.5(i)] that this implies there is an extension

$$\{0\} \longrightarrow \mathscr{W}(C(K_0)) \longrightarrow \mathscr{B}(C(K_0)) \longrightarrow C(K_0) \longrightarrow \{0\} \tag{4.2.9}$$

which splits strongly. Now for any infinite compact space K, $C(K)$ has an infinite set of characters, namely the point evaluations, so Helemskii's Global Dimension Theorem 4.2.5 implies that $\operatorname{db} C(K_0) \geqslant 2$. Thus, by Proposition 4.2.8, there is a singular, admissible extension of $C(K_0)$ which does not split strongly. Now Corollary 4.2.9 and Proposition 4.2.8 yield the result.

(iii) Let $X = E_{\mathrm{DLW}}$. By [28, Proposition 3.5] and [28, p. 208], we have an extension

$$\{0\} \longrightarrow \mathscr{J}(E_{\mathrm{DLW}}) \longrightarrow \mathscr{B}(E_{\mathrm{DLW}}) \longrightarrow \ell_\infty(\mathbb{Z}) \longrightarrow \{0\} \tag{4.2.10}$$

which splits strongly, where $\mathscr{J}(E_{\mathrm{DLW}})$ is the closed ideal of $\mathscr{B}(E_{\mathrm{DLW}})$ defined in [28, Definition 3.4]. It is standard that $\ell_\infty(\mathbb{Z}) \cong C(\beta\mathbb{Z})$, where $\beta\mathbb{Z}$ denotes the Stone–Čech compactification of the integers, and so there is an extension of $\ell_\infty(\mathbb{Z})$ which is singular and admissible, but does not split strongly, as explained in (ii). Again, Corollary 4.2.9 and Proposition 4.2.8 give the result. $\square$

The exact value of $\operatorname{db} \mathscr{B}(X)$ for the Banach spaces in Theorem 4.2.11 is unknown. Most techniques for bounding the value from above apply only when the Banach algebra is biprojective, and, as mentioned above, unital biprojective Banach algebras are contractible, so $\operatorname{db} \mathscr{B}(X)$ is not biprojective for these Banach spaces. Thus new ideas are probably required to make progress.

The corresponding result for the Calkin algebras of these three spaces is easily deduced. For brevity we focus on $E_{\mathcal{R}}$, but the other cases are exactly analogous, replacing the weakly compact operators with the relevant ideal.

Corollary 4.2.12. *For each closed ideal J of $\mathscr{B}(E_{\mathcal{R}})$ contained in $\mathscr{W}(E_{\mathcal{R}})$, there is a singular, admissible extension of $\mathscr{B}(E_{\mathcal{R}})/J$ which does not split strongly. Therefore $\operatorname{db} \mathscr{B}(E_{\mathcal{R}})/\mathscr{K}(E_{\mathcal{R}})$ and $\operatorname{db} \mathscr{B}(E_{\mathcal{R}})/\mathscr{W}(E_{\mathcal{R}})$ are at least two.*

Proof. Combining Corollary 4.2.9 with the fact that the extension (3.3.3) splits strongly, we see that it is enough to find an extension of $\ell_2(\mathbb{N})^{\sim}$ which is singular and admissible, but does not split strongly. We have already achieved this in Theorem 4.2.11(i), so the result follows. $\square$

We have proved that in certain cases there are elements of $\mathcal{H}^2(\mathscr{B}(X), \ker\delta)$ which are non-zero. The next proposition shows that our pullback method produces, in a sense, lots of these elements.

Proposition 4.2.13. *Let B be a Banach algebra. Suppose that there are a Banach algebra C and an extension:*

$$\{0\} \longrightarrow \ker\beta \xrightarrow{\iota} B \xrightarrow{\beta} C \longrightarrow \{0\} \tag{4.2.11}$$

which splits strongly. Suppose also that there are a Banach algebra A and a singular, admissible extension of C:

$$\{0\} \longrightarrow \ker\alpha \xrightarrow{\iota} A \xrightarrow{\alpha} C \longrightarrow \{0\}. \tag{4.2.12}$$

Then there are a singular, admissible extension of B:

$$\{0\} \longrightarrow \ker\delta \xrightarrow{\iota} D \xrightarrow{\delta} B \longrightarrow \{0\} \tag{4.2.13}$$

and a bounded linear embedding

$$\mathcal{H}^2(C, \ker\alpha) \hookrightarrow \mathcal{H}^2(B, \ker\delta).$$

Proof. By assumption there is a continuous linear map $\sigma : C \to A$ such that $\alpha \circ \sigma = \mathrm{id}_C$. From Theorem 3.2.5 we obtain an extension

$$\{0\} \longrightarrow \ker\delta \xrightarrow{\iota} D \xrightarrow{\delta} B \longrightarrow \{0\}$$

which is singular and admissible, with admissible map $Q : B \to D$, given by $Q(a) = (\sigma\beta(a), a)$ for each $a \in B$.

Since the extensions (4.2.12) and (4.2.13) are singular and admissible, Proposition 3.1.7(iii) implies that $\ker\delta$ and $\ker\alpha$ are Banach bimodules of B and C, respectively, when equipped with the module maps $a \cdot x = Q(a)x$, $x \cdot a = xQ(a)$ and $c \cdot y = \sigma(c)y$, $y \cdot c = y\sigma(c)$ (for $a \in B$, $x \in \ker\delta$, $c \in C$, $y \in \ker\alpha$).

First we shall show that there is a bounded linear embedding $\mathcal{Z}^2(C, \ker\alpha) \hookrightarrow \mathcal{Z}^2(B, \ker\delta)$. Set $\psi = \gamma|_{\ker\delta} : \ker\delta \to \ker\alpha$, which is an isomorphism of Banach algebras by the proof of Theorem 3.2.5(i). Take $R \in \mathcal{Z}^2(C, \ker\alpha)$ and define $T_R : B \times B \to \ker\delta$ by

$$T_R(a, b) = \psi^{-1}R\big(\beta a, \beta b\big) = \big(R(\beta a, \beta b), 0\big) \qquad (a, b \in B). \tag{4.2.14}$$

It is quickly checked that T_R is bounded and bilinear, since R is. Now for all $a, b, c \in B$ we have

$$\begin{aligned}
&a \cdot T_R(b, c) - T_R(ab, c) + T_R(a, bc) - T_R(a, b) \cdot c \\
&= Qa\big(R(\beta b, \beta c), 0\big) - \big(R(\beta ab, \beta c), 0\big) + \big(R(\beta a, \beta bc), 0\big) - \big(R(\beta a, \beta b), 0\big)Qc \\
&= \big(\sigma\beta a, a\big)\big(R(\beta b, \beta c), 0\big) - \big(R(\beta ab, \beta c), 0\big) + \big(R(\beta a, \beta bc), 0\big) - \big(R(\beta a, \beta b), 0\big)\big(\sigma\beta c, c\big) \\
&= \Big(\beta a \cdot R(\beta b, \beta c) - R(\beta a\beta b, \beta c) + R(\beta a, \beta b\beta c) - R(\beta a, \beta b) \cdot \beta c\ , 0\Big) \\
&= (0, 0)
\end{aligned}$$

because $R \in \mathcal{Z}^2(C, \ker\alpha)$. Thus $T_R \in \mathcal{Z}^2(B, \ker\delta)$ by (4.1.1).

Now define the map

$$\varphi : \mathcal{Z}^2(C, \ker\alpha) \to \mathcal{Z}^2(B, \ker\delta), \quad R \mapsto T_R.$$

This is clearly bounded and linear. Suppose that $R \in \ker\varphi$ and take $c, d \in C$. Then, since β is surjective, there exist $a, b \in B$ such that $\beta a = c$ and $\beta b = d$. So

$$\big(R(c, d), 0\big) = \big(R(\beta a, \beta b), 0\big) = T_R(a, b) = (0, 0),$$

which implies that $R = 0$; hence φ is injective. Therefore $\varphi : \mathcal{Z}^2(C, \ker\alpha) \hookrightarrow \mathcal{Z}^2(B, \ker\delta)$ is a bounded linear embedding.

Next we shall show that

$$\varphi^{-1}(\mathcal{N}^2(B, \ker\delta)) \subseteq \mathcal{N}^2(C, \ker\alpha). \tag{4.2.15}$$

Let $R \in \mathcal{Z}^2(C, \ker\alpha)$, and suppose that $\varphi(R) \in \mathcal{N}^2(B, \ker\delta)$. Then there exists

$S \in \mathscr{B}(B, \ker \delta)$ such that

$$T_R(a,b) = Q(a)S(b) - S(ab) + S(a)Q(b) \qquad (a, b \in B). \tag{4.2.16}$$

Let $\theta : B \to C$ be a continuous splitting homomorphism for (4.2.11). Then $\psi S\theta \in \mathscr{B}(C, \ker \alpha)$, and for each $c, d \in C$:

$$c \cdot \psi S\theta(d) = \sigma(c)\psi S\theta(d) = \psi\big((\sigma\beta\theta c, c)\big(\psi S\theta d, 0\big)\big) = \psi\big(Q\theta(c)S\theta(d)\big).$$

Similarly $\psi S\theta(c) \cdot d = \psi(S\theta(c)Q\theta(d))$. Therefore for each $c, d \in C$:

$$\begin{aligned}
&c \cdot \psi S\theta(d) - \psi S\theta(cd) + \psi S\theta(c) \cdot d = \psi\big(Q\theta(c)S\theta(d)\big) - \psi S\theta(cd) + \psi\big(S\theta(c)Q\theta(d)\big) \\
&= \psi\big(Q\theta(c)S\theta(d) - S\theta(cd) + S\theta(c)Q\theta(d)\big) = \psi T_R(\theta c, \theta d) \qquad \text{by (4.2.16)} \\
&= R(\beta\theta c, \beta\theta d) = R(c, d) \qquad \text{by (4.2.14).}
\end{aligned}$$

It follows that $R \in \mathcal{N}^2(C, \ker \alpha)$.

To finish, we need a bounded linear embedding $\mathcal{H}^2(C, \ker \alpha) \hookrightarrow \mathcal{H}^2(B, \ker \delta)$. There exists a subspace H of $\mathcal{Z}^2(C, \ker \alpha)$ such that $H \boxplus \mathcal{N}^2(C, \ker \alpha) = \mathcal{Z}^2(C, \ker \alpha)$, and H is isomorphic as a seminormed space to $\mathcal{H}^2(C, \ker \alpha)$. Therefore we identify $\mathcal{H}^2(C, \ker \alpha)$ with H, and write $\pi_B : \mathcal{Z}^2(B, \ker \delta) \to \mathcal{H}^2(B, \ker \delta)$ for the quotient map. Then by (4.2.15), $\ker \pi_B|_{\varphi(H)} \circ \varphi|_H = \{0\}$. Thus $\pi_B|_{\varphi(H)} \circ \varphi|_H : \mathcal{H}^2(C, \ker \alpha) \hookrightarrow \mathcal{H}^2(B, \ker \delta)$ is a bounded linear embedding. $\square$

A specific consequence of this result is that there are second Banach cohomology groups of $\mathscr{B}(E_{\mathcal{R}})$ which are very non-zero, in the sense that we can embed an enormous Banach space into them.

Corollary 4.2.14. *There is a one-dimensional Banach $\mathscr{B}(E_{\mathcal{R}})$-bimodule Y and a bounded linear injection of $\mathscr{B}(\ell_2(\mathbb{N}))$ into $\mathcal{H}^2(\mathscr{B}(E_{\mathcal{R}}), Y)$.*

Proof. We use Proposition 4.2.13. Set $B = \mathscr{B}(E_{\mathcal{R}})$ and $C = \ell_2(\mathbb{N})^\sim$. Then (4.2.11) is satisfied by Theorem 1.3.4. Let μ be a non-zero bounded bilinear functional on $\ell_2(\mathbb{N})$. As in the proof of Theorem 4.2.11(i), form the Banach algebra $\ell_2(\mathbb{N}) \oplus \mathbb{K}$ with the product $(\lambda, r)(\zeta, s) = (0, \mu(\lambda, \zeta))$ for $\lambda, \zeta \in \ell_2(\mathbb{N})$ and $r, s \in \mathbb{K}$. Then $\ell_2(\mathbb{N}) \oplus \mathbb{K}$ is a Banach algebra, and we have an extension

$$\{0\} \longrightarrow \{0\} \oplus \mathbb{K} \longrightarrow \ell_2(\mathbb{N}) \oplus \mathbb{K} \xrightarrow{\alpha_0} \ell_2(\mathbb{N}) \longrightarrow \{0\}$$

which is singular and admissible, where $\alpha_0(\lambda, r) = \lambda$ for each $\lambda \in \ell_2(\mathbb{N}), r \in \mathbb{K}$. By Lemma 3.2.7 there is an extension

$$\{0\} \longrightarrow (\{0\} \oplus \mathbb{K}) \oplus \{0\} \longrightarrow (\ell_2(\mathbb{N}) \oplus \mathbb{K})^\sim \xrightarrow{\alpha} \ell_2(\mathbb{N})^\sim \longrightarrow \{0\}$$

which is singular and admissible, so that (4.2.12) is fulfilled.

Set $Y = \ker \delta \simeq \ker \alpha = \{0\} \oplus \mathbb{K} \oplus \{0\}$, where δ is from (4.2.13). By Proposition 4.2.13 it is enough to find a bounded linear injection $\Upsilon : \mathscr{B}(\ell_2(\mathbb{N})) \to \mathcal{H}^2\big(\ell_2(\mathbb{N})^\sim, \ker \alpha\big)$.

Take $T \in \mathscr{B}(\ell_2(\mathbb{N}))$. We write $\langle \cdot, \cdot \rangle : \ell_2(\mathbb{N}) \times \ell_2(\mathbb{N}) \to \mathbb{K}$ for the bounded bilinear functional $\langle \lambda, \zeta \rangle = \sum_{i=1}^{\infty} \lambda_i \zeta_i$ where $\lambda = (\lambda_i)$ and $\zeta = (\zeta_i)$ are in $\ell_2(\mathbb{N})$. Define $\Upsilon_0(T) : \ell_2(\mathbb{N})^\sim \times \ell_2(\mathbb{N})^\sim \to \ker \alpha$ by

$$\Upsilon_0(T)\big(\lambda + r1_{\ell_2(\mathbb{N})^\sim}, \zeta + s1_{\ell_2(\mathbb{N})^\sim}\big) = \big(0, \langle T\lambda, \zeta \rangle\big) + 0(1_{\ell_2(\mathbb{N})^\sim})$$

for $\lambda, \zeta \in \ell_2(\mathbb{N})$ and $r, s \in \mathbb{K}$. Then because $\ell_2(\mathbb{N})$ has the trivial product it is easy to check that $\Upsilon_0(T) \in \mathcal{Z}^2\big(\ell_2(\mathbb{N})^\sim, \ker \alpha\big)$, and that

$$\Upsilon_0(T) \in \mathcal{N}^2\big(\ell_2(\mathbb{N})^\sim, \ker \alpha\big) \iff T = 0.$$

Therefore $\Upsilon : T \mapsto \Upsilon_0(T) + \mathcal{N}^2\big(\ell_2(\mathbb{N})^\sim, \ker \alpha\big)$ is a bounded linear injection from $\mathscr{B}(\ell_2(\mathbb{N}))$ into $\mathcal{H}^2\big(\ell_2(\mathbb{N})^\sim, \ker \alpha\big)$. The result follows. $\square$

4.3 Weak bidimension

The concept of homological bidimension is a fruitful one, as is clear from Helemskii's book [52]. One of its weaknesses, however, is that it is difficult to calculate, as evidenced by the fact that its value is unknown for many common Banach algebras. Examples include $C[0,1]$ and $\mathscr{K}(H)$ as well as those considered in the previous section. Moreover, the exact value is unknown for essentially all Banach algebras except the biprojective ones and those built from them. Part of the reason for this is that it requires knowledge about every Banach bimodule of the Banach algebra.

A related concept is that of the *weak bidimension* of a Banach algebra, which again has its roots in the purely algebraic theory (in this setting one simply replaces the group Ext with Tor, see [52, p. 163] and [77]). The topological definition involves restricting to a smaller class of Banach bimodules, the idea being that it may still capture the homological deficiencies of the Banach algebra, whilst being easier to calculate than the homological bidimension.

One reason in support of this approach is the success of the study of *amenability* for Banach algebras. Johnson introduced this notion, which involves studying cohomology groups with respect to a class of Banach bimodules known as dual modules, in [59]. Take a Banach algebra B and a Banach B-bimodule Y. Then the dual Banach space Y^* is a Banach B-bimodule as well, with respect to the maps

$$\langle y, b \cdot y^* \rangle = \langle y \cdot b, y^* \rangle \quad \text{and} \quad \langle y, y^* \cdot b \rangle = \langle b \cdot y, y^* \rangle \qquad (y \in Y, y^* \in Y^*, b \in B). \tag{4.3.1}$$

Such a Banach B-bimodule is called a *dual module.* Whenever we refer to the dual of a Banach B-bimodule as a bimodule without specifying the relevant maps, these are the ones implied. The study of amenability and its many generalisations has been a hugely successful area of research in the last 45 years, and so one suspects that, given the following definition, calculating the weak bidimension may be a little easier than calculating the homological bidimension.

Definition 4.3.1. Let B be a Banach algebra. The *weak bidimension of* B is

$$\mathrm{db_w}\, B = \min\{n \in \mathbb{N}_0 : \mathcal{H}^{n+1}(B, Y^*) = \{0\} \text{ for every Banach } B\text{-bimodule } Y\}$$

when this is finite. If there is no such non-negative integer, set $\mathrm{db_w}\, B = \infty$. If $\mathrm{db_w}\, B = 0$ then B is *amenable.* It is clear that $\mathrm{db_w}\, B \leqslant \mathrm{db}\, B$ for every Banach algebra B.

We would like to consider the possible values for $\mathrm{db_w}\, \mathscr{B}(X)$, just as we did for the homological bidimension. An amenable Banach algebra is in some sense 'small', and when X is an infinite-dimensional Banach space, $\mathscr{B}(X)$ is 'big'. In his memoir [59, 10.4], Johnson raised the formal question: for an infinite-dimensional Banach space X, is $\mathscr{B}(X)$ ever amenable? The intuition that when X is infinite-dimensional $\mathscr{B}(X)$ is too big to be amenable was shown to be false in a spectacular way by Argyros and Haydon [6] in 2011. We mentioned the Argyros–Haydon space in Chapter 2, but let us now discuss it in a little more detail. They constructed a Banach space X_{AH} whose only operators are scalar multiples of the identity plus a compact operator, answering a long-open question in Banach space theory (see, *e.g.*, Lindenstrauss' list of problems from 1976 [73]). More precisely, they proved the following.

Theorem 4.3.2 (Argyros-Haydon)**.** *There exists a Banach space* X_{AH} *such that* $\mathscr{B}(X_{\mathrm{AH}}) = \mathscr{K}(X_{\mathrm{AH}}) \oplus \mathbb{K} I_{X_{\mathrm{AH}}}$.

The corollary concerning amenability was first pointed out by Dales; it is recorded at the end of [6].

Corollary 4.3.3. *The Banach algebra* $\mathscr{B}(X_{\mathrm{AH}})$ *is amenable. So* $\mathrm{db_w}\, \mathscr{B}(X_{\mathrm{AH}}) = 0$.

Proof. A Banach algebra is amenable if and only if its unitisation is amenable [24, Proposition 2.8.58(i)]. So it is enough to show that $\mathscr{K}(X_{\mathrm{AH}})$ is amenable (since amenability is preserved by isomorphisms). Argyros and Haydon show that X_{AH}

is a so-called $\mathscr{L}_\infty$ space, and it is proved in [48] that if X is a $\mathscr{L}_\infty$ space, then $\mathscr{K}(X)$ is amenable. $\square$

Having said this, we expect $\mathscr{B}(X)$ not to be amenable for most Banach spaces. It is true that $\mathscr{B}(\ell_p)$ is not amenable for $1 \leqslant p \leqslant \infty$ (see [97], which includes a nice history of the problem, and good references), but proving this was not at all easy. The 'simplest' case, $\mathscr{B}(\ell_2)$, needed some deep facts about C^*-algebras. Some years later, Read gave a clever proof that $\mathscr{B}(\ell_1)$ is not amenable [93], but other techniques were needed before the general case could be obtained. In [97] Runde also shows that $\mathscr{B}(L_p[0,1])$ is not amenable for $1 < p < \infty$. Another (unpublished) example was given by G. A. Willis who showed that $\mathscr{B}(\ell_p \oplus \ell_q)$ is not amenable for $p, q \in (1, \infty)$, $p \neq q$. For these examples, $\mathrm{db}_{\mathrm{w}}\, \mathscr{B}(X) \geqslant 1$.

Thus far the weak bidimension of $\mathscr{B}(X)$ seems quite similar to the usual bidimension, except that we can have $\mathrm{db}_{\mathrm{w}}\, \mathscr{B}(X) = 0$, and it appears harder to give a lower bound (*cf.* Theorem 4.2.3). Therefore it seems reasonable to ask Lykova's question in this context as well.

Question 7. Does there exist a Banach space X such that $\mathrm{db}_{\mathrm{w}}\, \mathscr{B}(X) \geqslant 2$?

Before we answer this question we consider a generalisation of amenability. A Banach algebra B is naturally a Banach B-bimodule over itself, with the module maps given by multiplication. Therefore by our preliminary remarks, B^* is also a Banach B-bimodule.

Definition 4.3.4. A Banach algebra B is *weakly amenable* if $\mathcal{H}^1(B, B^*) = \{0\}$.

It is more common that $\mathscr{B}(X)$ is weakly amenable. Firstly, it is known that all C^*-algebras are weakly amenable [50], and so $\mathscr{B}(\ell_2)$ is. Dales, Ghahramani and Grønbæk [25, Proposition 5.7] observe that this is also true for $\mathscr{B}(\ell_p)$, $1 < p < \infty$, and Blanco [15] has shown that even $\mathscr{B}(J_2)$ and $\mathscr{B}(T)$ (where T is the Tsirelson space) are weakly amenable. In the converse direction, Blanco constructed an example of a Banach space Y with an unconditional basis such that $\mathscr{B}(Y)$ is not weakly amenable [15, Example 3.7]. Nevertheless, we perhaps expect $\mathscr{B}(X)$ to be weakly amenable for most Banach spaces. In the case of Read's space, $\mathscr{B}(E_{\mathcal{R}})$ is not weakly amenable since it has a continuous point derivation at a character (see [24, Theorem 2.8.63(ii), Proposition 2.7.11]). Of course the character corresponds to the ideal I of codimension one. The following result shows that $\mathscr{B}(E_{\mathcal{R}})$ is somehow further from being weakly amenable than this.

Theorem 4.3.5. $\mathrm{db}_{\mathrm{w}}\, \mathscr{B}(E_{\mathcal{R}}) \geqslant 2$.

We remark that the reduction of dimension formula (4.1.3) is not enough to prove Theorem 4.3.5 immediately, even though $\mathscr{B}(E_{\mathcal{R}})$ is not weakly amenable;

this is because we have two distinct module structures on $\mathscr{B}(E_{\mathcal{R}})^* = \mathscr{B}(\mathscr{B}(E_{\mathcal{R}}), \mathbb{K})$: one is the natural dual module structure, and the other comes from (4.1.3). Before giving the proof we need two elementary lemmas.

Lemma 4.3.6. *Let B be a Banach algebra and let E and F be Banach B-bimodules. Suppose that E and F are isomorphic as Banach B-bimodules. Then $\mathcal{H}^2(B, E) \simeq \mathcal{H}^2(B, F)$ as complete seminormed spaces.*

Proof. Denote by $\varphi : E \to F$ the continuous bimodule isomorphism. Then it is easily checked that the map

$$T + \mathcal{N}^2(B, E) \mapsto (\varphi \circ T) + \mathcal{N}^2(B, F), \quad \mathcal{H}^2(B, E) \to \mathcal{H}^2(B, F), \quad (T \in \mathcal{Z}^2(B, E))$$

is a bijective linear homeomorphism. □

Lemma 4.3.7. *Let B be a Banach algebra and E a Banach B-bimodule. Then E^{**} is a Banach B-bimodule with respect to the maps from* (4.3.1)*, and the canonical embedding $\kappa : E \to E^{**}$ is a bimodule homomorphism.* □

Proof of Theorem 4.3.5. As remarked above, $\mathscr{B}(E_{\mathcal{R}})$ is not weakly amenable (and therefore not amenable), so $\mathrm{db_w}\, \mathscr{B}(E_{\mathcal{R}}) \geqslant 1$. Therefore we must show that $\mathcal{H}^2(\mathscr{B}(E_{\mathcal{R}}), Y^*) \neq \{0\}$ for some Banach $\mathscr{B}(E_{\mathcal{R}})$-bimodule Y (in fact there is an analogue of (4.1.3) for dual modules [24, Corollary 2.8.34], so this is already enough without knowing that $\mathscr{B}(E_{\mathcal{R}})$ fails to be amenable). This is easy, provided we chase through the isomorphisms.

In the proof of Theorem 4.2.11(i), by using the pullback method, we demonstrated that there is an extension of $\mathscr{B}(E_{\mathcal{R}})$ of the following form:

$$\{0\} \longrightarrow \ker\delta \longrightarrow D \xrightarrow{\delta} \mathscr{B}(E_{\mathcal{R}}) \longrightarrow \{0\}$$

which is singular and admissible, but does not split strongly. Therefore $\ker\delta$ is a Banach $\mathscr{B}(E_{\mathcal{R}})$-bimodule by Proposition 3.1.7(iii). Proposition 4.2.8 shows that $\mathcal{H}^2(\mathscr{B}(E_{\mathcal{R}}), \ker\delta) \neq \{0\}$. Moreover, the proof of Theorem 3.2.5(i) tells us that $\ker\delta \cong \{0\} \oplus \mathbb{K} \oplus \{0\}$ (recalling that we have to unitise). Hence $\ker\delta$ is one-dimensional, and so $\kappa : \ker\delta \to (\ker\delta)^{**}$ is a Banach $\mathscr{B}(E_{\mathcal{R}})$-bimodule isomorphism by Lemma 4.3.7, when $(\ker\delta)^{**}$ is equipped with the natural module maps. Thus $\mathcal{H}^2(\mathscr{B}(E_{\mathcal{R}}), (\ker\delta)^{**}) \neq \{0\}$ by Lemma 4.3.6. Since $(\ker\delta)^{**}$ is a dual module, setting $Y = (\ker\delta)^*$ proves the result. □

In the other examples from Theorem 4.2.11, it is not clear that the particular modules for which $\mathcal{H}^2(\mathscr{B}(X), Y) \neq \{0\}$ are dual modules. So we cannot conclude that the weak bidimension is at least two straight away.

Chapter 5

Read's Banach Space $E_{\mathcal{R}}$

We now come to the promised exposition of Read's space $E_{\mathcal{R}}$. Chapter 1 has explained the motivation for the construction as well as its key properties, and Chapters 2, 3 and 4 have shown some of its interesting applications. Our main aim is to prove Theorem 1.3.4, and we shall need a thorough understanding of Read's work to do so. We roughly follow the structure of Read's paper [90] and the reader may find it useful to have alongside as we go. Our notation and approach is mostly the same. The author would like to thank his supervisor for several ideas which simplified parts of this chapter, especially the review of the construction of $E_{\mathcal{R}}$.

In the first section we shall make some preliminary observations concerning the weak Calkin algebra of a general Banach space, and give several other useful lemmas. The second section gives the details of the construction. We shall see that there are three 'layers' to it: the James-like spaces defined over Lorentz sequence spaces, the direct sum of such spaces, and a quotient of this by a cleverly-chosen subspace.

Section three provides an analysis of $E_{\mathcal{R}}$. Here our notation and perspective differs a little from [90], but hopefully the approach we take is instructive. The main point is to show that $E_{\mathcal{R}}^{**}/E_{\mathcal{R}}$ is in fact a separable Hilbert space H and identify an orthonormal basis for it explicitly. The key result in Read's paper is [90, Lemma 4.1]; its complex proof consists of several pages. The idea of the lemma is that we can represent $\mathscr{B}(E_{\mathcal{R}})/\mathscr{W}(E_{\mathcal{R}})$ on H and describe its image almost completely. We give a detailed proof for the benefit of the reader.

Having reviewed Read's work, and set some of it in a slightly different context, in the fourth section we prove some results of our own. Our main result is Theorem 1.3.4, which generalises Read's Theorem 1.3.2. An easy corollary of this theorem is that the weakly compact operators on $E_{\mathcal{R}}$ have a complement in $\mathscr{B}(E_{\mathcal{R}})$ which is easy to describe. In fact the complement is isomorphic as a Banach algebra to $\ell_2(\mathbb{N})^{\sim}$, the unitisation of $\ell_2(\mathbb{N})$ with the trivial product. This fourth section is joint work with N. J. Laustsen, and appears in a similar form in [72].

In the final section we present another application of our work, this time concerning commutators in $\mathscr{B}(E_{\mathcal{R}})$.

5.1 Preliminary results

All results in [90] are stated for complex scalars only; however, the proofs carry over verbatim to the real case, so we shall consider both cases simultaneously. A sequence of scalars, therefore, refers to a sequence of real or complex scalars, and a vector space may be real or complex unless specified. In this section we record some standard results that will be useful later. In some cases we give a proof, usually because the concepts or notation will come up in subsequent sections.

Proposition 5.1.1. *Let Z be a closed subspace of a Banach space X and let $Q_Z : X \to X/Z$ be the quotient map. Then the image under Q_Z of the open unit ball of X is the open unit ball of X/Z.*

Proof. For example, [78, Lemma 1.7.11]. $\square$

The following proposition, which is where Read begins [90, Lemma 1.1], hints at how we might demonstrate Theorem 1.3.2(ii). It shows how to represent $\mathscr{B}(X)/\mathscr{W}(X)$ on another Banach space, with the hope being to determine whether it is infinite-dimensional.

Proposition 5.1.2. *Let X be a non-reflexive Banach space. Then there is a unital algebra homomorphism $\Theta_0 : \mathscr{B}(X) \to \mathscr{B}(X^{**}/X)$ which is contractive and has $\ker \Theta_0 = \mathscr{W}(X)$. The map is given by $\Theta_0(T) = \overline{T^{**}}$ for each $T \in \mathscr{B}(X)$, where $\overline{T^{**}}(x^{**} + X) = T^{**}(x^{**}) + X$ for $x^{**} \in X^{**}$.*

Proof. For each $T \in \mathscr{B}(X)$, the linear map $\overline{T^{**}}$ is well-defined because $T^{**}(X) \subseteq X$. Using Proposition 5.1.1 we see that

$$||\Theta_0(T)|| = ||\overline{T^{**}}|| = ||T||,$$

so $\overline{T^{**}}$ is bounded and Θ_0 is contractive. Using the fact that $T \mapsto T^{**}$ is an algebra homomorphism, it follows that Θ_0 is too. Moreover, $I_X^{**} = I_{X^{**}}$ and so Θ_0 is unital because X is non-reflexive. Now $T \in \ker \Theta_0$ if and only if $\overline{T^{**}}(x^{**} + X) = 0 + X$ for every $x^{**} + X \in X^{**}/X$, if and only if $T^{**}(x^{**}) \in X$ for every $x^{**} \in X^{**}$. This is equivalent to saying that T is weakly compact by Gantmacher's Theorem 1.2.3(ii). $\square$

Thus we shall be interested in characterising the image of Θ_0, as it tells us about the properties of $\mathscr{B}(X)/\mathscr{W}(X)$.

Definition 5.1.3. Let Y be a Banach space and N a subset of Y. The *polar* of N is the set $N^\circ = \{y^* \in Y^* : \sup_{x \in N} |y^*(x)| \leqslant 1\}$ and the *annihilator* of N is the set $\{y^* \in Y^* : y^*(x) = 0 \text{ for each } x \in N\}$.

When N is a subspace of Y, the polar of N is a closed subspace of Y^* which is equal to the annihilator of N.

Read's space is a quotient of a direct sum of James-type spaces, and when dealing with quotients of Banach spaces it is often helpful to be able to dualise. This is the content of our next three results. Both 5.1.4 and 5.1.5 are standard, but it is useful to state and prove them in the specific form we require. Read uses the following lemma on [90, p. 312].

Lemma 5.1.4. *Let Y be a Banach space and $N \subseteq Y$ a closed subspace. Write $E = Y/N$ for the quotient space and $Q_N : Y \to E$ for the quotient map. Then*

(i) *$Q_N^* : E^* \to N^\circ$ is an isometric isomorphism;*

(ii) *$\varphi : Y^{**}/N^{\circ\circ} \to E^{**}$ given by $\varphi(y^{**} + N^{\circ\circ}) = Q_N^{**}(y^{**})$ for $y^{**} \in Y^{**}$ is an isomorphism.*

Proof. (i) The adjoint $Q_N^* : E^* \to Y^*$ is a linear isometry, using Proposition 5.1.1, so it is enough to show that $Q_N^*(E^*) = N^\circ$.

Pick $f \in N^\circ$. Then $f \in Y^*$ and $f|_N = 0$ so $N \subseteq \ker f$. By the Fundamental Isomorphism Theorem 1.2.4 there exists a bounded linear map $g : E \to \mathbb{K}$ such that the diagram:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "Q_N"'] & \mathbb{K} \\ E \arrow[ur, "g"] & \end{tikzcd} \tag{5.1.1}$$

is commutative. Then for an element $y \in Y$,

$$\langle y, Q_N^*(g) \rangle = \langle Q_N(y), g \rangle = g \circ Q_N(y) = f(y)$$

so $Q_N^*(g) = f$ and hence $N^\circ \subseteq Q_N^*(E^*)$.

For the converse choose $h \in E^*$. Then for each $n \in N$,

$$\langle n, Q_N^*(h) \rangle = \langle Q_N(n), h \rangle = 0.$$

Therefore $Q_N^*(h) \in N^\circ$, and so $Q_N^*(E^*) = N^\circ$. We conclude that $Q_N^* : E^* \to N^\circ$ is an isometric isomorphism.

(ii) The double adjoint $Q_N^{**} : Y^{**} \to E^{**}$ is a surjection because Q_N^* is an isometry.

Now $k \in \ker Q_N^{**}$ if and only if $0 = \langle e^*, Q_N^{**}(k)\rangle = \langle Q_N^*(e^*), k\rangle$ for every $e^* \in E^*$, if and only if $k \in Q_N^*(E^*)^\circ$. Hence $\ker Q_N^{**} = Q_N^*(E^*)^\circ = N^{\circ\circ}$ by (i). It follows from the Fundamental Isomorphism Theorem 1.2.4 that there is an isomorphism $\varphi : Y^{**}/N^{\circ\circ} \to E^{**}$ of the form stated. □

Our final lemma in this section again relates to duality: when checking weak* convergence of a bounded sequence, it is enough to do it on a total subset.

Lemma 5.1.5. *Let X be a Banach space, $D \subseteq X$ such that $\overline{\operatorname{span}}\, D = X$, $(f_j)_{j=1}^\infty$ a bounded sequence in X^*, and $f \in X^*$. Then $f_j \xrightarrow{w^*} f$ if and only if $\langle x, f_j\rangle \to \langle x, f\rangle$ for every $x \in D$.*

Proof. ($\Rightarrow$) Suppose that $f_j \xrightarrow{w^*} f$ as $j \to \infty$. This means that for every $x \in X$, $\langle x, f_j\rangle \to \langle x, f\rangle$ as $j \to \infty$, so certainly also for all $x \in D$.

($\Leftarrow$) Suppose that, for every $x \in D$, $\langle x, f_j\rangle \to \langle x, f\rangle$ as $j \to \infty$. Then by taking linear combinations, $\langle x, f_j\rangle \to \langle x, f\rangle$ for every $x \in \operatorname{span} D$. Pick $x_0 \in X$ and $\varepsilon > 0$. Since (f_j) is a bounded sequence, we can define $C = \sup\{||f_j||, ||f|| : j \in \mathbb{N}\} < \infty$. Let $\delta = \varepsilon/3C$. Then we may choose $x \in \operatorname{span} D$ such that $||x_0 - x|| \leqslant \delta$, since $\overline{\operatorname{span}}\, D = X$. Next, for each $j \in \mathbb{N}$

$$|\langle x_0, f_j\rangle - \langle x_0, f\rangle| \leqslant |\langle x_0 - x, f_j\rangle| + |\langle x, f_j - f\rangle| + |\langle x - x_0, f\rangle| \tag{5.1.2}$$

while $|\langle x_0 - x, f_j\rangle| \leqslant ||x_0 - x||\, ||f_j|| \leqslant C||x_0 - x||$ and $|\langle x_0 - x, f\rangle| \leqslant C||x_0 - x||$. Now choose j_0 such that $|\langle x, f_j - f\rangle| \leqslant \varepsilon/3$ for all $j \geqslant j_0$. By (5.1.2) this implies that for all $j \geqslant j_0$

$$|\langle x_0, f_j\rangle - \langle x_0, f\rangle| \leqslant 2C\delta + \varepsilon/3 = \varepsilon$$

and the result follows. □

Definition 5.1.6. Let $(X_n)_{n\in\mathbb{N}_0}$ be a sequence of Banach spaces over the same scalar field ($\mathbb{R}$ or $\mathbb{C}$). Define

$$\left(\bigoplus_{n=0}^{\infty} X_n\right)_{\ell_2} = \left\{(x_n)_{n=0}^\infty \;:\; \forall\, n \in \mathbb{N}_0 :\; x_n \in X_n \text{ and } \sum_{n=0}^{\infty} ||x_n||_{X_n}^2 < \infty\right\}.$$

Endowed with pointwise addition and scalar multiplication, and the norm $\|(x_n)\| = (\sum_{n=0}^\infty ||x_n||_{X_n}^2)^{\frac{1}{2}}$, $\big(\bigoplus_{n=0}^\infty X_n\big)_{\ell_2}$ is a Banach space, called the *ℓ_2-direct sum of* (X_n).

The dual of an infinite direct sum is naturally the infinite direct sum of the duals (see *e.g.*, [1, p. 286]).

Proposition 5.1.7. *Let $(X_n)_{n\in\mathbb{N}_0}$ be a sequence of Banach spaces over the same scalar field. Then there is an isometric isomorphism:*

$$\left(\bigoplus_{n=0}^{\infty} X_n\right)^*_{\ell_2} \simeq \left(\bigoplus_{n=0}^{\infty} X_n^*\right)_{\ell_2}$$

and the duality is given by:

$$\langle x, f\rangle = \sum_{n=0}^{\infty}\langle x_n, f_n\rangle$$

where $x = (x_n) \in (\bigoplus_{n=0}^{\infty} X_n)_{\ell_2}$ and $f = (f_n) \in (\bigoplus_{n=0}^{\infty} X_n^*)_{\ell_2}$. □

5.2 The construction of $E_{\mathcal{R}}$

5.2.1 James-like spaces

The fundamental building blocks of $E_{\mathcal{R}}$ are James-type spaces. Let us recall the definition of the classical James space constructed in [55] before we generalise. The James space can be equipped with several equivalent norms; the one we use will be most similar to Read's generalisation.

Definition 5.2.1. The *James space* J_2 is the Banach space of sequences $(a_i)_{i=1}^{\infty} \in c_0$ such that

$$||(a_i)_{i=1}^{\infty}||_{J_2} = \sup\left\{\left(\sum_{i=1}^{n-1}|a_{p_i}-a_{p_{i+1}}|^2+|a_{p_n}|^2\right)^{\frac{1}{2}} : p_1 < \cdots < p_n,\ n\in\mathbb{N},\ n\geqslant 2\right\} < \infty.$$

Definition 5.2.2. A Schauder basis $(e_n)_{n=1}^{\infty}$ for a Banach space A is *symmetric* if for each permutation π of $\mathbb{N}$ and each sequence of scalars $(\alpha_n)_{n=1}^{\infty}$ we have

$$\sum_{n=1}^{\infty}\alpha_n e_n \ \text{ converges } \iff \sum_{n=1}^{\infty}\alpha_n e_{\pi(n)} \text{ converges.}$$

The basis (e_n) is *1-symmetric* if for each $m\in\mathbb{N}$, each permutation π of $\mathbb{N}$, and scalars $\alpha_1,\dots\alpha_m,\beta_1,\dots,\beta_m$ we have

$$\left\|\sum_{i=1}^{m}\alpha_i\beta_i e_{\pi(i)}\right\|_A \leqslant \max\{|\beta_1|,\dots,|\beta_m|\}\left\|\sum_{i=1}^{m}\alpha_i e_i\right\|_A. \tag{5.2.1}$$

Suppose that a Banach space A has a symmetric basis. Then by passing to an equivalent norm on A, we may assume that the basis is 1-symmetric [74, p. 113].

Definition 5.2.3 (Read)**.** Let $(A, ||\cdot||_A)$ be a Banach space with a normalised, 1-symmetric basis (e_n). The *James-like space* JA is the Banach space of sequences $(a_i)_{i=1}^{\infty} \in c_0$ such that

$$||(a_i)_{i=1}^{\infty}||_{JA} = \sup\left\{ \left\|\sum_{i=1}^{n-1}(a_{p_i} - a_{p_{i+1}})^2 e_i + a_{p_n}^2 e_n\right\|_A^{\frac{1}{2}} : p_1 < \ldots < p_n,\ n \in \mathbb{N},\ n \geqslant 2\right\} < \infty. \tag{5.2.2}$$

Read defines the James-like spaces for A with a symmetric basis [90, Definition 1.2]. As remarked above, there is no generality lost by assuming it to be 1-symmetric; we do so because the calculations become a little easier.

Examples of Banach spaces A with a 1-symmetric basis include c_0, ℓ_p for $1 \leqslant p < \infty$, the Lorentz sequence spaces $d(w,p)$ for $1 \leqslant p < \infty$, which we shall see later, and Orlicz sequence spaces h_M. See, for example, [74, Chapter 3] for proofs of these facts.

If we take $A = \ell_1$ then we retrieve $JA = J_2$, the James space. Read remarks [90, p. 307] that it is easily verified that JA is a normed space, with no further proof, and proceeds to check completeness. The fact that $||\cdot||_{JA}$ is subadditive is not obvious, however, due to the squaring of the coefficients.

There are several ways to verify that it is indeed a norm. The first is a neat proof shown to us by Dr Graham Jameson, covering all A. The second is to consider another (earlier) generalisation of the James space, studied by Casazza and Lohman [20], and show that, in the case we need, Read's definition is equivalent. We shall cover both of these here. Upon request, Read kindly provided us with a third proof of the subadditivity of the norm, but we omit this for the sake of space, choosing instead to present Jameson's attractive general approach.

Lemma 5.2.4 (Jameson)**.** *Let V be a vector space over $\mathbb{K}$. Let $\nu : V \to [0,\infty)$ have the following properties for all $x, y \in V$ and $\lambda, \mu \in \mathbb{K}$:*

(i) $\nu(\lambda x) = |\lambda|\nu(x)$;

(ii) $\nu(x), \nu(y) \leqslant 1 \implies \nu(x+y) \leqslant 2$;

(iii) *if* $\lambda_n \to \lambda$ *and* $\mu_n \to \mu$ *in* $\mathbb{K}$ *as* $n \to \infty$, *then* $\nu(\lambda_n x + \mu_n y) \to \nu(\lambda x + \mu y)$ *as* $n \to \infty$;

(iv) $\nu(x) = 0 \implies x = 0$.

Then $\nu(x+y) \leqslant \nu(x) + \nu(y)$ for all $x, y \in V$, so that ν is a norm on V.

Proof. Take $x, y \in V$ with $\nu(x), \nu(y) \leqslant 1$. We firstly *claim* that for every $n \in \mathbb{N}$ and for every $r \in \{0, 1, \ldots, 2^n\}$,

$$\nu\left(\frac{r}{2^n}x + \left(1 - \frac{r}{2^n}\right)y\right) \leqslant 1.$$

We proceed by induction on $n \in \mathbb{N}$. Notice that for any $n \in \mathbb{N}$, the cases $r = 0$ and $r = 2^n$ are true by assumption. Thus for $n = 1$, we need only consider $r = 1$. Then, using (i) and (ii)

$$\nu\left(\frac{1}{2}x + \frac{1}{2}y\right) = \frac{1}{2}\nu(x+y) \leqslant \frac{1}{2} \times 2 = 1$$

which establishes the basis of the induction.

Suppose that the result is true for $k \in \mathbb{N}$. This means that for all $r \in \{0, \ldots, 2^k\}$, we have $\nu\big(\frac{r}{2^k}x + (1 - \frac{r}{2^k})y\big) \leqslant 1$. We must show that

$$\nu\big(\frac{r}{2^{k+1}}x + (1 - \frac{r}{2^{k+1}})y\big) \leqslant 1$$

for all $r \in \{0, \ldots, 2^{k+1}\}$. Let us split the problem into two parts. Firstly, suppose that $r \in \{0, \ldots, 2^k\}$. One can quickly check using (i) that

$$\nu\left(\frac{r}{2^{k+1}}x + \big(1 - \frac{r}{2^{k+1}}\big)y\right) = \frac{1}{2}\nu\left(\frac{r}{2^k}x + \big(1 - \frac{r}{2^k}\big)y + y\right).$$

Notice that since $r \in \{0, \ldots, 2^k\}$, $\nu\big(\frac{r}{2^k}x + (1 - \frac{r}{2^k})y\big) \leqslant 1$ by hypothesis. Also, we assumed that $\nu(y) \leqslant 1$. Therefore by (ii)

$$\nu\left(\frac{r}{2^{k+1}}x + \big(1 - \frac{r}{2^{k+1}}\big)y\right) = \frac{1}{2}\nu\left(\frac{r}{2^k}x + \big(1 - \frac{r}{2^k}\big)y + y\right) \leqslant \frac{1}{2} \times 2 = 1.$$

Next consider $r \in \{2^k + 1, \ldots, 2^{k+1}\}$. Then $r - 2^k \in \{1, \ldots, 2^k\}$. Now we see that

$$\frac{r}{2^{k+1}}x + \big(1 - \frac{r}{2^{k+1}}\big)y = \frac{1}{2}x - \frac{1}{2}y + \frac{r - 2^k}{2^{k+1}}x + \big(1 - \frac{r - 2^k}{2^{k+1}}\big)y.$$

But since $r - 2^k \in \{1, \ldots, 2^k\}$, $\nu\big(\frac{r - 2^k}{2^{k+1}}x + (1 - \frac{r - 2^k}{2^{k+1}})y\big) \leqslant 1$ by the first part, and $\nu(\frac{1}{2}x - \frac{1}{2}y) = \frac{1}{2}\nu(x - y) \leqslant 1$ by (i) and (ii). Therefore by (ii),

$$\nu\left(\frac{r}{2^{k+1}}x + \big(1 - \frac{r}{2^{k+1}}\big)y\right) \leqslant 1.$$

Hence the result is true for $k + 1$, and so the claim holds by induction.

Now we want to show that for every $\lambda \in (0, 1)$, $\nu(\lambda x + (1 - \lambda)y) \leqslant 1$. Choose $\lambda \in (0, 1)$. Then by 'dyadic rational approximation' there exists a sequence $\lambda_n = \frac{r_n}{2^n}$ with $r_n \in \{0, \ldots, 2^n\}$ such that $\lambda_n \to \lambda$ as $n \to \infty$. Using (iii) we see that $\nu(\lambda_n x + (1 - \lambda_n)y) \to \nu(\lambda x + (1 - \lambda)y)$ as $n \to \infty$. By the claim, $\nu(\lambda_n x + (1 - \lambda_n)y) \leqslant 1$, and so $\nu(\lambda x + (1 - \lambda)y) \leqslant 1$.

Now let $x, y \in V$ be arbitrary. We want to show that $\nu(x + y) \leqslant \nu(x) + \nu(y)$, so we may suppose that $x, y \neq 0$ by (i). Hence by (iv),

$$\lambda := \frac{\nu(y)}{(\nu(x) + \nu(y))} \in (0, 1).$$

Thus

$$\lambda\nu(x) = \frac{\nu(x)\nu(y)}{\nu(x) + \nu(y)} = \big(1 - \frac{\nu(y)}{\nu(x) + \nu(y)}\big)\nu(y) = (1 - \lambda)\nu(y). \tag{5.2.3}$$

Let $x' = \frac{1}{1-\lambda}x$ and $y' = \frac{1}{\lambda}y$. Now (5.2.3) and (i) give $\nu(x') = \frac{1}{1-\lambda}\nu(x) = \frac{1}{\lambda}\nu(y) = \nu(y')$. Write $k = \nu(y') = \nu(x')$. We have $\nu\big(\frac{x'}{k}\big) = \nu\big(\frac{y'}{k}\big) = 1$ by (i). And so by our earlier work

$$1 \geqslant \nu\left((1 - \lambda)\frac{x'}{k} + \lambda\frac{y'}{k}\right) = \nu\left(\frac{x}{k} + \frac{y}{k}\right) = \frac{1}{k}\nu(x + y).$$

Finally, $\nu(x + y) \leqslant k = \lambda k + (1 - \lambda)k = \lambda\nu(y') + (1 - \lambda)\nu(x') = \nu(x) + \nu(y)$. □

Definition 5.2.5. A Schauder basis (x_n) for a Banach space X is *1-unconditional* if, for every $m \in \mathbb{N}$ and for all m-tuples of scalars $(\alpha_1, \ldots, \alpha_m), (\beta_1, \ldots, \beta_m)$ such that $|\alpha_j| \leqslant |\beta_j|$ for all $j \in \{1, \ldots, m\}$, we have $|| \sum_{j=1}^m \alpha_j x_j|| \leqslant || \sum_{j=1}^m \beta_j x_j||$.

Remark 5.2.6. A 1-symmetric basis for a Banach space is 1-unconditional. To see this, let (x_n) be a 1-symmetric basis for a Banach space X, let $m \in \mathbb{N}$, and let $(\alpha_1, \ldots, \alpha_m), (\beta_1, \ldots, \beta_m)$ be m-tuples of scalars such that $|\alpha_j| \leqslant |\beta_j|$ for all $j \in \{1, \ldots, m\}$. We may assume that $\beta_j \neq 0$ for all $j \in \{1, \ldots, m\}$. Then

$$\left\|\sum_{j=1}^m \alpha_j x_j\right\| = \left\|\sum_{j=1}^m \beta_j (\frac{\alpha_j}{\beta_j}) x_j\right\| \leqslant \max\left\{\frac{|\alpha_1|}{|\beta_1|}, \ldots, \frac{|\alpha_m|}{|\beta_m|}\right\}\left\|\sum_{j=1}^m \beta_j x_j\right\| \leqslant \left\|\sum_{j=1}^m \beta_j x_j\right\|,$$

so that (x_n) is 1-unconditional.

We require a piece of notation before the next result: c_{00} denotes the vector space of scalar sequences with only finitely many non-zero entries.

Proposition 5.2.7 (Jameson)**.** *Let X be a Banach space with a 1-unconditional basis (x_n). For each $a = (\alpha_1, \ldots, \alpha_n, 0, 0, \ldots) \in c_{00}$, define $\nu(a) = || \sum_{j=1}^n |\alpha_j|^2 x_j||_X^{\frac{1}{2}}$. Then ν is a norm on c_{00}.*

Proof. We check the conditions of Lemma 5.2.4. It is simple to check (i). Choose throughout $a = (\alpha_1, \ldots, \alpha_n, 0, 0, \ldots)$ and $b = (\beta_1, \ldots, \beta_n, 0, 0, \ldots)$ with $\nu(a), \nu(b) \leqslant 1$.

(ii) We will need the fact that for any $\alpha, \beta \in \mathbb{K}$, $|\alpha + \beta|^2 \leqslant 2(|\alpha|^2 + |\beta|^2)$. Indeed,

$$|\alpha + \beta|^2 \leqslant (|\alpha| + |\beta|)^2 = |\alpha|^2 + |\beta|^2 + 2|\alpha||\beta| \leqslant 2(|\alpha|^2 + |\beta|^2) \tag{5.2.4}$$

because $0 \leqslant (|\alpha| - |\beta|)^2 = |\alpha|^2 + |\beta|^2 - 2|\alpha||\beta|$, so that $2|\alpha||\beta| \leqslant |\alpha|^2 + |\beta|^2$. Now using (5.2.4), and the 1-unconditionality of the basis,

$$\begin{aligned}\nu(a+b)^2 &= \left\| \sum_{j=1}^{n} |\alpha_j + \beta_j|^2 x_j \right\|_X \leqslant \left\| \sum_{j=1}^{n} 2(|\alpha_j|^2 + |\beta_j|^2) x_j \right\|_X \\ &\leqslant 2\left\| \sum_{j=1}^{n} |\alpha_j|^2 x_j \right\|_X + 2\left\| \sum_{j=1}^{n} |\beta_j|^2 x_j \right\|_X = 2(\nu(a)^2 + \nu(b)^2) \leqslant 2.2 = 4\end{aligned}$$

so (ii) holds.

(iii) Suppose that we have $\lambda_m \to \lambda$ and $\mu_m \to \mu$ as $m \to \infty$ in $\mathbb{K}$. Fix $n \in \mathbb{N}$ and $j \in \{1, \dots n\}$. Then $|\lambda_m \alpha_j + \mu_m \beta_j|^2 \to |\lambda \alpha_j + \mu \beta_j|^2$ in $\mathbb{K}$ as $m \to \infty$. Also $|\lambda_m \alpha_j + \mu_m \beta_j|^2 x_j \to |\lambda \alpha_j + \mu \beta_j|^2 x_j$ as $m \to \infty$ in X. Now varying j, we see that $\sum_{j=1}^{n} |\lambda_m \alpha_j + \mu_m \beta_j|^2 x_j \to \sum_{j=1}^{n} |\lambda \alpha_j + \mu \beta_j|^2 x_j$ as $m \to \infty$ in X. This implies further that

$$\left\| \sum_{j=1}^{n} |\lambda_m \alpha_j + \mu_m \beta_j|^2 x_j \right\|_X \to \left\| \sum_{j=1}^{n} |\lambda \alpha_j + \mu \beta_j|^2 x_j \right\|_X$$

as $m \to \infty$. Hence $\nu(\lambda_m a + \mu_m b) \to \nu(\lambda a + \mu b)$ as $m \to \infty$, and so (iii) is satisfied.

(iv) Suppose that $\nu(a) = 0$. Then $\sum_{j=1}^{n} |\alpha_j|^2 x_j = 0$ since $||\cdot||_X$ is a norm. Since (x_n) is a basis, $x_1, \dots x_n$ are linearly independent; whence $\alpha_j = 0$ for $j = 1, \dots, n$, so that $a = 0$.

Therefore ν is a norm on c_{00} by Lemma 5.2.4. □

Proposition 5.2.8. *Let A be a Banach space with a normalised 1-symmetric basis (e_n). Then the James-like space JA is a Banach space.*

Proof. We must check that the function $||\cdot||_{JA}$ from Definition 5.2.3 is a complete norm. Let $(a_i)_{i=1}^{\infty}$ be a sequence of scalars such that $|a_i| \to 0$ as $i \to \infty$. It is immediate that $||(a_i)||_{JA} = 0$ if $(a_i) = 0$. Suppose that $||(a_i)||_{JA} = 0$. Then for each $i \in \mathbb{N}$, setting $n = 2, p_1 = i, p_2 = i+1$, we obtain $(a_i - a_{i+1})^2 e_1 + a_{i+1}^2 e_2 = 0$, and so $a_i = 0$ because the basis vectors are linearly independent. Hence $(a_i) = 0$. It is also easy to check that $||(\lambda a_i)||_{JA} = |\lambda| ||(a_i)||_{JA}$ for all $\lambda \in \mathbb{K}$.

Next, the triangle inequality; this is where we use our preparation. By Remark 5.2.6, since (e_n) is 1-symmetric, it is 1-unconditional. Then Proposition 5.2.7 tells us that the function $\nu : c_{00} \to [0, \infty)$ given by $\nu(a) = || \sum_{j=1}^{n} |\alpha_j|^2 e_j ||_A^{\frac{1}{2}}$ for each $a = \sum_{j=1}^{n} \alpha_j e_j$ in c_{00} is a norm, which therefore satisfies the triangle inequality. Let $n \in \mathbb{N}$ such that $n \geqslant 2$, pick an increasing sequence $p_1 < \cdots < p_n$, and let

$(a_i), (b_i) \in JA$. Then, since (e_n) is 1-unconditional, we have

$$\begin{aligned}
&\Big\|\sum_{i=1}^{n-1}(a_{p_i}+b_{p_i}-a_{p_{i+1}}-b_{p_{i+1}})^2e_i+(a_{p_n}+b_{p_n})^2e_n\Big\|_A^{\frac{1}{2}}\\
&=\Big\|\sum_{i=1}^{n-1}|a_{p_i}+b_{p_i}-a_{p_{i+1}}-b_{p_{i+1}}|^2e_i+|a_{p_n}+b_{p_n}|^2e_n\Big\|_A^{\frac{1}{2}}\\
&=\nu\big((a_{p_1}+b_{p_1}-a_{p_2}-b_{p_2},\ldots,a_{p_n}+b_{p_n},0,0,\ldots)\big)\\
&=\nu\big((a_{p_1}-a_{p_2},\ldots,a_{p_n},0,\ldots)+(b_{p_1}-b_{p_2},\ldots,b_{p_n},0,\ldots)\big)\\
&\leqslant\nu\big((a_{p_1}-a_{p_2},\ldots,a_{p_n},0,\ldots)\big)+\nu\big((b_{p_1}-b_{p_2},\ldots,b_{p_n},0,\ldots)\big)\\
&=\Big\|\sum_{i=1}^{n-1}(a_{p_i}-a_{p_{i+1}})^2e_i+a_{p_n}^2e_n\Big\|_A^{\frac{1}{2}}+\Big\|\sum_{i=1}^{n-1}(b_{p_i}-b_{p_{i+1}})^2e_i+b_{p_n}^2e_n\Big\|_A^{\frac{1}{2}}.
\end{aligned}$$

Taking the supremum over all such finite increasing sequences $p_1 < p_2 < \cdots < p_n$, $n \geqslant 2$ shows that $||(a_i) + (b_i)||_{JA} \leqslant ||(a_i)||_{JA} + ||(b_i)||_{JA}$ as required.

To show that JA is complete with respect to $||\cdot||_{JA}$, take a Cauchy sequence $(a^j)_{j=1}^{\infty}$ in JA, and for each $j \in \mathbb{N}$ write $a^j = (a_i^j)_{i=1}^{\infty}$. Now fix $i \in \mathbb{N}$ and $\varepsilon > 0$ and take $p_1 = 1, p_2 = i$ (unless $i = 1$ in which case let $p_1 = 1, p_2 = 2$). Then, since A has a normalised 1-symmetric basis, there exists n_0 such that for all $n, m \geqslant n_0$

$$|a_i^m - a_i^n| \leqslant ||(a_1^m - a_1^n - a_i^m + a_i^n)^2 e_1 + (a_i^m - a_i^n)^2 e_2||_A^{\frac{1}{2}} < \varepsilon$$

using (5.2.2). Hence for each $i \in \mathbb{N}$, $(a_i^j)_{j=1}^{\infty}$ is Cauchy in $\mathbb{K}$. Write $a_i = \lim_{j\to\infty} a_i^j$ and $a = (a_1, a_2, \ldots)$. Straightforward calculations involving (5.2.2) show that $a \in JA$ and that $||a^j - a||_{JA} \to 0$ as $j \to \infty$. Therefore JA is a Banach space. $\square$

Having proved that James-like spaces are indeed Banach spaces, we would like to know what other nice properties they possess. Section 2 of Read's paper [90] establishes properties of JA that correspond to those of the James space, for example having a monotone shrinking basis, and JA being quasi-reflexive (provided A contains no copy of c_0). We shall soon see that, for the particular spaces A we will be interested in, these results follow from earlier work of Casazza and Lohman [20]. Therefore we shall postpone any further discussion until we have introduced these important spaces A.

5.2.2 Lorentz sequence spaces

Although the James-like spaces JA are defined for any Banach space A with a 1-symmetric basis, the case that we shall need for the construction of $E_{\mathcal{R}}$ is when A is a Lorentz sequence space.

Definition 5.2.9. Write $S(\mathbb{N})$ for the set of permutations of $\mathbb{N}$. Let $p \in [1, \infty)$ and let $w = (w_n)$ be a sequence $1 = w_1 \geqslant w_2 \geqslant \ldots > 0$ of real numbers called a *weight* such that $w_n \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} w_n = \infty$ (for example $w = (\frac{1}{n})_{n=1}^{\infty}$). The *Lorentz p-sequence space* $d(w, p)$ is the Banach space of scalar sequences:

$$d(w,p) = \left\{ (\beta_n)_{n=1}^{\infty} : \sup_{\pi \in S(\mathbb{N})} \left(\sum_{n=1}^{\infty} |\beta_{\pi(n)}|^p w_n \right)^{\frac{1}{p}} < \infty, \ \beta_n \in \mathbb{K} \right\} \tag{5.2.5}$$

with the usual pointwise vector space operations, and the norm:

$$||(\beta_n)||_{d(w,p)} = \sup_{\pi \in S(\mathbb{N})} \left(\sum_{n=1}^{\infty} |\beta_{\pi(n)}|^p w_n \right)^{\frac{1}{p}}. \tag{5.2.6}$$

Let $(\beta_n) \in d(w,p)$. If we take a permutation $\pi_0 : \mathbb{N} \to \mathbb{N}$ such that $|\beta_{\pi_0(1)}| \geqslant |\beta_{\pi_0(2)}| \geqslant \cdots$, a decreasing rearrangement, then we can calculate the norm of (β_n) explicitly:

$$||(\beta_n)||_{d(w,p)} = \left(\sum_{n=1}^{\infty} |\beta_{\pi_0(n)}|^p w_n \right)^{\frac{1}{p}}.$$

Lorentz sequence spaces were introduced in [75] in connection with some problems of harmonic analysis and interpolation theory. A nice account of their geometric properties is given in [74, §4.e], including the fact that they possess a normalised, monotone, 1-symmetric basis (e_n) given by

$$e_n = (0, \ldots, 0, 1, 0, \ldots) \qquad (n \in \mathbb{N}), \tag{5.2.7}$$

that is, with 1 in the nth place and zeros elsewhere.

We can now demonstrate the relation between James-like spaces and Casazza–Lohman's generalisation of the James space. In [20], their technique is as follows.

Let $(B, ||\cdot||_B)$ be a Banach space with a normalised monotone basis (b_j). Define

$$J_{CL}(B) = \left\{ (a_i)_{i=1}^{\infty} \in c_0 : \sup_{\substack{p_1 < \cdots < p_n \\ n \geqslant 2}} \left\| \sum_{i=1}^{n-1} (a_{p_i} - a_{p_{i+1}}) b_i + a_{p_n} b_n \right\|_B < \infty \right\} \tag{5.2.8}$$

with norm

$$||(a_i)||_{J_{CL}(B)} = \sup_{\substack{p_1 < \cdots < p_n \\ n \geqslant 2}} \left\| \sum_{i=1}^{n-1} (a_{p_i} - a_{p_{i+1}}) b_i + a_{p_n} b_n \right\|_B.$$

Then $(J_{CL}(B), || \cdot ||_{J_{CL}(B)})$ is a Banach space [20, Theorem 1] (using an equivalent norm). Note that subadditivity is easy to see here. If we let $B = \ell_2$ then we get the classical James space back.

Proposition 5.2.10. *Let $A = d(w, 1)$, a Lorentz 1-sequence space with weight w, and let $B = d(w, 2)$. Then $J_{CL}(B) = JA$, and so JA is a Banach space.*

Proof. Let $(a_i) \in c_0$ and take $p_1 < \cdots < p_n$, with $n \geqslant 2$. Write (b_i) for the basis of $d(w, 2)$ and (e_i) for the basis of $d(w, 1)$. Then

$$\begin{aligned}
\left\| \sum_{i=1}^{n-1} (a_{p_i} - a_{p_{i+1}}) b_i + a_{p_n} b_n \right\|_{d(w,2)} &= ||(a_{p_1} - a_{p_2}, a_{p_2} - a_{p_3}, \ldots, a_{p_n}, 0, 0, \ldots)||_{d(w,2)} \\
&= \sup_{\pi \in S(\mathbb{N})} \left(|a_{p_1} - a_{p_2}|^2 w_{\pi^{-1}(1)} + \cdots + |a_{p_n}|^2 w_{\pi^{-1}(n)} \right)^{\frac{1}{2}} \\
&= \left(\sup_{\pi \in S(\mathbb{N})} \left(|a_{p_1} - a_{p_2}|^2 w_{\pi^{-1}(1)} + \cdots + |a_{p_n}|^2 w_{\pi^{-1}(n)} \right) \right)^{\frac{1}{2}} \\
&= ||((a_{p_1} - a_{p_2})^2, (a_{p_2} - a_{p_3})^2, \ldots, a_{p_n}^2, 0, 0, \ldots)||_{d(w,1)}^{\frac{1}{2}} \\
&= \left\| \sum_{i=1}^{n-1} (a_{p_i} - a_{p_{i+1}})^2 e_i + a_{p_n}^2 e_n \right\|_{d(w,1)}^{\frac{1}{2}}.
\end{aligned}$$

Therefore, comparing (5.2.2) and (5.2.8), $J_{CL}(B) = JA$ as sets, and hence JA is a Banach space. ☐

Let $1 \leqslant p < \infty$. A basis (b_j) for a Banach space B is *block p-Hilbertian* if there exists $K > 0$ such that for each norm-bounded block basic sequence (z_k) of (b_j), and each sequence (α_k) of scalars we have

$$\left\| \sum_{k=1}^{m} \alpha_k z_k \right\|_B \leqslant (K \sup_{k \in \mathbb{N}} ||z_k||) \left(\sum_{k=1}^{m} |\alpha_k|^p \right)^{\frac{1}{p}}$$

for each $m \in \mathbb{N}$.

Casazza and Lohman proved that if B is a reflexive Banach space with a symmetric, monotone, block p-Hilbertian basis (b_j), then $J_{CL}(B)$ has a monotone shrinking basis, $J_{CL}(B)$ has codimension 1 in $J_{CL}(B)^{**}$, and $J_{CL}(B)$ is isomorphic to $J_{CL}(B)^{**}$ as a Banach space [20, Theorems 3,10,12]. We note that for each weight w, $B = d(w, 2)$ is reflexive and has a symmetric, monotone, block 2-Hilbertian basis [20, Remark 8], and consequently $J_{CL}(B) = JA$ has all the properties we require for Section 2 of Read's paper, as long as we remain in the case $A = d(w, 1)$. The following theorem summarises these remarks. For a James-like space JA, write JA^* and JA^{**} for its dual space and bidual space, respectively (this means $(JA)^*$, $(JA)^{**}$ and never $J(A^*)$ or $J(A^{**})$).

Theorem 5.2.11 (Read)**.** *Let $A = d(w, 1)$, a Lorentz 1-sequence space with weight w. Then*

(i) $(e_n)_{n=1}^{\infty}$ *is a monotone, shrinking Schauder basis for JA;*

(ii) *JA is quasi-reflexive;*

(iii) *JA is isomorphic to JA^{**};*

(iv) *JA has codimension 1 in JA^{**}.*

There is perhaps some ambiguity in denoting the bases of A and JA both by (e_n); the reason for this is they actually consist of the same elements of c_{00}, just considered in different spaces. The main difference is that (e_n) is not necessarily normalised as a basis of JA. From now on the context should always prevent confusion.

Read not only wants a specific space JA, he wants a whole sequence of spaces, in order to take the direct sum of them. But they have to relate in a very particular way.

For each $i \in \mathbb{N}_0$, let $(A_i, ||\cdot||_{A_i})$ be a Banach space with a normalised, 1-symmetric basis, which we write as (e_n), regardless of i. Again, this should not cause confusion since for the specific A_i we will soon define, the bases will be the same, namely the unit vector basis of c_{00}.

Definition 5.2.12. The sequence $(A_i)_{i\in\mathbb{N}_0}$ is *incomparable* if for every $i \in \mathbb{N}_0$, and for every $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that:

$$\left\|\sum_{j=1}^{N} e_j\right\|_{A_i} \leqslant \frac{1}{n} \inf_{k\neq i} \left\|\sum_{j=1}^{N} e_j\right\|_{A_k} \tag{5.2.9}$$

where $k \in \mathbb{N}_0$.

Read observes [90, Definition 3.2] that an incomparable sequence $(A_i)_{i\in\mathbb{N}_0}$ of such Banach spaces, each containing no copy of c_0, exists, but omits the details. Our aim is to fill in the details by giving a specific sequence of spaces. The corresponding sequence of James-like Banach spaces $(JA_i)_{i=0}^{\infty}$ will be key when building our direct sum in the next section. The idea of [90, Definition 3.2] is to take $A_i = d(w_i, 1)$ for $i \in \mathbb{N}_0$, where each w_i is a carefully chosen weight. We make this precise in the next lemma.

Lemma 5.2.13. *For each $i \in \mathbb{N}_0$, there exists a sequence of real numbers $w_i = (\alpha_{i,j})_{j=1}^{\infty}$ satisfying:*

(i) $1 = \alpha_{i,1} \geqslant \alpha_{i,2} \geqslant \alpha_{i,3} \geqslant \cdots > 0$*;*

(ii) $\sum_{j=1}^{\infty} \alpha_{i,j} = \infty$*;*

(iii) $\alpha_{i,j} \to 0$ *as* $j \to \infty$*;*

(iv) *for every $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that:*

$$\left\| \sum_{j=1}^{N} e_j \right\|_{d(w_i,1)} \leqslant \frac{1}{n} \inf_{k \neq i} \left\| \sum_{j=1}^{N} e_j \right\|_{d(w_k,1)}$$

where $(e_j)_{j=1}^{\infty}$ denotes the unit vector basis common to each Lorentz sequence space.

Moreover, the sequence $(A_i)_{i \in \mathbb{N}_0} = (d(w_i, 1))_{i \in \mathbb{N}_0}$ is incomparable and no A_i contains c_0.

Proof. We verify the 'moreover' statement first, assuming that the preceding statement holds true. Conditions (i)–(iii) ensure that each w_i is a weight, so that there are corresponding Lorentz 1-sequence spaces $d(w_i, 1)$ for every $i \in \mathbb{N}_0$. Thus condition (iv) makes sense. In particular, (iv) shows that the sequence $(A_i)_{i \in \mathbb{N}_0} = (d(w_i, 1))_{i \in \mathbb{N}_0}$ is incomparable by (5.2.9).

A standard result about Lorentz sequence spaces [74, Proposition 4.c.3] says that for any weight w, every infinite-dimensional closed subspace of $d(w, 1)$ contains a complemented copy of ℓ_1. So if $d(w, 1)$ contained a copy of c_0 we would find a copy of ℓ_1 inside c_0. Since this is impossible [3, Corollary 2.1.6], the result follows.

We need to write down a matrix $(\alpha_{i,j})_{i \in \mathbb{N}_0, j \in \mathbb{N}}$ with properties (i)-(iv).

Fix $i \in \mathbb{N}_0$. Firstly, note that

$$\left\| \sum_{j=1}^{N} e_j \right\|_{A_i} = \sum_{j=1}^{N} \alpha_{i,j} \tag{5.2.10}$$

so property (iv) becomes: for every $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that, for all $k \in \mathbb{N}_0 \backslash \{i\}$:

$$\sum_{j=1}^{N} \alpha_{i,j} \leqslant \frac{1}{n} \sum_{j=1}^{N} \alpha_{k,j}.$$

Choose a function $\sigma : \mathbb{N}_0 \to \mathbb{N}_0$ such that $\sigma^{-1}(\{j\})$ is infinite for each $j \in \mathbb{N}_0$, and such that $\sigma(n) \neq \sigma(n+1)$ for each $n \in \mathbb{N}_0$. For example, take $\sigma(n) =$ sum of digits of n. Then $\sigma^{-1}(\{j\})$ is infinite for each $j \in \mathbb{N}_0$ because we may add zeros after the final digit. Pick $n \in \mathbb{N}$ and write out the digits as $n = a_1 \cdots a_m$ for some $m \in \mathbb{N}$, $a_1, \ldots, a_m \in \mathbb{N}_0$. Then $\sigma(n+1) = \sigma(n) + 1$ unless $a_m = 9$, in which case there exists $j \in \mathbb{N}$ such that $n + 1 = a_1 \cdots a_j 99 \cdots 9$ and $a_j \neq 9$. Then $\sigma(n+1) = a_1 \cdots (a_j + 1) 00 \cdots 00 \neq \sigma(n)$. Hence $\sigma(n+1) \neq \sigma(n)$ for every $n \in \mathbb{N}_0$, so this function has the required properties.

Next define a sequence $(S_k)_{k=1}^{\infty}$ recursively by taking $S_1 = 1$ and for each $k \in \mathbb{N}$

$$S_{k+1} = (k+1)(S_k + 1). \tag{5.2.11}$$

This is a strictly increasing sequence. The first few terms are $(S_1, S_2, S_3, S_4, \ldots) = (1, 4, 15, 64, \ldots)$.

We would now like to define another function $\tau : \mathbb{N}_0 \to \mathbb{N}_0$ recursively. Take $\tau(0) = 0$, $\tau(1) = 1$ and $\tau(2) = 4$. For every $n \geqslant 2$ choose

$$\tau(n+1) = S_{n+1}\tau(n) - S_{n+1}\tau(n-1) + \frac{1}{n}S_n\tau(n-1) - \tau(n)S_{n-1}. \tag{5.2.12}$$

We prove that $\tau(n+1) > \tau(n)$ for every $n \in \mathbb{N}_0$. Indeed, it holds for $n = 0, 1, 2$. Let $n \geqslant 2$. We have $\frac{1}{n}S_n - 1 = S_{n-1}$, so by (5.2.12)

$$\tau(n+1) = (S_{n+1} - S_{n-1})(\tau(n) - \tau(n-1)) + \tau(n-1) \tag{5.2.13}$$

which, since $1 < S_{n+1} - S_{n-1}$, implies that $\tau(n+1) > \tau(n)$. It follows that τ is injective, and moreover, an easy induction argument shows that $\tau(n+1) > n\tau(n)$ for each $n \in \mathbb{N}_0$.

From this we define three sequences $(t_n), (u_n), (v_n)$ of real numbers. Let $t_1 = u_1 = v_1 = 1$. Then define

$$t_{n+1} = \frac{S_{n+1} - \frac{1}{n}S_n}{\tau(n+1) - \tau(n)} \qquad u_{n+1} = \frac{S_{n+1} - S_n}{\tau(n+1) - \tau(n)} \qquad v_{n+1} = \frac{1}{\tau(n+1) - \tau(n)} \tag{5.2.14}$$

for every integer $n \geqslant 1$. A quick check using (5.2.12) shows that

$$1 = t_1 \geqslant u_1 \geqslant v_1 \geqslant t_2 \geqslant u_2 \geqslant v_2 \geqslant t_3 \geqslant \cdots > 0 \tag{5.2.15}$$

and explains our choice of τ. Then by defining (for $j \in \mathbb{N}$ and $r \in \mathbb{N}_0$)

$$\alpha_{i,j} = \begin{cases} t_{r+1} & \text{if } i = \sigma(r) \text{ and } \ \tau(r) < j \leqslant \tau(r+1), \\ u_{r+1} & \text{if } i \neq \sigma(r), \sigma(r+1) \ \text{ and } \ \tau(r) < j \leqslant \tau(r+1), \\ v_{r+1} & \text{if } i = \sigma(r+1) \ \text{ and } \ \tau(r) < j \leqslant \tau(r+1), \end{cases} \tag{5.2.16}$$

we *claim* that we obtain the required weight w_i. Notice first that, given $j \in \mathbb{N}$, $r \in \mathbb{N}_0$ is uniquely determined because τ is injective. Thus the definition makes sense.

For all $n \in \mathbb{N}$ we have

$$\sum_{j=1}^{\tau(n)} \alpha_{i,j} = \begin{cases} \frac{1}{n}S_n & \text{if } i = \sigma(n) \\ S_n & \text{otherwise.} \end{cases} \tag{5.2.17}$$

We prove this by induction on $n \in \mathbb{N}$. The statement is clearly true for $n = 1$. Suppose that it holds for some $k \in \mathbb{N}$, so that

$$\sum_{j=1}^{\tau(k)} \alpha_{i,j} = \begin{cases} \frac{1}{k} S_k & \text{if } i = \sigma(k) \\ S_k & \text{otherwise} \end{cases}$$

Now we check the statement for $k+1$. In the first case, suppose that $i = \sigma(k+1)$. Then by a property of σ, $\sigma(k) \neq i$. So

$$\sum_{j=1}^{\tau(k+1)} \alpha_{i,j} = \sum_{j=1}^{\tau(k)} \alpha_{i,j} + \sum_{j=\tau(k)+1}^{\tau(k+1)} \alpha_{i,j} = S_k + \sum_{j=\tau(k)+1}^{\tau(k+1)} \alpha_{i,j}$$

using the inductive hypothesis. By (5.2.14) and (5.2.16) we obtain

$$\sum_{j=1}^{\tau(k+1)} \alpha_{i,j} = S_k + v_{k+1}(\tau(k+1) - \tau(k)) = \frac{1}{k+1} S_{k+1}$$

as required. In the second case, suppose that $\sigma(k) = i$. Then by the inductive hypothesis, (5.2.14) and (5.2.16), we have

$$\sum_{j=1}^{\tau(k+1)} \alpha_{i,j} = \frac{1}{k} S_k + \sum_{j=\tau(k)+1}^{\tau(k+1)} \alpha_{i,j} = \frac{1}{k} S_k + t_{k+1}(\tau(k+1) - \tau(k)) = S_{k+1}.$$

Finally suppose that $\sigma(k) \neq i$ and $\sigma(k) \neq i+1$. As before, the inductive hypothesis together with (5.2.14) and (5.2.16) imply that

$$\sum_{j=1}^{\tau(k+1)} \alpha_{i,j} = S_k + u_{k+1}(\tau(k+1) - \tau(k)) = S_{k+1}.$$

Therefore (5.2.17) holds for all $n \in \mathbb{N}$ by induction.

Before we formally finish the proof of the claim, let us give an intuition as to why it should work. In the abstract setting the idea for our matrix $(\alpha_{i,j})$ is given by the following diagram (for $n \in \mathbb{N}$):

$$
\begin{array}{rc}
 & \begin{matrix} & & \text{column} & & & & & & \\ & & \tau(n) & & & & & \tau(n+1) & \end{matrix} \\
\begin{matrix} \\ \\ \\ \\ \text{row } \sigma(n) \\ \\ \\ \sigma(n+1) \\ \\ \\ \\ \end{matrix} &
\begin{pmatrix}
1 & \cdots & | & & & & & | & \\
\vdots & & \vdots & & & & & \vdots & \\
1 & & S_n & \cdots & u_{n+1} & u_{n+1} & \cdots & S_{n+1} & \\
1 & \cdots & | & & & & & | & \cdots \\
1 & & \frac{1}{n}S_n & \cdots & t_{n+1} & t_{n+1} & \cdots & S_{n+1} & \\
1 & & | & & & & & | & \\
\vdots & & \vdots & & & & & \vdots & \cdots \\
1 & & S_n & \cdots & v_{n+1} & v_{n+1} & \cdots & \frac{1}{n+1}S_{n+1} & \\
1 & & | & & & & & | & \\
1 & & | & & & & & | & \\
\vdots & & \vdots & & & & & \vdots & \ddots
\end{pmatrix}
\end{array}
$$

where S_n means the *sum* of the row up to and including the marked column (that is, S_n is *not* the value of the matrix entry there, it is simply for illustration). At the column $\tau(n)$, every row has the same sum, S_n, except for row $\sigma(n)$ (the lines mean these rows have sum S_n, S_{n+1} respectively). By the time we reach column $\tau(n+1)$ we want to boost the sum of row $\sigma(n)$ so that it has sum S_{n+1}, but at the same time drop the sum of row $\sigma(n+1)$ so that it only has sum $\frac{1}{n+1}S_{n+1}$. The numbers t_{n+1}, u_{n+1}, v_{n+1} are therefore carefully chosen to do just that.

By filling in some entries things become a little more transparent.

$$
\begin{array}{rc}
 & \begin{matrix} \tau(1) & 2 & 3 & \tau(2) & 5 & 6 & \cdots & \tau(3) & 44 & 45 & \cdots \end{matrix} \\
\begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \\ \\ \end{matrix} &
\begin{pmatrix}
u_1 & u_2 & u_2 & u_2 & u_3 & u_3 & & u_3 & u_4 & u_4 & \\
v_1 & t_2 & t_2 & t_2 & u_3 & u_3 & & u_3 & u_4 & u_4 & \\
u_1 & v_2 & v_2 & v_2 & t_3 & t_3 & \cdots & t_3 & u_4 & u_4 & \\
u_1 & u_2 & u_2 & u_2 & v_3 & v_3 & & v_3 & t_4 & t_4 & \\
u_1 & u_2 & u_2 & u_2 & u_3 & u_3 & \cdots & u_3 & v_4 & v_4 & \cdots \\
u_1 & u_2 & u_2 & u_2 & u_3 & u_3 & & u_3 & u_4 & u_4 & \\
 & & & & & & & & & & \\
\vdots & & \vdots & & & & & \vdots & & & \ddots
\end{pmatrix}
\end{array}
\tag{5.2.18}
$$

Now let us observe some possible values for our sequences

$$\begin{aligned}
&(\tau(0), \tau(1), \tau(2), \tau(3), \tau(4), \ldots) = (0, 1, 4, 43, 2344, \ldots) \\
&(t_1, t_2, t_3, \ldots) = (1, 1, \frac{1}{3}, \ldots) \\
&(u_1, u_2, u_3, \ldots) = (1, 1, \frac{11}{39}, \ldots) \\
&(v_1, v_2, v_3, \ldots) = (1, \frac{1}{3}, \frac{1}{39}, \ldots)
\end{aligned}$$

Our matrix looks like

$$(\alpha_{i,j}) = \begin{pmatrix} 1 & 1 & 1 & 1 & \frac{11}{39} & & \\ 1 & 1 & 1 & 1 & \frac{11}{39} & & \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots & \\ 1 & 1 & 1 & 1 & \frac{1}{39} & & \\ 1 & 1 & 1 & 1 & \frac{11}{39} & & \\ \vdots & & & & & \ddots & \end{pmatrix}$$

where the i^{th} row corresponds to the weight w_i.

We can now prove the claim. We must check that $w_i = (\alpha_{i,j})_{j=1}^{\infty}$ from (5.2.16) satisfies properties (i)–(iv).

(i) We have $\alpha_{i,1} = 1$. Choose $j \in \mathbb{N}$ with $\tau(r) < j \leqslant \tau(r+1)$ for some unique $r \in \mathbb{N}_0$. If $\tau(r) < j+1 \leqslant \tau(r+1)$, then by (5.2.16) we have $\alpha_{i,j} = \alpha_{i,j+1}$. Otherwise $\tau(r+1) < j+1 \leqslant \tau(r+2)$, but then $\alpha_{i,j} \geqslant \alpha_{i,j+1}$ by (5.2.15) and (5.2.16).

(ii) Let M be a natural number greater than 1. We must show there exists n_0 such that $\sum_{j=1}^{n_0} \alpha_{i,j} > M$. If $i \neq \sigma(M)$, then by (5.2.17) we have

$$\sum_{j=1}^{\tau(M)} \alpha_{i,j} = S_M = M(S_{M-1} + 1) > M.$$

If $i = \sigma(M)$, then $i \neq \sigma(M+1)$, and so, by (5.2.17),

$$\sum_{j=1}^{\tau(M+1)} \alpha_{i,j} = S_{M+1} = (M+1)(S_M + 1) > M.$$

(iii) Take an integer $n \geqslant 2$. By (5.2.13) we have

$$\tau(n+1) - \tau(n) = (S_{n+1} - S_{n-1} - 1)(\tau(n) - \tau(n-1)).$$

Also, by (5.2.11)

$$S_{n+1} - S_{n-1} = (n+1)^2 + (n^2+n-1)S_{n-1}.$$

Since $\tau(n+1) > n\tau(n)$ for each $n \in \mathbb{N}_0$, we have $\tau(n+1) - \tau(n) > n-1$ for $n \in \mathbb{N}_0$; this implies that $\tau(n+1) - \tau(n) \to \infty$ as $n \to \infty$, and so $v_n \to 0$ as $n \to \infty$. Hence $\alpha_{i,j} \to 0$ as $j \to \infty$ by (5.2.15).

(iv) Take $n \in \mathbb{N}$. Since $\sigma^{-1}(\{i\})$ is infinite there exists $m \in \mathbb{N}$ such that $i = \sigma(n+m)$. Now let $N = \tau(n+m)$. Then using (5.2.10) and (5.2.17),

$$\left\|\sum_{j=1}^{N} e_j\right\|_{A_i} = \sum_{j=1}^{\tau(n+m)} \alpha_{i,j} = \frac{1}{n+m} S_{n+m}$$

and

$$\left\|\sum_{j=1}^{N} e_j\right\|_{A_k} = S_{n+m} \qquad \text{if } \; k \neq i$$

so indeed

$$\left\|\sum_{j=1}^{N} e_j\right\|_{A_i} = \frac{1}{n+m} \inf_{k \neq i} \left\|\sum_{j=1}^{N} e_j\right\|_{A_k} \leqslant \frac{1}{n} \inf_{k \neq i} \left\|\sum_{j=1}^{N} e_j\right\|_{A_k},$$

which implies that (iv) holds. □

5.2.3 Direct sums of Banach spaces and the space $E_{\mathcal{R}}$

Consider the sequence of Banach spaces $(JA_i)_{i=0}^{\infty}$, where $(A_i)_{i=0}^{\infty}$ is the incomparable sequence defined in Lemma 5.2.13. We actually need the sequence to be a little more complicated, so we require some further definitions.

Define the set

$$\mathbb{I} = \{2\} \cup \{i \in \mathbb{N}_0 : i \equiv 0 \bmod 6,\ i \equiv 4 \bmod 6, \text{ or } i \equiv 5 \bmod 6\}. \qquad (5.2.19)$$

Write $\varphi : \mathbb{I} \to \mathbb{N}_0$ for the order isomorphism taking $\mathbb{I}$ onto $\mathbb{N}_0$. Then for each $i \in \mathbb{N}_0$, let

$$B_i = \begin{cases} A_{\varphi(i)} & \text{if } i \in \mathbb{I} \\ A_0 & \text{if } i \equiv 1 \bmod 6 \\ A_1 & \text{if } i \equiv 2 \bmod 6 \text{ or } i \equiv 3 \bmod 6. \end{cases} \qquad (5.2.20)$$

This yields a sequence of James-like spaces with some repeats:

$$(JB_i)_{i=0}^{\infty} = (JA_0, JA_0, JA_1, JA_1, JA_2, JA_3, JA_4, JA_0, JA_1, JA_1, JA_5, JA_6, \ldots)$$

where each B_i is a Lorentz 1-sequence space. Since the sequence $(B_i)_{i\in\mathbb{I}}$ is just a relabelling of $(A_i)_{i=0}^{\infty}$, the weights are chosen so that $(B_i)_{i\in\mathbb{I}}$ is incomparable, that is, for every $i \in \mathbb{I}$, and for every $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that:

$$\left\| \sum_{j=1}^{N} e_j \right\|_{B_i} \leqslant \frac{1}{n} \inf_{k\neq i} \left\| \sum_{j=1}^{N} e_j \right\|_{B_k} \tag{5.2.21}$$

where $k \in \mathbb{I}$.

The next step in Read's construction is to 'stick the spaces together' in an infinite direct sum.

Definition 5.2.14. We define the Banach space $Y = \left(\bigoplus_{i=0}^{\infty} JB_i \right)_{\ell_2}$.

Before we finish the definition of $E_{\mathcal{R}}$ we note some useful properties of Y. By Theorem 5.2.11(i), when $A = d(w, 1)$, JA has a shrinking basis so the sequence of coordinate functionals (e_n^*) is a basis for JA^*. Consider the map

$$\Phi : \mathrm{span}\{e_n^* : n \in \mathbb{N}\} \subset JA^* \to \mathbb{K}$$

given by $\langle e_n^*, \Phi \rangle = 1$ for each $n \in \mathbb{N}$, extended by linearity. This is bounded and linear, with norm equal to 1. To see this, let $\sum_{n=1}^{N} \lambda_n e_n^*$ be in the domain $(\lambda_1, \ldots, \lambda_n \in \mathbb{K})$. Then $|\Phi(\sum_{n=1}^{N} \lambda_n e_n^*)| = |\sum_{n=1}^{N} \lambda_n|$. Clearly $\sum_{n=1}^{N} e_n$ is a unit vector in JA and $|\sum_{n=1}^{N} \lambda_n e_n^*(\sum_{n=1}^{N} e_n)| = |\sum_{n=1}^{N} \lambda_n|$. Thus

$$|\Phi(\sum_{n=1}^{N} \lambda_n e_n^*)| \leqslant \left\| \sum_{n=1}^{N} \lambda_n e_n^* \right\|_{JA^*},$$

and in fact $||\Phi|| = 1$ because $|\langle e_n^*, \Phi \rangle| = 1$ for each $n \in \mathbb{N}$. Since Φ is densely defined, it extends uniquely to a continuous linear map of the same norm, also denoted by Φ, on JA^*. Thus $\Phi \in JA^{**}$ and $||\Phi||_{JA^{**}} = 1$.

Recall that JA is quasi-reflexive by Theorem 5.2.11(iii). The next result is a special case of the duality for direct sums of Banach spaces (Proposition 5.1.7) when the spaces are quasi-reflexive. We adopt the following slight abuse of notation: for each $i \in \mathbb{N}_0$ there is a map $\Phi \in JB_i^{**}$, defined as above; we denote it by Φ, regardless of the index i.

Proposition 5.2.15. *There are isometric isomorphisms:*

$$Y^* \simeq \left(\bigoplus_{i=0}^{\infty} JB_i^*\right)_{\ell_2} \qquad Y^{**} \simeq \left(\bigoplus_{i=0}^{\infty} JB_i^{**}\right)_{\ell_2} \simeq \left(\bigoplus_{i=0}^{\infty} JB_i \oplus \mathbb{K}\Phi\right)_{\ell_2}.$$

Under these identifications the dual actions are as follows. For $y = (y_0, y_1, \ldots) \in Y$, $y^* = (y_0^*, y_1^*, \ldots) \in Y^*$ and $y^{**} = (y_0^{**}, y_1^{**}, \ldots) \in Y^{**}$ we have:

$$\langle y, y^* \rangle = \sum_{i=0}^{\infty} \langle y_i, y_i^* \rangle \qquad \text{and} \qquad \langle y^*, y^{**} \rangle = \sum_{i=0}^{\infty} \langle y_i^*, y_i^{**} \rangle$$

Thus from now on we identify $Y^* = (\bigoplus_{i=0}^{\infty} JB_i^*)_{\ell_2}$ and $Y^{**} = (\bigoplus_{i=0}^{\infty} JB_i \oplus \mathbb{K}\Phi)_{\ell_2}$.

We are well on our way to defining $E_{\mathcal{R}}$.

Definition 5.2.16. Read considers the Hilbert space $\overline{B} = \ell_2(\mathbb{N}_0)$ and relabels its standard orthonormal basis $(b_n)_{n \in \mathbb{N}_0}$ as follows:

$$\begin{aligned} \alpha_n &= b_{6n}, & \beta_n &= b_{6(n-1)+1}, & \gamma_n &= b_{6(n-1)+2}, \\ \delta_n &= b_{6(n-1)+3}, & x_n &= b_{6(n-1)+4}, & y_n &= b_{6(n-1)+5} \end{aligned} \qquad (n \in \mathbb{N}). \tag{5.2.22}$$

Let us introduce additional symbols for the following important linear combinations of these basis vectors:

$$\begin{aligned} \alpha_n' &= \alpha_n - (x_n - y_n), & \beta_n' &= \beta_n - (x_n + y_n), \\ \gamma_n' &= \gamma_n - (x_n + y_n), & \delta_n' &= \delta_n - \left(\frac{b_0}{2^n} - x_n + y_n\right) \end{aligned} \qquad (n \in \mathbb{N}). \tag{5.2.23}$$

Remark 5.2.17. Our definition of β_n' corrects a small mistake in [90, Definition 3.4(b)] where the sign of y_n is incorrect. This can be seen by comparison with the second line of the displayed equations at the bottom of [90, p. 313], and the seventh displayed equation of [90, p. 319].

Before stating the final definitions of Read, we need to explain his tensor notation, which is a convenient way of expressing elements of Y, Y^* and Y^{**}.

Definition 5.2.18 (Tensor Notation)**.** Recall that (e_n) denotes the 1-symmetric basis of JB_i (and B_i) for each $i \in \mathbb{N}_0$. For $n \in \mathbb{N}$ and $\xi = \sum_{i=0}^{\infty} \xi_i b_i \in \overline{B}$ we let

$$e_n \otimes \xi = (\xi_i e_n)_{i=0}^{\infty} = (\xi_0 e_n, \xi_1 e_n, \ldots) \in Y.$$

This extends by linearity to tensors $x \otimes \xi$ for $x \in c_{00}$, that is,

$$x \otimes \xi = (\xi_i x)_{i=0}^{\infty} = (\xi_0 x, \xi_1 x, \ldots) \in Y.$$

Why is $e_n \otimes \xi$ an element of Y? For each $n \in \mathbb{N}$ and $i \in \mathbb{N}_0$, using the definition of the JA norm we obtain:

$$||e_n||_{JB_i} \leqslant ||e_1 + e_2||_{B_i} \leqslant 2 \tag{5.2.24}$$

because each B_i has a normalised 1-symmetric basis. This uniform bound implies that

$$\begin{aligned} ||e_n \otimes \xi||_Y = ||(\xi_0 e_n, \xi_1 e_n, \ldots)||_Y &= \left(\sum_{i=0}^{\infty} \|\xi_i e_n\|_{JB_i}^2 \right)^{\frac{1}{2}} \\ &\leqslant 2 \left(\sum_{i=0}^{\infty} |\xi_i|^2 \right)^{\frac{1}{2}} = 2||\xi||_{\overline{B}} < \infty, \end{aligned} \tag{5.2.25}$$

so that $e_n \otimes \xi$ is an element of Y. By the same reasoning we obtain that $x \otimes \xi \in Y$ for $x \in c_{00}$.

We can also use this idea in Y^*. For each $n \in \mathbb{N}$ and $j \in \mathbb{N}_0$, the coordinate functional e_n^* is in JB_j^*. For $k \in \mathbb{N}$, denote by P_k the k^{th} basis projection for the basis (e_n) of JB_j (again abusing notation). Then

$$\begin{aligned} ||e_n^*||_{JB_j^*} &= \sup\{|\langle a, e_n^* \rangle| : ||a||_{JB_j} = 1, a \in JB_j\} \\ &\leqslant \sup\{||(P_n - P_{n-1})(a)|| : ||a|| = 1, a \in JB_j\} = ||P_n - P_{n-1}|| \leqslant 2 \end{aligned} \tag{5.2.26}$$

since (e_n) is a monotone basis (by convention let P_0 be the zero map). Thus for each $\xi = \sum_{i=0}^{\infty} \xi_i b_i \in \overline{B}$, we can define

$$e_n^* \otimes \xi = (\xi_i e_n^*)_{i=0}^{\infty} = (\xi_0 e_n^*, \xi_1 e_n^*, \ldots) \in Y^* \tag{5.2.27}$$

which is well-defined, as in (5.2.25). Again this leads to the definition of $x^* \otimes \xi \in Y^*$ for $x^* \in \operatorname{span}\{e_n^* : n \in \mathbb{N}\}$.

We can further extend this tensor notation to the bidual by defining

$$\Phi \otimes \xi = (\xi_i \Phi)_{i=0}^{\infty} \in Y^{**}, \tag{5.2.28}$$

which makes sense because $||\Phi||_{JB_j^{**}} = 1$ for each $j \in \mathbb{N}_0$. The tensor notation is intuitive because the elements work in a bilinear way, so we can manipulate them like usual tensors.

Example 5.2.19 (Tensor Duality)**.** Combining Proposition 5.2.15 with the tensor notation, we obtain the following tensor duality. Let $n \in \mathbb{N}$, and $\xi = (\xi_i)$, $\eta = (\eta_i)$, and $\chi = (\chi_i) \in \overline{B}$. Then

$$\begin{aligned}\langle e_n \otimes \xi, e_n^* \otimes \eta \rangle &= \big\langle (\xi_i e_n)_{i=0}^\infty, (\eta_i e_n^*)_{i=0}^\infty \big\rangle = \sum_{i=0}^\infty \langle \xi_i e_n, \eta_i e_n^* \rangle \\ &= \sum_{i=0}^\infty \xi_i \eta_i \langle e_n, e_n^* \rangle = \sum_{i=0}^\infty \xi_i \eta_i = (\xi | \overline{\eta}),\end{aligned}$$

where $\overline{\eta}$ means the pointwise complex conjugate, that is, $\overline{\eta} = (\overline{\eta_i})_{i=0}^\infty$. This extends naturally to sums of tensors, and also to the dual where

$$\begin{aligned}\langle e_n^* \otimes \eta, \Phi \otimes \chi \rangle &= \big\langle (\eta_i e_n^*)_{i=0}^\infty, (\chi_i \Phi)_{i=0}^\infty, \big\rangle = \sum_{i=0}^\infty \langle \eta_i e_n^*, \chi_i \Phi \rangle \\ &= \sum_{i=0}^\infty \eta_i \chi_i \langle e_n^*, \Phi \rangle = \sum_{i=0}^\infty \eta_i \chi_i = (\eta | \overline{\chi}).\end{aligned}$$

We are now in good shape to define Read's Banach space $E_{\mathcal{R}}$.

Definition 5.2.20. Denote

$$\begin{aligned}S &= \{\alpha_n', \beta_n', \gamma_n', \delta_n' : n \in \mathbb{N}\}, \\ V &= \overline{\operatorname{span}}\, S \subseteq \overline{B}, \\ N &= \overline{\operatorname{span}}\, \{e_n \otimes s : n \in \mathbb{N},\ s \in S\} \subseteq Y.\end{aligned}$$

Then define

$$E_{\mathcal{R}} = Y/N, \tag{5.2.29}$$

which is a Banach space with respect to the quotient norm.

In summary, the construction of $E_{\mathcal{R}}$ had three distinct stages. We began with the James-like spaces defined over Lorentz sequence spaces, then formed the direct sum of them in the sense of ℓ_2, and then took a quotient of this using certain relations on the coordinates. The assumption of incomparability on the family $(B_i)_{i \in \mathbb{I}}$ will turn out to be important, as will the particular subspace N that we chose. We gave an explicit example of an incomparable family, and this generates what we consider to be *the* Read space in this thesis. However, any example of such a family will produce a space with the same properties.

We are now ready to analyse the space $E_{\mathcal{R}}$ in some detail.

5.3 Analysing the space $E_{\mathcal{R}}$

Read proves two important facts about the space $E_{\mathcal{R}}$ in [90, §3]. The first is that there is an isomorphism between $E_{\mathcal{R}}^{**}/E_{\mathcal{R}}$ and $\overline{B}/V$ (so that $E_{\mathcal{R}}^{**}/E_{\mathcal{R}}$ is a Hilbert space) [90, Equation (3.6.5)]. The second says that it is possible to explicitly identify an 'almost orthonormal basis' for $\overline{B}/V$ [90, Lemma 3.7], and so $E_{\mathcal{R}}^{**}/E_{\mathcal{R}}$ has an 'almost orthonormal basis' too. Then the key result of the whole paper is [90, Lemma 4.1]. In this section we shall prove versions of the first two facts which suit our later purposes, and then give a detailed account of Read's Lemma 4.1. By refining the two facts slightly the proof of Lemma 4.1 becomes more transparent, and we are also able to avoid having to use the 'almost' language, above.

We begin by summarising Read's Lemma 3.6 and its subsidiaries 3.6.1, 3.6.2 and 3.6.4 [90, pp. 312–313]. The reader should recall the tensor notation from Definition 5.2.18.

Lemma 5.3.1. *For every $n \in \mathbb{N}$ the maps*

$$\begin{aligned} R_n : z &\mapsto \big(\langle z_i, e_n^* \rangle\big)_{i=0}^{\infty}, \qquad Y \to \overline{B} \qquad (z = (z_i)_{i=0}^{\infty} \in Y) \\ S_n : z^* &\mapsto \big(\langle e_n, z_i^* \rangle\big)_{i=0}^{\infty}, \qquad Y^* \to \overline{B} \qquad (z^* = (z_i^*)_{i=0}^{\infty} \in Y^*) \end{aligned}$$

are well-defined. Consequently, for every $z \in Y$, $z^* \in Y^*$ and $z^{**} \in Y^{**}$ there exist unique $y \in Y$ and $\xi \in \overline{B}$ such that

$$z = \sum_{n=1}^{\infty} e_n \otimes R_n z, \qquad z^* = \sum_{n=1}^{\infty} e_n^* \otimes S_n z^*, \qquad z^{**} = y + \Phi \otimes \xi.$$

Moreover, the following hold true:

(i) *$z \in N$ if and only if $R_n z \in V$ for every $n \in \mathbb{N}$;*

(ii) *$z^* \in N^\circ$ if and only if $S_n z^* \in V^\perp$ for every $n \in \mathbb{N}$;*

(iii) *$z^{**} \in N^{\circ\circ}$ if and only if $\xi \in V$ and $R_n y \in V$ for every $n \in \mathbb{N}$.*

Proof. Choose $n \in \mathbb{N}$, $z = (z_i)_{i=0}^{\infty} \in Y$, $z^* = (z_i^*)_{i=0}^{\infty} \in Y^*$ and $z^{**} \in Y^{**}$. Then by (5.2.24) and (5.2.26)

$$\begin{aligned} ||R_n z||^2 &= ||\big(\langle z_i, e_n^* \rangle\big)_{i=0}^{\infty}||_{\overline{B}}^2 = \sum_{i=0}^{\infty} |\langle z_i, e_n^* \rangle|^2 \leqslant \sum_{i=0}^{\infty} ||z_i||_{JB_i}^2 \, ||e_n^*||_{JB_i^*}^2 \leqslant 4||z||_Y^2 < \infty \\ ||S_n z^*||^2 &= ||\big(\langle e_n, z_i^* \rangle\big)_{i=0}^{\infty}||_{\overline{B}}^2 = \sum_{i=0}^{\infty} |\langle e_n, z_i^* \rangle|^2 \leqslant \sum_{i=0}^{\infty} ||e_n||_{JB_i}^2 ||z_i^*||_{JB_i^*}^2 \leqslant 4||z^*||_{Y^*}^2 < \infty \end{aligned}$$

so that R_n and S_n are well-defined.

To prove the series expansions, take $\varepsilon > 0$. Since $z \in Y$, there exists $L \in \mathbb{N}$ such that $\sum_{i=L+1}^{\infty} ||z_i||_{JB_i}^2 \leqslant \varepsilon^2/8$. Also, we may choose $M \in \mathbb{N}$ such that

$$||z_i - P_m z_i||_{JB_i} \leqslant \varepsilon/\sqrt{2(L+1)}$$

for $i \in \{0, \dots, L\}$ and $m \geqslant M$ (where P_m denotes the m^{th} basis projection on JB_i).

Now pick $m \geqslant M$. Then by the definition of the tensor notation

$$\sum_{n=1}^{m} e_n \otimes R_n z = \sum_{n=1}^{m} \big(\langle z_i, e_n^* \rangle e_n\big)_{i \in \mathbb{N}_0} = \left(\sum_{n=1}^{m} \langle z_i, e_n^* \rangle e_n\right)_{i \in \mathbb{N}_0} = (P_m z_i)_{i \in \mathbb{N}_0}.$$

Thus, since each JB_i has a monotone basis by Theorem 5.2.11, we have

$$\begin{aligned}
&\left\|z - \sum_{n=1}^{m} e_n \otimes R_n z\right\|_Y^2 = ||(z_i - P_m z_i)_{i \in \mathbb{N}_0}||_Y^2 = \sum_{i=0}^{\infty} ||z_i - P_m z_i||_{JB_i}^2 \\
&= \sum_{i=0}^{L} ||z_i - P_m z_i||_{JB_i}^2 + \sum_{i=L+1}^{\infty} ||I_{JB_i} - P_m||^2 ||z_i||_{JB_i}^2 \\
&\leqslant (L+1)(\varepsilon/\sqrt{2(L+1)})^2 + 4\varepsilon^2/8 = \varepsilon^2.
\end{aligned}$$

Hence $z = \sum_{n=1}^{\infty} e_n \otimes R_n z$. A related argument yields $z^* = \sum_{n=1}^{\infty} e_n^* \otimes S_n z^*$.

Next we want the expansion of z^{**}. We have $Y^{**} = (\bigoplus_{i=0}^{\infty} JB_i \oplus \mathbb{K}\Phi)_{\ell_2}$, so we may write $z^{**} = (w_0 + \lambda_0 \Phi, w_1 + \lambda_1 \Phi, \dots)$, where $w_k \in JB_k$ and $\lambda_k \in \mathbb{K}$ for each $k \in \mathbb{N}_0$. Then since $||\Phi|| = 1$, there is $C > 0$ such that

$$||z^{**}||_{Y^{**}}^2 = \sum_{i=0}^{\infty} ||w_i + \lambda_i \Phi||_{JB_i^{**}}^2 \geqslant \sum_{i=0}^{\infty} C\big(||w_i||_{JB_i} + |\lambda_i|\big)^2 \geqslant C \sum_{i=0}^{\infty} ||w_i||_{JB_i}^2 + C \sum_{i=0}^{\infty} |\lambda_i|^2.$$

This implies that $y = (w_0, w_1, \dots) \in Y$ and $\xi = (\lambda_0, \lambda_1, \dots) \in \overline{B}$, and that

$$z^{**} = (w_0, w_1, \dots) + (\lambda_0 \Phi, \lambda_1 \Phi, \dots) = y + \Phi \otimes \xi.$$

To see that the expansion is unique, suppose that there exist $\xi' = (\lambda_0', \lambda_1', \dots) \in \overline{B}$ and $y' = (w_0', w_1', \dots)$ such that $y + \Phi \otimes \xi = y' + \Phi \otimes \xi'$. Then $(y - y') + \Phi \otimes (\xi - \xi') = 0$ which implies that

$$\big(w_0 - w_0' + (\lambda_0 - \lambda_0')\Phi, w_1 - w_1' + (\lambda_1 - \lambda_1')\Phi, \dots\big) = 0.$$

Hence $\xi = \xi'$ and $y = y'$ because each coordinate has a direct sum decomposition.

(i) Suppose that there exists $n \in \mathbb{N}$ such that $R_n z \notin V$. Then we can take $\eta \in V^{\perp}$ such that $(R_n z | \eta) = 1$. Consider $e_n^* \otimes \overline{\eta}$, where $\overline{\eta}$ means the pointwise complex conjugate, that is, $\overline{\eta} = \big(\overline{\eta_0}, \overline{\eta_1}, \dots\big)$. Observe that $v \in V$ if and only if $\overline{v} \in V$ because $V = \overline{\operatorname{span}}\, S$ and S has this property. It follows that $u \in V^{\perp}$ if and only if $\overline{u} \in V^{\perp}$ because for each $u \in V^{\perp}$ and $v \in V$ we have $(\overline{u}|v) = \overline{(u|\overline{v})} = 0$.

Hence $\overline{\eta} \in V^\perp$. By the tensor duality from Proposition 5.2.15 we have

$$N = \overline{\operatorname{span}}\,\{e_m \otimes s : s \in S, m \in \mathbb{N}\} \subseteq \ker e_n^* \otimes \overline{\eta},$$

but $\langle z, e_n^* \otimes \overline{\eta}\rangle = (R_n z|\eta) = 1$, so that $z \notin N$. Contrapositively, the forward implication holds.

On the other hand, suppose that $R_n z \in V$ for every $n \in \mathbb{N}$. Let $v \in V = \overline{\operatorname{span}}\, S$. Then we can write $v = \sum_{n=1}^\infty \lambda_n s_n$ for some $s_n \in S$ and $\lambda_n \in \mathbb{K}$. So for $j, M \in \mathbb{N}$ by (5.2.25) we have

$$\begin{aligned}\left\| \sum_{n=1}^M \lambda_n (e_j \otimes s_n) - e_j \otimes \sum_{n=1}^\infty \lambda_n s_n \right\|_Y &= \left\| e_j \otimes \sum_{n=1}^M \lambda_n s_n - e_j \otimes \sum_{n=1}^\infty \lambda_n s_n \right\|_Y \\ = \left\| e_j \otimes \left(\sum_{n=1}^M \lambda_n s_n - \sum_{n=1}^\infty \lambda_n s_n \right) \right\|_Y &\leqslant 2 \left\| \sum_{n=1}^M \lambda_n s_n - \sum_{n=1}^\infty \lambda_n s_n \right\|_{\overline{B}}, \end{aligned} \tag{5.3.1}$$

which tends to zero as $M \to \infty$. Hence $e_j \otimes v = \sum_{n=1}^\infty \lambda_n (e_j \otimes s_n) \in N$.

We conclude that for each $j \in \mathbb{N}$ and $v \in V$, $e_j \otimes v \in N$. Now we can write $z = \sum_{n=1}^\infty e_n \otimes R_n z$, and so $z \in N$ because N is a closed subspace.

(ii) Suppose that there is $n \in \mathbb{N}$ such that $S_n z^* \notin V^\perp$. Take $\zeta \in V$ such that $(S_n z^*|\zeta) = 1$. Then by (5.3.1), $e_n \otimes \overline{\zeta} \in N$. By duality

$$\left\langle e_n \otimes \overline{\zeta}, z^* \right\rangle = \left\langle e_n \otimes \overline{\zeta}, \sum_{m=1}^\infty e_m^* \otimes S_m z^* \right\rangle = \left(\overline{\zeta} | \overline{S_n z^*}\right) = \left(S_n z^* | \zeta\right) = 1$$

so $z^* \notin N^\circ$.

Conversely, suppose that for every $n \in \mathbb{N}$, $S_n z^* \in V^\perp$. Again, write $z^* = \sum_{n=1}^\infty e_n^* \otimes S_n z^*$. Now fix $j \in \mathbb{N}$. Then for every $s \in S$ and $m \in \mathbb{N}$ we have

$$\langle e_m \otimes s, e_j^* \otimes S_j z^* \rangle = \begin{cases} \left(S_j z^* | \overline{s}\right) = 0 & \text{if } m = j \\ 0 & \text{if } m \neq j, \end{cases}$$

from which it follows that $e_j^* \otimes S_j z^* \in N^\circ$. Since N° is a closed subspace of Y^*, this implies that $z^* \in N^\circ$.

(iii) Write $z^{**} = y + \Phi \otimes \xi$. Suppose firstly that $\xi \notin V$. Then we can choose $\chi \in V^\perp$ such that $(\xi|\chi) = 1$. Next, for each $j \in \mathbb{N}$

$$\begin{aligned}\langle e_j^* \otimes \overline{\chi}, z^{**} \rangle &= \left\langle e_j^* \otimes \overline{\chi}, \sum_{n=1}^\infty e_n \otimes R_n y + \Phi \otimes \xi \right\rangle \\ &= (R_j y|\chi) + (\xi|\chi) = (R_j y|\chi) + 1. \end{aligned} \tag{5.3.2}$$

Since $\sum_{n=1}^\infty e_n \otimes R_n y$ converges, we have $||e_n \otimes R_n y||_Y \to 0$ as $n \to \infty$. Also, for each $j \in \mathbb{N}$ and $i \in \mathbb{N}_0$, $||e_j||_{JB_i} \geqslant 1$ so that

$$||e_j \otimes R_j y||^2 = ||(\langle y_0, e_j^* \rangle e_j, \langle y_1, e_j^* \rangle e_j, \ldots)||^2 = \sum_{i=0}^{\infty} |\langle y_i, e_j^* \rangle|^2 \, ||e_j||_{JB_i}^2$$

$$\geqslant \sum_{i=0}^{\infty} |\langle y_i, e_j^* \rangle|^2 = ||R_j y||_B^2$$

so we can choose $n \in \mathbb{N}$ such that $||R_n y|| < \frac{1}{||\chi||}$. Then by (5.3.2)

$$|\langle e_n^* \otimes \overline{\chi}, z^{**} \rangle| = |(R_n y|\chi) + 1| \geqslant 1 - |(R_n y|\chi)| \geqslant 1 - ||R_n y|| \, ||\chi|| > 0.$$

Note that $e_n^* \otimes \overline{\chi} \in N^\circ$ by the tensor duality. Hence $z^{**} \notin N^{\circ\circ}$.

Now suppose that $z^{**} \in N^{\circ\circ}$ and take $n \in \mathbb{N}$ and $\chi \in V^\perp$. By the previous calculation $\xi \in V$, and by duality $e_n^* \otimes \overline{\chi} \in N^\circ$. Then as in (5.3.2)

$$0 = \langle e_n^* \otimes \overline{\chi}, z^{**} \rangle = (R_n y|\chi) + (\xi|\chi) = (R_n y|\chi).$$

Therefore $R_n y \in V$, so the forward implication holds.

To prove the reverse implication, suppose that $\xi \in V$ and $R_n y \in V$ for every $n \in \mathbb{N}$. Then by (i), $y \in N \subseteq N^{\circ\circ}$. Take $w^* \in N^\circ$. Then $w^* = \sum_{n=1}^{\infty} e_n^* \otimes S_n w^*$ and $S_n w^* \in V^\perp$ by (ii). Thus we obtain

$$\langle w^*, z^{**} \rangle = \langle w^*, y + \Phi \otimes \xi \rangle = \left\langle \sum_{n=1}^{\infty} e_n^* \otimes S_n w^*, \Phi \otimes \xi \right\rangle$$

$$= \sum_{n=1}^{\infty} \left(S_n w^* | \overline{\xi} \right) = 0$$

which proves that $z^{**} \in N^{\circ\circ}$. □

Next we show how $E_{\mathcal{R}}^{**}/E_{\mathcal{R}} \simeq \overline{B}/V$.

Proposition 5.3.2. *The following diagram is commutative:*

$$\begin{tikzcd}
\overline{B} \arrow[rr, two heads, "Q_V"] \arrow[d, "R_0"'] & & \overline{B}/V \arrow[d, "\widehat{R_0}", "\simeq"'] \\
Y^{**} \arrow[r, two heads, "Q_N^{**}"] & E_{\mathcal{R}}^{**} \arrow[r, two heads, "\pi_{E_{\mathcal{R}}}"] & E_{\mathcal{R}}^{**}/E_{\mathcal{R}}
\end{tikzcd} \tag{5.3.3}$$

where $R_0 : \xi \mapsto (\xi_i \Phi)_{i=0}^{\infty}$ is a linear isometry, Q_V, Q_N and $\pi_{E_{\mathcal{R}}}$ are the quotient maps, and $\widehat{R_0} : \xi + V \mapsto \pi_{E_{\mathcal{R}}} Q_N^{**} R_0(\xi)$ is an isomorphism of Banach spaces.

Proof. Throughout the proof we consider Y and $E_{\mathcal{R}}$ as subspaces of Y^{**} and $E_{\mathcal{R}}^{**}$, respectively. We first show that $R_0 : \xi \mapsto (\xi_i \Phi)_{i=0}^{\infty}$ is well-defined, where $\Phi \in JB_i^{**}$ is the functional from Proposition 5.2.15. Noticing that

$$||(\xi_i \Phi)_{i=0}^{\infty}||_{Y^{**}} = ||\Phi||\,||\xi||_{\overline{B}} < \infty$$

we see that the map is indeed well-defined. Now because $||\Phi||_{JB_i} = 1$ for every $i \in \mathbb{N}_0$, R_0 is an isometry; it is also clearly linear.

We aim to use the Fundamental Isomorphism Theorem 1.2.4 to obtain the claimed commutative diagram. To do this we need to show that $\ker \pi_{E_{\mathcal{R}}} Q_N^{**} R_0 = V$, and that $\pi_{E_{\mathcal{R}}} Q_N^{**} R_0$ is surjective.

To begin, observe that the proof of Lemma 5.1.4(ii) demonstrates that $\ker Q_N^{**} = N^{\circ\circ}$. Next we see that $Q_N^{**}(Y) = E_{\mathcal{R}}$ because $Q_N^{**}|_Y = Q_N$, and so $\ker \pi_{E_{\mathcal{R}}} Q_N^{**} = (Q_N^{**})^{-1}(E_{\mathcal{R}}) \supseteq Y + N^{\circ\circ}$. For the reverse inclusion, take $y^{**} \in (Q_N^{**})^{-1}(E_{\mathcal{R}})$. Then there exists $y \in Y$ such that $Q_N^{**}(y^{**}) = Q_N(y)$. This implies that $y^{**} - y \in \ker Q_N^{**} = N^{\circ\circ}$, and so $y^{**} \in Y + N^{\circ\circ}$. Hence $\ker \pi_{E_{\mathcal{R}}} Q_N^{**} = Y + N^{\circ\circ}$.

Choose $v \in V$. Then $R_0(v) = \Phi \otimes v \in N^{\circ\circ}$ by Lemma 5.3.1(ii), which implies that $\ker \pi_{E_{\mathcal{R}}} Q_N^{**} R_0 \supseteq V$ by the above. Conversely, suppose that $\xi \in \ker \pi_{E_{\mathcal{R}}} Q_N^{**} R_0$. Then $\Phi \otimes \xi \in \ker \pi_{E_{\mathcal{R}}} Q_N^{**} = Y + N^{\circ\circ}$. Hence there exists $y \in Y$ such that $\Phi \otimes \xi - y \in N^{\circ\circ}$. Now Lemma 5.3.1(iii) implies that $\xi \in V$, and so we conclude that $\ker \pi_{E_{\mathcal{R}}} Q_N^{**} R_0 = V$.

Secondly, we must show that $\pi_{E_{\mathcal{R}}} Q_N^{**} R_0$ is surjective. Lemma 5.3.1 shows that $Y^{**} = Y + R_0[\overline{B}]$ and hence

$$E_{\mathcal{R}}^{**}/E_{\mathcal{R}} = \pi_{E_{\mathcal{R}}} Q_N^{**}[Y^{**}] = \pi_{E_{\mathcal{R}}} Q_N^{**} R_0[\overline{B}]$$

because $\pi_{E_{\mathcal{R}}}$ and Q_N^{**} are surjective and $Y \subseteq \ker \pi_{E_{\mathcal{R}}} Q_N^{**}$. We conclude that $\pi_{E_{\mathcal{R}}} Q_N^{**} R_0$ is surjective.

Now the Fundamental Isomorphism Theorem 1.2.4 implies that there exists an isomorphism $\widehat{R_0} : \overline{B}/V \to E_{\mathcal{R}}^{**}/E_{\mathcal{R}}$ making the diagram commutative. $\square$

When working with Hilbert spaces it is usually easier to deal with orthogonal complements than quotients. Read considers $\overline{B}/V$, but we prefer to use $V^{\perp}$ (the orthogonal complement of V in $\overline{B}$). Of course these are isometrically isomorphic. This leads to a form of [90, Lemma 3.7]. To state the result succinctly we introduce some further notation.

Definition 5.3.3. Define the Hilbert space $H = \overline{\operatorname{span}}\,\{b_0, x_n, y_n : n \in \mathbb{N}\} \subset \overline{B}$.

Lemma 5.3.4. *Let $P_{V^{\perp}} : \overline{B} \to V^{\perp}$ be the orthogonal projection onto the closed subspace $V^{\perp}$. Then $P_{V^{\perp}}|_H : H \to V^{\perp}$ is an isomorphism and $||P_{V^{\perp}}|_H^{-1}|| \leqslant \sqrt{21}$.*

Proof. Being an orthogonal projection with closed range, $||P_{V^\perp}|| = 1$. One of the things we want to show is that $P_{V^\perp}(H) = V^\perp$; a good starting point is to prove that $P_{V^\perp}(H)$ is at least dense in $V^\perp$. First we want to show that $V^\perp = \overline{\operatorname{span}}\{P_{V^\perp}(b_i) : i \in \mathbb{N}_0\}$. The inclusion $\supseteq$ is clear. Take $v' \in V^\perp$. Since $v' \in \overline{B}$ we can write it as $v' = \sum_{i=0}^{\infty} \lambda_i b_i$ for some sequence (λ_i) of scalars. Then $v' = P_{V^\perp}(v') = P_{V^\perp}(\sum_{i=0}^{\infty} \lambda_i b_i) = \sum_{i=0}^{\infty} \lambda_i P_{V^\perp}(b_i)$, so we have the reverse inclusion.

If we can show that for every $i \in \mathbb{N}_0$ there exists $h_i \in H$ such that $P_{V^\perp}(b_i) = P_{V^\perp}(h_i)$, then $\operatorname{span}\{P_{V^\perp}(b_i) : i \in \mathbb{N}_0\} \subseteq P_{V^\perp}(H)$ so that $P_{V^\perp}(H)$ is dense in $V^\perp$. So pick $i \in \mathbb{N}_0$. It is enough to find $h_i \in H$ such that $h_i - b_i \in \ker P_{V^\perp} = V$. Recalling Definition 5.2.16, if $i = 0$ or $b_i = x_n$ or $b_i = y_n$ for some $n \in \mathbb{N}$, then simply take $h_i = b_0, x_n, y_n$ respectively. If $b_i = \alpha_n$ then choose $h_i = x_n - y_n$. If $b_i = \beta_n, \gamma_n$ choose $h_i = x_n + y_n$; for $b_i = \delta_n$ take $h_i = 2^{-n} b_0 - x_n + y_n$. Therefore $P_{V^\perp}(H)$ is dense in $V^\perp$.

Next we *claim* that for every $x \in H$, $||P_{V^\perp} x|| \geqslant \frac{1}{\sqrt{21}} ||x||$. This implies that $P_{V^\perp}|_H$ is injective and has closed range. By the argument in the previous paragraph it follows that $\operatorname{im} P_{V^\perp}|_H = V^\perp$, and moreover,

$$||P_{V^\perp}|_H^{-1} P_{V^\perp} x|| = ||x|| \leqslant \sqrt{21} ||P_{V^\perp} x|| \qquad (x \in H)$$

so that $||P_{V^\perp}|_H^{-1}|| \leqslant \sqrt{21}$. Therefore establishing the claim proves the lemma.

Take a typical element $x = \lambda_0 b_0 + \sum_{n=1}^{\infty} (\lambda_n x_n + \mu_n y_n) \in H$ for some $\lambda_j, \mu_j \in \mathbb{K}$, and we may as well choose $||x|| = 1$. Then $1 = ||x|| = |\lambda_0|^2 + \sum_{n=1}^{\infty} (|\lambda_n|^2 + |\mu_n|^2)$. By the definition of an orthogonal projection we have

$$||P_{V^\perp} x|| = \inf_{z \in V} ||x - z|| = \inf_{z \in \operatorname{span} S} ||x - z||.$$

Let

$$z = \sum_{n=1}^{N} \left(a'_n \alpha'_n + b'_n \beta'_n + c'_n \gamma'_n + d'_n \delta'_n \right) \qquad (a'_n, b'_n, c'_n, d'_n \in \mathbb{K})$$

be an arbitrary element in $\operatorname{span} S$. We may suppose that $||z|| \leqslant 1$, for, if not, let Q_z denote the orthogonal projection onto $\mathbb{K}z \subseteq \overline{B}$. Then $||Q_z(x)|| \leqslant 1$ where $Q_z(x) \in \operatorname{span} S$ and $||x - Q_z(x)|| = ||(I_{\overline{B}} - Q_z)(x - z)|| \leqslant ||x - z||$, so that we get a better approximation to $||P_{V^\perp} x||$.

Since H is a closed subspace of $\overline{B}$ it follows that

$$H^\perp = \overline{\operatorname{span}}\{\alpha_n, \beta_n, \gamma_n, \delta_n : n \in \mathbb{N}\}.$$

Take $\Pi : \overline{B} \to H^\perp$ to be the orthogonal projection along H. Then

$$\Pi z = \sum_{n=1}^{N} a'_n \alpha_n + b'_n \beta_n + c'_n \gamma_n + d'_n \delta_n.$$

Therefore $||\Pi z||^2 = \sum_{n=1}^{N} |a'_n|^2 + |b'_n|^2 + |c'_n|^2 + |d'_n|^2$. Also note that $\Pi x = 0$. Using Pythagoras' Theorem we get (recalling that $||z|| \leqslant 1$)

$$\begin{aligned} ||x - z||^2 &= ||\Pi(x - z)||^2 + ||(I_{\overline{B}} - \Pi)(x - z)||^2 = ||\Pi z||^2 + ||x - z + \Pi z||^2 \\ &\geqslant ||\Pi z||^2 + \big(||x|| - ||z - \Pi z||\big)^2 \geqslant ||\Pi z||^2 + \big(1 - ||z||\big)^2, \end{aligned} \tag{5.3.4}$$

where $I_{\overline{B}}$ denotes the identity operator on $\overline{B}$.

Rewrite $z = \sum_{n=1}^{N} w_n - (\sum_{n=1}^{N} 2^{-n} d_n) b_0$, so that for each $n \in \{1, \dots N\}$ we have $w_n \in \operatorname{span}\{x_k, y_k, \alpha_k, \beta_k, \gamma_k, \delta_k : k \in \mathbb{N}\}$, and $(w_m|w_n) = 0$ for all $m \neq n$. By the triangle inequality and Pythagoras' Theorem

$$||z|| \leqslant \left\|\sum_{n=1}^{N} w_n\right\| + \left\|\sum_{n=1}^{N} 2^{-n} d_n b_0\right\| = \left(\sum_{n=1}^{N} ||w_n||^2\right)^{\frac{1}{2}} + \left|\sum_{n=1}^{N} 2^{-n} d_n\right|. \tag{5.3.5}$$

We also obtain

$$||w_n|| \leqslant \sqrt{3}(|a'_n| + |b'_n| + |c'_n| + |d'_n|)$$

because $||\alpha'_n|| = ||\beta'_n|| = ||\gamma'_n|| = ||\delta'_n + 2^{-n} b_0|| = \sqrt{3}$ by a further application of Pythagoras' Theorem. Putting this into (5.3.5) we see that

$$\begin{aligned} ||z|| &\leqslant \left(\sum_{n=1}^{N} 3\big(|a'_n| + |b'_n| + |c'_n| + |d'_n|\big)^2\right)^{\frac{1}{2}} + \max_{1 \leqslant n \leqslant N} |d'_n| \\ &\leqslant \left(\sum_{n=1}^{N} 12\big(|a'_n|^2 + |b'_n|^2 + |c'_n|^2 + |d'_n|^2\big)\right)^{\frac{1}{2}} + \max_{1 \leqslant n \leqslant N} |d'_n| \end{aligned}$$

by the Cauchy–Schwarz inequality. This gives $||z|| \leqslant (1 + 2\sqrt{3})||\Pi z||$. It follows from (5.3.4) that

$$\begin{aligned} ||x - z||^2 \geqslant ||\Pi z||^2 + \big(1 - ||z||\big)^2 &\geqslant \frac{1}{(1 + 2\sqrt{3})^2}||z||^2 + \big(1 - ||z||\big)^2 \\ &= \left(\frac{1}{(1 + 2\sqrt{3})^2} + 1\right)||z||^2 - 2||z|| + 1 \end{aligned}$$

so if we let $r = ||z||$ then this is just a quadratic polynomial

$$p(r) = \left(\frac{1}{(1 + 2\sqrt{3})^2} + 1\right) r^2 - 2r + 1.$$

By the standard method for minimising polynomials we find that the minimum is at $r_0 = \frac{13+4\sqrt{3}}{14+4\sqrt{3}}$, which implies that $p(r_0)$ is approximately $0.04778 \geqslant \frac{1}{21}$. Hence $||x - z|| \geqslant \frac{1}{\sqrt{21}}$ and so for every $x \in H$, $||P_{V^\perp} x|| \geqslant \frac{1}{\sqrt{21}} ||x||$ as required. $\square$

We may summarise the link between [90, Lemma 3.7] and Lemma 5.3.4 using the following commutative diagram

$$\begin{tikzcd} H \arrow[r, "P_{V^\perp}|_H", "\simeq"'] \arrow[d, hook, "\iota"'] & V^\perp \arrow[d, "\simeq"', "Q_V|_{V^\perp}"] \\ \overline{B} \arrow[r, two heads, "Q_V"] & \overline{B}/V \end{tikzcd} , \tag{5.3.6}$$

where $\iota : H \to \overline{B}$ is the inclusion and $Q_V : \overline{B} \to \overline{B}/V$ is the quotient map. Read's result says that $Q_V(H) = \overline{B}/V$.

5.3.1 Summary of Section 3 of Read's paper

Combining (5.3.3) and (5.3.6), we can now identify an explicit isomorphism

$$U : H \to E_{\mathcal{R}}^{**}/E_{\mathcal{R}}$$

which will allow us to state Read's Lemma 4.1 succinctly:

$$\begin{tikzcd} H \arrow[rr, "P_{V^\perp}|_H", "\simeq"'] \arrow[d, hook, "\iota"'] \arrow[rrdd, dashed, "U"] & & V^\perp \arrow[d, "\simeq"', "Q_V|_{V^\perp}"] \\ \overline{B} \arrow[rr, two heads, "Q_V"] \arrow[d, "R_0"'] & & \overline{B}/V \arrow[d, "\simeq"', "\widehat{R}_0"] \\ Y^{**} \arrow[r, two heads, "Q_N^{**}"] & E_{\mathcal{R}}^{**} \arrow[r, two heads, "\pi_{E_{\mathcal{R}}}"] & E_{\mathcal{R}}^{**}/E_{\mathcal{R}}. \end{tikzcd} \tag{5.3.7}$$

This isomorphism U induces a continuous algebra isomorphism via

$$\operatorname{Ad} U : \mathscr{B}(E_{\mathcal{R}}^{**}/E_{\mathcal{R}}) \to \mathscr{B}(H), \qquad T \mapsto U^{-1}TU.$$

By Proposition 5.1.2 there is an contractive algebra homomorphism

$$\Theta_0 : \mathscr{B}(E_{\mathcal{R}}) \to \mathscr{B}(E_{\mathcal{R}}^{**}/E_{\mathcal{R}})$$

given by $\Theta_0(T) = \overline{T^{**}}$ for each $T \in \mathscr{B}(E_{\mathcal{R}})$, where $\overline{T^{**}}(x^{**} + E_{\mathcal{R}}) = T^{**}(x^{**}) + E_{\mathcal{R}}$

for $x^{**} \in E^{**}_{\mathcal{R}}$. Since $\ker \Theta_0 = \mathscr{W}(E_{\mathcal{R}})$ the Fundamental Isomorphism Theorem 1.2.4 implies there is also a continuous algebra homomorphism $\Theta : \mathscr{B}(E_{\mathcal{R}})/\mathscr{W}(E_{\mathcal{R}}) \to \mathscr{B}(E^{**}_{\mathcal{R}}/E_{\mathcal{R}})$ given by $\Theta(T + \mathscr{W}(E_{\mathcal{R}})) = \Theta_0(T)$ for each $T \in \mathscr{B}(E_{\mathcal{R}})$. Whence the following diagram is commutative:

$$\begin{tikzcd}
\mathscr{B}(E_{\mathcal{R}}) \arrow[r, "\Theta_0"] \arrow[d, "\pi"'] & \mathscr{B}(E^{**}_{\mathcal{R}}/E_{\mathcal{R}}) \arrow[r, "\operatorname{Ad} U", "\cong"'] & \mathscr{B}(H) \\
\mathscr{B}(E_{\mathcal{R}})/\mathscr{W}(E_{\mathcal{R}}) \arrow[ur, "\Theta"] & &
\end{tikzcd}$$

where π is the quotient map, and $\operatorname{im} \Theta = \operatorname{im} \Theta_0$. Read considers the operator Θ frequently, whereas we prefer to use Θ_0.

5.3.2 The central lemma

The content of Read's Lemma 4.1 is that only operators of a very special type can belong to the image of $\operatorname{Ad} U \circ \Theta_0$. To make this precise, we introduce some further notation.

Definition 5.3.5. For each $\xi \in H$, let $\tau_\xi \in \mathscr{B}(H)$ be the rank-one operator given by $\tau_\xi : \eta \mapsto (\eta|b_0)\xi$. Define $H_0 = \overline{\operatorname{span}}\,\{x_n + y_n : n \in \mathbb{N}\} \subseteq H$, and $\mathscr{T} = \{\tau_\xi : \xi \in H_0\}$.

Lemma 5.3.6 (Read)**.** *Let $\tau \in \operatorname{im}(\operatorname{Ad} U \circ \Theta_0)$, and suppose that, with respect to the orthonormal basis $(a_i)_{i=0}^\infty = (b_0, x_1, y_1, \ldots)$ of H (where $a_0 = b_0, a_{2i-1} = x_i, a_{2i} = y_i$ for all $i \in \mathbb{N}$), τ has matrix*

$$M = (\eta_{ij})_{i,j=0}^\infty$$

where $\eta_{ij} = (\tau a_i|a_j)$. Then:

(a) *$\eta_{ii} = \eta_{jj}$ for all i, j;*

(b) *if $j \neq 0$ and $i \neq j$, then $\eta_{ij} = 0$;*

(c) *for each $n \in \mathbb{N}$, $\eta_{2n-1,0} = \eta_{2n,0}$.*

Hence M has the form

$$\begin{pmatrix} \lambda & 0 & 0 & 0 & & \\ \mu_1 & \lambda & 0 & 0 & \cdots & \\ \mu_1 & 0 & \lambda & 0 & & \\ \mu_2 & 0 & 0 & \lambda & & \\ \mu_2 & & & & \lambda & \\ \vdots & & & & & \ddots \end{pmatrix} \tag{5.3.8}$$

for scalars $\mu_j, \lambda \in \mathbb{K}$, and $j \in \mathbb{N}$.

More succinctly, the following inclusion holds:

$$\operatorname{Ad} U \circ \Theta_0(\mathscr{B}(E_{\mathcal{R}})) \subseteq \mathscr{T} + \mathbb{K} I_H. \tag{5.3.9}$$

We have made the identifications between H, $\overline{B}/V$ and $E_{\mathcal{R}}^{**}/E_{\mathcal{R}}$ in the first two lines of [90, Lemma 4.1] more explicit. Note that we have also corrected a small error in [90, Lemma 4.1(c)] where the first μ_3 should be μ_2, as is clear from (a)–(c).

Let us postpone the proof until we have discussed the application of this powerful result. The reason that this is the key technical step towards understanding $\mathscr{B}(E_{\mathcal{R}})$ is that it allows us to identify the codimension 1 ideal I from Theorem 1.3.2 as the preimage $(\operatorname{Ad} U \circ \Theta_0)^{-1}(\mathscr{T})$. Theorem 1.3.2 is then quickly deduced (*cf.* [90, Corollary 4.2]). We shall defer the proof of Theorem 1.3.2 until the end of Section 4, because it will follow neatly from our more general Theorem 1.3.4 (also to be proved in Section 4).

Before presenting the proof of Lemma 5.3.6 we give another preliminary lemma.

Lemma 5.3.7. *For each $n \in \mathbb{N}$, let $\sigma_n = e_1 + \cdots + e_n \in c_{00}$. Then for every $\xi \in \overline{B}$, $\sigma_n \otimes \xi \xrightarrow{w^*} \Phi \otimes \xi$ in Y^{**} as $n \to \infty$.*

Proof. Choose $\xi = (\xi_j)_{j=0}^{\infty} \in \overline{B}$. For each $i \in \mathbb{N}_0$, Theorem 5.2.11(i) says that (e_j) is a shrinking basis for JB_i, so $(e_j^*)_{j=1}^{\infty}$ is a basis for JB_i^*. Thus we have $Y^* = \overline{\operatorname{span}}\,\{e_n^* \otimes b_i : n \in \mathbb{N}, i \in \mathbb{N}_0\}$. Applying Lemma 5.1.5 with $X = Y^*$, $f_n = \sigma_n \otimes \xi$, $f = \Phi \otimes \xi$, and $D = \{e_n^* \otimes b_i : n \in \mathbb{N}, i \in \mathbb{N}_0\}$, it is enough to show that for every $m \in \mathbb{N}$ and $i \in \mathbb{N}_0$, $\langle e_m^* \otimes b_i, \sigma_n \otimes \xi\rangle \to \langle e_m^* \otimes b_i, \Phi \otimes \xi\rangle$ as $n \to \infty$.

So fix $m \in \mathbb{N}$ and $i \in \mathbb{N}_0$. We recall what the tensors mean (Definition 5.2.18), and obtain:

$$\langle e_m^* \otimes b_i, \sigma_n \otimes \xi\rangle = \langle \sigma_n, e_m^*\rangle\langle \xi, b_i\rangle = \begin{cases} \xi_i & \text{if } m \leqslant n \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we see that $\langle e_m^* \otimes b_i, \Phi \otimes \xi\rangle = \xi_i \langle e_m^*, \Phi\rangle = \xi_i$. Therefore as $n \to \infty$, $\langle e_m^* \otimes b_i, \sigma_n \otimes \xi\rangle \to \langle e_m^* \otimes b_i, \Phi \otimes \xi\rangle$. The result follows. $\square$

Proof of Lemma 5.3.6. Let $\tau \in \operatorname{im} \operatorname{Ad} U \circ \Theta_0$. The big aim is to show that τ has the matrix form given in (5.3.8). By definition, $\operatorname{Ad} U \circ \Theta_0 : \mathscr{B}(E_{\mathcal{R}}) \to \mathscr{B}(H)$, so there exists $T \in \mathscr{B}(E_{\mathcal{R}})$ such that $\tau = \operatorname{Ad} U \circ \Theta_0(T) \in \mathscr{B}(H)$.

Let $\xi \in H$. Then

$$\begin{aligned}\tau(\xi) = \operatorname{Ad} U \circ \Theta_0(T)(\xi) = U^{-1}\Theta_0(T)U(\xi) = U^{-1}\overline{T^{**}}U(\xi) &\implies U\tau(\xi) = \overline{T^{**}}U(\xi) \\ \implies \pi_{E_\mathcal{R}} Q_N^{**}(\Phi \otimes \tau(\xi)) = \overline{T^{**}}(\pi_{E_\mathcal{R}} Q_N^{**}(\Phi \otimes \xi)) \quad \text{using (5.3.7)} \\ \implies Q_N^{**}(\Phi \otimes \tau(\xi)) - T^{**}Q_N^{**}(\Phi \otimes \xi) \in \ker \pi_{E_\mathcal{R}} = E_\mathcal{R} = Q_N(Y)\end{aligned}$$

since $\overline{T^{**}}\pi_{E_\mathcal{R}} = \pi_{E_\mathcal{R}} T^{**}$. This implies that there exists $y' \in Y$ such that

$$Q_N^{**}(\Phi \otimes \tau(\xi)) - T^{**}Q_N^{**}(\Phi \otimes \xi) = Q_N(y') = Q_N^{**}(y').$$

Hence there exists $y = -y' \in Y$ such that

$$Q_N^{**}(\Phi \otimes \tau(\xi) + y) = T^{**}Q_N^{**}(\Phi \otimes \xi). \tag{5.3.10}$$

Lemma 5.3.7 implies that $\sigma_n \otimes \xi \xrightarrow{w^*} \Phi \otimes \xi$ in Y^{**} as $n \to \infty$. Every dual operator is weak* continuous, and so it follows that

$$TQ_N(\sigma_n \otimes \xi) = T^{**}Q_N^{**}(\sigma_n \otimes \xi) \xrightarrow{w^*} T^{**}Q_N^{**}(\Phi \otimes \xi) \quad \text{in } E_\mathcal{R}^{**} \text{ as } n \to \infty.$$

Therefore by (5.3.10),

$$TQ_N(\sigma_n \otimes \xi) \xrightarrow{w^*} Q_N^{**}(\Phi \otimes \tau(\xi) + y) \quad \text{in } E_\mathcal{R}^{**} \text{ as } n \to \infty.$$

By the definition of weak* convergence this means that for each $f \in E_\mathcal{R}^*$ we have

$$\langle TQ_N(\sigma_n \otimes \xi), f\rangle \longrightarrow \langle Q_N^* f, \Phi \otimes \tau(\xi) + y\rangle \quad \text{as } n \to \infty. \tag{5.3.11}$$

Next choose an element $u \in V^\perp \subset \overline{B}$ and fix $i \in \mathbb{N}$. Then $e_i^* \otimes u \in N^\circ$. To see this, recall that $N = \overline{\operatorname{span}}\,\{e_n \otimes s : s \in S, n \in \mathbb{N}\}$, and $V = \overline{\operatorname{span}}\, S$. Then for all $n \in \mathbb{N}$, by the standard duality $\langle e_n \otimes s, e_i^* \otimes u\rangle = \langle e_n, e_i^*\rangle(u|\bar{s}) = 0$ ($\bar{s}$ means the coordinatewise complex conjugate) and this is therefore true on all of N. So indeed $e_i^* \otimes u \in N^\circ$. By Lemma 5.1.4(i), $N^\circ = Q_N^*(E_\mathcal{R}^*)$, and hence $e_i^* \otimes u = Q_N^*(f_i)$ for some $f_i \in E_\mathcal{R}^*$. By (5.3.11) we have

$$\begin{aligned}\langle TQ_N(\sigma_n \otimes \xi), f_i\rangle \longrightarrow \langle Q_N^* f_i, \Phi \otimes \tau(\xi) + y\rangle &= \langle e_i^* \otimes u, \Phi \otimes \tau(\xi) + y\rangle \\ &= (\tau(\xi)|\bar{u}) + \langle e_i^* \otimes u, y\rangle \end{aligned} \tag{5.3.12}$$

as $n \to \infty$, using bilinearity of the duality bracket, the duality of Y^* and Y, and the fact that $\langle e_i^*, \Phi\rangle = 1$.

For any $i, n \in \mathbb{N}$ write $\lambda_{i,n} = \langle TQ_N(\sigma_n \otimes \xi), f_i\rangle$ and $\nu_i = \langle e_i^* \otimes u, y\rangle$. Rewriting

(5.3.12) we have

$$\lambda_{i,n} \to (\tau(\xi)|\bar{u}) + \nu_i \qquad \text{as } n \to \infty. \tag{5.3.13}$$

We *claim* two things about these numbers:

(1) $\nu_i \to 0$ as $i \to \infty$;

(2) for fixed $n \in \mathbb{N}$, $\lambda_{i,n} \to 0$ as $i \to \infty$.

To prove (1) we firstly write out the coordinates $u = (u^j)_{j=0}^\infty \in \overline{B}$ and $y = (y^j)_{j=0}^\infty \in Y$, where $u^j \in \mathbb{K}$ such that $\sum_{j=0}^\infty |u^j|^2 < \infty$, and $y^j \in JB_j$ such that $\sum_{j=0}^\infty ||y^j||_{JB_j}^2 < \infty$. Then for each $i \in \mathbb{N}$ we consider the duality

$$\nu_i = \langle e_i^* \otimes u, y\rangle = \langle (u^j e_i^*)_{j=0}^\infty, (y^j)_{j=0}^\infty\rangle = \sum_{j=0}^\infty u^j \langle y^j, e_i^*\rangle. \tag{5.3.14}$$

Using Hölder's inequality we obtain

$$\sum_{j=0}^\infty |u^j|\, ||y^j||_{JB_j} \leqslant ||u||_{\overline{B}} ||y||_Y < \infty.$$

So, given $\varepsilon > 0$, there exists $L \in \mathbb{N}$ such that $\sum_{j=L+1}^\infty |u^j|\, ||y^j|| < \varepsilon$. For each fixed $j \in \mathbb{N}_0$, $\langle y^j, e_i^*\rangle \to 0$ as $i \to \infty$ because $y^j \in c_0$. So there exists $i_0 \in \mathbb{N}$ such that $\sum_{j=0}^L |u^j||\langle y^j, e_i^*\rangle| < \varepsilon$ for every $i \geqslant i_0$. Thus for $i \geqslant i_0$, (5.3.14) implies that

$$\begin{aligned} |\nu_i| &\leqslant \sum_{j=0}^L |u^j||\langle y^j, e_i^*\rangle| + \sum_{j=L+1}^\infty |u^j||\langle y^j, e_i^*\rangle| \\ &\leqslant \varepsilon + \sum_{j=L+1}^\infty |u^j|\, ||y^j||\, ||e_i^*||_{JB_i^*} \leqslant 3\varepsilon, \end{aligned}$$

which is as small as we like. This proves (1).

For (2) we fix $n \in \mathbb{N}$. By definition we have $\lambda_{i,n} = \langle TQ_N(\sigma_n \otimes \xi), f_i\rangle$. The element $TQ_N(\sigma_n \otimes \xi)$ is in $E_{\mathcal{R}} = Y/N$ and so there exists $z \in Y$ such that $Q_N(z) = TQ_N(\sigma_n \otimes \xi)$. Write $z = (z^j)_{j=0}^\infty$. Then

$$\lambda_{i,n} = \langle Q_N(z), f_i\rangle = \langle z, Q_N^* f_i\rangle = \langle z, e_i^* \otimes u\rangle = \langle (z^j)_{j=0}^\infty, (u^j e_i^*)_{j=0}^\infty\rangle = \sum_{j=0}^\infty u^j \langle z^j, e_i^*\rangle.$$

Compare this to (5.3.14). By the same argument as for (1) we obtain that $\lambda_{i,n} \to 0$ as $i \to \infty$. This proves (2).

The next step in our proof is a gliding hump style argument. We want to inductively choose three sequences of natural numbers, $(M_j), (N_j), (K_j)$, such that $M_1 < N_1 < K_1 < M_2 < N_2 < K_2 < \cdots$, and such that for every $r \in \mathbb{N}$:

(i) $|\nu_i| \leqslant 2^{-r}$ for all $i \geqslant M_r$;

(ii) $|\lambda_{i,n} - (\tau(\xi)|\bar{u}) - \nu_i| < 2^{-r}$ for all $i \leqslant M_r$ and $n \geqslant N_r$;

(iii) $|\lambda_{i,n}| < 2^{-r}$ for all $n \leqslant N_r$ and $i \geqslant K_r$.

More precisely, for each $r \in \mathbb{N}$, let P_r be the statement

$$P_r : \quad \exists\, M_1 < N_1 < K_1 < \cdots < M_r < N_r < K_r \quad \text{such that (i), (ii) and (iii) hold.}$$

We proceed by induction. For the base step let $r = 1$. By part (1) of our claim above, $\nu_i \to 0$ as $i \to \infty$, so we can find M_1 such that (i) holds. By (5.3.13) we can find $N_1 > M_1$ such that (ii) holds. And because of (2), for each fixed n, $\lambda_{i,n} \to 0$ as $i \to \infty$. Therefore we can find $K_1 > N_1$ such that (iii) is satisfied. Thus the statement is true for $r = 1$.

Now for the induction step, assume the statement is true for some natural number r. By (1) we can pick $M_{r+1} > K_r$ such that (i) is satisfied. Using (5.3.13), for each $i \leqslant M_{r+1}$ we can find $N_{r+1}^{(i)}$ such that $|\lambda_{i,n} - (\tau(\xi)|\bar{u}) - \nu_i| < 2^{-r}$ for all $n \geqslant N_{r+1}^{(i)}$. Then choosing $N_{r+1} = \max\{M_{r+1}+1, N_{r+1}^{(i)} : i \leqslant M_{r+1}\}$, we fulfill (ii). Similarly we can find K_{r+1} that works for (iii), by repeated applications of (2). Hence the result is true for all $r \in \mathbb{N}$ by induction, and our sequences are defined.

Now that we have chosen such sequences, we begin to estimate the norm of T. Choose an odd natural number R, and define

$$\rho = \sum_{j=1}^{R} (-1)^{j-1} \sigma_{N_j} \in c_{00}, \tag{5.3.15}$$

where we recall that $\sigma_{N_j} = \sum_{k=1}^{N_j} e_k$. Next, for any $i \in \mathbb{N}$, the bilinearity of the tensors implies that

$$\langle TQ_N(\rho \otimes \xi), f_i\rangle = \sum_{j=1}^{R} (-1)^{j-1} \langle TQ_N(\sigma_{N_j} \otimes \xi), f_i\rangle = \sum_{j=1}^{R} (-1)^{j-1} \lambda_{i,N_j}. \tag{5.3.16}$$

Define $s_1 = \langle TQ_N(\rho\otimes\xi), f_{M_1}\rangle - (\tau(\xi)|\bar{u}) - \nu_{M_1}$ and for natural numbers $1 < r \leqslant R$ set

$$s_r = \sum_{j=1}^{r-1} (-1)^{j-1} \lambda_{M_r,N_j} + \sum_{j=r}^{R} (-1)^{j-1} \big(\lambda_{M_r,N_j} - (\tau(\xi)|\bar{u}) - \nu_{M_r}\big). \tag{5.3.17}$$

Therefore from (5.3.16)

$$s_r = \begin{cases} \langle TQ_N(\rho \otimes \xi), f_{M_r}\rangle & \text{if } r \text{ is even,} \\ \langle TQ_N(\rho \otimes \xi), f_{M_r}\rangle - (\tau(\xi)|\bar{u}) - \nu_{M_r} & \text{if } r \text{ is odd ,} \end{cases} \tag{5.3.18}$$

which comes from considering when we have cancellation. Now set

$$\varepsilon_r' = \begin{cases} s_r & \text{if } r \text{ is even,} \\ s_r + \nu_{M_r} & \text{if } r \text{ is odd.} \end{cases} \tag{5.3.19}$$

Fix $1 < r \leqslant R$. Then we have $M_r > K_{r-1}$, so $|\lambda_{M_r,n}| < 2^{-(r-1)}$ for $n \leqslant N_{r-1}$ by (iii), and so $|\lambda_{M_r,N_j}| \leqslant 2^{-(r-1)}$ for natural numbers $j < r$. For $r \leqslant j \leqslant R$, (ii) gives

$$|\lambda_{M_r,N_j} - (\tau(\xi)|\bar{u}) - \nu_{M_r}| \leqslant 2^{-j}.$$

Using the triangle inequality

$$\begin{aligned} |s_r| &\leqslant \sum_{j=1}^{r-1} |\lambda_{M_r,N_j}| + \sum_{j=r}^{R} |\lambda_{M_r,N_j} - (\tau(\xi)|\bar{u}) - \nu_{M_r}| \\ &\leqslant (r-1)2^{-(r-1)} + \sum_{j=r}^{R} 2^{-j} \leqslant (r-1)2^{-r+1} + 2^{-r}(\sum_{i=0}^{\infty} 2^{-i}) = r2^{-r+1}. \end{aligned}$$

Suppose that r is even. Then we have $|\varepsilon_r'| = |s_r| \leqslant r2^{-r+1}$; if r is odd we obtain $|\varepsilon_r'| \leqslant |s_r| + |\nu_{M_r}| \leqslant r2^{-r+1} + 2^{-r} \leqslant (r+1)2^{-r+1}$ by (i).

For every $1 \leqslant r \leqslant R$ we have obtained

$$|\varepsilon_r'| \leqslant (r+1)2^{-r+1}. \tag{5.3.20}$$

Define

$$f = \sum_{k=1}^{R} (-1)^{k-1} f_{M_k} \in E_{\mathcal{R}}^*. \tag{5.3.21}$$

Then

$$Q_N^*(f) = \sum_{k=1}^{R} (-1)^{k-1} Q_N^*(f_{M_k}) = \sum_{k=1}^{R} (-1)^{k-1} (e_{M_k}^* \otimes u) = \left(\sum_{k=1}^{R} (-1)^{k-1} e_{M_k}^* \right) \otimes u,$$

so that $||f||_{E_{\mathcal{R}}^*} = || \sum_{k=1}^{R} (-1)^{k-1} e_{M_k}^* \otimes u||_{Y^*}$ since Q_N^* is an isometry.

Let $K = (R+1)/2$. Thus using (5.3.18) and (5.3.19), we obtain

$$\begin{aligned} \langle TQ_N(\rho \otimes \xi), f \rangle &= \sum_{k=1}^{R} (-1)^{k-1} \langle TQ_N(\rho \otimes \xi), f_{M_k} \rangle \\ &= \sum_{t=1}^{K} \langle TQ_N(\rho \otimes \xi), f_{M_{2t-1}} \rangle - \sum_{t=1}^{K-1} \langle TQ_N(\rho \otimes \xi), f_{M_{2t}} \rangle \\ &= \sum_{t=1}^{K} (\varepsilon_{2t-1}' + (\tau(\xi)|\bar{u})) - \sum_{t=1}^{K-1} \varepsilon_{2t}' = K(\tau(\xi)|\bar{u}) + \sum_{k=1}^{R} (-1)^{k-1} \varepsilon_k'. \end{aligned}$$

We can find a lower bound for $|\langle TQ_N(\rho\otimes\xi), f\rangle|$ by (5.3.20)

$$\begin{aligned}|\langle TQ_N(\rho\otimes\xi), f\rangle| &\geqslant K|(\tau(\xi)|\bar{u})| - \sum_{k=1}^{R}|\varepsilon'_k| \\ &\geqslant K|(\tau(\xi)|\bar{u})| - \sum_{k=1}^{\infty}(k+1)2^{1-k} \geqslant K|(\tau(\xi)|\bar{u})| - 6, \qquad (5.3.22)\end{aligned}$$

using the fact that $\sum_{k=1}^{\infty}(k+1)2^{1-k}$ is a standard infinite sum equal to 6.

Next, we reformulate $||\rho\otimes b_k||_Y$ for $k\in\mathbb{N}_0$ as follows.

$$\begin{aligned}||\rho\otimes b_k||_Y = ||(0,0,\ldots,0,\rho,0,\ldots)||_Y = ||\rho||_{JB_k} &= \left\|\sum_{j=1}^{R}(-1)^{j-1}\sigma_{N_j}\right\|_{JB_k} \\ = ||\sigma_{N_1}-\sigma_{N_2}+\sigma_{N_3}-\cdots+\sigma_{N_R}||_{JB_k} &= \left\|\sum_{j=1}^{R}e_j\right\|_{B_k}^{\frac{1}{2}} = ||\sigma_R||_{B_k}^{\frac{1}{2}} \qquad (5.3.23)\end{aligned}$$

by calculating the maximum possible number of 'jumps' from 1 to 0, and using the fact that B_k has a 1-symmetric basis. This is perhaps most easily seen by re-writing $\rho=\sum_{k=0}^{K-1}\chi_{[N_{2k}+1,N_{2k+1}]}$ where $\chi_{[a,b]}=\sum_{j=a}^{b}e_j$ (and $N_0=0$).

A further requirement is to estimate $||f||_{E^*_{\mathcal{R}}}$. Writing $z^*=\sum_{k=1}^{R}(-1)^{k-1}e^*_{M_k}\in c_{00}$ we recall that $Q^*_N(f)=z^*\otimes u$ and $||f||_{E^*_{\mathcal{R}}}=||z^*\otimes u||_{Y^*}$. Write $u=\sum_{j=0}^{\infty}u^jb_j\in V^{\perp}$. The meaning of the tensors implies that $z^*\otimes u=(u^jz^*)_{j=0}^{\infty}\in Y^*$, so

$$||f||_{E^*_{\mathcal{R}}} = ||z^*\otimes u||_{Y^*} = \left(\sum_{j=0}^{\infty}|u^j|^2||z^*||^2_{JB_j}\right)^{\frac{1}{2}}. \qquad (5.3.24)$$

Now fix $j\in\mathbb{N}_0$ and take $v=\sum_{k=1}^{\infty}v_ke_k\in c_{00}\subset JB_j$ such that $||v||_{JB_j}\leqslant 1$. Then since $z^*\in c_{00}\subset JB^*_j$ we calculate

$$\begin{aligned}|\langle v,z^*\rangle| &= |\sum_{k=1}^{R}(-1)^{k-1}v_{M_k}| = |v_{M_1}-v_{M_2}+v_{M_3}-\cdots+v_{M_R}| \\ &\leqslant \sqrt{K}\left(\sum_{k=1}^{K-1}|v_{M_{2k-1}}-v_{M_{2k}}|^2+|v_{M_R}|^2\right)^{\frac{1}{2}} \qquad \text{by Cauchy–Schwarz} \\ &= \sqrt{K}\left\langle\sum_{k=1}^{K-1}|v_{M_{2k-1}}-v_{M_{2k}}|^2e_k+|v_{M_R}|^2e_K, \sum_{s=1}^{K}e^*_s\right\rangle^{\frac{1}{2}} \\ &\leqslant \sqrt{K}\left\|\sum_{s=1}^{K}e^*_s\right\|_{B^*_j}^{\frac{1}{2}}\left\|\sum_{k=1}^{K-1}|v_{M_{2k-1}}-v_{M_{2k}}|^2e_k+|v_{M_R}|^2e_K\right\|_{B_j}^{\frac{1}{2}} \\ &\leqslant \sqrt{K}\left\|\sum_{s=1}^{K}e^*_s\right\|_{B^*_j}^{\frac{1}{2}}||v||_{JB_j}. \qquad (5.3.25)\end{aligned}$$

A general fact about 1-symmetric bases, proved in [74, Proposition 3.a.6] for example, says that

$$\left\|\sum_{s=1}^{K} e_s^*\right\|_{B_j^*} = \frac{K}{||\sigma_K||_{B_j}}.$$

We conclude from (5.3.25) that $|\langle v, z^*\rangle| \leqslant \dfrac{K}{||\sigma_K||_{B_j}^{\frac{1}{2}}}||v||_{JB_j}$ and so $||z^*||_{JB_j^*} \leqslant \dfrac{K}{||\sigma_K||_{B_j}^{\frac{1}{2}}}$, as it is enough to consider a dense subset of the unit ball. Hence by (5.3.24) we can bound $||f||$ as follows:

$$||f||_{E_{\mathcal{R}}^*} \leqslant \left(\sum_{j=0}^{\infty} |u^j|^2 \frac{K^2}{||\sigma_K||_{B_j}}\right)^{\frac{1}{2}} = K\left(\sum_{j=0}^{\infty} \frac{|u^j|^2}{||\sigma_K||_{B_j}}\right)^{\frac{1}{2}}. \tag{5.3.26}$$

Recall that we began the proof with $\tau \in \operatorname{im} \operatorname{Ad} U \circ \Theta_0 \subset \mathscr{B}(H)$. Thus far we have worked with a completely general $\xi \in H$, to observe the action of τ on ξ. We have also considered a general element $u \in V^\perp$ and the inner product $(\tau(\xi)|\bar{u})$, and have obtained some useful inequalities. The idea of the following important assertion is that if we restrict to specific $\xi \in H$ and $u \in V^\perp$ we can say a lot more.

Claim. *Suppose that there exists $k \in \mathbb{N}_0$ such that $\xi - b_k \in V$, and such that $(u|b_j) = 0$ for each $j \in \mathbb{N}_0$ with $B_j = B_k$. Then $(\tau(\xi)|\bar{u}) = 0$.*

To begin the proof of this claim take $\xi \in H$ and suppose that there exists $k \in \mathbb{N}_0$ such that $\xi - b_k \in V = \overline{\operatorname{span}}\, S$. Then for any $x \in c_{00}$, $x \otimes (\xi - b_k) \in N$, just as in the proof of Lemma 5.3.1. In particular, this implies that $\rho \otimes (\xi - b_k) \in N$, where ρ is the vector from (5.3.15).

Assume towards a contradiction that there exists $u \in V^\perp$ such that $(u|b_j) = 0$ for each $j \in \mathbb{N}_0$ with $B_j = B_k$, but $(\tau(\xi)|\bar{u}) \neq 0$. By replacing u with su for suitable $s \in \mathbb{K}$ we may suppose that $(\tau(\xi)|\bar{u}) = 7$. Then $(\tau(\xi)|\bar{u}) - \frac{6}{K} = 7 - \frac{6}{K} \geqslant 1$.

Our assumptions give

$$\sum_{j=0}^{\infty} \frac{|u^j|^2}{||\sigma_K||_{B_j}} = \sum_{\substack{j=0\\ B_j \neq B_k}}^{\infty} \frac{|u^j|^2}{||\sigma_K||_{B_j}} \leqslant \big(\inf_{B_j \neq B_k} ||\sigma_K||_{B_j}\big)^{-1} ||u||_{\overline{B}}^2.$$

And so by (5.3.26)

$$||f||_{E_{\mathcal{R}}^*} \leqslant K||u||_{\overline{B}}\big(\inf_{B_j \neq B_k} ||\sigma_K||_{B_j}\big)^{-\frac{1}{2}}. \tag{5.3.27}$$

We know that $\rho \otimes (\xi - b_k) \in N$, and so $Q_N(\rho \otimes \xi) = Q_N(\rho \otimes b_k)$. Also, for each $j \in \mathbb{N}_0$, the 1-symmetry (and therefore 1-unconditionality) of the basis (e_n) for B_j

implies that

$$||\sigma_L||_{B_j} \leqslant ||\sigma_{2L}||_{B_j} \leqslant 2||\sigma_L||_{B_j} \qquad (L \in \mathbb{N}). \tag{5.3.28}$$

Therefore by (5.3.22), (5.3.23), (5.3.27) and (5.3.28),

$$\begin{aligned} K|(\tau(\xi)|\bar{u})| - 6 &\leqslant |\langle TQ_N(\rho \otimes \xi), f\rangle| \leqslant ||T||\,||Q_N(\rho \otimes b_k)||_Y ||f||_{E^*_{\mathcal{R}}} \\ &\leqslant ||T||\,||\rho \otimes b_k||_Y ||f||_{E^*_{\mathcal{R}}} \\ &\leqslant \sqrt{2}K||T||\,||\sigma_K||^{\frac{1}{2}}_{B_k} ||u||_{\overline{B}} \big(\inf_{B_j \neq B_k} ||\sigma_K||_{B_j}\big)^{-\frac{1}{2}} \end{aligned}$$

and so

$$\sqrt{2}||T||\,||u||_{\overline{B}} \geqslant \left(|(\tau(\xi)|\bar{u})| - \frac{6}{K}\right)\left(\frac{\inf_{B_j \neq B_k} ||\sigma_K||_{B_j}}{||\sigma_K||_{B_k}}\right)^{\frac{1}{2}} \geqslant \left(\frac{\inf_{B_j \neq B_k} ||\sigma_K||_{B_j}}{||\sigma_K||_{B_k}}\right)^{\frac{1}{2}}. \tag{5.3.29}$$

Just prior to (5.3.15) we fixed the odd number R and subsequently defined $K = (R+1)/2$. Now allowing R to vary, we see that (5.3.29) is actually true for *every* natural number K.

Recall from (5.2.19) that $\mathbb{I} = \{2\} \cup \{i \in \mathbb{N}_0 : i \equiv 0, 4, 5 \bmod 6\}$. Define $k_0 \in \mathbb{I}$ by

$$k_0 = \begin{cases} k & \text{if } k \in \mathbb{I} \\ 0 & \text{if } k \equiv 1 \bmod 6 \\ 2 & \text{if } k \equiv 2, 3 \bmod 6. \end{cases}$$

By comparing this definition with (5.2.20), we observe that $B_k = B_{k_0}$, from which it follows that $\inf_{B_j \neq B_k} ||\sigma_K||_{B_j} = \inf_{j \in \mathbb{I} \setminus \{k_0\}} ||\sigma_K||_{B_j}$ for any $K \in \mathbb{N}$.
Using (5.3.29) we see that for every $K \in \mathbb{N}$

$$||\sigma_K||_{B_{k_0}} \geqslant \frac{1}{2||T||^2\,||u||^2_{\overline{B}}} \inf_{j \in \mathbb{I} \setminus \{k_0\}} ||\sigma_K||_{B_j},$$

which contradicts the incomparability of the family $(B_i)_{i \in \mathbb{I}}$ (take $0 < \frac{1}{n} < \frac{1}{2||T||^2\,||u||^2}$ in (5.2.21)). Thus the claim is proved.

Equipped with this result we can finish the proof by splitting into a number of cases, depending on $k \in \mathbb{N}_0$.

Case 1 (x_n, y_n). Suppose that $k \equiv 4 \bmod 6$ or $k \equiv 5 \bmod 6$. Let

$$\xi = b_k = \begin{cases} x_n & \text{if } k = 6(n-1)+4 \\ y_n & \text{if } k = 6(n-1)+5 \end{cases} \tag{5.3.30}$$

for some $n \in \mathbb{N}$. Then $k \in \mathbb{I}$ and $B_j = B_k$ if and only if $j = k$. Hence, by the claim, for each $u \in V^\perp$ such that $(u|b_k) = 0$ we have $(\tau(b_k)|\bar{u}) = 0$. To begin to ascertain the matrix form of $\tau \in \mathscr{B}(H)$, we want to show that $\tau(b_k) \in \mathbb{K}b_k$. The proof amounts to picking good choices of $u \in V^\perp$.

Recall that $H = \overline{\operatorname{span}}\{b_0, x_n, y_n : n \in \mathbb{N}\}$. So with the standard basis expansion,

$$\tau(b_k) = (\tau(b_k)|b_0)b_0 + \sum_{j=1}^{\infty}(\tau(b_k)|x_j)x_j + (\tau(b_k)|y_j)y_j. \tag{5.3.31}$$

Firstly, let $u = b_0 + \sum_{j=1}^{\infty} 2^{-j}\delta_j$. To check that $u \in V^\perp$, it is enough to check that $u \perp S$. By (5.2.23), we have $u \perp \alpha'_j, \beta'_j, \gamma'_j$ for every $j \in \mathbb{N}$. For $m \in \mathbb{N}$ observe that

$$\begin{aligned}(u|\delta'_m) &= \left(b_0 \,\middle|\, \delta_m - (2^{-m}b_0 - x_m + y_m)\right) + \left(\sum_{j=1}^{\infty} 2^{-j}\delta_j \,\middle|\, \delta_m - (2^{-m}b_0 - x_m + y_m)\right)\\ &= -2^{-m} + \sum_{j=1}^{\infty} 2^{-j}(\delta_j|\delta_m) = -2^{-m} + 2^{-m} = 0.\end{aligned}$$

Therefore $u \in V^\perp$. Since by (5.3.30) we also have $(u|b_k) = 0$, it follows that $(\tau(b_k)|\bar{u}) = 0$ by the claim. Thus by (5.3.31)

$$\begin{aligned}0 &= \left((\tau(b_k)|b_0)b_0 + \sum_{j=1}^{\infty}(\tau(b_k)|x_j)x_j + (\tau(b_k)|y_j)y_j \,\middle|\, b_0 + \sum_{j=1}^{\infty} 2^{-j}\delta_j\right)\\ &= \left((\tau(b_k)|b_0)b_0 \,\middle|\, b_0\right) = (\tau(b_k)|b_0).\end{aligned}$$

We are left with

$$\tau(b_k) = \sum_{j=1}^{\infty}(\tau(b_k)|x_j)x_j + (\tau(b_k)|y_j)y_j.$$

Fix $i \in \mathbb{N}$ and make a second choice of $u = x_i + \alpha_i + \beta_i + \gamma_i - \delta_i$. It is easily checked that $u \in V^\perp$, and $(u|b_k) = 0$ unless $b_k = x_n$ and $i = n$. So if $b_k = y_n$ or $i \neq n$ then

$$\begin{aligned}0 &= (\tau(b_k)|\bar{u})\\ &= \left(\sum_{j=1}^{\infty}(\tau(b_k)|x_j)x_j + (\tau(b_k)|y_j)y_j \,\middle|\, x_i + \alpha_i + \beta_i + \gamma_i - \delta_i\right) = (\tau(b_k)|x_i).\end{aligned}$$

Therefore $(\tau(y_n)|x_i) = 0$ for every $i \in \mathbb{N}$, and $(\tau(x_n)|x_i) = 0$ for every $i \neq n$. This leaves us with

$$\tau(x_n) = (\tau(x_n)|x_n)x_n + \sum_{j=1}^{\infty}(\tau(x_n)|y_j)y_j \qquad \text{and} \qquad \tau(y_n) = \sum_{j=1}^{\infty}(\tau(y_n)|y_j)y_j. \tag{5.3.32}$$

Thirdly, fix $i \in \mathbb{N}$ and choose $u = y_i - \alpha_i + \beta_i + \gamma_i + \delta_i$, which is in $V^\perp$. Also $(u|b_k) = 0$ unless $b_k = y_n$ and $i = n$. Using (5.3.32), the claim implies that

$0 = (\tau(x_n)|y_i)$ for all $i \in \mathbb{N}$, and $0 = (\tau(y_n)|y_i)$ for $i \neq n$.

In conclusion $\tau(x_n) = (\tau(x_n)|x_n)x_n \in \mathbb{K}x_n$ and $\tau(y_n) = (\tau(y_n)|y_n)y_n \in \mathbb{K}y_n$. By evaluating at the basis vectors x_n, y_n for each $n \in \mathbb{N}$, this tells us that the matrix of τ has the form

$$\begin{array}{cc}
 & \begin{matrix} b_0 & x_1 & y_1 & x_2 & \cdots & & \end{matrix} \\
\begin{matrix} b_0 \\ x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \end{matrix} &
\begin{pmatrix}
* & 0 & 0 & 0 & & & \\
* & * & 0 & 0 & \cdots & & \\
* & 0 & * & 0 & & & \\
* & 0 & 0 & * & & & \\
* & & & & * & & \\
\vdots & & & & & \ddots &
\end{pmatrix}
\end{array}$$

with respect to the basis $(b_0, x_1, y_1, \ldots)$ of H. We have yet to determine the first column, and the exact form of the main diagonal. These are covered in the remaining three cases.

Case 2 (α_n). Suppose that $k \neq 0$ and $k \equiv 0 \bmod 6$, so that $b_k = \alpha_n$ where $n = \frac{k}{6}$. So $k \in \mathbb{I}$ and $B_k = B_j$ if and only if $j = k$. We wish to show that $(\tau(y_n)|y_n) = (\tau(x_n)|x_n)$. We need $\xi - \alpha_n \in V$ so let $\xi = x_n - y_n$. Let $u = x_n + y_n + 2\beta_n + 2\gamma_n \in V^{\perp}$. Clearly $(u|b_k) = 0$. Combining the claim, Case 1, and the fact that $\tau(\xi) \in H$, we obtain

$$0 = (\tau(\xi)|x_m + y_m + 2\beta_m + 2\gamma_m) = (\tau(x_m)|x_m) - (\tau(y_m)|y_m). \tag{5.3.33}$$

Hence the elements in the main diagonal of the matrix of τ are pairwise the same.

Case 3 (b_0). Let $k = 0$. Note that $B_j = B_0$ if and only if $j = 0$ or $j \equiv 1 \bmod 6$. Take $\xi = b_0 \in H$; then $\xi - b_0 = 0 \in V$. Now take $m \in \mathbb{N}$ and set $u = x_m - y_m + 2\alpha_m - 2\delta_m \in V^{\perp}$. Then $(u|b_0) = (u|\beta_i) = 0$ for every $i \in \mathbb{N}$. Using the fact that $\tau(\xi) \in H$ and the claim, we obtain

$$0 = (\tau(b_0)|x_m - y_m + 2\alpha_m - 2\delta_m) = (\tau(b_0)|x_m) - (\tau(b_0)|y_m).$$

We have now deduced that our matrix has the form

$$\begin{array}{c} \\ b_0 \\ x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} b_0 & x_1 & y_1 & x_2 & \cdots \end{array} \\ \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & & & \\ \mu_1 & \lambda_2 & 0 & 0 & \cdots & & \\ \mu_1 & 0 & \lambda_2 & 0 & & & \\ \mu_2 & 0 & 0 & \lambda_3 & & & \\ \mu_2 & & & & \lambda_3 & & \\ \vdots & & & & & \ddots & \end{pmatrix} \end{array}$$

for some scalars $\lambda_m, \mu_m \in \mathbb{K}$.

Case 4 (δ_n). Consider $k \equiv 3 \bmod 6$; note that $B_j = B_k = B_2$ if and only if $j \equiv 2, 3 \bmod 6$. We have $b_k = \delta_n$ where $k = 6(n-1)+3$. Set $\xi = 2^{-n}b_0 - x_n + y_n$. Then $\xi - b_k \in V$. Choose

$$u = b_0 + \sum_{j=1}^{\infty} \big(\frac{1}{2^{j+1}}(x_j - y_j) + \frac{1}{2^j}\alpha_j\big).$$

One can easily see that $u \in V^\perp$ and that $(u|\delta_i) = (u|\gamma_i) = 0$ for each $i \in \mathbb{N}$. From the other cases and the claim we obtain

$$\begin{aligned} 0 &= \left(\tau(2^{-n}b_0 - x_n + y_n) \;\middle|\; b_0 + \sum_{j=1}^{\infty} \big(\frac{1}{2^{j+1}}(x_j - y_j) + \frac{1}{2^j}\alpha_j\big)\right) \\ &= 2^{-n}(\tau(b_0)|b_0) - 2^{-(n+1)}(\tau(x_n)|x_n) - 2^{-(n+1)}(\tau(y_n)|y_n). \end{aligned}$$

Hence $(\tau(b_0)|b_0) = (\tau(y_n)|y_n) = (\tau(x_n)|x_n)$ using (5.3.33), giving τ the required matrix form of

$$\begin{array}{c} \\ b_0 \\ x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} b_0 & x_1 & y_1 & x_2 & \cdots \end{array} \\ \begin{pmatrix} \lambda & 0 & 0 & 0 & & & \\ \mu_1 & \lambda & 0 & 0 & \cdots & & \\ \mu_1 & 0 & \lambda & 0 & & & \\ \mu_2 & 0 & 0 & \lambda & & & \\ \mu_2 & & & & \lambda & & \\ \vdots & & & & & \ddots & \end{pmatrix} \end{array}$$

for some $\lambda, \mu_m \in \mathbb{K}$. This completes the proof. □

5.4 The algebra of bounded operators on Read's space

We have worked hard to gain a thorough understanding of Read's results, and it is now time to prove some of our own. Theorem 1.3.2 tells us quite a lot about $\mathscr{B}(E_{\mathcal{R}})$, in particular, that it has an important and interesting codimension 1 ideal I.

Our main original result, Theorem 1.3.4, tells us more about $\mathscr{B}(E_{\mathcal{R}})$ than Read proved in his paper. To be specific, our theorem shows that $\mathscr{B}(E_{\mathcal{R}})$ decomposes as the direct sum of the weakly compact operators on $E_{\mathcal{R}}$ and a pleasingly simple closed subalgebra. The proof of Theorem 1.3.4 builds on Read's work in two ways—we expand the conclusion of Lemma 5.3.6 to include a larger image, and then 'go back' to $\mathscr{B}(E_{\mathcal{R}})$ in the manner of [90, Corollary 4.2].

The diligent reader of this chapter so far will find the proofs in this section of a similar nature to the preceding ones. Those who have skipped ahead to this point may need to keep looking back to the earlier sections.

Our first result extends Lemma 5.3.6 by showing that the inclusion (5.3.9) in the statement of Lemma 5.3.6 is in fact an equality. Recall the important space $H = \overline{\operatorname{span}}\,\{b_0, x_n, y_n : n \in \mathbb{N}\} \subset \overline{B}$.

Theorem 5.4.1. *For Read's space $E_{\mathcal{R}}$*

$$\operatorname{Ad} U \circ \Theta_0(\mathscr{B}(E_{\mathcal{R}})) = \mathscr{T} + \mathbb{K} I_H.$$

Before justifying this statement, let us prove a simple lemma.

Lemma 5.4.2. *For operators $W, W' \in \mathscr{T} + \mathbb{K} I_H \subseteq \mathscr{B}(H)$, $W = W'$ if and only if $W(b_0) = W'(b_0)$.*

Proof. $(\Rightarrow)$ This is clear since it is true for all elements of H.
$(\Leftarrow)$ Suppose that $W(b_0) = W'(b_0)$. Write $W = \tau_\eta + \lambda I_H$ and $W' = \tau_\nu + \mu I_H$ for some $\eta, \nu \in H_0$ and $\lambda, \mu \in \mathbb{K}$. Then $W(b_0) = (\tau_\eta + \lambda I_H)(b_0) = \eta + \lambda b_0$ and similarly $W'(b_0) = \nu + \mu b_0$, which implies that $\eta + \lambda b_0 = \nu + \mu b_0$. So for every $\zeta \in H_0$, $(\eta - \nu|\zeta) = ((\mu - \lambda) b_0|\zeta) = 0$ since b_0 is orthogonal to H_0. Therefore $\eta - \nu \in H_0 \cap H_0^\perp = \{0\}$ so $\eta = \nu$. This implies that $\lambda b_0 = \mu b_0$ so that $\lambda = \mu$; hence $W = W'$. $\square$

Proof of Theorem 5.4.1. From Lemma 5.3.6 we know that $\operatorname{Ad} U \circ \Theta_0(\mathscr{B}(E_{\mathcal{R}})) \subseteq \mathscr{T} + \mathbb{K} I_H$, so it is enough to show that $\mathscr{T} + \mathbb{K} I_H \subseteq \operatorname{Ad} U \circ \Theta_0(\mathscr{B}(E_{\mathcal{R}}))$.

Choose $\tau_\xi + \lambda I_H \in \mathscr{T} + \mathbb{K} I_H$, for some $\xi \in H_0$ and $\lambda \in \mathbb{K}$. Write $\xi = \sum_{n=1}^\infty \xi_n (x_n + y_n)$ so that $||\xi||_{\overline{B}} = \sqrt{2}\big(\sum_{n=1}^\infty |\xi_n|^2\big)^{\frac{1}{2}}$. We seek to define an operator

$T_\xi \in \mathscr{B}(Y)$ and then show that it induces an operator $\widehat{T_\xi} \in \mathscr{B}(E_{\mathcal{R}})$ such that

$$\operatorname{Ad} U \circ \Theta_0(\widehat{T_\xi}) = \tau_\xi. \tag{5.4.1}$$

Then, since Θ_0 is unital, $\operatorname{Ad} U \circ \Theta_0(\widehat{T_\xi} + \lambda I_{E_{\mathcal{R}}}) = \tau_\xi + \lambda I_H$, so that $\tau_\xi + \lambda I_H \in \operatorname{Ad} U \circ \Theta_0(\mathscr{B}(E_{\mathcal{R}}))$, which would complete the proof.

To this end, take an element $y = (y(i))_{i=0}^\infty \in Y$, where $y(i) \in JB_i$ for each $i \in \mathbb{N}_0$. (We use this notation for the coordinates of y to steer clear of confusion with the basis vectors y_i from Definition 5.2.16).

By the definition of the particular JB_i spaces in (5.2.20), we see that $JB_{6(n-1)+1} = JB_0$ for each $n \in \mathbb{N}$. By Definition 5.2.16, for each $n \in \mathbb{N}$, $y(0) \otimes \beta_n$ is the element of Y whose $(6(n-1)+1)^{\text{st}}$ coordinate is $y(0)$ and all other coordinates are zero. Then the series $\sum_{n=1}^\infty \xi_n y(0) \otimes \beta_n$ is convergent in Y; it has norm $\frac{1}{\sqrt{2}}||\xi||_{\overline{B}}||y(0)||_{JB_0}$ because

$$\begin{aligned}\left\|\sum_{n=1}^\infty \xi_n y(0) \otimes \beta_n\right\|_Y &= ||(0,\ldots,\xi_1 y(0), 0, \ldots, 0, \xi_2 y(0), 0, \ldots)||_Y \\ = \left(\sum_{n=1}^\infty ||\xi_n y(0)||^2_{JB_{6(n-1)+1}}\right)^{\frac{1}{2}} &= \left(\sum_{n=1}^\infty ||\xi_n y(0)||^2_{JB_0}\right)^{\frac{1}{2}} = ||y(0)||_{JB_0}\big(\sum_{n=1}^\infty |\xi_n|^2\big)^{\frac{1}{2}}.\end{aligned}$$

Again using (5.2.20), it holds that $JB_{6(n-1)+2} = JB_{6(n-1)+3} = JB_2$ for each $n \in \mathbb{N}$, and the series $\sum_{n=1}^\infty y(6(n-1)+3)/2^n$ in JB_2 converges absolutely because for every $N \in \mathbb{N}$

$$\sum_{n=1}^N ||y(6(n-1)+3)/2^n||_{JB_2} = \sum_{n=1}^N \frac{1}{2^n}||y(6(n-1)+3)||_{JB_2} \leqslant ||y||_Y \sum_{n=1}^\infty \frac{1}{2^n} \leqslant ||y||_Y.$$

Temporarily denote the sum of the series by x; we have $||x||_{JB_2} \leqslant ||y||_Y$.

For each $n \in \mathbb{N}$, $x \otimes \gamma_n$ is the member of Y with zeros everywhere except the $(6(n-1)+2)^{\text{nd}}$ coordinate, which is x. Then the series $\sum_{n=1}^\infty \xi_n\, x \otimes \gamma_n$ converges in Y and

$$\left\|\sum_{n=1}^\infty \xi_n\, x \otimes \gamma_n\right\|_Y^2 = \sum_{n=1}^\infty ||\xi_n x||^2_{JB_2} = \frac{1}{2}||x||^2_{JB_2}||\xi||^2_{\overline{B}},$$

which implies that $||\sum_{n=1}^\infty \xi_n\, x \otimes \gamma_n||_Y \leqslant \frac{1}{\sqrt{2}}\|\xi\|_{\overline{B}}\, \|y\|_Y$.

Now we may define a map $T_\xi : Y \to Y$ by

$$\begin{aligned}T_\xi y &= \sum_{n=1}^\infty \xi_n y(0) \otimes \beta_n + \sum_{n=1}^\infty \xi_n\, x \otimes \gamma_n \\ &= \sum_{n=1}^\infty \xi_n \Bigg(y(0) \otimes \beta_n + \Bigg(\sum_{m=1}^\infty \frac{y(6(m-1)+3)}{2^m}\Bigg) \otimes \gamma_n\Bigg).\end{aligned} \tag{5.4.2}$$

Observe that

$$||T_\xi y|| \leqslant \frac{1}{\sqrt{2}}||\xi||_{\overline{B}}||y(0)||_Y + \frac{1}{\sqrt{2}}||\xi||_{\overline{B}}||y||_Y \leqslant \sqrt{2}||\xi||_{\overline{B}}||y||_Y,$$

and so T_ξ is bounded, with norm at most $\sqrt{2}||\xi||_{\overline{B}}$. The tensor notation works in the usual way, so T_ξ is also linear, and therefore $T_\xi \in \mathscr{B}(Y)$.

We want to show that T_ξ induces an operator on $E_{\mathcal{R}}$, so the next step is to check that $T_\xi[N] \subseteq N$ where N is the subspace of Y from Definition 5.2.20. Because T_ξ is bounded and linear, it is enough that $T_\xi(e_n \otimes s) \in N$ for each $n \in \mathbb{N}$ and $s \in S$.

It quickly follows from Definition 5.2.16 and the definition of T_ξ that $T_\xi(e_n \otimes \eta) = 0$ for each $\eta \in \{\alpha_m, \beta_m, \gamma_m, x_m, y_m : m \in \mathbb{N}\}$, and therefore $T_\xi(e_n \otimes s) = 0 \in N$ for each $s \in \{\alpha'_m, \beta'_m, \gamma'_m : m \in \mathbb{N}\}$ by (5.2.23). For the final case, we have

$$\begin{aligned} T_\xi(e_n \otimes \delta'_m) &= T_\xi(e_n \otimes \delta_m) - \frac{1}{2^m}T_\xi(e_n \otimes b_0) = \sum_{k=1}^\infty \xi_k \frac{e_n}{2^m} \otimes \gamma_k - \frac{1}{2^m}\sum_{k=1}^\infty \xi_k\, e_n \otimes \beta_k \\ &= \frac{1}{2^m}\sum_{k=1}^\infty \xi_k\, e_n \otimes (\gamma_k - \beta_k) = \frac{1}{2^m}\sum_{k=1}^\infty \xi_k\, e_n \otimes (\gamma'_k - \beta'_k) \\ &= \frac{1}{2^m}\sum_{k=1}^\infty \xi_k\, (e_n \otimes \gamma'_k - e_n \otimes \beta'_k) \in N \qquad (m \in \mathbb{N}), \end{aligned}$$

which completes the proof that $T_\xi[N] \subseteq N$.

Now the Fundamental Isomorphism Theorem 1.2.4 tells us that there is a unique operator $\widehat{T}_\xi \in \mathscr{B}(E_{\mathcal{R}})$ given by $\widehat{T}_\xi(y + N) = T_\xi(y) + N$ for $y \in Y$, thus making the following diagram commutative

$$\begin{tikzcd} Y \arrow[r, "T_\xi"] \arrow[d, "Q_N"'] & Y \arrow[d, "Q_N"] \\ E_{\mathcal{R}} \arrow[r, dashed, "\widehat{T}_\xi"] & E_{\mathcal{R}}. \end{tikzcd} \tag{5.4.3}$$

Also,

$$||\widehat{T}_\xi|| = ||\widehat{T}_\xi Q_N|| = ||Q_N T_\xi|| \leqslant \sqrt{2}||\xi||_{\overline{B}} \tag{5.4.4}$$

by Proposition 5.1.1.

The remaining point in question is whether $\operatorname{Ad} U \circ \Theta_0(\widehat{T}_\xi) = \tau_\xi$. This requires a little more work. By definition $\operatorname{Ad} U \circ \Theta_0(\widehat{T}_\xi)$ and τ_ξ are both elements of $\mathscr{T} + \mathbb{K}I_H$, so Lemma 5.4.2 implies that it is enough to show that $\operatorname{Ad} U \circ \Theta_0(\widehat{T}_\xi)(b_0) = \tau_\xi(b_0)$, or equivalently, that

$$\Theta_0(\widehat{T}_\xi)Ub_0 = U\xi. \tag{5.4.5}$$

For each $m \in \mathbb{N}$ and $i \in \mathbb{N}_0$, $\sigma_m = \sum_{j=1}^m e_j$ is a unit vector in JB_i (since it has only one 'jump'). Let $\eta \in \overline{B}$. Then Lemma 5.3.7 assures us that $\sigma_m \otimes \eta \overset{w^*}{\to} \Phi \otimes \eta$ in Y^{**}. Recall that $\xi = \sum_{n=1}^{\infty} \xi_n(x_n + y_n)$, and so by (5.4.2) and (5.2.23)

$$\begin{aligned} Q_N T_\xi(\sigma_m \otimes b_0) &= Q_N\left(\sum_{n=1}^{\infty} \xi_n\, \sigma_m \otimes \beta_n\right) \\ &= \sum_{n=1}^{\infty} \xi_n Q_N(\sigma_m \otimes \beta_n) = \sum_{n=1}^{\infty} \xi_n Q_N(\sigma_m \otimes (x_n + y_n)) \\ &= Q_N\left(\sigma_m \otimes \sum_{n=1}^{\infty} \xi_n(x_n + y_n)\right) = Q_N(\sigma_m \otimes \xi) \end{aligned}$$

for each $m \in \mathbb{N}$. From this, it follows that

$$Q_N^{**} T_\xi^{**}(\Phi \otimes b_0) = \mathrm{w}^*\text{-}\lim_{m\to\infty} Q_N T_\xi(\sigma_m \otimes b_0) = \mathrm{w}^*\text{-}\lim_{m\to\infty} Q_N(\sigma_m \otimes \xi) = Q_N^{**}(\Phi \otimes \xi)$$

since Q_N^{**} and $(Q_N T_\xi)^{**}$ are weak* continuous, and $(Q_N T_\xi)^{**}(y) = Q_N T_\xi(y)$ for each $y \in Y$ (these are general properties of bidual operators).

The time has come to prove (5.4.5), using the diagrams (5.3.7) and (5.4.3), together with Proposition 5.1.2:

$$\begin{aligned} \Theta_0(\widehat{T}_\xi) U b_0 &= \Theta_0(\widehat{T}_\xi)\pi_{E_\mathrm{R}} Q_N^{**}(\Phi \otimes b_0) = \pi_{E_\mathrm{R}}(\widehat{T}_\xi Q_N)^{**}(\Phi \otimes b_0) \\ &= \pi_{E_\mathrm{R}} Q_N^{**} T_\xi^{**}(\Phi \otimes b_0) = \pi_{E_\mathrm{R}} Q_N^{**}(\Phi \otimes \xi) = U\xi. \end{aligned}$$

This completes the proof. □

Remark 5.4.3. We observe that we have also proved Read's Lemma 4.2 (which we have not stated). For, let $\mu = (\mu_1, \mu_2 \ldots) \in c_{00} \subseteq \overline{B}$. Then by Theorem 5.4.1 there is $T \in \mathscr{B}(E_\mathcal{R})$ such that $\operatorname{Ad} U \circ \Theta_0(T) = \tau_\mu$, and τ_μ has the required matrix form. This is not coincidental—indeed the proof of Theorem 5.4.1 was inspired by Read's Lemma 4.2.

We have finally arrived at our main theorem from Chapter 1! For convenience we choose to state the result in an equivalent form by equipping the separable Hilbert spaces H_0 and $\ell_2(\mathbb{N})$ with the trivial product and then identifying them as Banach algebras.

Theorem 1.3.4. *There exists a continuous, unital, surjective algebra homomorphism β from $\mathscr{B}(E_\mathcal{R})$ onto $\widetilde{H_0}$, with $\ker\beta = \mathscr{W}(E_\mathcal{R})$, such that the extension*

$$\{0\} \longrightarrow \mathscr{W}(E_\mathcal{R}) \xrightarrow{\iota} \mathscr{B}(E_\mathcal{R}) \xrightarrow{\beta} \widetilde{H_0} \longrightarrow \{0\} \qquad (5.4.6)$$

splits strongly.

Proof. Firstly, we want to show that the map

$$\Upsilon : \xi \mapsto \tau_\xi, \qquad H_0 \to \mathscr{T} \tag{5.4.7}$$

is an isometric algebra isomorphism. Indeed, if $\xi, \nu \in H_0$, $h \in H$ and $\lambda \in \mathbb{K}$ then by Definition 5.3.5

$$\begin{aligned}
&\tau_\xi \tau_\nu(h) = (h|b_0)(\nu|b_0)\xi = 0 = \tau_0(h) = \tau_{\xi\nu}(h) \quad \text{since } b_0 \text{ is orthogonal to } H_0,\\
&(\tau_\xi + \tau_\nu)(h) = (h|b_0)\xi + (h|b_0)\nu = \tau_{\xi+\nu}(h),\\
&(\lambda\tau_\xi)(h) = \lambda(h|b_0)\xi = \tau_{\lambda\xi}(h),\\
&||\tau_\xi(h)|| = ||(h|b_0)\xi|| = |(h|b_0)|\,||\xi|| \leqslant ||h||\,||\xi||,
\end{aligned}$$

so that Υ is a contractive algebra homomorphism. It is also surjective by Definition 5.3.5, and moreover, for each $\xi \in H_0$, $\tau_\xi(b_0) = \xi$, so that $||\tau_\xi|| = ||\xi||$. Therefore Υ is an isometric algebra isomorphism.

Hence the map

$$\widetilde{\Upsilon} : \xi + \lambda 1_{\widetilde{H_0}} \mapsto \Upsilon(\xi) + \lambda I_H, \qquad \widetilde{H_0} \to \mathscr{T} + \mathbb{K} I_H,$$

is a continuous, unital algebra isomorphism. By (5.3.9) we may set

$$\beta = \widetilde{\Upsilon}^{-1} \circ \operatorname{Ad} U \circ \Theta_0 : \mathscr{B}(E_{\mathcal{R}}) \to \widetilde{H_0}. \tag{5.4.8}$$

Then β is a continuous, unital algebra homomorphism, and β is surjective by Theorem 5.4.1. Moreover, since $\operatorname{Ad} U$ and $\widetilde{\Upsilon}^{-1}$ are isomorphisms,

$$\ker \beta = \ker \Theta_0 = \mathscr{W}(E_{\mathcal{R}}).$$

Thus we obtain the extension (5.4.6). To complete the proof, we must demonstrate that (5.4.6) splits strongly.

In the proof of Theorem 5.4.1 we showed that for every $\xi \in H_0$ there exists a bounded operator $\widehat{T}_\xi \in I \subseteq \mathscr{B}(E_{\mathcal{R}})$ such that $\operatorname{Ad} U \circ \Theta_0(\widehat{T}_\xi) = \tau_\xi$. Define

$$\rho_0 : \xi \mapsto \widehat{T}_\xi, \qquad H_0 \to I. \tag{5.4.9}$$

Then ρ_0 is linear, and continuous because $||\widehat{T}_\xi|| \leqslant \sqrt{2}||\xi||_{\overline{B}}$ by (5.4.4). It remains to check that ρ_0 is an algebra homomorphism; since H_0 has the trivial product, we need to show that $\widehat{T}_\xi \widehat{T}_\eta = 0$ for $\xi, \eta \in H_0$. By the diagram (5.4.3) it is enough to prove that $T_\xi T_\eta = 0 \in \mathscr{B}(Y)$.

To do this, pick $y = (y(i))_{i=0}^\infty \in Y$. We may write $z = T_\xi y = (z(i))_{i=0}^\infty$ and

$\xi = \sum_{n=1}^{\infty} \xi_n(x_n + y_n)$. Then, by (5.4.2),

$$z(i) = \begin{cases} \xi_n y(0) & \text{if } i = 6(n-1)+1 \text{ for some } n \in \mathbb{N} \\ \xi_n \displaystyle\sum_{m=1}^{\infty} \frac{y(6(m-1)+3)}{2^m} & \text{if } i = 6(n-1)+2 \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \qquad (i \in \mathbb{N}_0),$$

so that $z(0) = 0 = z(6(m-1)+3)$ for each $m \in \mathbb{N}$. Applying (5.4.2) again yields $0 = T_\eta z = T_\eta T_\xi y$, as required.

Therefore the map

$$\rho : \xi + \lambda 1_{\widetilde{H_0}} \mapsto \rho_0(\xi) + \lambda I_{E_{\mathcal{R}}}, \qquad \widetilde{H_0} \to \mathscr{B}(E_{\mathcal{R}})$$

is a continuous algebra homomorphism. Finally, we must verify that ρ is a right inverse of β. Take $\xi + \lambda 1_{\widetilde{H_0}} \in \widetilde{H_0}$, for some $\lambda \in \mathbb{K}$ and $\xi \in H_0$. Then since Θ_0 and β are unital, and using (5.4.1), (5.4.7) and (5.4.8), we obtain

$$\beta \circ \rho(\xi + \lambda 1_{\widetilde{H_0}}) = \widetilde{\Upsilon}^{-1} \circ \operatorname{Ad} U \circ \Theta_0(\widehat{T_\xi} + \lambda I_{E_{\mathcal{R}}}) = \xi + \lambda 1_{\widetilde{H_0}}$$

which proves that (5.4.6) splits strongly. □

Corollary 5.4.4. *There exists a closed subalgebra C of $\mathscr{B}(E_{\mathcal{R}})$ such that $\mathscr{B}(E_{\mathcal{R}}) = \mathscr{W}(E_{\mathcal{R}}) \oplus C$. Moreover, $C \cong \ell_2(\mathbb{N})^\sim$ where $\ell_2(\mathbb{N})^\sim$ denotes the unitisation of $\ell_2(\mathbb{N})$ with the trivial product.*

Proof. There is a continuous algebra homomorphism $\rho : \ell_2(\mathbb{N})^\sim \to \mathscr{B}(E_{\mathcal{R}})$ given by the unitisation of (5.4.9). Let $C = \operatorname{im} \rho$. This is a closed subalgebra of $\mathscr{B}(E_{\mathcal{R}})$ because ρ is a right inverse of the map β from (5.4.8). It follows that $\rho : \ell_2(\mathbb{N})^\sim \to C$ is an isomorphism. In addition, $\mathscr{B}(E_{\mathcal{R}}) = \mathscr{W}(E_{\mathcal{R}}) \oplus C$ because $\mathscr{W}(E_{\mathcal{R}}) = \ker \beta$. □

We now have several routes to the proof of Read's main theorem, but let us deduce it from Theorem 1.3.4.

Proof of Theorem 1.3.2. Recall that $I = (\operatorname{Ad} U \circ \Theta_0)^{-1}(\mathscr{T})$. Note that by (5.4.9), ρ_0 is a continuous right inverse of $\beta|_I$ and so $\operatorname{im} \rho_0$ is a closed subalgebra with the trivial product. Also, from (5.4.9) it is clear that $I \supseteq \mathscr{W}(E_{\mathcal{R}}) \oplus \operatorname{im} \rho_0$ because $\mathscr{W}(E_{\mathcal{R}}) = \ker \Theta_0$, and for each $T \in I$ we have $T = \rho_0\beta(T) + (T - \rho_0\beta(T))$ where $T - \rho_0\beta(T) \in \ker \beta = \mathscr{W}(E_{\mathcal{R}})$. Therefore

$$I = (\operatorname{Ad} U \circ \Theta_0)^{-1}(\mathscr{T}) = \mathscr{W}(E_{\mathcal{R}}) \oplus \operatorname{im} \rho_0$$

and so I is a closed ideal of $\mathscr{B}(E_{\mathcal{R}})$. Corollary 5.4.4 now implies that

$$\mathscr{B}(E_{\mathcal{R}}) = \mathscr{W}(E_{\mathcal{R}}) \oplus \operatorname{im}\rho = \mathscr{W}(E_{\mathcal{R}}) \oplus \operatorname{im}\rho_0 \oplus \mathbb{K}I_{E_{\mathcal{R}}} = I \oplus \mathbb{K}I_{E_{\mathcal{R}}}. \tag{5.4.10}$$

Thus I has codimension 1 in $\mathscr{B}(E_{\mathcal{R}})$. Conditions (i), (ii) and (iii) follow immediately. □

5.5 Commutators on $E_{\mathcal{R}}$

We end the chapter by giving an application of Corollary 5.4.4 to commutators in the unital Banach algebra $\mathscr{B}(E_{\mathcal{R}})$. An element a of an algebra A is a *commutator* if there exist $b, c \in A$ such that $a = bc - cb$. Given a (non-commutative) algebra A, one may ask what the set of commutators in A looks like. Are there elements which are not commutators? Can we characterise the ones which are? In particular, we may focus on the case where A is a unital normed algebra.

Wintner proved an important early result in this direction [111], showing that the identity operator on an infinite-dimensional, separable Hilbert space is not a commutator. By a different method, Wielandt observed that the result remains valid in an arbitrary unital normed algebra [109].

Theorem 5.5.1 (Wielandt). *Let A be a unital normed algebra. Then the identity 1_A is not a commutator.*

Corollary 5.5.2 (Halmos, [51]). *Let M be a proper closed ideal of a unital normed algebra A. Then for $m \in M$ and λ a non-zero scalar, $\lambda 1_A + m$ is not a commutator.*

Proof. Apply Wielandt's Theorem to the unital normed algebra A/M. □

Let us specialise to the unital Banach algebra $A = \mathscr{B}(X)$ for some infinite-dimensional Banach space X. A commutator in $\mathscr{B}(X)$ (or a commutator *on* X) is a bounded operator T such that there are $A, B \in \mathscr{B}(X)$ with $T = AB - BA$. Here, consideration of commutators is linked to the study of derivations from $\mathscr{B}(X)$.

For the case of $X = \ell_2$, Brown and Pearcy [16] showed that the only bounded operators on ℓ_2 which are not commutators are of the form $\lambda I_{\ell_2} + K$, where $K \in \mathscr{K}(\ell_2)$ and $\lambda \in \mathbb{K}\backslash\{0\}$. Later, Apostol [4], [5] complemented their result by proving that the only bounded operators on ℓ_p $(1 < p < \infty)$ and c_0 which are not commutators are of the form $\lambda I + K$, where K is compact and λ is a non-zero scalar. Some thirty years passed before further progress was made. In 2009 Dosev [30] proved that the analogous result was true for $X = \ell_1$, and then, together with Johnson [31], demonstrated that the only non-commutators on ℓ_∞ are of the form $\lambda I_{\ell_\infty} + S$ where S is strictly singular and λ is non-zero.

Given this evidence, as well as further work of their own, Chen, Johnson and Zheng made the following definition in [22], and the subsequent conjecture.

Definition 5.5.3. A Banach space X is a *Wintner space* if the only elements of $\mathscr{B}(X)$ which are not commutators are operators of the form $\lambda I_X + T$, where λ is a non-zero scalar and T is contained in a proper closed ideal of $\mathscr{B}(X)$.

Conjecture 5.5.4 (Chen, Johnson, Zheng, 2011)**.** Every infinite-dimensional Banach space is a Wintner space.

The conjecture is false (as anticipated by the conjecturers, who termed it 'wild'!). This was first shown by Tarbard [104, Lemma 3.2.3], using his Banach space X_2 which we initially encountered in Chapter 2. Bounded operators on X_2 have a unique decomposition as $\lambda I_{X_2}+\alpha K+\beta S$ for some $\lambda, \alpha, \beta \in \mathbb{R}$, $K \in \mathscr{K}(X_2)$, and S a strictly singular, non-compact operator such that $S^2 = 0$ [104, Theorem 3.1.4]. Because of the uniqueness of the decomposition the operator S is not a commutator, and it cannot be of the form $\lambda I_{X_2} + T$ where $\lambda \neq 0$ and T is contained in a proper closed ideal. Therefore X_2 is not a Wintner space.

We now show that $E_{\mathcal{R}}$ is also not a Wintner space, using essentially the same method as Tarbard. So even though this is not the first example, it is still an instructive one. Tarbard's space is very different to Read's as it uses the impressive machinery of the Argyros–Haydon construction; hence the class of non-Wintner spaces is quite varied.

We shall need the fact that, by (5.4.10), there is a closed subalgebra $D = \operatorname{im}\rho_0$ of $\mathscr{B}(E_{\mathcal{R}})$ with the trivial product such that $\mathscr{B}(E_{\mathcal{R}}) = \mathscr{W}(E_{\mathcal{R}}) \oplus D \oplus \mathbb{K}I_{E_{\mathcal{R}}}$.

Proposition 5.5.5. *Every commutator on $E_{\mathcal{R}}$ is weakly compact. Moreover, every non-zero element of D is not a commutator, and is not of the form $\lambda I_{E_{\mathcal{R}}}+T$, where $\lambda \in \mathbb{K}\backslash\{0\}$ and T is contained in a proper closed ideal.*

Proof. Suppose that S is a commutator on $E_{\mathcal{R}}$. Then there are $T_1, T_2 \in \mathscr{B}(E_{\mathcal{R}})$ such that $S = T_1T_2 - T_2T_1$. By the above remark we can uniquely write $T_1 = \lambda_1 I_{E_{\mathcal{R}}} + d_1 + W_1$ and $T_2 = \lambda_2 I_{E_{\mathcal{R}}} + d_2 + W_2$ for some $\lambda_1, \lambda_2 \in \mathbb{K}$, $d_1, d_2 \in D$ and $W_1, W_2 \in \mathscr{W}(E_{\mathcal{R}})$. Then

$$\begin{aligned}
S &= T_1T_2 - T_2T_1\\
&= (\lambda_1 I_{E_{\mathcal{R}}} + d_1 + W_1)(\lambda_2 I_{E_{\mathcal{R}}} + d_2 + W_2) - (\lambda_2 I_{E_{\mathcal{R}}} + d_2 + W_2)(\lambda_1 I_{E_{\mathcal{R}}} + d_1 + W_1)\\
&= (\lambda_1\lambda_2 I_{E_{\mathcal{R}}} + \lambda_1 d_2 + \lambda_1 W_2 + \lambda_2 d_1 + d_1W_2 + \lambda_2 W_1 + W_1 d_2 + W_1W_2)\\
&- (\lambda_1\lambda_2 I_{E_{\mathcal{R}}} + \lambda_1 d_2 + \lambda_1 W_2 + \lambda_2 d_1 + W_2 d_1 + \lambda_2 W_1 + d_2 W_1 + W_2W_1)\\
&= d_1W_2 + W_1d_2 + W_1W_2 - W_2d_1 - d_2W_1 - W_2W_1 \in \mathscr{W}(E_{\mathcal{R}}).
\end{aligned}$$

So every commutator on $E_{\mathcal{R}}$ is weakly compact. Choose a non-zero $d \in D$. Then d is not a commutator, since it is not weakly compact. Suppose that $d = \lambda I_{E_{\mathcal{R}}} + T$ for some scalar $\lambda \neq 0$, and T contained in a proper closed ideal M. Then $T = d - \lambda I_{E_{\mathcal{R}}}$. But then, using the fact that $d^2 = 0$ we have $(\lambda I_{E_{\mathcal{R}}} + d)T = (\lambda I_{E_{\mathcal{R}}} + d)(d - \lambda I_{E_{\mathcal{R}}}) = -\lambda^2 I_{E_{\mathcal{R}}} \in M$. It follows that $I_{E_{\mathcal{R}}} \in M$, a contradiction since M is a proper ideal. □

Corollary 5.5.6. *Read's space $E_{\mathcal{R}}$ is not a Wintner space.*

It would be interesting to know whether every weakly compact operator on $E_{\mathcal{R}}$ is a commutator.

We next make a further link to Chapter 2: the Argyros–Motakis space is not a Wintner space either.

Proposition 5.5.7. *Let X_{AM} be the Argyros–Motakis space. Then every element of $\mathscr{S}(X_{\mathrm{AM}}) \backslash \mathscr{K}(X_{\mathrm{AM}})$ is not a commutator, and is not of the form $\lambda I_{X_{\mathrm{AM}}} + T$, where $\lambda \in \mathbb{K} \backslash \{0\}$ and T is contained in a proper closed ideal.*

Proof. This is similar to the previous proposition. Let R be a commutator on X_{AM} and take $T_1, T_2 \in \mathscr{B}(X_{\mathrm{AM}})$ such that $R = T_1T_2 - T_2T_1$. By Theorem 2.2.2(ii) there are S_1, S_2 strictly singular and $\lambda_1, \lambda_2 \in \mathbb{K}$ satisfying $T_1 = \lambda_1 I_{X_{\mathrm{AM}}} + S_1$ and $T_2 = \lambda_2 I_{X_{\mathrm{AM}}} + S_2$. Therefore by Theorem 2.2.2(iii)

$$\begin{aligned} R &= T_1T_2 - T_2T_1 \\ &= (\lambda_1 I_{X_{\mathrm{AM}}} + S_1)(\lambda_2 I_{X_{\mathrm{AM}}} + S_2) - (\lambda_2 I_{X_{\mathrm{AM}}} + S_2)(\lambda_1 I_{X_{\mathrm{AM}}} + S_1) \\ &= S_1S_2 - S_2S_1 \in \mathscr{K}(X_{\mathrm{AM}}). \end{aligned}$$

So every $S \in \mathscr{S}(X_{\mathrm{AM}}) \backslash \mathscr{K}(X_{\mathrm{AM}})$ is not a commutator.

Suppose that $S = \lambda I_{X_{\mathrm{AM}}} + T$, where $\lambda \in \mathbb{K} \backslash \{0\}$ and T is contained in a non-zero proper closed ideal M. Then

$$T(\lambda I_{X_{\mathrm{AM}}} + S) = (S - \lambda I_{X_{\mathrm{AM}}})(\lambda I_{X_{\mathrm{AM}}} + S) = -\lambda^2 I_{X_{\mathrm{AM}}} + S^2 \in M.$$

Now X_{AM} has a basis by Theorem 2.2.2(i), and so $\mathscr{A}(X_{\mathrm{AM}}) = \mathscr{K}(X_{\mathrm{AM}})$. Therefore $\mathscr{K}(X_{\mathrm{AM}}) \subseteq M$ because of the general fact that every non-zero closed ideal contains the approximable operators [24, Theorem 2.5.8(ii)]. Hence $I_{X_{\mathrm{AM}}} \in M$ because S^2 is compact, but this contradicts the fact that M is proper. □

Corollary 5.5.8. *The Argyros–Motakis space X_{AM} is not a Wintner space.*

Proof. By the proof of Theorem 2.2.3 the quotient space $\mathscr{S}(X_{\mathrm{AM}})/\mathscr{K}(X_{\mathrm{AM}})$ is infinite-dimensional, so there are lots of strictly singular non-compact operators on X_{AM}. Hence the result follows from Proposition 5.5.7. □

Proposition 5.5.7 shows that any finite sum of commutators on X_{AM} is compact. It would be interesting to know whether, conversely, every compact operator can be written as a finite sum of commutators.

Chapter 6

Weakly Inessential Operators

It follows from Theorem 1.3.4 that $\mathscr{B}(E_{\mathcal{R}})/\mathscr{W}(E_{\mathcal{R}}) \cong \ell_2(\mathbb{N})^{\sim}$, where $\ell_2(\mathbb{N})$ has the trivial product, and an immediate consequence of this is that the weak Calkin algebra of $E_{\mathcal{R}}$ has a large radical, in the sense that it has codimension one and is isomorphic to $\ell_2(\mathbb{N})$. The preimage of this large radical is the codimension one ideal I of $\mathscr{B}(E_{\mathcal{R}})$, which we have seen is central to the surprising properties of $\mathscr{B}(E_{\mathcal{R}})$. In this chapter we deduce that I actually coincides with an operator ideal—the so-called *weakly inessential operators* on $E_{\mathcal{R}}$, whose definition is explained below. It is natural to examine how this class of operators contributes to the properties of $\mathscr{B}(E_{\mathcal{R}})$, and to consider the form it takes for other Banach spaces.

Therefore in this chapter we study the preimage of the radical of the weak Calkin algebra for a general Banach space X. The preimage is a closed ideal of $\mathscr{B}(X)$, and we call an operator *weakly inessential* if it is an element of this ideal. It is clear that weakly compact operators are weakly inessential, so we can rephrase the fact that $\mathscr{B}(E_{\mathcal{R}})/\mathscr{W}(E_{\mathcal{R}})$ has a large radical by saying that there are many operators which are weakly inessential but not weakly compact. Our intuition is that this is an odd phenomenon, and that on classical Banach spaces most weakly inessential operators should be weakly compact.

This intuition partly comes from the related class of inessential operators on Banach spaces. As we mentioned briefly in Chapter 2, Kleinecke [65] (following Yood) introduced the notion of an inessential operator as an element in the preimage of the radical of the Calkin algebra. He showed that such operators have much in common with compact ones; indeed for many Banach spaces, every inessential operator is compact. Of course this is not true in general, otherwise Chapter 2 would be rather redundant! The Argyros–Motakis space X_{AM} is an example of the extreme opposite situation: $\mathscr{K}(X_{\mathrm{AM}})$ is a separable subset of $\mathscr{B}(X_{\mathrm{AM}})$, whereas $\mathscr{E}(X_{\mathrm{AM}})$ is non-separable (since strictly singular operators are inessential).

The connections between inessential operators and Fredholm theory have made them an enduring object of study. In contrast, weakly inessential operators have been virtually ignored in the literature, in part due to a lack of good examples. Our aim is to demonstrate that this is an important and interesting class of operators, and that there are many examples if one is prepared to look carefully. We show that on most classical Banach spaces every weakly inessential operator is weakly compact, but that this is not the case if we pass to direct sums of classical spaces. A particular example is the direct sum of James' spaces; this hints at the case of Read's space, since it is an infinite direct sum of James-like spaces.

We begin at a further level of abstraction. Consider Banach spaces X and Y. There are many natural ideals of $\mathscr{B}(X)$, for example the finite rank, compact and weakly compact operators. Now $\mathscr{B}(X,Y)$ is a Banach space, and $\mathscr{F}(X,Y)$, the finite rank operators from X to Y, forms a linear subspace (as do the other types). But by looking at operators between different Banach spaces, we have apparently lost the information coming from the ideal structure of $\mathscr{F}(X)$ because there is no multiplication. The concept of an *operator ideal*, due to Pietsch, partly remedies this. It shows that many of the ideals of $\mathscr{B}(X)$ remain 'ideals' of $\mathscr{B}(X,Y)$ in the sense of being closed under composition from the left and right by arbitrary operators (as we shall make precise, below). Following Kleinecke's initial work, Pietsch showed that the ideal of inessential operators on a single Banach space could be turned into an operator ideal between two spaces (by which we mean that we return to Kleinecke's definition when the two spaces are the same). He later demonstrated a general procedure for forming the radical of an operator ideal, and proved that forming the radical of the compact operators yields the inessentials.

We study the radical of the operator ideal of weakly compact operators, termed the weakly inessential operators, and denoted by $\mathscr{W}\mathscr{E}(X,Y)$. This makes sense because it agrees with our earlier definition when $X=Y$. Although the radical is defined for an arbitrary operator ideal, $\mathscr{W}\mathscr{E}(X,Y)$ has not been studied in detail. We shall show that $\mathscr{W}\mathscr{E}(X,Y)$ is distinct from other common operator ideals, and shall place it within the standard hierarchy. For the convenience of the reader, and to inform our later proofs, we also prove some of Pietsch's theorems. In addition, we give a description of the weakly inessential operators on various pairs of Banach spaces X and Y, particularly when $X=Y$.

In the second section we restrict to operators on a single Banach space. Kleinecke's main theorem in [65] relates the set of Riesz operators on a Banach space X and the ideal of inessential operators $\mathscr{E}(X)$. We follow his reasoning to prove an analogous version for the ideal of weakly inessential operators $\mathscr{W}\mathscr{E}(X)$ and the (appropriately defined) set of weakly Riesz operators. This Kleinecke-type theorem then yields a different proof that $\mathscr{W}\mathscr{E}(E_{\mathcal{R}})=I$, using only Read's work (*i.e.* not using Theorem 1.3.4).

The third section contains a brief introduction to the Dunford–Pettis property for Banach spaces and its links to $\mathscr{W}\mathscr{E}(X, Y)$. We introduce a property weaker than the Dunford–Pettis property and then conclude the chapter by examining it for certain spaces.

6.1 The operator ideal of weakly inessential operators

Definition 6.1.1. An *operator ideal* is an assignment $\mathscr{I}$ which associates to each pair of Banach spaces (X, Y) a linear subspace $\mathscr{I}(X, Y)$ of $\mathscr{B}(X, Y)$ satisfying:

(i) $\mathscr{I}(X_0, Y_0)$ is non-zero for some Banach spaces X_0 and Y_0;

(ii) for any Banach spaces W, X, Y, Z, and bounded operators $R \in \mathscr{B}(W, X)$, $S \in \mathscr{I}(X, Y)$, and $T \in \mathscr{B}(Y, Z)$, we have $TSR \in \mathscr{I}(W, Z)$.

We write $\mathscr{I}(X)$ instead of $\mathscr{I}(X, X)$; note that this forms an ideal of $\mathscr{B}(X)$. Let $\mathscr{I}$ be an operator ideal and X, Y be Banach spaces. Write $\overline{\mathscr{I}(X, Y)}$ for the closure in the operator norm of $\mathscr{I}(X, Y)$ in $\mathscr{B}(X, Y)$. Then $\overline{\mathscr{I}}$ is also an operator ideal, called the *closure* of $\mathscr{I}$; we say that $\mathscr{I}$ is *closed* if $\mathscr{I} = \overline{\mathscr{I}}$. An operator ideal $\mathscr{I}_1$ is contained in an operator ideal $\mathscr{I}_2$, written $\mathscr{I}_1 \subseteq \mathscr{I}_2$, if $\mathscr{I}_1(X, Y) \subseteq \mathscr{I}_2(X, Y)$ for each pair of Banach spaces (X, Y).

Definition 6.1.2. Let X and Y be Banach spaces and let $T \in \mathscr{B}(X, Y)$. Then T is:

(i) *strictly singular* if for each infinite-dimensional closed subspace M of X, $T|_M^{T(M)} : M \to T(M)$ is not an isomorphism;

(ii) *completely continuous* if (Tx_n) converges in norm in Y whenever (x_n) converges weakly in X;

(iii) *inessential* if for every $S \in \mathscr{B}(Y, X)$, $I_X + ST$ is a Fredholm operator. Recall that $R \in \mathscr{B}(X, Y)$ is *Fredholm* if $\dim \ker R < \infty$ and $\operatorname{codim} \operatorname{im} R < \infty$.

Let $\mathscr{F}(X, Y), \mathscr{A}(X, Y), \mathscr{K}(X, Y), \mathscr{W}(X, Y), \mathscr{S}(X, Y), \mathscr{V}(X, Y)$ and $\mathscr{E}(X, Y)$ denote the sets of finite rank, approximable, compact, weakly compact, strictly singular, completely continuous and inessential operators, respectively. Then the assignment $\mathscr{F}$ is an operator ideal, and the respective assignments $\mathscr{A}, \mathscr{K}, \mathscr{W}, \mathscr{S}, \mathscr{V}$ and $\mathscr{E}$ are closed operator ideals.

Note that when $X = Y$ our definitions of strictly singular and inessential operators are equivalent to the ones in Chapter 2 (the proof for inessential operators requires Atkinson's Theorem 6.1.6).

Definition 6.1.3. Let $\mathscr{I}$ be an operator ideal. A bounded operator $T \in \mathscr{B}(X,Y)$ between Banach spaces X and Y belongs to the *radical* $\mathscr{I}^{\mathrm{rad}}$ if for every $S \in \mathscr{B}(Y,X)$ there exist $U \in \mathscr{B}(X)$ and $V \in \mathscr{I}(X)$ such that $U(I_X + ST) = I_X + V$.

Theorem 6.1.4 (Pietsch)**.** *For each operator ideal $\mathscr{I}$, the radical $\mathscr{I}^{\mathrm{rad}}$ is a closed operator ideal and $\mathscr{I} \subseteq \mathscr{I}^{\mathrm{rad}}$.*

Proof. [85, 4.3.2 & 4.3.4]. We check the axioms. Let X, Y be Banach spaces, let $\lambda \in \mathbb{K}$ and let $T_1, T_2 \in \mathscr{I}^{\mathrm{rad}}(X,Y)$. Take $S \in \mathscr{B}(Y,X)$. Then there exist $U_1, U_2 \in \mathscr{B}(X)$ and $V_1, V_2 \in \mathscr{I}(X)$ such that $U_1(I_X + ST_1) = I_X + V_1$ and $U_2(I_X + \lambda U_1 S T_2) = I_X + V_2$. Thus

$$U_2 U_1 (I_X + S(T_1 + \lambda T_2)) = U_2(I_X + V_1 + \lambda U_1 S T_2) = I_X + V_2 + U_2 V_1.$$

Since $V_2 + U_2 V_1 \in \mathscr{I}(X)$ we see that $T_1 + \lambda T_2 \in \mathscr{I}^{\mathrm{rad}}(X,Y)$. Therefore $\mathscr{I}^{\mathrm{rad}}(X,Y)$ is a linear subspace of $\mathscr{B}(X,Y)$.

(i) The assignment is non-zero because $\mathscr{I} \subseteq \mathscr{I}^{\mathrm{rad}}$, to be proved below.

(ii) Choose Banach spaces W, X, Y, Z, and operators $R \in \mathscr{B}(W,X)$, $S \in \mathscr{I}^{\mathrm{rad}}(X,Y)$, and $T \in \mathscr{B}(Y,Z)$. Given $L \in \mathscr{B}(Z,W)$ there are $U_0 \in \mathscr{B}(X)$ and $V \in \mathscr{I}(X)$ such that $U_0(I_X + RLTS) = I_X + V$. Define $U = I_W - LTSU_0R \in \mathscr{B}(W)$. Then

$$\begin{aligned} U(I_W + LTSR) &= I_W + LTSR - LTSU_0(I_X + RLTS)R \\ &= I_W + LTSR - LTS(I_X + V)R = I_W + (-LTSVR). \end{aligned}$$

Since $-LTSVR \in \mathscr{I}(W)$ we conclude that $TSR \in \mathscr{I}(W,Z)^{\mathrm{rad}}$. Therefore $\mathscr{I}^{\mathrm{rad}}$ is an operator ideal.

The next task is to show $\mathscr{I}^{\mathrm{rad}}(X,Y)$ is closed in the operator norm; we shall prove that $\overline{\mathscr{I}^{\mathrm{rad}}(X,Y)} \subseteq \mathscr{I}^{\mathrm{rad}}(X,Y)$. To this end choose $T \in \overline{\mathscr{I}^{\mathrm{rad}}(X,Y)}$ and $S \in \mathscr{B}(Y,X)$, and we may suppose that $S \neq 0$. By the definition of the closure there exists $T_0 \in \mathscr{I}^{\mathrm{rad}}(X,Y)$ such that $||T - T_0|| < \frac{1}{||S||}$. Therefore

$$||S(T - T_0)|| \leqslant ||S||\,||T - T_0|| < 1$$

and so $I_X + S(T - T_0)$ is invertible in $\mathscr{B}(X)$ by the Neumann criterion for invertibility [78, Corollary 3.3.15].

Now $[I_X + S(T - T_0)]^{-1} S T_0 \in \mathscr{I}^{\mathrm{rad}}(X)$ and so there exist $U_0 \in \mathscr{B}(X)$ and $V_0 \in \mathscr{I}(X)$ such that $U_0(I_X + [I_X + S(T - T_0)]^{-1} S T_0) = I_X + V_0$. Set

$$U := U_0[I_X + S(T - T_0)]^{-1}.$$

Then we have

$$\begin{aligned} U(I_X + ST) &= U_0[I_X + S(T - T_0)]^{-1}[I_X + S(T - T_0) + ST_0] \\ &= U_0(I_X + [I_X + S(T - T_0)]^{-1}ST_0) = I_X + V_0. \end{aligned}$$

Thus $T \in \mathscr{I}^{\mathrm{rad}}(X, Y)$ and so $\mathscr{I}^{\mathrm{rad}}(X, Y)$ is closed.

Finally, to show that $\mathscr{I} \subseteq \mathscr{I}^{\mathrm{rad}}$, we need only observe that for $T \in \mathscr{I}(X, Y)$ and $S \in \mathscr{B}(Y, X)$, $ST \in \mathscr{I}(X)$ since $\mathscr{I}$ is an operator ideal, and so we can take $U = I_X$ and $V = ST$ in the definition. □

It it useful to observe that the definition of the radical is symmetric [85, 4.3.8].

Lemma 6.1.5. *Let $\mathscr{I}$ be an operator ideal. Let X and Y be Banach spaces and let $T \in \mathscr{I}^{\mathrm{rad}}(X, Y)$. Then for every $S \in \mathscr{B}(Y, X)$ there exist $U \in \mathscr{B}(X)$ and $V, V' \in \mathscr{I}(X)$ such that $U(I_X + ST) = I_X + V$ and $(I_X + ST)U = I_X + V'$.*

Proof. Take $T \in \mathscr{I}^{\mathrm{rad}}(X, Y)$. Then for every $S \in \mathscr{B}(Y, X)$ there exist $U \in \mathscr{B}(X)$ and $V \in \mathscr{I}(X)$ such that $U(I_X + ST) = I_X + V$ by definition, so it remains to demonstrate the existence of V' with the claimed properties.

We have $L := I_X - U = UST - V \in \mathscr{I}^{\mathrm{rad}}(X)$ because $\mathscr{I}^{\mathrm{rad}}$ is an operator ideal containing $\mathscr{I}$ by Theorem 6.1.4. So there exist $V_0 \in \mathscr{I}(X)$ and $U_0 \in \mathscr{B}(X)$ such that $U_0(I_X - L) = I_X + V_0$. Thus $U_0 U = I_X + V_0$. Now if we set

$$V' := V_0(I_X - U - STU) + U_0 V U \in \mathscr{I}(X)$$

we obtain

$$\begin{aligned} I_X + V' &= I_X + V_0(I_X - U - STU) + U_0 V U = I_X + V_0 - V_0 U - V_0 STU + U_0 V U \\ &= U_0 U - V_0 U - V_0 STU + U_0 V U = (U_0 - V_0 - V_0 ST + U_0 V)U \\ &= (U_0(I_X + V) - V_0(I_X + ST))U = (U_0 U(I_X + ST) - V_0(I_X + ST))U \\ &= (U_0 U - V_0)(I_X + ST)U = (I_X + ST)U. \end{aligned}$$

The result follows. □

Theorem 6.1.6 (Atkinson's Theorem (classical version))**.** *Let X and Y be Banach spaces and $T \in \mathscr{B}(X, Y)$. Then T is Fredholm if and only if there exist operators $A, B \in \mathscr{B}(Y, X)$, $Q \in \mathscr{F}(X)$ and $R \in \mathscr{F}(Y)$ such that $AT = I_X + Q$ and $TB = I_Y + R$.*

Proof. A good reference is [85, Theorem 26.3.2] (note that Pietsch calls Fredholm operators Φ-isomorphisms). □

Proposition 6.1.7 (Pietsch)**.** *$\mathscr{K}^{\mathrm{rad}} = \mathscr{E}$.*

Proof. Let X and Y be Banach spaces. We must show that $\mathscr{K}^{\text{rad}}(X,Y) = \mathscr{E}(X,Y)$; the strategy is to make good use of Atkinson's Theorem. Indeed, the inclusion $\mathscr{K}^{\text{rad}}(X,Y) \supseteq \mathscr{E}(X,Y)$ is clear from Atkinson's Theorem, since every finite rank operator is compact.

For the reverse inclusion let $T \in \mathscr{K}^{\text{rad}}(X,Y)$. Then for every $S \in \mathscr{B}(Y,X)$ there exist $U \in \mathscr{B}(X)$ and $K \in \mathscr{K}(X)$ such that $U(I_X + ST) = I_X + K$. Next we require Riesz' classical result, which says that for any $K \in \mathscr{K}(X)$, $I_X + K$ is Fredholm [85, Theorem 26.3.3]. Hence $U(I_X + ST)$ is Fredholm and so Atkinson's Theorem yields $A \in \mathscr{B}(X)$ such that $AU(I_X + ST) = I_X + Q$ for some $Q \in \mathscr{F}(X)$.

By Lemma 6.1.5 there is $K' \in \mathscr{K}(X)$ such that $(I_X + ST)U = I_X + K'$. A second appeal to Atkinson's Theorem gives $B \in \mathscr{B}(X)$ such that $(I_X + ST)UB = I_X + R$ for some $R \in \mathscr{F}(X)$, and so, appealing to Atkinson's Theorem once again we see that $I_X + ST$ is Fredholm. Thus $T \in \mathscr{E}(X,Y)$. □

Definition 6.1.8. Let X and Y be Banach spaces, and $T \in \mathscr{B}(X,Y)$. Then T is *weakly inessential* if for every $S \in \mathscr{B}(Y,X)$ there exist $U \in \mathscr{B}(X)$ and $W \in \mathscr{W}(X)$ such that $U(I_X + ST) = I_X + W$. The set of all weakly inessential operators from X to Y is denoted by $\mathscr{W\!E}(X,Y)$.

Proposition 6.1.9. *$\mathscr{W\!E} = \mathscr{W}^{\text{rad}}$ and so $\mathscr{W\!E}$ is a closed operator ideal containing $\mathscr{W}$.*

Proof. By the definition of the radical of an operator ideal $\mathscr{W\!E} = \mathscr{W}^{\text{rad}}$, and so we apply Theorem 6.1.4. □

We see quickly that $\mathscr{W\!E}$ often coincides with $\mathscr{W}$. Recall that a Banach space X is reflexive if and only if $\mathscr{W}(X) = \mathscr{B}(X)$ if and only if $I_X \in \mathscr{W}(X)$ [78, Propositions 3.5.4 and 3.5.6].

Proposition 6.1.10. *If X or Y is reflexive then $\mathscr{W\!E}(X,Y) = \mathscr{W}(X,Y)$.*

Proof. This follows from the fact that if X or Y is reflexive then $\mathscr{W}(X,Y) = \mathscr{B}(X,Y)$, and that $\mathscr{W}(X,Y) \subseteq \mathscr{W\!E}(X,Y)$ by Proposition 6.1.9. □

Proposition 6.1.11. *Let X be a Banach space. Then $\mathscr{W\!E}(X) = \mathscr{B}(X)$ if and only if X is reflexive.*

Proof. If X is reflexive then $\mathscr{B}(X) = \mathscr{W}(X) = \mathscr{W\!E}(X)$ by Proposition 6.1.10. Conversely, suppose that $\mathscr{W\!E}(X) = \mathscr{B}(X)$. Then $I_X \in \mathscr{W\!E}(X)$ so there exist $U \in \mathscr{B}(X)$ and $W \in \mathscr{W}(X)$ such that $U(I_X + (-I_X)I_X) = I_X + W$. Therefore $I_X \in \mathscr{W}(X)$ and hence X is reflexive. □

But there are cases when $\mathscr{W\!E}$ and $\mathscr{W}$ are distinct.

Theorem 6.1.12.

(i) $\mathscr{E}$ *is a proper subclass of* $\mathscr{W}\mathscr{E}$*;*

(ii) $\mathscr{W}$ *is a proper subclass of* $\mathscr{W}\mathscr{E}$*;*

(iii) $\mathscr{V}$ *is incomparable with* $\mathscr{W}\mathscr{E}$*.*

Proof. Let X, Y be Banach spaces.

(i) Let $T \in \mathscr{E}(X,Y)$, and take $S \in \mathscr{B}(Y,X)$. By Proposition 6.1.7, $\mathscr{K}^{\mathrm{rad}} = \mathscr{E}$ and so there are $U \in \mathscr{B}(X)$ and $K \in \mathscr{K}(X)$ such that $U(I_X + ST) = I_X + K$. Since every compact operator is weakly compact it follows that $T \in \mathscr{W}\mathscr{E}(X,Y)$. To see that $\mathscr{E}$ is a *proper* subclass of $\mathscr{W}\mathscr{E}$, consider the Hilbert space ℓ_2. Proposition 6.1.10 implies that $\mathscr{W}\mathscr{E}(\ell_2) = \mathscr{B}(\ell_2)$ since ℓ_2 is reflexive, and a standard fact about inessential operators says that $\mathscr{E}(X) = \mathscr{B}(X)$ if and only if X is finite-dimensional. So $\mathscr{E}(\ell_2) \subsetneq \mathscr{W}\mathscr{E}(\ell_2) = \mathscr{B}(\ell_2)$.

(ii) Proposition 6.1.9 ensures that $\mathscr{W}$ is a subclass of $\mathscr{W}\mathscr{E}$. To see that it is proper let $T : \ell_1 \to c_0$ be a bounded surjection (which exists by the Banach–Mazur Theorem [3, Theorem 2.3.1]). This is inessential because every bounded operator from ℓ_1 to c_0 is [43, Theorem 1], and thus weakly inessential by (i). But it is not weakly compact because its range is closed but not reflexive [78, Proposition 3.5.6]. Therefore $\mathscr{W}(\ell_1, c_0) \subsetneq \mathscr{W}\mathscr{E}(\ell_1, c_0)$.

(iii) We have $\mathscr{W}\mathscr{E}(\ell_1) \subsetneq \mathscr{V}(\ell_1) = \mathscr{B}(\ell_1)$ since ℓ_1 has the Schur property [3, Theorem 2.3.6] and is not reflexive (using Proposition 6.1.11). On the other hand, by (i) we know that $\mathscr{W}\mathscr{E}(\ell_2) = \mathscr{B}(\ell_2)$, but $\mathscr{V}(\ell_2) = \mathscr{K}(\ell_2) \subsetneq \mathscr{B}(\ell_2)$ by [78, Theorem 3.4.37] which shows that completely continuous operators are compact on reflexive spaces. So neither $\mathscr{V} \subseteq \mathscr{W}\mathscr{E}$ nor $\mathscr{W}\mathscr{E} \subseteq \mathscr{V}$ hold in general. □

Corollary 6.1.13. $\mathscr{F}$*,* $\mathscr{A}$*,* $\mathscr{K}$ *and* $\mathscr{S}$ *are proper subclasses of* $\mathscr{W}\mathscr{E}$*.*

Proof. It is standard that these operator ideals are contained in $\mathscr{E}$ [85, Theorem 26.7.3], and so the result follows by Theorem 6.1.12(i). □

We can now place the weakly inessential operators in the standard hierarchy of operator ideals. The picture looks like this (where the arrows point from the smaller operator ideal to the larger one):

$$
\begin{tikzcd}
 & \mathscr{B}(X,Y) & \\
 & \mathscr{W}\mathscr{E}(X,Y) \arrow[u] & \\
\mathscr{V}(X,Y) \arrow[uur] & \mathscr{E}(X,Y) \arrow[u] & \\
 & \mathscr{S}(X,Y) \arrow[u] & \mathscr{W}(X,Y) \arrow[uul] \\
 & \mathscr{K}(X,Y) \arrow[uul] \arrow[u] \arrow[ur] & \\
 & \mathscr{A}(X,Y) \arrow[u] & \\
 & \mathscr{F}(X,Y) \arrow[u] & \\
 & \{0\} \arrow[u] &
\end{tikzcd}
$$

Definition 6.1.14. Let X, Y and X_0 be Banach spaces. An operator ideal $\mathscr{I}$ is *injective* if for every closed subspace Y_0 of Y and $T \in \mathscr{B}(X, Y_0)$ with $\iota T \in \mathscr{I}(X,Y)$, we have $T \in \mathscr{I}(X,Y)$ (where $\iota : Y_0 \to Y$ is the canonical embedding), and *surjective* if for any surjection $Q \in \mathscr{B}(X, X_0)$ and each $T \in \mathscr{B}(X_0, Y)$, we have $T \in \mathscr{I}(X_0, Y)$ whenever $TQ \in \mathscr{I}(X,Y)$.

The operator ideals $\mathscr{F}, \mathscr{K}, \mathscr{W}$, and $\mathscr{V}$ are injective but $\mathscr{A}$ is not, while $\mathscr{F}, \mathscr{K}$, and $\mathscr{W}$ are surjective, but $\mathscr{A}$ and $\mathscr{V}$ are not.

Proposition 6.1.15. *The operator ideal $\mathscr{W}\mathscr{E}$ is neither injective nor surjective.*

Proof. The inclusion map $\iota : c_0 \to \ell_\infty$ is inessential by a result of Pełczyński [84] (noting that strictly cosingular operators are inessential [85, Theorem 26.7.3]). Thus ι is weakly inessential by Theorem 6.1.12(i). Define T to be the identity operator on c_0. Then ιT is weakly inessential since $\mathscr{W}\mathscr{E}$ is an operator ideal, but T is not weakly inessential because $\mathscr{W}\mathscr{E}(c_0) \neq \mathscr{B}(c_0)$ by Proposition 6.1.11. Thus $\mathscr{W}\mathscr{E}$ is not injective.

As in Theorem 6.1.12(ii), the surjection $Q : \ell_1 \to c_0$ is weakly inessential. Again, let T be the identity operator on c_0. Then $TQ \in \mathscr{W}\mathscr{E}(\ell_1, c_0)$ but T is not weakly inessential. So $\mathscr{W}\mathscr{E}$ is not surjective. $\square$

So $\mathscr{W}$ is injective and surjective but $\mathscr{W}\mathscr{E}$ is neither; hence these operator ideals are fundamentally different. Note that in the above proposition we have also shown that $\mathscr{E}$ is neither injective nor surjective, which is well known.

We now focus on the case of a single non-zero Banach space. Kleinecke's original definition of $\mathscr{E}(X)$ was in terms of the radical of the Calkin algebra. The next proposition demonstrates the analogous result for $\mathscr{W\!E}(X)$; mainly this is showing that Pietsch's definition of the radical of an operator ideal is the right generalisation of the inessential operators.

Proposition 6.1.16. *Let X be a Banach space. Then*

$$\mathscr{W\!E}(X) = \{T \in \mathscr{B}(X) : T + \mathscr{W}(X) \in \operatorname{rad} \mathscr{B}(X)/\mathscr{W}(X)\}.$$

Proof. If X is reflexive then $\operatorname{rad} \mathscr{B}(X)/\mathscr{W}(X) = \operatorname{rad}\{0\} = \{0\}$ so the result is true by Proposition 6.1.11. Thus we may suppose that X is not reflexive, and hence $\mathscr{B}(X)/\mathscr{W}(X)$ is unital.

Let $T \in \mathscr{W\!E}(X)$. Then for each $S \in \mathscr{B}(X)$ we must show that $I_X + ST + \mathscr{W}(X) \in \operatorname{inv} \mathscr{B}(X)/\mathscr{W}(X)$ by the characterisation of the radical for unital algebras. But this is clear from the definition and Lemma 6.1.5. Therefore T is contained in the right hand side.

Conversely, suppose that $T \in \mathscr{B}(X)$ such that $T + \mathscr{W}(X) \in \operatorname{rad} \mathscr{B}(X)/\mathscr{W}(X)$. Then for each $S \in \mathscr{B}(X)$ there is $U + \mathscr{W}(X)$ which is the inverse of $I_X + ST + \mathscr{W}(X)$ in the weak Calkin algebra. It follows that $U(I_X + ST) = I_X + W$ for some $W \in \mathscr{W}(X)$, and so $T \in \mathscr{W\!E}(X)$. □

As mentioned in the introduction, our intuition is that for many examples the ideal of weakly inessential operators should coincide with the weakly compacts. This is true for finite-dimensional spaces, and, more generally, for reflexive spaces as Proposition 6.1.11 shows. We can extend this result a little to the class of quasi-reflexive Banach spaces.

Proposition 6.1.17. *Let X be a quasi-reflexive Banach space. Then $\mathscr{W}(X) = \mathscr{W\!E}(X)$.*

Proof. Suppose that $\dim(X^{**}/X) = 1$. By a result of Edelstein and Mityagin [34], $\mathscr{B}(X) = \mathscr{W}(X) \oplus \mathbb{K}I_X$, and so $\operatorname{rad} \mathscr{B}(X)/\mathscr{W}(X) \cong \operatorname{rad}(\mathbb{K}) = \{0\}$. Therefore $\mathscr{W}(X) = \mathscr{W\!E}(X)$ by Proposition 6.1.16. □

Example 6.1.18. We shall now present some further examples of Banach spaces X such that $\mathscr{W}(X) = \mathscr{W\!E}(X)$.

(i) $X = \ell_1$. Since ℓ_1 is not reflexive, $\mathscr{W\!E}(\ell_1)$ is a proper closed ideal of $\mathscr{B}(\ell_1)$ by Proposition 6.1.11. A fundamental result of Feldman, Gohberg and Markus [42] says that $\mathscr{K}(\ell_1)$ is the only non-trivial proper closed ideal of $\mathscr{B}(\ell_1)$. Thus $\mathscr{K}(\ell_1) = \mathscr{W}(\ell_1) = \mathscr{W\!E}(\ell_1)$.

(ii) $X = c_0$. Feldman, Gohberg and Markus' result also applies to c_0, and since c_0 is not reflexive, $\mathscr{K}(c_0) = \mathscr{W}(c_0) = \mathscr{WE}(c_0)$.

(iii) $X = \ell_\infty$. Again this is not reflexive so $\mathscr{WE}(\ell_\infty)$ is a proper closed ideal of $\mathscr{B}(\ell_\infty)$. Loy and Laustsen [70] observed that $\mathscr{W}(\ell_\infty)$ is the unique maximal ideal of $\mathscr{B}(\ell_\infty)$ and so $\mathscr{W}(\ell_\infty) = \mathscr{WE}(\ell_\infty)$.

(iv) $X = C[0, \omega_1]$, because

$$\mathscr{K}(C[0, \omega_1]) = \mathscr{W}(C[0, \omega_1]) = \mathscr{E}(C[0, \omega_1]) = \mathscr{WE}(C[0, \omega_1]),$$

as in Chapter 2.

(v) $X = X_{\mathrm{AH}}$ the Argyros-Haydon space. Since $\mathscr{B}(X_{\mathrm{AH}}) = \mathscr{K}(X_{\mathrm{AH}}) \oplus \mathbb{K} I_{X_{\mathrm{AH}}}$ and X_{AH} is not reflexive we have $\mathscr{K}(X_{\mathrm{AH}}) = \mathscr{W}(X_{\mathrm{AH}}) = \mathscr{WE}(X_{\mathrm{AH}})$.

Recall that one of our motivations for looking at weakly inessential operators was that $\mathscr{W}(E_{\mathcal{R}}) \subsetneq \mathscr{WE}(E_{\mathcal{R}})$, and this seemed to be connected to the unusual properties exhibited by $\mathscr{B}(E_{\mathcal{R}})$. Therefore we would like to find some more examples of weakly inessential operators which are not weakly compact. As we have seen, finding these is not so easy. To start, we will need to take direct sums of Banach spaces.

Let X_1 and X_2 be Banach spaces. Then $X_1 \oplus X_2$ becomes a Banach space when given the max norm $||(x_1, x_2)||_\infty = \max\{||x_1||_{X_1}, ||x_2||_{X_2}\}$. For $k = 1, 2$, write $J_k : X_k \to X_1 \oplus X_2$ for the natural inclusion and $P_k : X_1 \oplus X_2 \to X_k$ for the canonical projection. Then there is a natural bijection between the elements of $\mathscr{B}(X_1 \oplus X_2)$ and operator-valued 2×2 matrices, given by:

$$\mathscr{B}(X_1 \oplus X_2) \ni T \longleftrightarrow \begin{pmatrix} T_{11} : X_1 \to X_1 & T_{12} : X_2 \to X_1 \\ T_{21} : X_1 \to X_2 & T_{22} : X_2 \to X_2 \end{pmatrix}$$

where $T_{jk} = P_j T J_k$ for $j, k \in \{1, 2\}$. Let $\mathscr{I}$ be an operator ideal. Then

$$T \in \mathscr{I}(X_1 \oplus X_2) \iff T_{jk} \in \mathscr{I}(X_k, X_j) \qquad (j, k \in \{1, 2\}).$$

Example 6.1.19. We shall now present some examples of Banach spaces X such that $\mathscr{W}(X) \subsetneq \mathscr{WE}(X)$.

(i) Let Y be a non-reflexive, separable Banach space which does not contain a copy of ℓ_1. Take a bounded surjection $T : \ell_1 \to Y$. Then T is not weakly compact because its (closed) range is not reflexive [78, Proposition 3.5.6], but T is strictly singular because ℓ_1 is ℓ_1-saturated (that is, every infinite-dimensional closed subspace contains a copy of ℓ_1), yet Y does not contain a copy of ℓ_1. Hence T is weakly inessential by Corollary 6.1.13. So the operator on $\ell_1 \oplus Y$ corresponding to the matrix $\begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}$ is in $\mathscr{W}\mathscr{E}(\ell_1 \oplus Y)$ but not $\mathscr{W}(\ell_1 \oplus Y)$.

We may take, for example, $Y = c_0$ or $Y = J_p$ for $1 < p < \infty$ (both contain no copy of ℓ_1 because their dual spaces are separable).

(ii) $c_0 \oplus \ell_\infty$. The inclusion map $\iota : c_0 \to \ell_\infty$ is weakly inessential (see the proof of Proposition 6.1.15) and has closed range. But since its (closed) range is not reflexive, ι cannot be weakly compact. Thus the operator $\begin{pmatrix} 0 & 0 \\ \iota & 0 \end{pmatrix}$ on $c_0 \oplus \ell_\infty$ is weakly inessential but not weakly compact.

(iii) $J_p \oplus J_q$ for $1 < p < q < \infty$. The formal inclusion $j : J_p \to J_q$ is strictly singular but not weakly compact. The fact that it is not weakly compact was noted by Loy and Willis [76, p. 344], and it is shown to be strictly singular in [92, Lemma 1.1]. Therefore it is weakly inessential, and so by the same principle as before $\begin{pmatrix} 0 & 0 \\ j & 0 \end{pmatrix}$ is weakly inessential but not weakly compact.

(iv) X_k for $k \in \mathbb{N}, n \geqslant 2$, the Tarbard spaces. As noted in the list in Chapter 2, $\mathscr{K}(X_k) = \mathscr{W}(X_k)$ since $X_k^* \simeq \ell_1$, and there is a strictly singular operator on X_k which is not compact.

(v) X_{KL}, Kania and Laustsen's space from Chapter 2. This is similar to the Tarbard spaces. We have $\mathscr{K}(X_{\mathrm{KL}}) = \mathscr{W}(X_{\mathrm{KL}})$ since $X_{\mathrm{KL}}^* \simeq \ell_1$. In [64] Kania and Laustsen prove that there are inessential operators on X_{KL} which are not compact; hence the result.

(vi) $E_{\mathcal{R}}$, as observed at the start of the chapter. We shall give a different proof of this fact in Theorem 6.2.11.

It seems that a previous lack of examples where $\mathscr{W}(X) \subsetneq \mathscr{W}\mathscr{E}(X)$ is the reason that weakly inessential operators have had so little attention in the literature. Given these new examples, hopefully the reader is now convinced that this is an operator ideal worthy of a little more attention.

Read's space is distinctive within the list because it is not clear whether $\mathscr{W}\mathscr{E}(X)$ coincides with any common operator ideal (most coincide with the strictly singular operators). It would be interesting to have further examples with this property.

6.2 A version of Kleinecke's Theorem

Throughout this section we work with complex Banach spaces to ensure that our spectra are non-empty. The monograph of Caradus, Pfaffenburger and Yood [19] is a good source for more detail on the topics covered.

Definition 6.2.1. Let X be an infinite-dimensional Banach space and let $T \in \mathscr{B}(X)$. The *essential spectrum* of T is $\sigma_e(T) = \sigma(T + \mathscr{K}(X))$, and T is a *Riesz operator* if $\sigma_e(T) = \{0\}$. Denote the set of Riesz operators on X by $\mathscr{R}(X)$.

Clearly every compact operator is a Riesz operator; in fact the definition is inspired by Riesz' Theorem describing the spectrum of a compact operator [78, Theorem 3.4.27]. There is a standard characterisation of Riesz operators in terms of Fredholm operators which will be useful for us.

Lemma 6.2.2. *Let X be an infinite-dimensional Banach space. Then $T \in \mathscr{R}(X)$ if and only if $\lambda I_X - T$ is Fredholm for each $\lambda \in \mathbb{C}\backslash\{0\}$.*

Proof. ($\Rightarrow$) Let $T \in \mathscr{R}(X)$, and take $\lambda \in \mathbb{C}\backslash\{0\}$. Since $\sigma(T + \mathscr{K}(X)) = \{0\}$, there exist $S \in \mathscr{B}(X)$ and $K, K' \in \mathscr{K}(X)$ such that $S(\lambda I_X - T) = I_X + K$ and $(\lambda I_X - T)S = I_X + K'$. By Riesz' classical result, this implies that $S(\lambda I_X - T)$ and $(\lambda I_X - T)S$ are Fredholm, and so by Theorem 6.1.6 there are $A, B \in \mathscr{B}(X)$ and $Q, R \in \mathscr{F}(X)$ such that $(AS)(\lambda I_X - T) = I_X + Q$ and $(\lambda I_X - T)(SB) = I_X + R$. A second appeal to Atkinson's Theorem yields that $\lambda I_X - T$ is Fredholm.

($\Leftarrow$) Take $\lambda \in \mathbb{C}\backslash\{0\}$. By hypothesis $\lambda I_X - T$ is Fredholm, so Theorem 6.1.6 implies that there exist $A \in \mathscr{B}(X)$ and $Q \in \mathscr{F}(X)$ satisfying $A(\lambda I_X - T) = I_X + Q$. By Lemma 6.1.5 there exists $R \in \mathscr{F}(X)$ such that $(\lambda I_X - T)A = I_X + R$. Hence $\lambda I_X - T \in \operatorname{inv} \mathscr{B}(X)/\mathscr{K}(X)$. It follows that T is Riesz. ☐

We will require two important theorems involving Riesz operators.

Theorem 6.2.3 (Atkinson's Theorem (advanced version))**.** *Let X and Y be infinite-dimensional Banach spaces and take $T \in \mathscr{B}(X, Y)$. Then T is Fredholm if and only if there exist bounded operators $A, B \in \mathscr{B}(Y, X)$, $Q \in \mathscr{R}(X)$ and $R \in \mathscr{R}(Y)$ such that $AT = I_X + Q$ and $TB = I_Y + R$.*

Proof. ($\Rightarrow$) This follows from the classical version of Atkinson's Theorem since finite rank operators are Riesz.

($\Leftarrow$) Suppose that there exist bounded operators $A, B \in \mathscr{B}(Y, X)$, $Q \in \mathscr{R}(X)$ and $R \in \mathscr{R}(Y)$ such that $AT = I_X + Q$ and $TB = I_Y + R$. Then $AT - I_X$ and $TB - I_Y$ are Riesz. So by Lemma 6.2.2, AT and TB are Fredholm operators. Thus T is Fredholm by another application of Atkinson's Theorem. ☐

Theorem 6.2.4 (Kleinecke's Theorem)**.** *Let X be an infinite-dimensional Banach space, and let $\mathscr{I}$ be a non-zero ideal in $\mathscr{B}(X)$ such that $\mathscr{I}$ is contained in the set of Riesz operators. Then*

$$\mathscr{E}(X) = \{T \in \mathscr{B}(X) : T + \mathscr{I} \in \operatorname{rad}(\mathscr{B}(X)/\mathscr{I})\}.$$

Proof. Since X is infinite-dimensional, I_X is not a Riesz operator, and so $\mathscr{B}(X)/\mathscr{I}$ is a unital Banach algebra.

Denote the quotient map by $\pi : \mathscr{B}(X) \to \mathscr{B}(X)/\mathscr{I}$, and let $T \in \mathscr{E}(X)$. Then for each $S \in \mathscr{B}(X)$, $I_X + ST$ is Fredholm. By the classical version of Atkinson's Theorem there are $A \in \mathscr{B}(X)$ and $F \in \mathscr{F}(X)$ such that $A(I_X + ST) = I_X + F$. By Lemma 6.1.5 there also exists $F' \in \mathscr{F}(X)$ satisfying $(I_X + ST)A = I_X + F'$. Since $\mathscr{I}$ is a non-zero ideal in $\mathscr{B}(X)$ it contains the finite rank operators, so in particular $\pi(I_X + ST) \in \operatorname{inv} \mathscr{B}(X)/\mathscr{I}$. By the standard characterisation of the radical this implies that $T + \mathscr{I} \in \operatorname{rad}(\mathscr{B}(X)/\mathscr{I})$.

Conversely, suppose that $T \in \mathscr{B}(X)$ such that $T + \mathscr{I} \in \operatorname{rad}(\mathscr{B}(X)/\mathscr{I})$. Then for each $S \in \mathscr{B}(X)$, $\pi(I_X + ST) \in \operatorname{inv} \mathscr{B}(X)/\mathscr{I}$. It follows that there exist $A \in \mathscr{B}(X)$ and $J, J' \in \mathscr{I}$ such that $A(I_X+ST) = I_X+J$ and $(I_X+ST)A = I_X+J'$. Since $\mathscr{I} \subseteq \mathscr{R}(X)$, J and J' are Riesz operators, and so an appeal to the advanced version of Atkinson's Theorem gives the result. □

This leads to two well-known applications.

Corollary 6.2.5. *For each infinite-dimensional Banach space X, $\mathscr{I} = \mathscr{E}(X)$ is the maximum ideal in $\mathscr{B}(X)$ such that $\mathscr{I}$ is contained in $\mathscr{R}(X)$.*

Proof. The inessential operators form an operator ideal, and so in particular $\mathscr{E}(X)$ is an ideal of $\mathscr{B}(X)$. Let $T \in \mathscr{E}(X)$ and take $\lambda \neq 0$. By definition $I_X - \frac{1}{\lambda}T$ is Fredholm, and from this it is clear that $\lambda I_X - T$ is Fredholm. Lemma 6.2.2 implies that T is Riesz; hence $\mathscr{E}(X) \subseteq \mathscr{R}(X)$.

To show that the inessentials form the maximum such ideal, take an ideal $\mathscr{I}$ in $\mathscr{B}(X)$ such that $\mathscr{I}$ is contained in $\mathscr{R}(X)$. If $\mathscr{I} = \{0\}$ the result is clearly true. In the non-zero case, by applying Kleinecke's Theorem we see that

$$\mathscr{E}(X) = \{T \in \mathscr{B}(X) : T + \mathscr{I} \in \operatorname{rad}(\mathscr{B}(X)/\mathscr{I})\} \supseteq \mathscr{I}$$

and so the proof is complete. □

Corollary 6.2.6. *Let X and Y be infinite-dimensional Banach spaces, let $T : X \to Y$ be a Fredholm operator, and let $S : X \to Y$ be an inessential operator. Then $T + S$ is a Fredholm operator.*

Proof. Let $T \in \mathscr{B}(X, Y)$ be a Fredholm operator and let $S \in \mathscr{E}(X, Y)$. By the classical version of Atkinson's Theorem there exist $A, B \in \mathscr{B}(Y, X)$, $Q \in \mathscr{F}(X)$, and $R \in \mathscr{F}(Y)$ such that $AT = I_X + Q$ and $TB = I_Y + R$. Then $I_X - A(T + S) = I_X - AT - AS = -Q - AS \in \mathscr{E}(X) \subseteq \mathscr{R}(X)$ by Corollary 6.2.5. This implies that $A(T + S)$ is Fredholm by Lemma 6.2.2. Similarly, $(T + S)B$ is Fredholm and so $T + S$ is Fredholm by Theorem 6.1.6. □

We would like to produce a result similar to Kleinecke's Theorem for the weakly inessential operators. To do this we need a notion to replace that of a Riesz operator.

Definition 6.2.7. Let X be a non-reflexive Banach space and let $T \in \mathscr{B}(X)$. The *weak essential spectrum* of T is $\sigma_w(T) = \sigma(T + \mathscr{W}(X))$, and T is a *weakly Riesz operator* if $\sigma_w(T) = \{0\}$. Denote by $\mathscr{W}\mathscr{R}(X)$ the set of weakly Riesz operators on X.

Lemma 6.2.8. *Every weakly inessential operator on a non-reflexive Banach space X is a weakly Riesz operator.*

Proof. Since X is non-reflexive, $\mathscr{B}(X)/\mathscr{W}(X)$ is a unital Banach algebra. Let $T : X \to X$ be weakly inessential and take a complex number $\lambda \neq 0$. Then by Proposition 6.1.16, $I_X + (-\frac{1}{\lambda} I_X)T + \mathscr{W}(X) \in \operatorname{inv} \mathscr{B}(X)/\mathscr{W}(X)$. Thus $(\lambda I_X - T) + \mathscr{W}(X) \in \operatorname{inv} \mathscr{B}(X)/\mathscr{W}(X)$, and therefore $\lambda \notin \sigma_w(T)$. Thus $\sigma_w(T) \subseteq \{0\}$. We can now use the fact that the spectrum is non-empty [24, Theorem 2.2.41] to conclude that $\sigma_w(T) = \{0\}$, but in fact there is an alternative easy argument.

Assume that $0 \notin \sigma_w(T)$. Then there exists $S \in \mathscr{B}(X)$ such that $ST + \mathscr{W}(X) = I_X + \mathscr{W}(X)$, so $I_X - ST \in \mathscr{W}(X)$. But since T is weakly inessential, Proposition 6.1.16 implies that $I_X - ST + \mathscr{W}(X) \in \operatorname{inv} \mathscr{B}(X)/\mathscr{W}(X)$, a contradiction. We conclude that $\sigma_w(T) = \{0\}$ □

Proposition 6.2.9. *Let X be a non-reflexive Banach space, and let $\mathscr{I}$ be an ideal in $\mathscr{B}(X)$ such that $\mathscr{W}(X) \subseteq \mathscr{I} \subseteq \mathscr{W}\mathscr{R}(X)$. Then*

$$\mathscr{W}\mathscr{E}(X) = \{T \in \mathscr{B}(X) : T + \mathscr{I} \in \operatorname{rad}(\mathscr{B}(X)/\mathscr{I})\}.$$

Proof. By assumption X is not reflexive, so I_X is not weakly Riesz. Hence $\mathscr{I}$ is a proper ideal.

'$\supseteq$' Let $T \in \mathscr{B}(X)$ such that $T + \mathscr{I} \in \operatorname{rad} \mathscr{B}(X)/\mathscr{I}$. Then for each $S \in \mathscr{B}(X)$ there exist $U \in \mathscr{B}(X)$ and $R \in \mathscr{I}$ such that $U(I_X + ST) = I_X + R$. Since $\mathscr{I} \subseteq \mathscr{W}\mathscr{R}(X)$ we know that $\sigma(R + \mathscr{W}(X)) = \{0\}$. Therefore $-I_X - R + \mathscr{W}(X)$ is invertible in $\mathscr{B}(X)/\mathscr{W}(X)$ by the definition of the spectrum. So there exist $Q \in \mathscr{B}(X)$ and $W \in \mathscr{W}(X)$ such that $Q(I_X + R) = I_X + W$. Hence

$$QU(I_X + ST) = Q(I_X + R) = I_X + W$$

and so $T \in \mathscr{W}\mathscr{E}(X)$.

'$\subseteq$' Let $T \in \mathscr{W}\mathscr{E}(X)$. Then for every $S \in \mathscr{B}(X)$, there exist $U \in \mathscr{B}(X)$ and $W_1 \in \mathscr{W}(X)$ such that $U(I_X + ST) = I_X + W_1$. Now $I_X - U = UST - W_1 \in \mathscr{W}\mathscr{E}(X)$ so there exist $U_0 \in \mathscr{B}(X)$ and $W_0 \in \mathscr{W}(X)$ such that $U_0 U = I_X + W_0$. Set

$W_2 = W_0(I_X - U - STU) + U_0W_1U \in \mathscr{W}(X)$. Then $(I_X + ST)U = I_X + W_2$, as is easily calculated. Thus $(I_X + ST + \mathscr{I})(U + \mathscr{I}) = I_X + \mathscr{I}$ and

$$(U + \mathscr{I})(I_X + ST + \mathscr{I}) = U(I_X + ST) + \mathscr{I} = I_X + W_1 + \mathscr{I} = I_X + \mathscr{I}$$

since $\mathscr{W}(X) \subseteq \mathscr{I}$. So $I_X + ST + \mathscr{I} \in \operatorname{inv} \mathscr{B}(X)/\mathscr{I}$ which implies that $T + \mathscr{I} \in \operatorname{rad} \mathscr{B}(X)/\mathscr{I}$. □

Corollary 6.2.10. *For each non-reflexive Banach space X, $\mathscr{I} = \mathscr{W}\mathscr{E}(X)$ is the maximum ideal in $\mathscr{B}(X)$ such that $\mathscr{W}(X) \subseteq \mathscr{I} \subseteq \mathscr{W}\mathscr{R}(X)$.*

Proof. By Lemma 6.2.8 and Theorem 6.1.12(ii) we know that $\mathscr{W}(X) \subseteq \mathscr{W}\mathscr{E}(X) \subseteq \mathscr{W}\mathscr{R}(X)$, and that $\mathscr{W}\mathscr{E}(X)$ is a closed ideal of $\mathscr{B}(X)$. Let $\mathscr{I}$ be an ideal of $\mathscr{B}(X)$ such that $\mathscr{W}(X) \subseteq \mathscr{I} \subseteq \mathscr{W}\mathscr{R}(X)$. Then Proposition 6.2.9 yields

$$\mathscr{W}\mathscr{E}(X) = \{T \in \mathscr{B}(X) : T + \mathscr{I} \in \operatorname{rad}(\mathscr{B}(X)/\mathscr{I})\} \supseteq \mathscr{I}$$

which completes the proof. □

As an application of these results we prove a theorem about the gap between the weakly inessentials and weakly compacts. The difference between $\mathscr{W}(X)$ and $\mathscr{W}\mathscr{E}(X)$ can be 'as big as possible', as the next result shows. Note that we are assuming, as we may, that the scalar field for $E_{\mathcal{R}}$ is the complex numbers. For the proof we only need Read's main theorem along with our Kleinecke-type result.

Theorem 6.2.11. *Let $E_{\mathcal{R}}$ be Read's Banach space. Then $\mathscr{W}\mathscr{E}(E_{\mathcal{R}}) = I$ has codimension 1 in $\mathscr{B}(E_{\mathcal{R}})$, but $\mathscr{W}(E_{\mathcal{R}})$ has infinite codimension in $\mathscr{B}(E_{\mathcal{R}})$.*

Proof. We first point out that I has codimension 1 and $\mathscr{W}(E_{\mathcal{R}})$ has infinite codimension in $\mathscr{B}(E_{\mathcal{R}})$ by Theorem 1.3.2; in particular, $E_{\mathcal{R}}$ is non-reflexive. Our strategy is to show that every bounded operator in I is weakly Riesz, and apply Corollary 6.2.10.

So take $T \in I$ and $\lambda \neq 0$; note by Theorem 1.3.2(iii) that $T^2 \in \mathscr{W}(E_{\mathcal{R}})$. Then

$$\big(\frac{1}{\lambda}I_{E_{\mathcal{R}}} + \frac{1}{\lambda^2}T\big)\big(\lambda I_{E_{\mathcal{R}}} - T\big) = I_{E_{\mathcal{R}}} - \frac{1}{\lambda^2}T^2 = \big(\lambda I_{E_{\mathcal{R}}} - T\big)\big(\frac{1}{\lambda}I_{E_{\mathcal{R}}} + \frac{1}{\lambda^2}T\big)$$

and so $\lambda I_{E_{\mathcal{R}}} - T + \mathscr{W}(E_{\mathcal{R}})$ is invertible in $\mathscr{B}(E_{\mathcal{R}})/\mathscr{W}(E_{\mathcal{R}})$. Thus $\sigma_w(T) \subseteq \{0\}$. Suppose that 0 does not belong to $\sigma_w(T)$. Then there exist $R \in \mathscr{B}(E_{\mathcal{R}})$ and $W \in \mathscr{W}(E_{\mathcal{R}})$ such that $RT = I_{E_{\mathcal{R}}} + W$. But then $I_{E_{\mathcal{R}}} = RT - W \in I$ because I contains the weakly compacts. Contrapositively, $\sigma_w(T) = \{0\}$ because I is a proper ideal. Therefore $T \in \mathscr{W}\mathscr{R}(E_{\mathcal{R}})$, which implies that $I \subseteq \mathscr{W}\mathscr{R}(E_{\mathcal{R}})$.

Corollary 6.2.5 now ensures that $I \subseteq \mathscr{W}\mathscr{E}(E_{\mathcal{R}})$ because $\mathscr{W}(E_{\mathcal{R}}) \subseteq I$. Proposition 6.1.11 implies that $\mathscr{W}\mathscr{E}(E_{\mathcal{R}}) \subsetneq \mathscr{B}(E_{\mathcal{R}})$, and therefore $I = \mathscr{W}\mathscr{E}(E_{\mathcal{R}})$ because I is a maximal ideal. This proves the theorem. □

6.3 The weak inessential property

In this section we explore the relationship between the operator ideals $\mathscr{E}$ and $\mathscr{W}\mathscr{E}$ in more detail. For example, for which pairs of Banach spaces X and Y are they equal?

Definition 6.3.1. A Banach space X has the *Dunford–Pettis property* (DPP) if for every Banach space Y, $\mathscr{W}(X,Y) \subseteq \mathscr{V}(X,Y)$.

Example 6.3.2. Banach spaces having the Dunford–Pettis property include all finite-dimensional spaces, $L_1(\Omega, \Sigma, \mu)$ for any σ-finite measure space (Ω, Σ, μ), and $C(K)$ for a compact Hausdorff space K. In particular, this tells us that ℓ_1, c_0 and ℓ_∞ have (DPP).

If X^* has (DPP) then X has (DPP) (see *e.g.*, [78, Exercise 3.60]). This leads to some more unusual examples including $X_{\mathrm{KL}}, X_{\mathrm{AH}}$ and X_k for $k \in \mathbb{N}$, $k \geqslant 2$, because their dual spaces are isomorphic to ℓ_1 (see Chapter 2). Infinite-dimensional reflexive spaces do not have (DPP).

Aiena, González and Martínez-Abejón proved the following proposition [2, Proposition 3.3] in more generality. We give a proof to demonstrate the concepts in question.

Proposition 6.3.3. *Let X and Y be Banach spaces, and suppose that X has the Dunford–Pettis property (DPP). Then $\mathscr{E}(X,Y) = \mathscr{W}\mathscr{E}(X,Y)$.*

Proof. By Theorem 6.1.12(i) we have the inclusion $\mathscr{E}(X,Y) \subseteq \mathscr{W}\mathscr{E}(X,Y)$. So it is enough to demonstrate the converse.

Take $T \in \mathscr{W}\mathscr{E}(X,Y)$ and choose $S \in \mathscr{B}(Y,X)$. Then there exist $U \in \mathscr{B}(X)$ and $V \in \mathscr{W}(X)$ such that $U(I_X + ST) = I_X + V$. Since X has the Dunford–Pettis property; then V is completely continuous. Let (x_n) be a bounded sequence in X. Since V is weakly compact there is a subsequence (x_{n_k}) such that (Vx_{n_k}) converges weakly, and since V is also completely continuous, (VVx_{n_k}) converges in norm. Thus $V^2 \in \mathscr{K}(X)$. From this we see that

$$(U - VU)(I_X + ST) = (I_X - V)(I_X + V) = I_X + (-V^2)$$

Hence $T \in \mathscr{K}^{\mathrm{rad}}(X,Y)$, and so by Proposition 6.1.7, T is inessential. $\square$

We define a property related to (DPP) which captures the fact that $\mathscr{E} = \mathscr{W}\mathscr{E}$.

Definition 6.3.4. A Banach space X has the *weak inessential property* (WIP) if $\mathscr{W}(X) \subseteq \mathscr{E}(X)$.

All finite-dimensional spaces have (WIP), but infinite-dimensional reflexive spaces do not. The next proposition provides some more examples.

Proposition 6.3.5. *Let X be a Banach space. Then the following are equivalent:*

(a) *X has (WIP);*

(b) *$\mathscr{E}(X,Y) = \mathscr{W}\mathscr{E}(X,Y)$ for every Banach space Y;*

(c) *$\mathscr{W}(X,Y) \subseteq \mathscr{E}(X,Y)$ for every Banach space Y.*

Proof. (a) $\Rightarrow$ (b) Suppose that X has (WIP). Choose a Banach space Y, and $T \in \mathscr{W}\mathscr{E}(X,Y)$. By Theorem 6.1.12(i), it is enough to show that $T \in \mathscr{E}(X,Y)$. The implication is true if X is finite-dimensional, so we may suppose that X is infinite-dimensional. By definition, for every $S \in \mathscr{B}(Y,X)$ there are $U \in \mathscr{B}(X)$ and $W \in \mathscr{W}(X)$ such that $U(I_X + ST) = I_X + W$. Also, by Lemma 6.1.5 there is $W' \in \mathscr{W}(X)$ such that $(I_X + ST)U = I_X + W'$. Since $\mathscr{W}(X) \subseteq \mathscr{E}(X) \subseteq \mathscr{R}(X)$, the advanced version of Atkinson's Theorem (Theorem 6.2.3) says that $I_X + ST$ is Fredholm, so that $T \in \mathscr{E}(X,Y)$, as required.

(b) $\Rightarrow$ (c) For each Banach space Y we have $\mathscr{W}(X,Y) \subseteq \mathscr{W}\mathscr{E}(X,Y) = \mathscr{E}(X,Y)$ by Theorem 6.1.12(ii).

(c) $\Rightarrow$ (a) This is trivial. $\square$

Propositions 6.3.3 and 6.3.5 show that (DPP) implies (WIP) for all Banach spaces, giving various other examples of spaces with (WIP).

Intuitively (WIP) should be a much weaker condition than (DPP) because it is an internal property, depending only on the Banach space itself. However, it is not immediate to give an example which has (WIP) but lacks (DPP). With some work we can show that the Schreier space, defined below, satisfies this. The author would like to thank Dr Tomasz Kania for suggesting this example.

Definition 6.3.6. Let $\mathbb{C}^{\mathbb{N}}$ denote the vector space of all scalar sequences with pointwise operations, and let $1 \leqslant p < \infty$. For $x = (\alpha_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ the function

$$||x||_{Z_p} := \sup\left\{ \left(\sum_{j=1}^{k} |\alpha_{n_j}|^p \right)^{\frac{1}{p}} : k, n_1, \ldots, n_k \in \mathbb{N},\ k \leqslant n_1 < n_2 < \cdots < n_k \right\}$$

defines a complete norm on the subspace $Z_p := \{x \in \mathbb{C}^{\mathbb{N}} : ||x||_{Z_p} < \infty\}$ of $\mathbb{C}^{\mathbb{N}}$. We call the Banach space $(Z_p, ||\cdot||_{Z_p})$ the p^{th} *unrestricted Schreier space*. The sequence $(e_n)_{n\in\mathbb{N}}$, where $e_n = (0,\ldots,0,1,0,\ldots)$ is a normalised basic sequence in Z_p, and so $S_p = \overline{\operatorname{span}}\,\{e_n : n \in \mathbb{N}\} \subseteq Z_p$ is a Banach space with basis (e_n), called the p^{th} *Schreier space.*

The Banach space S_1 is usually known as *the* Schreier space in the literature. Bird and Laustsen [13] appear to have been the first to study S_p for general p.

It is well known that S_1 lacks the Dunford–Pettis property, but for $p > 1$ the result does not seem to be explicitly stated in the literature. Our proof uses a characterisation of S_p^{**} given by Bird and Laustsen.

Proposition 6.3.7. *For $1 \leqslant p < \infty$, the p^{th} Schreier space S_p fails the Dunford–Pettis property.*

Proof. Let $1 \leqslant p < \infty$. We recall an equivalent characterisation of (DPP), that is, X has (DPP) if and only if for every pair of weakly null sequences (x_n) in X and (x_n^*) in X^*, we have $\langle x_n, x_n^* \rangle \to 0$ as $n \to \infty$ [78, Theorem 3.5.18]. We shall prove that the canonical Schauder basis (e_n) for S_p is weakly null, and (e_n^*) (the sequence of coordinate functionals) in S_p^* is weakly null, but $\langle e_n, e_n^* \rangle \not\to 0$ as $n \to \infty$.

Firstly, Bird and Laustsen have shown that the basis (e_n) for S_p is shrinking [13, Proposition 3.10, Corollary 3.12] (this was known for $p = 1$, as noted by the authors). This implies that (e_n) is weakly null by a standard result about shrinking bases [3, Proposition 3.2.7].

They also succeeded in characterising the second dual S_p^{**} via the following commutative diagram [13, 3.14]:

$$
\begin{tikzcd}
S_p \arrow[r, "\kappa"] \arrow[d, "\iota"'] & S_p^{**} \arrow[d, "\Upsilon", "\simeq"'] \\
Z_p \arrow[r, equal] & \mathrm{bip}(S_p)
\end{tikzcd}
\tag{6.3.1}
$$

where ι is the natural inclusion, κ is the canonical embedding into the bidual, and $\mathrm{bip}(S_p) = \{(\alpha_n) \in \mathbb{C}^{\mathbb{N}} : \sup_m || \sum_{n=1}^m \alpha_n e_n || < \infty\}$, where bip stands for 'bounded initial projections'. For $F \in S_p^{**}$, the map $\Upsilon : F \mapsto \big(\langle e_n^*, F \rangle\big)_{n=1}^{\infty}$ is an isometric isomorphism because (e_n) is a monotone shrinking basis.

Take $F \in S_p^{**}$. Then by (6.3.1), $\Upsilon(F) = (\langle e_n^*, F \rangle)_{n=1}^{\infty} \in Z_p$. Now [13, Lemma 3.3(ii)] shows that $Z_p \subset c_0$ as a set, so that $\langle e_n^*, F \rangle \to 0$ as $n \to \infty$. This implies that (e_n^*) is weakly null and, since $\langle e_n, e_n^* \rangle = 1 \not\to 0$ as $n \to \infty$, we conclude that S_p fails the Dunford–Pettis property. $\square$

Example 6.3.8. Let $1 \leqslant p < \infty$. We shall show that S_p has (WIP), which means that $\mathscr{W}(S_p, Y) \subseteq \mathscr{E}(S_p, Y)$ for every Banach space Y. So take a Banach space Y and $W \in \mathscr{W}(S_p, Y)$. Then by the fundamental result of Davis–Figiel–Johnson–Pełczyński [29], W factors through a reflexive space, which means that there exist a reflexive Banach space Z and bounded operators $A : S_p \to Z$ and $B : Z \to Y$ such that $W = BA$. Now S_p is c_0-saturated [13, Corollary 5.4], which means that every infinite-dimensional closed subspace of S_p contains a copy of c_0.

Consider $A : S_p \to Z$; we claim it is strictly singular. Indeed, if M is an infinite-dimensional closed subspace then M is not reflexive since it contains a copy of c_0. So if $A|_M$ were an isomorphism, then Z would contain a non-reflexive subspace. However, this cannot be because Z is reflexive. Thus $A \in \mathscr{S}(S_p, Z)$. Since the strictly singular operators form an operator ideal this implies that $W \in \mathscr{S}(S_p, Y)$. Therefore $\mathscr{W}(S_1, Y) \subseteq \mathscr{E}(S_p, Y)$ by Corollary 6.1.13, and so S_p has (WIP).

Therefore (WIP) $\not\Rightarrow$ (DPP) in general.

The Schreier space S_1 does have the *weak Dunford–Pettis property* (wDPP), as shown by González and Gutiérrez [44]. The definition is as follows: a Banach space X has (wDPP) if for every uniformly weakly null sequence (x_n) in X, and every weakly null sequence (x_n^*) in X^*, $\langle x_n, x_n^* \rangle \to 0$. A sequence (x_n) is *uniformly weakly null* if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\operatorname{card}\{n \in \mathbb{N} : |\langle x_n, f \rangle| \geqslant \varepsilon\} \leqslant N$ for every $f \in X^*$ with $||f|| = 1$.

The question arises as to whether this is equivalent to (WIP); the following example shows not, although it is still open whether (WIP) implies (wDPP).

Example 6.3.9. The Tsirelson space T is infinite-dimensional and reflexive, so $\mathscr{E}(T) \subsetneq \mathscr{W}(T) = \mathscr{B}(T)$. Hence T lacks (WIP). But T has (wDPP) [44, p. 3].

There seems plenty of scope for future work. Some specific questions: are there other Banach spaces X with $\mathscr{W\!E}(X)$ having codimension one, but $\mathscr{W}(X)$ having infinite codimension in $\mathscr{B}(X)$? What properties would such an X have in common with $E_{\mathcal{R}}$? It would also be interesting to know if $\mathscr{W\!E}(E_{\mathcal{R}})$ is the *unique* maximal ideal of $\mathscr{B}(E_{\mathcal{R}})$.

In an abstract direction, one would like to know if weakly inessential operators can be applied to a 'weak Fredholm theory', just as inessential operators apply to Fredholm theory. We would also like to know if the weak inessential property is really a new (or indeed useful) concept, or if it is just an equivalent formulation of another.

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