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Crystal frameworks, symmetry and affinely periodic flexes

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Abstract. Symmetry equations are obtained for the rigidity matrices associated with various forms of infinitesimal flexibility for an idealised bond-node crystal framework $\mathcal{C}$ in $\mathbb{R}^d$. These equations are used to derive symmetry-adapted Maxwell–Calladine counting formulae for periodic self-stresses and affinely periodic infinitesimal mechanisms. The symmetry equations also lead to general Fowler–Guest formulae connecting the character lists of subrepresentations of the crystallographic space and point groups which are associated with bonds, nodes, stresses, flexes and rigid motions. A new derivation is also given for the Borcea–Streinu rigidity matrix and the correspondence between its nullspace and the space of affinely periodic infinitesimal flexes.

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1. Introduction

A finite bar-joint framework (G, p) is a graph $G = (V, E)$ together with a correspondence $v_i \to p_i$ between vertices and framework points in $\mathbb{R}^d$, the joints or nodes of (G, p). The framework edges are the line segments $[p_i, p_j]$ associated with the edges of G and these are viewed as inextensible bars. Accounts of the analysis of infinitesimal flexibility and the combinatorial rigidity of such frameworks can be found in Asimow and Roth [1], [2] and Graver, Servatius and Servatius [10].

In the presence of a spatial symmetry group $\mathcal{S}$ for (G, p), with isometric representation $\rho_{sp} : \mathcal{S} \to \mathrm{Isom}(\mathbb{R}^d)$, Fowler and Guest [8] considered certain

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unitary representations $\rho_n \otimes \rho_{sp}$ and ρ_e of $\mathcal{S}$ on the finite-dimensional vector spaces

$$\mathcal{H}_v = \sum_{\text{joints}} \oplus \mathbb{R}^d, \quad \mathcal{H}_e = \sum_{\text{bars}} \oplus \mathbb{R}.$$

These spaces contain, respectively, the linear subspace of infinitesimal flexes of (G, p), denoted $\mathcal{H}_{\text{fl}} \subseteq \mathcal{H}_v$, and the linear subspace of infinitesimal self-stresses, $\mathcal{H}_{\text{str}} \subseteq \mathcal{H}_e$. Also $\mathcal{H}_{\text{fl}}$ contains the space $\mathcal{H}_{\text{rig}}$ of rigid motion infinitesimal flexes while $\mathcal{H}_{\text{mech}} = \mathcal{H}_{\text{fl}} \ominus \mathcal{H}_{\text{rig}}$ is the space of infinitesimal mechanisms. It was shown that for the induced representations ρ_{mech} and ρ_{str} that there is a relationship between their associated character lists. Specifically,

$$[\rho_{\text{mech}}] - [\rho_{\text{str}}] = [\rho_{sp}] \circ [\rho_n] - [\rho_e] - [\rho_{\text{rig}}]$$

where, for example, $[\rho_{\text{mech}}]$ is the character list

$$[\rho_{\text{mech}}] = (tr(\rho_{\text{mech}}(g_1)), \ldots, tr(\rho_{\text{mech}}(g_n))),$$

for a fixed set of generators of $\mathcal{S}$, and where $[\rho_{sp}] \circ [\rho_n]$ indicates entry-wise product.

The Fowler–Guest formula can be viewed as a symmetry-adapted version of Maxwell's counting rule for finite rigid frameworks. Indeed, evaluating the formula at the identity element g_1 one recovers the more general Maxwell–Calladine equation, which in three dimensions takes the form

$$m - s = 3|V| - |E| - 6,$$

where m and s are the dimensions of the spaces of infinitesimal mechanisms and infinitesimal self-stresses, respectively. Moreover it has been shown in a variety of studies that the character equation for individual symmetries can lead to useful symmetry-adapted counting conditions for rigidity and isostaticity (stress-free rigidity). See, for example, Connelly, Guest, Fowler, Schulze and Whiteley [6], Fowler and Guest [9], Owen and Power [16] and Schulze [21], [22].

The notation above is taken from Owen and Power [16] where symmetry equations were given for the rigidity matrix $R(G, p)$ and a direct proof of the character equation obtained, together with a variety of applications. In particular a variant of the character equation was given for translationally periodic infinite frameworks and for strict periodicity. In the present development, which is self-contained, we combine the symmetry equation approach with a new derivation of the rigidity matrix, identified in Borcea and Streinu [4], which is associated with affinely periodic infinitesimal flexes. This derivation is given via infinite rigidity matrices and this perspective is well-positioned for the incorporation of space group symmetries. By involving such symmetries we obtain general symmetry-adapted affine Maxwell–Calladine equations for periodic frameworks (see Theorem 3.5 for example) and affine variants of the character list formula.

To be more precise, let $\mathcal{C}$ be a countably infinite bar-joint framework in $\mathbb{R}^d$ with discrete vertex set and with translational periodicity for a full

rank subgroup $\mathcal{T}$ of isometric translations. We refer to frameworks with this form of periodicity as *crystal frameworks*. An affinely periodic flex of $\mathcal{C}$ is one that, roughly speaking, allows periodicity relative to an affine transformation of the ambient space, including contractions, global rotations and sheering. The continuous and infinitesimal forms of this are given in Definition 2.2 and it is shown how the infinitesimal flexes of this type correspond to vectors in the null space of a finite rigidity matrix $R(\mathcal{M}, \mathbb{R}^{d^2})$ with $|F_e|$ rows and $d|F_v| + d^2$ columns. Here $\mathcal{M} = (F_v, F_e)$, a *motif* for $\mathcal{C}$, is a choice of a set F_v of vertices and a set F_e of edges whose translates partition the vertices and edges of $\mathcal{C}$. The "extra" d^2 columns correspond to the d^2 degrees of freedom present in the affine transformation.. When the affine matrices are restricted to a subspace $\mathcal{E}$ there is an associated rigidity matrix and linear transformation, denoted $R(\mathcal{M}, \mathcal{E})$. The case of strict periodicity corresponds to $\mathcal{E} = \{0\}$.

The symmetry-adapted Maxwell–Calladine formulæ for affinely periodic flexibility are derived from symmetry equations of independent interest that take the form

$$\pi_e(g)R(\mathcal{M}, \mathcal{E}) = R(\mathcal{M}, \mathcal{E})\pi_v(g),$$

where π_e and π_v are finite-dimensional representations of the space group $\mathcal{G}(\mathcal{C})$ of $\mathcal{C}$. As we discuss in Section 3, these in turn derive from more evident symmetry equations for the infinite rigidity matrix $R(\mathcal{C})$.

The infinite-dimensional vector space transformation perspective for the infinite matrix $R(\mathcal{C})$ is also useful in the consideration of quite general infinitesimal flexes including local flexes, diminishing flexes, supercell-periodic flexes and, more generally, phase-periodic flexes. In particular phase-periodic infinitesimal flexes for crystal frameworks are considered in Owen and Power [17] and Power [18] in connection with the analysis of rigid unit modes (RUMs) in material crystals, as described in Dove et al [7] and Wegner [24], for example. See also Badri, Kitson and Power [3].

In the final section we give some illustrative examples. The well-known kagome framework and a framework suggested by a Roman tiling of the plane, are contrasting examples which are both in Maxwell counting equilibrium. We also consider some overconstrained frameworks, including $\mathcal{C}_{\text{Hex}}$ which is formed as a regular network of triangular faced hexahedra, or bipyramids. In addition to such curious mathematical frameworks we note that there exist a profusion of polyhedral net frameworks which are suggested by the crystalline structure of natural materials, such as quartz, perovskite, and various aluminosilicates and zeolites. Some of these frameworks, such as the cubic sodalite framework, have pure mathematical forms. Examples of this type can be found in Borcea and Streinu [4], Dove et al [7], Kapko et al [12], Power [18] and Wegner [24] for example.

A number of recent articles also consider affinely periodic flexes and rigidity. Ross, Schulze and Whiteley [20] consider Borcea–Streinu style rigidity

matrices and orbit rigidity matrices to obtain a counting predictor for infinitesimal motion in the presence of space symmetries of separable type. (See also Corollary 3.6 below.) Using this they derive the many counting conditions in two and three dimensions which arise from the type of the separable space group in combination with the type of the affine axial velocity space $\mathcal{E}$. In a more combinatorial vein Borcea and Streinu [5] and Malestein and Theran [13] consider a detailed combinatorial rigidity theory for periodic frameworks. In particular they obtain combinatorial characterisations of periodic affine infinitesimal rigidity and isostaticity for generic periodic frameworks in $\mathbb{R}^2$.

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2. Crystal frameworks and rigidity matrices

2.1. Preliminaries. For convenience suppose first that d is equal to 3. The changes needed for the extension to $d = 2, 4, 5, \dots$ are essentially notational. Let $G = (V, E)$ be a countable simple graph with

$$V = \{v_1, v_2, \dots\}, \quad E \subseteq V \times V,$$

and let p be a sequence (p_i) where the $p_i = p(v_i)$ are corresponding points in $\mathbb{R}^3$. The pair (G, p) is said to be a *bar-joint framework* in $\mathbb{R}^3$ with framework vertices p_i (the joints or nodes) and framework edges (or bars) given by the straight line segments $[p_i, p_j]$ between p_i and p_j when (v_i, v_j) is an edge in E. We assume that the bars have positive lengths.

An *infinitesimal flex* of (G, p) is a vector $u = (u_i)$, with each $u_i \in \mathbb{R}^3$ regarded as a velocity applied to p_i, such that for each edge $[p_i, p_j]$

$$\langle p_i - p_j, u_i \rangle = \langle p_i - p_j, u_j \rangle.$$

This simply asserts that for each edge the components in the edge direction of the endpoint velocities are in agreement. Equivalently, an infinitesimal flex is a velocity vector $v = (v_i)$ for which the distance deviation

$$|p_i - p_j| - |(p_i + tu_i) - (p_j + tu_j)|$$

of each edge is of order t^2 as $t \to 0$.

An *isometry* of $\mathbb{R}^3$ is a distance-preserving map $T : \mathbb{R}^3 \to \mathbb{R}^3$. A *full rank translation group* $\mathcal{T}$ is a set of translation isometries $\{T_k : k \in \mathbb{Z}^3\}$ with $T_{k+l} = T_k + T_l$ for all k, l, $T_k \neq I$ if $k \neq 0$, such that the three vectors

$$a_1 = T_{e_1}0, \quad a_2 = T_{e_2}0, \quad a_3 = T_{e_3}0,$$

associated with the generators $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$ of $\mathbb{Z}^3$ are not coplanar. We refer to these vectors as the *period vectors* for $\mathcal{T}$ and $\mathcal{C}$. The following definitions follow the formalism of Owen and Power [17], [18].

Definition 2.1. A crystal framework $\mathcal{C} = (F_v, F_e, \mathcal{T})$ in $\mathbb{R}^3$, with full rank translation group $\mathcal{T}$ and finite motif (F_v, F_e), is a countable bar-joint framework with framework points $p_{\kappa,k}$, for $1 \leq \kappa \leq t, k \in \mathbb{Z}^3$, such that:

(i) $F_v = \{p_{\kappa,0} : 1 \leq \kappa \leq t\}$ and F_e is a finite set of framework edges.
(ii) For each κ and k the point $p_{\kappa,k}$ is the translate $T_k p_{\kappa,0}$.
(iii) The set $\mathcal{C}_v$ of framework points is the disjoint union of the sets $T_k(F_v), k \in \mathbb{Z}^3$.
(iv) The set $\mathcal{C}_e$ of framework edges is the disjoint union of the sets $T_k(F_e), k \in \mathbb{Z}^3$.

The finiteness of the motif and the full rank of $\mathcal{T}$ ensure that the periodic set $\mathcal{C}_v$ is a discrete set in $\mathbb{R}^3$ in the usual sense.

One might also view the motif as a choice of representatives for the translation equivalence classes of the vertices and the edges of $\mathcal{C}$. It is natural to take F_v as the vertices of $\mathcal{C}$ that lie in a distinguished *unit cell*, such as $[0,1)\times[0,1)\times[0,1)$ in the case that $\mathcal{T}$ is the standard cubic translation group for $\mathbb{Z}^3 \subseteq \mathbb{R}^3$. In this case one could take F_e to consist of the edges lying in the unit cell together with some number of cell-spanning edges $e = [p_{\kappa,0}, p_{\tau,\delta}]$ where $\delta = \delta(e) = (\delta_1, \delta_2, \delta_3)$ is nonzero. This can be a convenient assumption and with it we may refer to $\delta(e)$ as the *exponent* of the edge. The case of general motifs is discussed at the end of this section.

While we focus on discrete full rank periodic bar-joint frameworks, which appear in many mathematical models in applications, we remark that there are, of course, other interesting forms of infinite frameworks with infinite spatial symmetry groups. In particular there is the class of *cylinder frameworks* in $\mathbb{R}^d$ which are translationally periodic for a subgroup $\{T_k : k \in \mathbb{Z}^r\}$ of a full rank translation group $\mathcal{T}$. See, for example, the two-dimensional strip frameworks of [17] and the hexahedral tower in Section 4. On the other hand one can also consider infinite frameworks constrained to infinite surfaces and employ restricted framework rigidity theory [14].

2.2. Affinely periodic infinitesimal flexes. We shall give various definitions of affinely periodic infinitesimal flexes and we start with a finite matrix-data description and give a connection with certain continuous flexes. This notation takes the form (u, A) where A is an arbitrary $d \times d$ matrix and $u = (u_\kappa)_{\kappa \in F_v}$ is a real vector in $\mathbb{R}^{d|F_v|}$ composed of the infinitesimal flex vectors $u_\kappa = u_{\kappa,0}$ that are assigned to the framework vertices $p_{\kappa,0}$. The matrix A, an *affine velocity matrix*, is viewed both as a $d \times d$ real matrix and as a vector in $\mathbb{R}^{d^2}$ in which the columns of A are written in order. Thus, each pair (v, B) generates an assignment of displacement velocities to the framework points, and in some cases the resulting velocity vector will qualify as an affinely periodic infinitesimal flex in the sense given below. In Theorem 2.7 we obtain a three-fold characterisation of such flexes.

By an *affine flow* we mean a differentiable function $t \to A_t$ from $[0, t_1]$ to the set of nonsingular linear transformations of $\mathbb{R}^3$ such that $A_0 = I$ and

we write A for the derivative at $t = 0$. (See also Owen and Power [17].) In particular for any matrix A the function $t \to I + tA$ is an affine flow, for suitably small t_1.

Definition 2.2. Let $\mathcal{C}$ be a crystal framework.

(i) A flow-periodic flex of $\mathcal{C}$ for an affine flow $t \to A_t$ is a coordinate-wise differentiable path $p(t) = (p_{\kappa,k}(t))$, for $t \in [0, t_1]$ for some $t_1 > 0$, such that:

(a) The flow-periodicity condition holds;

$$p_{\kappa,k}(t) = A_t T_k A_t^{-1} p_{\kappa,0}(t), \quad \text{for all} \quad \kappa, k, t.$$

(b) For each edge $e = [p_{\kappa,k}, p_{\tau,k+\delta(e)}]$ the distance function

$$t \to d_e(t) := |p_{\kappa,k}(t) - p_{\tau,k+\delta(e)}(t)|$$

is constant.

(ii) An affinely periodic infinitesimal flex of $\mathcal{C}$ is a pair (u, A), with $u \in \mathbb{R}^{3|F_v|}$ and A a $d \times d$ real matrix such that for all motif edges $e = [p_{\kappa,0}, p_{\tau,\delta(e)}]$ in F_e,

$$\langle p_{\kappa,0} - p_{\tau,\delta(e)}, u_\kappa - u_\tau + A(p_{\tau,0} - p_{\tau,\delta(e)})\rangle = 0.$$

The notion of a flow-periodic flex, with finite path vertex motions, is quite intuitive and easily illustrated. One can imagine for example a periodic zig-zag bar framework (perhaps featuring as a subframework of a full rank framework) which flexes periodically by concertina-like contraction and expansion. Affinely periodic infinitesimal flexes are perhaps less intuitive. However we have the following proposition.

Proposition 2.3. *Let $p(t)$ be a flow-periodic flex of $\mathcal{C}$ for the flow $t \to A_t$ whose derivative at $t = 0$ is A. Then $(p'_{\kappa,0}(0), A)$ is an affinely periodic infinitesimal flex of $\mathcal{C}$.*

Proof. Differentiating the flow-periodicity condition

$$A_t^{-1} p_{\tau,\delta}(t) = T_\delta A_t^{-1} p_{\tau,0}(t)$$

gives

$$-Ap_{\tau,\delta}(0) + p'_{\tau,\delta}(0) = -Ap_{\tau,0}(0) + p'_{\tau,0}(0)$$

and so

$$p'_{\tau,\delta}(0) = p'_{\tau,0}(0) + A(p_{\tau,\delta}(0) - p_{\tau,0}(0)).$$

Differentiating and evaluating at zero the constant function

$$t \to \langle p_{\kappa,0}(t) - p_{\tau,\delta(e)}(t), p_{\kappa,0}(t) - p_{\tau,\delta(e)}(t)\rangle,$$

for the edge $e = [p_{\kappa,0}, p_{\tau,\delta(e)}]$ in F_e, gives

$$0 = \langle p_{\kappa,0}(0) - p_{\tau,\delta(e)}(0), p'_{\kappa,0}(0) - p'_{\tau,\delta(e)}(0)\rangle.$$

Thus, substituting, with $\delta(e) = \delta$, gives

$$0 = \langle p_{\kappa,0}(0) - p_{\tau,\delta(e)}(0), p'_{\kappa,0}(0) - p'_{\tau,0}(0) + A(p_{\tau,0}(0) - p_{\tau,\delta(e)}(0))\rangle,$$

as required. □

We remark that with a similar proof one can show the following related equivalence. The pair (u, A) is an affinely periodic infinitesimal flex of $\mathcal{C}$ if and only if for the affine flow $A_t = I + tA$ the distance deviation

$$|p_{\kappa,0} - p_{\tau,\delta(e)}| - |(p_{\kappa,0} + tu_\kappa) - A_t T_{\delta(e)} A_t^{-1}(p_{\tau,\delta(e)} + tu_\tau)|$$

of each edge is of order t^2 as $t \to 0$. This property justifies, to some extent, the terminology that (u, A) is an infinitesimal flex for $\mathcal{C}$. Note that an infinitesimal rotation flex of $\mathcal{C}$ (defined naturally, or by means of Proposition 2.3) provides an affine infinitesimal flex.

2.3. Rigidity matrices. We first define a finite matrix associated with the motif $\mathcal{M} = (F_v, F_e)$ which is essentially the rigidity matrix identified by Borcea and Streinu [4]. We write it as $R(\mathcal{M}, \mathbb{R}^{d^2})$ and also view it as a linear transformation from $\mathbb{R}^{d|F_v|} \oplus \mathbb{R}^{d^2}$ to $\mathbb{R}^{|F_e|}$. In the case $d = 3$ the summand space $\mathbb{R}^{d^2}$ is a threefold direct sum $\mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$. For $\mathcal{C}$ with cubic lattice structure with period vectors

$$a_1 = (1, 0, 0), \quad a_2 = (0, 1, 0), \quad a_3 = (0, 0, 1),$$

an *affine velocity matrix* $A = (a_{ij})$ provides a row vector in $\mathbb{R}^9$ in which the columns of A are written in order. It is shown in Theorem 2.7 that the composite vector (u, A) lies in the kernel of this rigidity matrix if and only if (u, A) is an affinely periodic infinitesimal flex in the sense of Definition 2.2(ii), and if and only if the related infinite vector $\tilde{u} = (u_\kappa - Ak)_{\kappa,k}$ is an infinitesimal flex of $\mathcal{C}$ in the usual bar-joint framework sense.

Definition 2.4. Let $\mathcal{C} = (F_v, F_e, \mathcal{T})$ be a crystal framework in $\mathbb{R}^d$ with isometric translation group $\mathcal{T}$ and motif $\mathcal{M} = (F_v, F_e)$. Then the affinely periodic rigidity matrix $R(\mathcal{M}, \mathbb{R}^{d^2})$ is the $|F_e| \times (d|F_v| + d^2)$ real matrix whose rows, labelled by the edges $e = [p_{\kappa,0}, p_{\tau,\delta(e)}]$ of F_e, have the form

$$[0 \cdots 0 \ v_e \ 0 \cdots 0 \ -v_e \ 0 \ \cdots \ 0 \ \delta_1 v_e \cdots \delta_d v_e],$$

where $v_e = p_{\kappa,0} - p_{\tau,\delta(e)}$ is the edge vector for e, distributed in the d columns for κ, where $-v_e$ appears in the columns for τ, and where $\delta(e) = (\delta_1, \ldots, \delta_d)$ is the exponent of e. If e is a reflexive edge in the sense that $\kappa = \tau$ then the entries in the d columns for κ are zero.

The transformation $R(\mathcal{M}, \mathbb{R}^{d^2})$ has the block form $[R(\mathcal{M}) \ X(\mathcal{M})]$ where $R(\mathcal{M})$ is the (strictly) periodic rigidity matrix, or motif rigidity matrix. We also define the linear transformations $R(\mathcal{M}, \mathcal{E})$ for linear subspaces $\mathcal{E} \subseteq \mathbb{R}^{d^2}$ of affine velocity matrices, by restricting the domain;

$$R(\mathcal{M}, \mathcal{E}) : \mathbb{R}^{3|F_v|} \oplus \mathcal{E} \to \mathbb{R}^{|F_e|}.$$

For example in three dimensions one could take $\mathcal{E}$ to be the three dimensional space of diagonal matrices and this would provide the rigidity matrix which detects the persistence of infinitesimal rigidity even in the presence of axial

expansion and contraction. (See also [20] for other interesting forms of relaxation of strict periodicity.)

For a general countably infinite bar-joint framework (G, p) one may define a rigidity matrix $R(G, p)$ in the same way as for finite bar-joint frameworks. For the crystal frameworks $\mathcal{C}$ it takes the following form.

Definition 2.5. Let $\mathcal{C} = (F_v, F_e, \mathcal{T})$ be a crystal framework in $\mathbb{R}^d$ as given in Definition 2.4. Then the infinite rigidity matrix for $\mathcal{C}$ is the real matrix $R(\mathcal{C})$ whose rows, labelled by the edges $e = [p_{\kappa,k}, p_{\tau,k+\delta(e)}]$ of $\mathcal{C}$, for $k \in \mathbb{Z}^d$, have the form

$$[\cdots 0 \cdots 0 \ v_e \ 0 \cdots 0 \ -v_e \ 0 \ \cdots \ 0 \cdots],$$

where $v_e = p_{\kappa,k} - p_{\tau,k+\delta(e)}$ is the edge vector for e, distributed in the d columns for κ, k and where $-v_e$ appears in the columns for $(\tau, k + \delta(e))$.

We view $R(\mathcal{C})$ as a linear transformation from $\mathcal{H}_v$ to $\mathcal{H}_e$ where $\mathcal{H}_v$ and $\mathcal{H}_e$ are the direct product vector spaces,

$$\mathcal{H}_v = \Pi_{\kappa,k}\mathbb{R}^3, \quad \mathcal{H}_{e,k} = \Pi_k\mathbb{R}.$$

In particular, as in the finite framework case, a velocity vector v in $\mathcal{H}_v$ is an infinitesimal flex of $\mathcal{C}$ if and only if v lies in the nullspace of $R(\mathcal{C})$.

The following theorem shows the role played by the rigidity matrices $R(\mathcal{M}, \mathbb{R}^{d^2})$ and $R(\mathcal{C})$ in locating affinely periodic infinitesimal flexes. For the general case we require the invertible transformation $Z : \mathbb{R}^d \to \mathbb{R}^d$ which maps the standard basis vectors $\gamma_1, \ldots, \gamma_d$ for $\mathbb{R}^d$ to the period vectors $a_1, \ldots, a_d$ of $\mathcal{C}$. In the next definition a finite vector-matrix pair (v, A) generates a velocity vector in $\mathcal{H}_v$ of affinely periodic type.

Definition 2.6. The space $\mathcal{H}_v^{\text{aff}}$ is the vector subspace of $\mathcal{H}_v$ consisting of *affinely periodic velocity vectors* $\tilde{v} = (\tilde{v}_{\kappa,k})$, each of which is determined by a finite vector $v = (v_\kappa)_{\kappa \in F_v}$ in $\mathbb{R}^{|F_v|}$ and a $d \times d$ real matrix A by the equations

$$\tilde{v}_{\kappa,k} = v_\kappa - AZk, \quad k \in \mathbb{Z}^d.$$

Note that $\mathcal{H}_v^{\text{aff}}$ is a linear subspace of $\mathcal{H}_v$. For if $\tilde{u}, \tilde{v}$ correspond to (u, A) and (v, B) respectively, then

$$\begin{aligned} \tilde{u}_{\kappa,k} + \tilde{v}_{\kappa,k} &= u_\kappa - AZk + v_\kappa - BZk \\ &= (u_\kappa + v_\kappa) - (A + B)Zk \end{aligned}$$

which defines the infinitesimal velocity corresponding to $(u + v, A + B)$.

In Definition 2.2(ii) an infinitesimal flex is denoted by a *vector-matrix pair* (u, A) with $u \in \mathbb{R}^{d|F_v|}$, corresponding to vertex displacement velocities in a unit cell, and a $d \times d$ matrix A corresponding to axis displacement velocities. In the next theorem we identify this vector-matrix data (u, A) with*vector-vector* data (u, AZ) in which AZ denotes a vector of length d^2 given by the d vectors $AZ\gamma_1, \ldots, AZ\gamma_d$.

Theorem 2.7. *Let $\mathcal{C}$ be a crystal framework in $\mathbb{R}^d$ with translation group $\mathcal{T}$ and period vector matrix Z. Then the restriction of the rigidity matrix transformation $R(\mathcal{C}) : \mathcal{H}_v \to \mathcal{H}_e$ to the finite-dimensional space $\mathcal{H}_v^{\mathrm{aff}}$ has representing matrix $R(\mathcal{M}, \mathbb{R}^{d^2})$. Moreover, the following statements are equivalent:*

(i) *The vector-matrix pair (u, A), with $u \in \mathbb{R}^{d|F_v|}$ and $A \in M_d(\mathbb{R})$, is an affinely periodic infinitesimal flex for $\mathcal{C}$.*
(ii) *The vector-vector pair (u, AZ) lies in the nullspace of $R(\mathcal{M}, \mathbb{R}^{d^2})$.*
(iii) *The vector $\tilde{u}$ lies in the nullspace of $R(\mathcal{C})$, where $\tilde{u} \in \mathcal{H}_v$ is the vector defined by the affinely periodic extension formula*

$$\tilde{u}_{\kappa,k} = u_\kappa - AZk, \quad k \in \mathbb{Z}^d.$$

Proof. Let $e = [p_{\kappa,0}, p_{\tau,\delta(e)}]$ be an edge in F_e and as before write v_e for the edge vector $p_{\kappa,0} - p_{\tau,\delta(e)}$. Note that the term $A(p_\tau - p_{\tau,\delta(e)})$ in Definition 2.2 is equal to $A(\delta_1 a_1 + \cdots + \delta_d a_d) = AZ(\delta_1\gamma_1 + \cdots + \delta_d\gamma_d)$. Let $\eta_e, e \in F_e$, be the standard basis vectors for $\mathbb{R}^{|F_e|}$. The inner product in Definition 2.2 may be written

$$\begin{aligned}
\langle p_{\kappa,0} - p_{\tau,\delta(e)}, & u_\kappa - u_\tau + A(p_{\tau,0} - p_{\tau,\delta(e)})\rangle \\
&= \langle v_e, u_\kappa - u_\tau\rangle + \langle v_e, \sum_i AZ(\delta_i\gamma_i)\rangle \\
&= \langle R(\mathcal{M})u, \eta_e\rangle + \sum_i \langle \delta_i v_e, AZ\gamma_i\rangle \\
&= \langle R(\mathcal{M})u, \eta_e\rangle + \langle (X(\mathcal{M})(AZ), \eta_e\rangle \\
&= \langle R(\mathcal{M}, \mathbb{R}^{d^2})(u, AZ), \eta_e\rangle.
\end{aligned}$$

Thus the equivalence of (i) and (ii) follows.

That the statements (i) and (ii) are equivalent to (iii) follows from the following calculation. Let $w = (w_\kappa)$ be a vector in $\mathbb{R}^{d|F_v|}$, let (w, B) be vertex-matrix data and let $\tilde{w}$ be the associated velocity vector in $\mathcal{H}_v^{\mathrm{aff}}$. Then, for e in F_e, given by $[p_{\kappa,0}, p_{\tau,\delta(e)}]$, the e, k coordinate of the vector $R(\mathcal{C})\tilde{w}$ in $\mathcal{H}_e$ is given by

$$\begin{aligned}
(R(\mathcal{C})\tilde{w})_{e,k} &= \langle v_e, \tilde{w}_{\kappa,k}\rangle - \langle v_e, \tilde{w}_{\tau,k+\delta(e)}\rangle \\
&= \langle v_e, \tilde{w}_{\kappa,k} - \tilde{w}_{\tau,k}\rangle + \langle v_e, BZ\delta(e)\rangle \\
&= \langle v_e, w_\kappa - w_\tau\rangle + \sum_i \langle \delta_i v_e, BZ\gamma_i\rangle \\
&= (R(\mathcal{M})w)_e + (X(\mathcal{M})(BZ))_e \\
&= (R(\mathcal{M}, \mathbb{R}^{d^2})(w, BZ))_e.
\end{aligned}$$

□

We say that a periodic bar-joint framework $\mathcal{C}$ is *affinely periodically infinitesimally rigid* if the only affinely periodic infinitesimal flexes are those in the finite-dimensional space $\mathcal{H}_{\mathrm{rig}} \subseteq \mathcal{H}_v$ of infinitesimal flexes induced by

isometries of $\mathbb{R}^d$. The following theorem was obtained by Borcea and Streinu [4] where the rigidity matrix was derived from a projective variety identification of the finite flexing space (configuration space) of the framework.

Theorem 2.8. *A periodic bar-joint framework $\mathcal{C}$ with motif $\mathcal{M}$ is affinely periodically infinitesimally rigid if and only if*

$$\operatorname{rank} R(\mathcal{M}, \mathbb{R}^{d^2}) = d|F_v| + d(d-1)/2.$$

Proof. In view of the description of affinely periodic flexes in Theorem 2.7 the rigidity requirement is equivalent to the equality

$$\operatorname{rank} R(\mathcal{M}, \mathbb{R}^{d^2}) = d|F_v| + d^2 - \dim \mathcal{H}_{rig}.$$

The space $\mathcal{H}_{\mathrm{rig}}$ has dimension $d(d+1)/2$, since a basis may be provided by d infinitesimal translations and $d(d-1)/2$ independent infinitesimal rotations, and so the proof is complete. □

The theorem above is generalised by the Maxwell–Calladine formula in Theorem 2.10 which incorporates the following space of self-stresses for $\mathcal{C}$ relative to $\mathcal{E}$. Symmetry adapted variants are given in Corollary 3.3 and Theorem 3.5.

Definition 2.9. An *infinitesimal self-stress* of a countable bar-joint framework (G, p), with vertices of finite degree, is a vector in $\mathcal{H}_e$ lying in the cokernel of the infinite rigidity matrix $R(G, p)$.

For a periodic framework $\mathcal{C}$ such vectors $w = (w_{e,k})$ are characterised, as in the case of finite frameworks, by a linear dependence row condition, namely

$$\sum_{e,k} w_{e,k} R(\mathcal{C})_{((e,k),(\kappa,\sigma,l))} = 0, \qquad \text{for all } \kappa, \sigma, l.$$

Alternatively this can be paraphrased in terms of local conditions

$$\sum_{(\tau,l):[p_{\kappa,k},p_{\tau,l}]\in\mathcal{C}_e} w_{\tau,l}(p_{\kappa,k} - p_{\tau,l}) = 0,$$

which may be interpreted as a balance of internal stresses.

From the calculation in Theorem 2.7 it follows that the rigidity matrix $R(\mathcal{C})$ maps affinely periodic displacement velocities to periodic vectors in $\mathcal{H}_e$. Write $\mathcal{H}_e^{\mathrm{per}}$ for the vector space of such vectors and $\mathcal{H}_{\mathrm{str}}^{\mathrm{aff}}$ (resp. $\mathcal{H}_{\mathrm{str}}^{\mathcal{E}}$) for the subspace of periodic self-stresses that correspond to vectors in the cokernel of $R(\mathcal{M}, \mathbb{R}^{d^2})$ (resp. $R(\mathcal{M}, \mathcal{E})$, and let $s_{\mathcal{E}}$ denote the vector space dimension of $\mathcal{H}_{\mathrm{str}}^{\mathcal{E}}$, Also let $f_{\mathcal{E}}$ denote the dimension of the space $\mathcal{H}_{\mathrm{rig}}^{\mathcal{E}}$ of infinitesimal affinely periodic rigid motions with data (u, A) with affine velocity matrix $A \in \mathcal{E}$, and let $m_{\mathcal{E}}$ denote the dimension of the space of infinitesimal mechanisms of the same form. Thus $m_{\mathcal{E}} = \dim \mathcal{H}_{\mathrm{mech}}^{\mathcal{E}}$ where $\mathcal{H}_{\mathrm{fl}}^{\mathcal{E}} = \mathcal{H}_{\mathrm{mech}}^{\mathcal{E}} \oplus \mathcal{H}_{\mathrm{rig}}^{\mathcal{E}}$.

For $\mathcal{E} = \mathbb{R}^{d^2}$ the following theorem is Proposition 3.11 of [4].

Theorem 2.10. *Let $\mathcal{C}$ be a crystal framework in $\mathbb{R}^d$ with given translational periodicity and let $\mathcal{E} \subset M_d(\mathbb{R})$ be a linear space of affine velocity matrices. Then*

$$m_{\mathcal{E}} - s_{\mathcal{E}} = d|F_v| + \dim \mathcal{E} - |F_e| - f_{\mathcal{E}}$$

where $m_{\mathcal{E}}$ is the dimension of the space of non rigid motion $\mathcal{E}$-affinely periodic infinitesimal flexes and where $s_{\mathcal{E}}$ is the dimension of the space of periodic self-stresses for $\mathcal{C}$ and $\mathcal{E}$.

Proof. By Theorem 2.7 we may identify $\mathcal{H}_v^{\text{aff}}$ with $\mathbb{R}^{d|F_v|} \oplus \mathbb{R}^{d^2}$ and we may identify $\mathcal{H}_v^{\mathcal{E}}$ with $\mathbb{R}^{d|F_v|} \oplus \mathcal{E}$, the domain space of $R(\mathcal{M}, \mathcal{E})$. We have

$$\mathbb{R}^{d|F_v|} \oplus \mathcal{E} = (\mathcal{H}_v^{\mathcal{E}} \ominus \mathcal{H}_{\text{mech}}^{\mathcal{E}}) \oplus \mathcal{H}_{\text{mech}}^{\mathcal{E}} \oplus \mathcal{H}_{\text{rig}}^{\mathcal{E}}$$

and

$$\mathbb{R}^{|F_e|} = (\mathcal{H}_e^{\text{per}} \ominus \mathcal{H}_{\text{str}}^{\mathcal{E}}) \oplus \mathcal{H}_{\text{str}}^{\mathcal{E}}.$$

With respect to this decomposition $R(\mathcal{M}, \mathcal{E})$ takes the block form

$$R(\mathcal{M}, \mathcal{E}) = \begin{bmatrix} R & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

with R an invertible matrix. Since R is square it follows that

$$d|F_v| + \dim \mathcal{E} - (m_{\mathcal{E}} + f_{\mathcal{E}}) = |F_e| - s_{\mathcal{E}},$$

as required. □

We remark that one may view an affinely periodic infinitesimal flex (u, A) as an infinitesimal flex of a finite framework located in the set $\mathbb{R}^3/\mathcal{T}$ regarded as a distortable torus. The edges of such a framework are determined by the motif and include "wrap-around" edges determined naturally by the edges with nontrivial exponent. Here, in effect, the opposing (parallel) faces of the unit cell parallelepiped are identified, so that one can define (face-to-face) periodicity in the natural way as the corresponding concept for translation periodicity. Similarly one can specify flex periodicity modulo an affine velocity impetus of the torus, given by the matrix A. See also Whiteley [25].

2.4. The general form of $R(\mathcal{M}, \mathbb{R}^{d^2})$. We have so far made the notationally simplifying assumption that the motif set F_e consists of edges with at least one vertex in the set F_v. One can always choose such motifs and the notion of the exponent of such edges is natural. However, more general motifs are natural should one wish to highlight polyhedral units or symmetry and we now note the minor adjustments needed for the general case. See also [18].

Let $e = [p_{\kappa,k}, p_{\tau,k'}]$ be an edge of F_e with both k and k' not equal to the zero multi-index and suppose first that $\kappa \neq \tau$. Then the column entries in row e for the κ label are, as before, the entries of the edge vector $v_e = p_{\kappa,k} - p_{\tau,k'}$, and the entries in the τ columns are, as before, those of $-v_e$. The final d^2 entries are modified using the generalised edge exponent $\delta(e) = k' - k$.

Note that this row is determined up to sign by the vertex order for the framework edge and this sign could be fixed by imposing an order on the edge. (See also the discussion in Section 4 of the various labelled graphs.) If e is a reflexive edge in the sense that $\kappa = \tau$ then once again the entry for the κ labelled columns are zero and the final d^2 entries similarly use the generalised exponent.

3. Symmetry equations and counting formulae

3.1. Finite-dimensional representations of the space group. We first obtain symmetry equations for the affine rigidity matrix $R(\mathcal{M}, \mathbb{R}^{d^2})$ with respect to representations of the space group of $\mathcal{C}$.

Let $\mathcal{G}(\mathcal{C})$ be the abstract crystallographic group of the crystal framework $\mathcal{C} = (F_v, F_e, \mathcal{T})$. This is the space group of isometric maps of $\mathbb{R}^d$ which map $\mathcal{C}$ to itself, viewed as an abstract group. This entails the following two assertions.

(i) Each symmetry element g in $\mathcal{G}(\mathcal{C})$ acts as a permutation of the framework vertex labels,

$$(\kappa, k) \to g \cdot (\kappa, k),$$

and this permutation induces a permutation of the framework edge labels,

$$(e, k) \to g \cdot (e, k).$$

(ii) There is a representation $\rho_{sp} : g \to T_g$ of $\mathcal{G}(\mathcal{C})$ in $\mathrm{Isom}(\mathbb{R}^d)$, the spatial representation, which extends the indexing map $\mathbb{Z}^d \to \mathcal{T}$ and is such that for each framework vertex

$$p_{g\cdot(\kappa,k)} = T_g(p_{\kappa,k}).$$

The permutation action on vertex labels simplifies when the symmetry g is *separable* with respect to $\mathcal{T}$. By this we mean that $g \cdot (\kappa, k) = (g \cdot \kappa, g \cdot k)$ where $\kappa \to g \cdot \kappa$ and $k \to g \cdot k$ are group actions on F_v and $\mathcal{T}$ for the group generated by g. This may occur, for example, for a translation group with orthonormal period vectors and symmetries of axial reflection, or rotation by $\pi/2$ or π. A simplification also occurs for a weaker property, namely when the symmetry g is *semiseparable* relative to the translation group (or motif). This requires only that there is an induced action $\kappa \to g \cdot \kappa$ on the vertex set of the motif, so that

$$g \cdot (\kappa, k) = (g \cdot \kappa, k')$$

where k' generally depends on g, κ and k. This is the case, for example, for the rotational symmetries of the kagome framework and the Roman tiling framework given in Section 4. The group $\mathcal{T}$ is taken to be the natural (maximal) translational symmetry group. In the case of the Roman tile framework and rotation by $\pi/6$ the action on F_v is free, whereas for this rotation symmetry and the kagome framework the action is not free.

In general a finite order symmetry g of $\mathcal{C}$ need not induce a permutation action on F_v. One need only consider rotation symmetries for the basic grid framework whose framework vertex set is $\mathbb{Z}^2$ together with the nonstandard translation group with vertex motif $F_v = \{(0,0),(0,1)\}$.

Assume that the group action of $\mathcal{G}(\mathcal{C})$ on F_v is semiseparable. On vertex coordinate labels define the corresponding action $g \cdot (\kappa, \sigma, k) = (g \cdot \kappa, \sigma, k')$, where σ, in case $d = 3$, denotes x, y or z. Let ρ_v be the permutation representation of $\mathcal{G}(\mathcal{C})$ on the velocity space $\mathcal{H}_v$, as (linear) vector space transformations, which is defined by

$$(\rho_v(g)u)_{\kappa,\sigma,k} = u_{g^{-1}\cdot(\kappa,\sigma,k)}$$

Similarly let ρ_e be the representation of $\mathcal{G}(\mathcal{C})$ on $\mathcal{H}_e$ such that

$$(\rho_e(g)w)_{e,k} = w_{g^{-1}\cdot(e,k)}.$$

These linear transformations may also be defined as forward shifts in the sense that, for $l \in \mathbb{Z}^d$,

$$\rho_v(l) : \xi_{\kappa,\sigma,k} \to \xi_{\kappa,\sigma,k+l},$$

$$\rho_e(l) : \eta_{e,k} \to \eta_{e,k+l}.$$

where $\{\xi_{\kappa,\sigma,k}\}$, $\{\eta_{e,k}\}$ are natural coordinate "bases". Here $\eta_{e,k}$ denotes the element of $\mathcal{H}_e$, given in terms of the Kronecker symbol, by

$$(\eta_{e,k})_{e',k'} = \delta_{e,e'}\delta_{k,k'}.$$

The totality of these vectors gives a normal vector space basis for the set of finitely nonzero vectors of $\mathcal{H}_v$, and a general vector in $\mathcal{H}_v$ has an associated well-defined representation as an infinite sum. The basis $\{\xi_{\kappa,\sigma,k}\}$ is similarly defined. Note that, for $d = 3$, $\rho_v(g)$ has multiplicity three, associated with the decomposition

$$\mathcal{H}_v = \prod_{\kappa \in F_v, k \in \mathbb{Z}^d} \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R},$$

whereas $\rho_e(g)$ has multiplicity one.

We now define a representation $\tilde{\rho}_v$ of the space group $\mathcal{G}(\mathcal{C})$ as affine maps of $\mathcal{H}_v$. If T is an isometry of $\mathbb{R}^d$ let $\tilde{T}$ be the affine map on $\mathcal{H}_v$ determined by coordinate-wise action of T and let

$$\tilde{\rho}_v(g) := \rho_v(g)\tilde{T}_g = \tilde{T}_g\rho_v(g)$$

where $g \to T_g$ is the affine isometry representation ρ_{sp}. One may also identify $\tilde{\rho}_v(\cdot)$ as the tensor product representation $\rho_n(\cdot) \otimes \rho_{\mathrm{sp}}(\cdot)$ where $\rho_n(\cdot)$ is the multiplicity one version of $\rho_v(\cdot)$ (where the subscript "n" stands for "node"). Here the transformation $\rho_n(g)$ is linear while $\rho_{\mathrm{sp}}(g)$ may be affine.

In the next lemma and the ensuing discussion we make explicit the action of the transformation $\tilde{\rho}_v(g)$ on the space of affinely periodic velocity vectors.

Lemma 3.1. *Let $\mathcal{C}$ be a crystal framework with motif (F_v, F_e) and let g be a semiseparable element of the crystallographic space group $\mathcal{G}(\mathcal{C})$. Let $T_g = TB$ be the translation-linear factorisation of the isometry $T_g = \rho_{\rm sp}(g)$, where $Tw = w - \gamma$ for some $\gamma \in \mathbb{R}^d$. Also let $\tilde{u}$ be a velocity vector in $\mathcal{H}_v^{\rm aff}$ which corresponds to the vertex-matrix pair (u, A) (as in Definition 2.2 and Theorem 2.7). Then $\tilde{\rho}_v(g)\tilde{u}$ is a velocity vector $\tilde{v}$ in $\mathcal{H}_v^{\rm aff}$ corresponding to a pair (v, BAB^{-1}).*

Proof. Let $(\kappa', k') = g \cdot (\kappa, k)$. Then

$$\begin{aligned}(\tilde{\rho}(g)\tilde{u})_{\kappa',k'} = T_g\tilde{u}_{g^{-1}\cdot(\kappa',k')} &= T_g\tilde{u}_{\kappa,k} = T_g(u_\kappa - AZk) \\ &= B(u_\kappa - AZk) - \gamma = w_1 - BAZk\end{aligned}$$

with $w_1 = Bu_\kappa - \gamma = T_g u_\kappa$.

On the other hand k may be related to k'. We have

$$p_{\kappa',0} + Zk' = p_{\kappa',k'} = T_g p_{\kappa,k} = T_g(p_{\kappa,0} + Zk)$$

and so, since $T_g^{-1} = B^{-1}T^{-1}$,

$$Zk = T_g^{-1}(Zk' + p_{\kappa',0}) - p_{\kappa,0} = B^{-1}(Zk' + p_{\kappa',0} + \gamma) - p_{\kappa,0}.$$

Thus Zk has the form $B^{-1}Zk' + w_2$ where $w_2 = T_g^{-1}p_{\kappa',0} - p_{\kappa,0}$ is independent of k'. Substituting,

$$(\tilde{\rho}(g)\tilde{u})_{\kappa',k'} = w_1 - BAZk = (w_1 - BAw_2) - BAB^{-1}Zk'.$$

Also, since g is semiseparable the vector $v = w_1 - BAw_2$ depends only on κ', and so the proof is complete. □

It follows from the lemma that if all symmetries are semiseparable for F_v then $\tilde{\rho}_v$ determines a finite-dimensional representation π_v of $\mathcal{G}(\mathcal{C})$ which is given by restriction to $\mathcal{H}_v^{\rm aff}$. We write this as $\pi_v(g) = \tilde{\rho}_v(g)|_{\mathcal{H}_v^{\rm aff}}$. The proof of the lemma also provides detail for a coordinatisation of this representation and we now give the details of this.

We first introduce what might be referred to as the unit cell representation of $\mathcal{G}(\mathcal{C})$ (or a subgroup of semiseparable symmetries). This is the finite-dimensional representation μ_v on $\mathbb{R}^{d|F_v|}$ given by

$$(\mu_v(g)u)_\kappa = T_g u_{g^{-1}\cdot\kappa}, \quad \kappa \in F_v.$$

That this is a representation follows from the observation that it is identifiable as a tensor product representation

$$\mu_v : g \to \nu_n(g) \otimes \rho_{\rm sp}(g)$$

where ν_n is the vertex class permutation representation on $\mathbb{R}^{|F_v|}$, so that $\mu_v = \nu_n \otimes Id_3$. This representation features as a subrepresentation of π_v as we see below.

There is a companion edge class representation for the range space $\mathcal{H}_e$. Define first the linear transformation representation ρ_e of $\mathcal{G}(\mathcal{C})$ on $\mathcal{H}_e$ given by

$$(\rho_e(g)w)_{f,k} = w_{g^{-1}\cdot(f,k)}, \qquad f \in F_e, k \in \mathbb{Z}^d.$$

This in turn provides a finite-dimensional representation π_e of $\mathcal{G}(\mathcal{C})$ on the finite-dimensional space $\mathcal{H}_e^{per}$ of *periodic* vectors, that is, the space of $\rho_e(l)$-periodic vectors for $l \in \mathbb{Z}^d$. These periodic vectors w are determined by the (scalar) values $w_{f,0}$, for $f \in F_e$. Writing $\eta_f, f \in F_e$, for the natural basis elements for $\mathbb{R}^{|F_e|}$ we have the edge class permutation matrix representation

$$\pi_e(g)\eta_f = \eta_{g^{-1}\cdot f}, \quad f \in F_e, \quad g \in \mathcal{G}(\mathcal{C}).$$

In the formalism preceding Theorem 2.7, we have identified $\mathcal{H}_v^{\text{aff}}$ with a finite-dimensional vector space which we now denote as $\mathcal{D}$. This is the domain of $R(\mathcal{M}, \mathbb{R}^{d^2})$;

$$\mathcal{D} := \mathbb{R}^{d|F_v|} \oplus \mathbb{R}^{d^2} = (\mathbb{R}^{|F_v|} \otimes \mathbb{R}^d) \oplus (\mathbb{R}^d \oplus \cdots \oplus \mathbb{R}^d).$$

Let $D : \mathcal{H}_v^{\text{aff}} \to \mathcal{D}$ be the linear identification in which the vector $D\tilde{u}$ corresponds to the matrix-data form (u, A) written as a row vector of columns. Thus

$$\pi_v(\cdot) = D\tilde{\rho}_v(\cdot)|_{\mathcal{H}_v^{\text{aff}}} D^{-1}.$$

Note that any transformation in $\mathcal{L}(\mathcal{D})$, the space of all linear transformations on $\mathcal{D}$, has a natural 2×2 block-matrix representation. With respect to this we have the block form

$$\pi_v(g) = \begin{bmatrix} \mu_v(g) & \Phi_1(g) \\ 0 & \Phi_2(g) \end{bmatrix}, \quad g \in \mathcal{G}(\mathcal{C}),$$

where $\Phi_2(g)(A) = BAB^{-1}$. To see this, return to the proof of Lemma 3.1 and note that in the correspondence

$$\tilde{\rho}_v(g) : (u, A) \to (v, BAB^{-1}),$$

if $A = 0$ then $v_{\kappa'} = T_g u_\kappa$ and so $v = \mu_v(g)u$ in this case.

To see the nature of the linear transformation Φ_1 note that from the lemma that

$$(\Phi_1(g)(A))_{\kappa'} = (-BAw_2)_{\kappa'} = -BA(T_g^{-1}p_{\kappa',0} - p_{\kappa,0}).$$

In particular if g is a fully separable symmetry for the periodicity, so that $T_g p_{\kappa,0}$ has the form $p_{\kappa',0}$ for all κ, then Φ_2 is the zero map and a simple block diagonal form holds for $\pi_v(g)$, namely,

$$\pi_v(g)(u, A) = (\mu_v(g)u, T_g A T_g^{-1}).$$

In fact it follows that we also have this block diagonal form if g is a symmetry such that T_g acts as a permutation of the motif set $F_v = \{p_{\kappa,0}\}$. Such a symmetry is necessarily of finite order. When this is the case we shall say that the symmetry g *acts on* F_v. (See Corollary 3.3, Theorem 3.5 and Corollary 3.6.)

3.2. Symmetry equations. The following theorem is a periodic framework version of the symmetry equation given in Owen and Power [16].

Theorem 3.2. *Let $\mathcal{C}$ be a crystal framework in $\mathbb{R}^d$ and let $g \to T_g$ be a representation of the space group $\mathcal{G}(\mathcal{C})$ as isometries of $\mathbb{R}^d$.*

(a) *For the rigidity matrix transformation $R(\mathcal{C}) : \mathcal{H}_v \to \mathcal{H}_e$,*

$$\rho_e(g)R(\mathcal{C}) = R(\mathcal{C})\tilde{\rho}_v(g), \quad g \in \mathcal{G}(\mathcal{C}).$$

(b) *Let $\mathcal{G} \subseteq \mathcal{G}(\mathcal{C})$ be a subgroup of semiseparable symmetries. Then, for the affinely periodic rigidity matrix $R(\mathcal{M}, \mathbb{R}^{d^2})$ determined by the motif $\mathcal{M} = (F_v, F_e)$,*

$$\pi_e(g)R(\mathcal{M}, \mathbb{R}^{d^2}) = R(\mathcal{M}, \mathbb{R}^{d^2})\pi_v(g), \quad g \in \mathcal{G},$$

where π_e is the edge class representation of $\mathcal{G}$ in $\mathbb{R}^{|F_e|}$ induced by ρ_e and where π_v is the representation of $\mathcal{G}$ in $(\mathbb{R}^{|F_v|} \otimes \mathbb{R}^d) \oplus \mathbb{R}^{d^2}$ induced by $\tilde{\rho}_v$.

Proof. By Theorem 2.7 and Lemma 3.1 we have an identification of the space $\mathcal{H}_v^{\text{aff}}$ of affinely periodic velocity vectors with the domain of $R(\mathcal{M}, \mathbb{R}^{d^2})$. It follows from the preceding discussion that the symmetry equation given in (b) follows from the symmetry equation in (a).

To verify (a) observe first that if (G, p) is the framework for $\mathcal{C}$ with labelling as given in Definition 2.5 then the framework for $(G, g \cdot p)$ with relabelled framework vector $g \cdot p$ with $(g \cdot p)_{\kappa,k} = p_{g\cdot(\kappa,k)}$ has rigidity matrix $R(G, g \cdot p)$. Examining matrix entries shows that this matrix is equal to the row and column permuted matrix $\rho_e(g)^{-1}R(G, p)\rho_v(g)$.

On the other hand the row for the edge $e = [p_{\kappa,k}, p_{\tau,l}]$ has entries in the columns for κ and for τ given by the coordinates of $T_g v_e$ and $-T_g v_e$ respectively, where $v_e = p_{\kappa,k} - p_{\tau,l}$. If $T_g = TB$ is the translation-linear factorisation, with B an orthogonal matrix, note that

$$T_g v_e = TBp_{\kappa,k} - TBp_{\tau,l} = Bv_e.$$

Here v_e is a column vector, while the κ column entries are given by the row vector transpose $(Bv_e)^t = v_e^t B^t = v_e^t B^{-1}$. Thus

$$R(G, g \cdot p) = R(G, p)(I \otimes B^{-1}).$$

Note also, that for any rigidity matrix R and any affine translation S we have $R(I \otimes S) = R$, as transformations from $\mathcal{H}_v$ to $\mathcal{H}_e$. Thus we also have

$$R(G, g \cdot p) = R(G, g \cdot p)(I \otimes T^{-1}) = R(G, p)(I \otimes B^{-1})(I \otimes T^{-1})$$

which is $R(G, p)(I \otimes T_g^{-1})$ and (a) follows. □

As an application we obtain general Fowler–Guest style Maxwell Calladine formulae relating the traces (characters) of symmetries g in various subrepresentations of π_v, π_e derived from flexes and stresses.

Let $\mathcal{E}$ be a fixed subspace of $d \times d$ axial velocity matrices A, let g be a semiseparable symmetry for which $T_g A = A T_g$ for all $A \in \mathcal{E}$, and let H be the cyclic subgroup generated by g. It follows from Lemma 3.1 that there is a restriction representation $\pi_v^{\mathcal{E}}$ of H on the space

$$\mathcal{H}_v^{\mathcal{E}} := \mathbb{R}^{d|F_v|} \oplus \mathcal{E}$$

and that there is an associated intertwining symmetry equation. This equation implies that the space of $\mathcal{E}$-affinely periodic infinitesimal flexes is an invariant subspace for the representation π_v of H. Thus there is an associated restriction representation of H, namely $\pi_{\mathrm{fl}}^{\mathcal{E}}$. We note the following subrepresentations of $\pi_v^{\mathcal{E}}$:

- $\pi_{\mathrm{fl}}^{\mathcal{E}}$ on the invariant subspace $\mathcal{H}_{\mathrm{fl}}^{\mathcal{E}} := \ker R(\mathcal{M}, \mathcal{E})$,
- $\pi_{\mathrm{rig}}^{\mathcal{E}}$ on the invariant subspace $\mathcal{H}_{\mathrm{rig}}^{\mathcal{E}} := \mathcal{H}_v^{\mathcal{E}} \cap \mathcal{H}_{\mathrm{rig}}$,
- $\pi_{\mathrm{mech}}^{\mathcal{E}}$ on the invariant subspace $\mathcal{H}_{\mathrm{mech}}^{\mathcal{E}} := \mathcal{H}_{\mathrm{fl}}^{\mathcal{E}} \ominus \mathcal{H}_{\mathrm{rig}}^{\mathcal{E}}$.

Also, we have a subrepresentation of π_e^{per}, namely:

- $\pi_{\mathrm{str}}^{\mathcal{E}}$ on the invariant subspace $\mathcal{H}_{\mathrm{str}}^{\mathcal{E}} := \operatorname{coker} R(\mathcal{M}, \mathcal{E})$.

Corollary 3.3. *Let $\mathcal{C}$ be a crystal framework and let g be a semiseparable space group symmetry or a symmetry which acts on F_v. Also, let $\mathcal{E} \subseteq \mathbb{R}^{d^2}$ be a space of affine velocity matrices which commute with T_g. Then*

$$tr(\pi_{\mathrm{mech}}^{\mathcal{E}}(g)) - tr(\pi_{\mathrm{str}}^{\mathcal{E}}(g)) = tr({\pi_v}^{\mathcal{E}}(g)) - tr(\pi_e(g)) - tr(\pi_{\mathrm{rig}}^{\mathcal{E}}(g)).$$

Proof. The rigidity matrix $R(\mathcal{M}, \mathcal{E})$ effects an equivalence between the subrepresentations of H on the spaces $\mathcal{H}_v^{\mathcal{E}} \ominus \mathcal{H}_{\mathrm{fl}}^{\mathcal{E}}$ and $\mathcal{H}_e \ominus \mathcal{H}_{\mathrm{str}}^{\mathcal{E}}$. Thus the traces of the representations of g on these spaces are equal and from this the identity follows. Indeed, with respect to the decompositions

$$\mathcal{H}_v^{\mathcal{E}} = (\mathcal{H}_v^{\mathcal{E}} \ominus \mathcal{H}_{\mathrm{fl}}^{\mathcal{E}}) \oplus \mathcal{H}_{\mathrm{mech}}^{\mathcal{E}} \oplus \mathcal{H}_{\mathrm{rig}}^{\mathcal{E}}$$
$$\mathcal{H}_e^{\mathcal{E}} = (\mathcal{H}_e \ominus \mathcal{H}_{\mathrm{str}}^{\mathcal{E}}) \oplus \mathcal{H}_{\mathrm{str}}^{\mathcal{E}}$$

the rigidity matrix has block form

$$R(\mathcal{M}, \mathcal{E}) = \begin{bmatrix} R_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with R_1 an invertible square matrix. Thus R_1 effects an equivalence between the representations on the first summands above. It follows that the traces of these representations of g are equal and so

$$tr(\pi_v^{\mathcal{E}}(g)) - tr(\pi_{\mathrm{mech}}^{\mathcal{E}}(g)) - tr(\pi_{\mathrm{rig}}^{\mathcal{E}}(g)) = tr(\pi_e(g)) - tr(\pi_{\mathrm{str}}(g))$$

as required. □

Recall that the *point group* of a crystal framework is the quotient group $\mathcal{G}_{\mathrm{pt}}(\mathcal{C}) = \mathcal{G}(\mathcal{C})/\mathbb{T}$ determined by a maximal subgroup $\mathcal{T} = \{T_l : l \in \mathbb{Z}^d\}$. In the next theorem we suppose that the space group is separable, so that $\mathcal{G}(\mathcal{C})$ is isomorphic to $\mathbb{Z}^d \times \mathcal{G}_{\mathrm{pt}}(\mathcal{C})$. Write $\dot{g}$ for the coset element in $\mathcal{G}_{\mathrm{pt}}(\mathcal{C})$ of an element g of the space group. Then, following an appropriate shift, one

can recoordinatise the bar-joint framwork $\mathcal{C}$ so that for the natural inclusion map $i : \mathcal{G}_{\rm pt}(\mathcal{C}) \to \mathcal{G}(\mathcal{C})$ the map $\dot g \to T_{i(\dot g)}$ is a representation by *linear* isometries. In this case we obtain from the restriction of π_v and π_e to $\mathcal{G}_{\rm pt}(\mathcal{C})$ the representations

$$\dot\pi_v : \mathcal{G}_{\rm pt}(\mathcal{C}) \to \mathcal{L}(\mathcal{D}), \quad \dot\pi_e : \mathcal{G}_{\rm pt}(\mathcal{C}) \to \mathcal{L}(\mathbb{R}^{|F_e|}).$$

In fact $\dot\pi_v$ has a block diagonal form with respect to the direct sum decomposition $\mathcal{D} = \mathbb{R}^{d|F_v|} \oplus \mathbb{R}^{d^2}$ and is well-defined by the recipe

$$\dot\pi_v(\dot g) = \begin{bmatrix} \mu_v(g) & 0 \\ 0 & \Phi_2(g) \end{bmatrix}, \quad g \in \dot g \in \mathcal{G}_{\rm pt}(\mathcal{C}),$$

where $\Phi_2(g) : A \to T_g A T_g^{-1}$. Also $\dot\pi_e$ is similarly explicit and agrees with the natural edge class permutation representation ν_e which is well-defined by

$$\nu_e(\dot g)\eta_f = \eta_{g^{-1}.f}, \quad f \in F_e, \quad g \in \dot g \in \mathcal{G}_{\rm pt}(\mathcal{C}).$$

Theorem 3.4. *For a crystal framework with separable space group $\mathcal{G}(\mathcal{C})$,*

$$\dot\pi_e(\dot g)R(\mathcal{M}, \mathbb{R}^{d^2}) = R(\mathcal{M}, \mathbb{R}^{d^2})\dot\pi_v(\dot g), \quad \dot g \in \mathcal{G}_{\rm pt}(\mathcal{C}).$$

Moreover for a fixed element $\dot g \in \mathcal{G}_{\rm pt}(\mathcal{C})$ the individual symmetry equation

$$\dot\pi_e(\dot g)R(\mathcal{M}, \mathcal{E}) = R(\mathcal{M}, \mathcal{E})\dot\pi_v(\dot g)$$

holds where $\mathcal{E}$ is any space of matrices which is invariant under the map $A \to T_{\dot g} A T_{\dot g}^{-1}$.

Proof. The equations follow from those of Theorem 3.2. □

3.3. Symmetry-adapted Maxwell–Calladine equations. We now give Maxwell–Calladine formulae for symmetry respecting mechanisms and self-stresses and which which derive from consideration of velocity vector spaces in which *all* vectors are fixed under an individual symmetry.

Suppose first that g is a space group element in $\mathcal{G}(\mathcal{C})$ which is of separable type or which acts on F_v. Then T_g is an orthogonal linear transformation and $\pi_v(g)$ takes the block diagonal form

$$\pi_v(g)(u, A) = (\mu_v(g)u, T_g A T_g^{-1}).$$

Let $\mathcal{H}_v^g$ be the subspace of vectors (v, A) which are fixed by the linear transformation $\pi_v(g)$. Then $\mathcal{H}_v^g$ splits as a direct sum which we write as $\mathcal{F}_g \oplus \mathcal{E}_g$. where $\mathcal{E}_g$ is the space of matrices commuting with T_g and $\mathcal{F}_g$ is the space of vectors v that are fixed by the linear transformation $\mu_v(g) = \nu_n(g) \otimes T_g$.

Let $\mathcal{H}_e^g$ be the subspace of vectors in $\mathbb{R}^{|F_e|}$ that are fixed by $\pi_e(g)$, so that $\mathcal{H}_e^g$ is naturally identifiable with $\mathbb{R}^{e_g}$ where e_g is the number of orbits of edges in F_e induced by g.

It follows from the symmetry equations in Theorem 3.4 that the rigidity matrix transformation $R(\mathcal{M}, \mathcal{E}_g)$ maps $\mathcal{H}_v^g$ to $\mathcal{H}_e^g$.

In the domain space write $\mathcal{H}^g_{\rm rig}$ for the space of ($\pi_v(g)$-invariant) rigid motions in $\mathcal{H}^g_v \subseteq \mathbb{R}^{d|F_v|} \oplus \mathcal{E}_g$. Similarly write $\mathcal{H}^g_{\rm fl}$ for the space of $\pi_v(g)$-invariant affinely periodic infinitesimal flexes (u, A) in $\mathcal{H}^g_v$ and write $\mathcal{H}^g_{\rm mech}$ for the orthogonal complement space $\mathcal{H}^g_{\rm fl} \ominus \mathcal{H}^g_{\rm rig}$.

In the co-domain space $\mathcal{H}^g_e$ of periodic $\pi_e(g)$-invariant vectors we simply have the subspace $\mathcal{H}^g_{\rm str}$ of $\pi_e(g)$-invariant periodic infinitesimal self-stresses.

Finally let

$$m_g = \dim \mathcal{H}^g_{\rm mech}, \quad s_g = \dim \mathcal{H}^g_{\rm str}, \quad e_g = \dim \mathcal{H}^g_e, \quad f_g = \mathcal{H}^g_{\rm rig}.$$

Noting that $\dim \mathcal{H}^g_v = \dim \mathcal{F}_g + \dim \mathcal{E}_g$ the following symmetry-adapted Maxwell–Calladine formula follows from Corollary 3.3.

Theorem 3.5. *Let $\mathcal{C}$ be a crystal framework and let g be a separable symmetry of $\mathcal{G}(\mathcal{C})$ or a symmetry which acts on F_v. Then*

$$m_g - s_g = \dim \mathcal{F}_g + \dim \mathcal{E}_g - e_g - f_g.$$

The right-hand side of this equation is readily computable as follows:

(i) $\dim \mathcal{F}_g$ is the dimension of the space of vectors in $\mathbb{R}^{|F_v|} \otimes \mathbb{R}^d$ which are fixed by the linear transformation $\nu_n(g) \otimes T_g$.

(ii) $\dim \mathcal{E}_g$ is the dimension of the commutant of T_g, that is of the linear space of $d \times d$ matrices A with $AT_g = T_g A$.

(iii) e_g is the number of orbits in the motif set F_e under the action of g.

(iv) f_g is the dimension of the subspace of infinitesimal rigid motions in $\mathcal{F}_g$.

For a general symmetry g in $\mathcal{G}(\mathcal{C})$, such as a glide reflection, the above applies except for the splitting of the space $\mathcal{H}^g_v$ of $\pi(g)$-fixed vectors. In this case we obtain the Maxwell–Calladine formula

$$m_g - s_g = \dim \mathcal{H}^g_v - e_g - f_g$$

and $\dim \mathcal{H}^g_v$ is computed by appeal to the full block triangular representation of the transformation $\pi_v(g)$.

For strictly periodic flexes rather than affinely periodic flexes, that is, for the case $\mathcal{E} = \{0\}$, there is a modified formula with replacement of f_g by the dimension, f^{per}_g say, of the subspace $\mathcal{H}^{per}_v$ of g-symmetric periodic velocity vectors corresponding to rigid motions. Thus for the identity symmetry g we obtain

$$m - s = d|V_f| - |V_e| - d$$

as already observed in Theorem 2.10.

We remark that the restricted transformation $R(\mathcal{M}, \mathcal{E}_g) : \mathcal{H}^g_v \to \mathcal{H}^g_e$ is a coordinate free form of the orbit rigidity matrix used by Ross, Schulze and Whiteley [20] (see also Schulze and Whiteley [23]). With this matrix they derive counting inequalities as predictors for infinitesimal mechanisms and we give a general such formula in the next corollary. The analysis in [20] also indicates how finite motions (rather than infinitesimal ones) arise if, for

example, the framework is sufficiently generic apart from the demands of a specific symmetry and periodicity.

Corollary 3.6. *Let $\mathcal{C}$ be a crystal framework and let g be a separable symmetry of $\mathcal{G}(\mathcal{C})$ or a symmetry which acts on F_v. If*

$$e_g < \dim \mathcal{F}_g + \dim \mathcal{E}_g - f_g$$

then there exists a non-rigid motion infinitesimal flex which is T_g-symmetric.

Proof. Immediate from Theorem 3.5. □

Finally we remark that if (G, p) is an infinite bar-joint framework in $\mathbb{R}^3$ which is generated by the translates of a finite motif under a translation *subgroup* $\{T_k : k \in \mathbb{Z} \subseteq \mathbb{Z}^3\}$ then one can similarly obtain symmetry equations for the rigidity matrices $R(G, p)$. One such example is the hexahedron tower illustrated in the next section.

4. Some crystal frameworks

We now examine some illustrative frameworks and note in particular that the various Maxwell–Calladine equations give readily computable information. This is a potentially useful tool given the formidable size of the rigidity matrix for even moderate crystal frameworks.

A motif $\mathcal{M} = (F_v, F_e)$ for $\mathcal{C}$, with specified period vectors, is a geometrical construct that carries all the ingredients necessary for determining of various rigidity matrices. It may be specifed in more combinatorial terms as a *labelled motif graph*, by which we mean the simple abstract graph $(V(F_e), F_e)$ (with $V(E_f)$ the set of *all* vertices from F_e) together with a vertex labelling given by the (κ, k) labelling and a distance specification $d : F_e \to \mathbb{R}_+$ given by the lengths of the edges. In our discussions, in contrast with [5], [13], [19], we have not been concerned with combinatorial characterisations, nor with realizability, and so we have had no need for this abstract form. However we note that there are equivalences between such doubly labelled graphs and the formalism of other authors, who incorporate gain graphs (Ross, Schulze, Whiteley [20]), coloured graphs (Malestein and Theran [13]), and quotients of periodic graphs (Borcea and Streinu [4]). For example the gain graph arises from the labelled motif graph $(V(F_e), F_e)$ by merging vertices with the same κ label and assigning to the new edges a direction to one of the vertices, coming from the ancestral vertices (κ, k), (τ, k'), say κ, and assigning to this edge a label, namely $k - k' \in \mathbb{Z}^d$ for the choice of κ (and $k' - k$ for the choice of τ).

4.1. The kagome framework $\mathcal{C}_{\mathbf{kag}}$. The kagome framework has been examined in a variety of engineering contexts. See [11] for example. It also features as slices in higher dimensional crystal frameworks such as the kagome

net framework of β–cristobalite. See, for example, [7], [18], [24]. Its hexagonal structure is implied by Figure 1 which also indicates a motif

$$\mathcal{M} = (F_v, F_e), \quad F_v = \{p_1, p_2, p_3\}, \quad F_e = \{e_1, \ldots, e_6\}.$$

where $e_1 = [p_{1,(0,0)}, p_{2,(0,0)}], e_2 = [p_{2,(0,0)}, p_{3,(0,0)}], e_3 = [p_{3,(0,0)}, p_{1,(0,0)}]$, $e_4 = [p_{1,(0,0)}, p_{2,(-1,0)}], e_5 = [p_{2,(0,0)}, p_{3,(1,-1)}], e_6 = [p_{3,(0,0)}, p_{1,(0,1)}]$.

Consider a scaling in which the period vectors are

$$a_1 = (1, 0), \quad a_2 = (1/2, \sqrt{3}/2),$$

so that the equilateral triangles of the framework have side length 1/2.

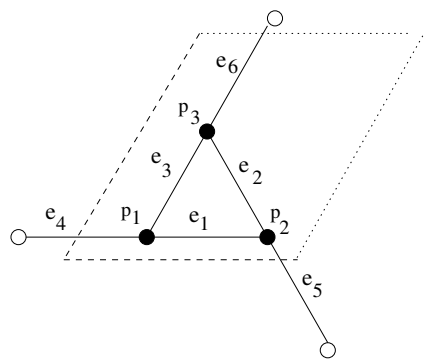

FIGURE 1. A motif for the kagome framework, $\mathcal{C}_{\text{kag}}$.

The motif rigidity matrix $R(\mathcal{M})$ is 6×6 while $R(\mathcal{M}, \mathbb{R}^{d^2})$ is 6×10. Examining the local condition for a self-stress (or examining the co-kernel of $R(\mathcal{M})$) one soon finds a basis of three vectors in $\mathbb{R}^{|F_e|}$ for the space of periodic self-stresses. In fact it is natural to take the three vectors which assign coefficients of 1 to a colinear pair of edges of F_e and the coefficient 0 to the other four edges.

For strictly periodic vertex displacement velocities and flexes, that is, for the case $\mathcal{E} = \{0\}$, the Maxwell–Calladine equation of Theorem 2.10 takes the form

$$m - s = 2|F_v| + 0 - |F_e| - 2 = -2.$$

Note in particular that for $\mathcal{E} = \{0\}$ there is a 2-dimensional space of periodic infinitesimal flexes of rigid motion type and so $f_{\mathcal{E}} = 2$. Since $s = 3$ we have $m = 1$. This infinitesimal mechanism corresponds to the unique (up to scalar) "alternating rotation mechanism", with the motif triangle infinitesimally rotating about its centre. We may write this in matrix data form as $(u_{\text{rot}}, 0)$

To compute the motif rigidity matrix $R(\mathcal{M})$ and the affine rigidity matrix

$$R(\mathcal{M}, \mathbb{R}^{d^2}) = [R(\mathcal{M}) \quad X(\mathcal{M})]$$

we note that

$$v_{e_1} = (-1/2, 0), \qquad v_{e_2} = (1/4, -\sqrt{3}/4), \qquad v_{e_3} = (1/4, \sqrt{3}/4),$$
$$v_{e_4} = (1/2, 0), \qquad v_{e_5} = (-1/4, \sqrt{3}/4), \qquad v_{e_6} = (-1/4, -\sqrt{3}/4),$$
$$\delta(e_4) = (-1, 0), \qquad \delta(e_5) = (1, -1), \qquad \delta(e_6) = (0, 1).$$

This leads to the identification of $R(\mathcal{M}, \mathbb{R}^9)$ as the matrix

$$\begin{bmatrix} -1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 & 0 & 0 & 0 & 0 \\ -1/4 & -\sqrt{3}/4 & 0 & 0 & 1/4 & \sqrt{3}/4 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & -1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ 1/4 & \sqrt{3}/4 & 0 & 0 & -1/4 & -\sqrt{3}/4 & 0 & 0 & -1/4 & -\sqrt{3}/4 \end{bmatrix}.$$

In the affine case, with $\mathcal{E} = M_2(\mathbb{R})$, we have $f_{\mathcal{E}} = 3$, since infinitesimal rotation is now admissible, and so

$$m_{\mathcal{E}} - s_{\mathcal{E}} = d|F_v| + \dim \mathcal{E} - |F_e| - f_{\mathcal{E}} = 6 + 4 - 6 - 3 = 1.$$

From the affine rigidity matrix it follows readily that there are no linear dependencies among the rows and so $s_{\mathcal{E}} = 0$. (Roughly speaking, no periodic self-stresses remain on admitting all affine motions.) Thus $m_{\mathcal{E}} = 1$ indicating that affine freedom confers no additional (linearly independent) affinely periodic infinitesimal flex beyond infinitesimal rotation. One can check in particular that while there is "purely affine" flex fixing the motif vertices, with matrix data $(0, A)$, this flex is a linear combination of rigid rotation and the alternation rotation $(u_{\mathrm{rot}}, 0)$.

Consider the rotation symmetry g whose isometry T_g is rotation by $\pi/3$ about the centre of the triangle in the motif. This symmetry acts on F_v. The space $\mathcal{E}_g \subseteq M_d(\mathbb{R})$, of the symmetry-adapted Maxwell equation in Theorem 3.5, has dimension 2. Also $f_g = 1$ since now only infinitesimal rotation is admissible as a rigid motion. The space $\mathcal{F}_g$ is the space of fixed vectors for the linear transformation $\mu_v(g) = \nu_n(g) \otimes T_g$ which has the form $S \otimes T_g$ where S is a cyclic shift of order 3. It follows that $\dim \mathcal{F}_g$ is 2. Thus

$$m_g - s_g = \dim \mathcal{F}_g + \dim \mathcal{E}_g - e_g - f_g = 2 + 2 - 2 - 1 = 1.$$

Since $m_g \leq 1$ it follows that $s_g = 0$. While there is a g-symmetric self-stress for strict periodicity, this stress is invalidated by the admission of affine motion and does not contribute to s_g.

4.2. A Roman tile framework. A rather different hexagonal framework is the crystal framework $\mathcal{C}_{\mathrm{rom}}$ based on the Roman tiling indicated in Figure 2. The framework is rich in infinitesimal flexes. For example note that there are linear "channels" of alternating squares and hexagons which are joined by edges that are all orthogonal to an infinite line. There are thus nontrivial infinitesimal flexes that fix the vertices in one half-plane of such a line and infinitesimally translate all the other vertices in the line direction.

Another evident infinitesimal flex effects identical infinitesimal rotations on every triangular unit. This is a (strictly) periodic infinitesimal mechanism with 6-fold rotational symmetry. We also remark that there are less evident *local flexes* whose support (nonzero velocities) lie in a 2-fold supercell of four parallelograms. Such a flex corresponds to a cycle of translational motions of the six triangular units of a ring of triangles.

To determine the affine rigidity matrix note that there is a natural parallelogram unit cell in Figure 2 which encloses 6 framework vertices. We take this to be the motif set F_v and to determine the translational symmetry group $\mathcal{T}$. The 8 internal edges of this unit cell together with an appropriate choice of 4 cell-spanning edges provides a possible set F_e. We may choose these edges to be the two westward edges and the two south-east edges. The resulting motif $\mathcal{M}$ determines a 12×16 affinely periodic rigidity matrix $R(\mathcal{M}, \mathbb{R}^{d^2})$. This in turn determines the affinely periodic infinitesimal flexes and the associated self-stresses.

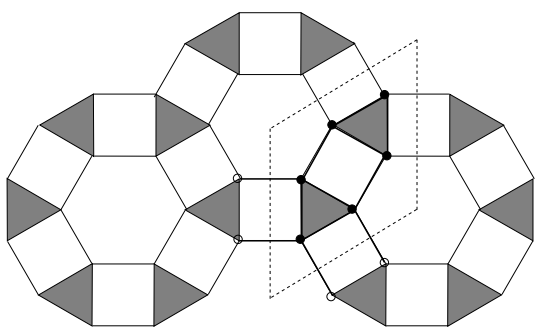

FIGURE 2. Part of the Roman tiling framework $\mathcal{C}_{\text{rom}}$.

From Theorem 2.10 the dimension m_{per} of the space of strictly periodic mechanisms ($\mathcal{E} = \{0\}$) satisfies

$$m_{per} - s = d|F_v| + \dim \mathcal{E} - |F_e| - 2 = 12 + 0 - 12 - 2 = -2$$

In view of the triangle rotation flex $m_{per} \geq 1$ and so there are at least 3 linearly independent periodic self-stresses.

On the other hand the unrestricted affine variant of the Maxwell–Calladine equation gives

$$m_{\text{aff}} - s_{\text{aff}} = d|F_v| + d^2 - |F_e| - f_{\text{rig}} = 12 + 4 - 12 - 3 = 1,$$

with $f_{\text{rig}} = 3$ since there are now 3 independent rigid motion flexes. It follows that there is at least a 1-dimensional space of linearly independent affinely periodic infinitesimal mechanisms.

Let g be the rotational symmetry of $\mathcal{C}_{\text{rom}}$ which has period 6. This is semiseparable with respect to the motif indicated in Figure 2, where F_v consists of the 6 framework points inside the unit cell parallelogram. However, the corresponding representation $\pi_v(\cdot)$ of g, and its generated cyclic

group, (see Theorem 3.2) does not block-diagonalise. Accordingly we consider a motif F'_v, for the same translation group, on which g acts. For this we may take F'_v to be the six framework points of a hexagon subframework, with F'_e the edges of the hexagon plus 6 further edges being the g-orbit of an additional edge incident to a point in F'_v. Since g acts on F'_v the symmetry adapted Maxwell–Calladine formula applies. In this case we have

$$m_g - s_g = \dim \mathcal{F}_g + \dim \mathcal{E}_g - e_g - f_g = 2 + 2 - 2 - 1 = 1.$$

Indeed, $\mu_v(g)$ has the form $S \otimes T_g$, which is a cyclic shift on $\mathbb{R}^{12}$ with a 2-dimensional fixed point space, while $\mathcal{E}_g$, the space of matrices commuting with T_g has dimension 2. It follows directly from this calculation that that there is an affinely periodic infinitesimal mechanism with rotational symmetry.

4.3. Rigid unit frameworks. The planar diagram of Figure 3 shows a template for a translationally periodic body-pin framework in the plane with kite-shaped rigid unit bodies. This framework is equivalent, from the point of view of rigidity, to a bar-joint framework formed by the edges of the quadrilaterals together with added internal edges. Figure 3 shows a motif (F_v, F_e) for a (nonmaximal) translation group such that an inversion symmetry g acts on F_v.

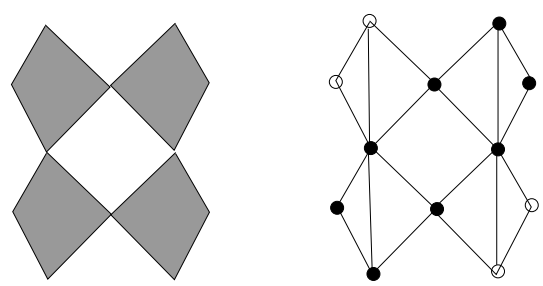

FIGURE 3. Template and motif for a rigid unit framework.

That there is a nontrivial infinitesimal flex which is periodic may be seen from the symmetry-adapted Maxwell–Calladine equation for g. The vertex motif vector space $\mathcal{F}_g$ is the sum of four 2 dimensional spaces. The commutant space $\mathcal{E}_g$ for the linear inversion $T_g : (x, y) \to (-x, -y)$ is the set of all matrices and so has dimension 4. The number e_g of edge classes (including two classes for the added vertical edges) is 10, while the number f_g of independent inversion symmetric rigid motions is 1, for rotation. It follows that

$$m_g - s_g = \dim \mathcal{F}_g + \dim \mathcal{E}_g - e_g - f_g = 8 + 4 - 10 - 1 = 1$$

and so $m_g \geq 1$.

By way of contrast note that the framework $\mathcal{C}_{\text{kite}}$ implied by Figure 4 has no inversion symmetry.

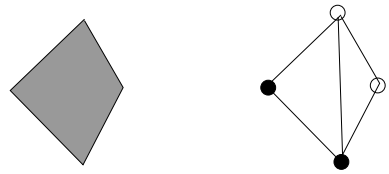

FIGURE 4. The rigid unit for kite framework $\mathcal{C}_{\text{kite}}$, and a minimal motif.

Although there are no nontrivial periodic flexes, for any translation group, there is an essentially unique affinely periodic infinitesimal flex for the maximal translation group. This flex is given by alternating infinitesimal rotations of the rigid units about the midpoints of the vertical diagonal edges. The affine nature of this flex is expressed in the geometrically diminishing velocities in the direction of the positive x-axis.

4.4. The hexahedron framework $\mathcal{C}_{\text{Hex}}$. A hexahedron, or triangular bipyramid, with equilateral triangle faces, may be formed by joining two tetrahedra at a common face. Let $\mathcal{C}_{\text{Hex}}$ be the hexahedron crystal framework in which bipyramids are vertically oriented and horizontally connected, in a regular triangular fashion, and where copies of such horizontal layers are stacked on top of each other. More precisely, $\mathcal{C}_{\text{Hex}}$ is determined by a single bipyramid and the period vectors

$$a_1 = (1, 0, 0), \quad a_2 = (1/2, \sqrt{3}/2, 0) \quad a_3 = (0, 0, 2h),$$

where $h = \sqrt{2}/\sqrt{3}$. A motif $\mathcal{M} = (F_v, F_e)$ is given by the vertex set $F_v = \{p_1, p_2\}$, with equatorial vertex $p_1 = (0, 0, 0)$ and south polar vertex $p_2 = (1/2, \sqrt{3}/6, -h)$, and the edge set F_e consisting of the 9 edges of the bipyramid. Thus the polar vertices have degree 6 while the equatorial vertices have degree 12. The motif rigidity matrix $R(\mathcal{M})$ is a 9×6 matrix and the affine rigidity matrix $R(\mathcal{M}, \mathbb{R}^9)$ is 9×15. The fully affine version of the Maxwell–Calladine equation in Theorem 2.10 for $\mathcal{E} = \mathbb{R}^9$ gives

$$m_{\mathcal{E}} - s_{\mathcal{E}} = d|F_v| + \dim \mathcal{E} - |F_e| - f_{\mathcal{E}} = 2 \times 3 + 9 - 9 - 6 = 0.$$

which is inconclusive on its own. On the other hand, the "internal" edge $[p_1, p_2]$ of the motif, with exponent $(0, 0, 0)$, gives a row of zeros to the associated row of the 9×9 submatrix $X(\mathcal{M})$ of the affine rigidity matrix. Thus $X(\mathcal{M})$ has rank less than 9 and so there is a nonzero vector $(0, AZ)$ in the kernel of $R(\mathcal{M}, \mathbb{R}^9)$. We conclude then that $m_{\mathcal{E}} \geq 1$ and $s_{\mathcal{E}} \geq 1$. In fact $\mathcal{C}_{\text{Hex}}$ has nontrivial infinitesimal flexes, analogous to the affine flex of the kite framework, in which the bipyramids undergo infinitesimal rotation about (parallel) lines through their centroids which are parallel to an equatorial edge.

4.5. The hexahedron tower. We define the hexahedron tower framework $\mathcal{G}_{\text{Hex}}$ as the one-dimensionally translationally periodic framework in

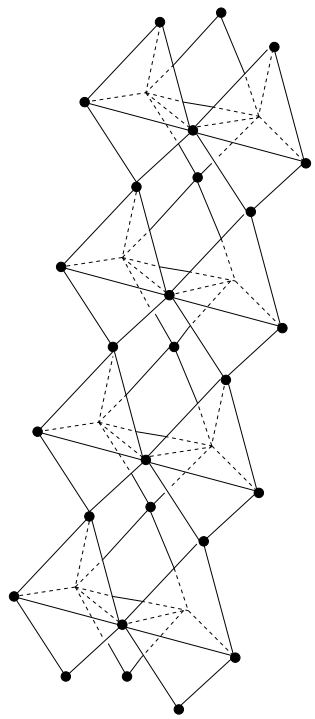

FIGURE 5. Part of a hexahedron tower.

$\mathbb{R}^3$ formed by stacking identical hexahedron 3-rings infinitely in both directions, as indicated in Figure 5. In particular $\mathcal{G}_{\text{Hex}}$ is a subframework of $\mathcal{C}_{\text{Hex}}$. A single hexahedron 3-ring has 3 degrees of freedom beyond the 6 isometric motion freedoms. To see this note that if the connecting triangle is fixed, removing all continuous spatial motion possibilities, then each of the three hexahedron units can flex independently. In view of the connections between the hexahedron 3-rings it follows that any finite tower also has 3 degrees of freedom in this sense.

The hexahedron tower has an evident rotationally symmetric continuous flex $p(t)$ in which the triples of degree 6 vertices alternately move a finite distance towards or away from the central axis. Viewing a single 3-ring as the unit cell this flex is flow-periodic with respect to 2-cell periodicity and with respect to affine contraction along the central axis. The associated infinitesimal flex $p'(0)$ is an affine (2-cell) periodic infinitesimal flex of contractive affine type. Such flexes are characterised by the property that the connecting vertex triples lie on planes which are parallel, as we see below.

For continuous flexes of the hexahedron tower we note the following curious dichotomy.

Proposition 4.1. *Let $p(t), t \in [0,1]$, be a continuous flex of the hexahedron tower $\mathcal{G}_{\mathrm{Hex}} = (G_{\mathrm{Hex}}, p)$ which fixes the connecting triangle of a single hexahedron 3-ring. Then either $(G_{\mathrm{Hex}}, p(t))$ is axially rotationally symmetric for all t or for some $s \in (0,1]$ the framework $(G_{\mathrm{Hex}}, p(s))$ is bounded.*

Proof. Assume first that the framework $(G_{\mathrm{Hex}}, p(s))$ has a hexahedron 3-ring whose polar triples determine planes which are not parallel. The position of one polar triple in $\mathbb{R}^3$ determines (uniquely) the position of the next triple while the subsequent triple is determined by reflection of the first triple in the plane through the second triple. From this reflection principle it follows that these planes and all subsequent planes, similarly identified, pass through a common line. Also it follows that $(G, p(s))$ is a bounded framework which circles around this line. (We are assuming here that self-intersections are admissible.) If the angle between consecutive planes happens to be a rational multiple of 2π then the vertices of the deformed tower occupy a finite number of positions with infinite multiplicity.

To complete the proof it suffices to show that if a single hexahedron 3-ring (H, p) is flexed to a position so that the plane of the three south pole vertices p'_1, p'_2, p'_3 is parallel to the plane of the three north pole vertices p'_4, p'_5, p'_6, then the 3-ring is rotationally symmetric (about the line through the centroid of the connecting triangle of equatorial vertices).

Consider the three-dimensional manifold of "flex positions" (H, p') for which $p'_1 = p_1$ is fixed, p'_2 lies on a fixed open line segment between p_1 and p_2, and p'_3 has two degrees of freedom in a small open disc in the fixed plane through p_1, p_2, p_3. The parallelism condition corresponds to the two equations

$$\langle p'_4 - p'_3, (p_2 - p_1) \times (p_3 - p_1) \rangle = 0, \quad \langle p'_5 - p'_3, (p_2 - p_1) \times (p_3 - p_1) \rangle = 0.$$

It follows that there is a one-dimensional submanifold of positions p' with parallel triple-point planes. Since there is a one-dimensional submanifold of rotationally symmetric positions these positions provide all the positions p', as required. □

By reversing the time parameter t in the proof above, and assuming a decrease of the angle to zero at time $t = 1$, one obtains a curious continuous "unwrapping flex", from a bounded "wrapped" hexahedron tower at time $t = 0$ to the unbounded tower $\mathcal{G}_{\mathrm{Hex}}$ at time $t = 1$, with bounded frameworks at each intermediate time.

From the observations above we can conclude that $\mathcal{C}_{\mathrm{Hex}}$ is continuously rigid. Indeed, there is no continuous flex during which a hexahedron tower subframework becomes a bounded framework. (This leads to a violation of the triangle inequality in neighbouring towers.) The only other possibility is that all hexahedron tower subframeworks undergo an axially symmetric continuous flex. However, because of their connectivity this is not possible.

Added in proof. We note that finite and infinite column frameworks are also considered in a recent paper of Whiteley [26].

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