# Quantum Random Walks

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This thesis is dedicated to my Parents and to my girlfriend Jayna.

Niniejszą pracę dedykuję moim Rodzicom oraz mojej dziewczynie Jaynie.

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# Author's statement

Hereby I declare that the present thesis was prepared by me and none of its contents was obtained by means that are against the law.

I also declare that the present thesis is a part of a PhD Programme at Lancaster University. The thesis has never before been a subject of any procedure of obtaining an academic degree.

The thesis contains research carried out jointly: Chapter 3 forms the basis of the paper [20] co-authored with A. C. R. Belton and J. M. Lindsay. Chapter 5 and Chapter 6 form the basis of the paper [21] co-authored with A. C. R. Belton and J. M. Lindsay. Hereby I declare that I made a full contribution to all aspects of this research and the writing of these papers.

April 30, 2014 Michał Gnacik

# Abstract

In this thesis we investigate the convergence of various quantum random walks to quantum stochastic cocycles defined on a Bosonic Fock space. We prove a quantum analogue of the Donsker invariance principle by invoking the so-called semigroup representation of quantum stochastic cocycles. In contrast to similar results by other authors our proof is relatively elementary. We also show convergence of products of ampliated random walks with different system algebras; in particular, we give a sufficient condition to obtain a cocycle via products of cocycles. The CCR algebra, its quasifree representations and the corresponding quasifree stochastic calculus are also described. In particular, we study in detail gauge-invariant and squeezed quasifree states.

We describe repeated quantum interactions between a 'small' quantum system and an environment consisting of an infinite chain of particles. We study different cases of interaction, in particular those which occur in weak coupling limits and low density limits. Under different choices of scaling of the interaction part we show that random walks, which are generated by the associated unitary evolutions of a repeated interaction system, strongly converge to unitary quantum stochastic cocycles. We provide necessary and sufficient conditions for such convergence. Furthermore, under repeated quantum interactions, we consider the situation of an infinite chain of identical particles where each particle is in an arbitrary faithful normal state. This includes the case of thermal Gibbs states. We show that the corresponding random walks converge strongly to unitary cocycles for which the driving noises depend on the state of the incoming particles. We also use conditional expectations to obtain a simple condition, at the level of generators, which suffices for the convergence of the associated random walks. Limit cocycles, for which noises depend on the state of the incoming particles, are also obtained by investigating what we refer to as 'compressed' random walks. Lastly, we show that the cocycles obtained via the procedure of repeated quantum interactions are quasifree, thus the driving noises form a representation of the relevant CCR algebra. Both gauge-invariant and squeezed representations are shown to occur.

# Contents

# Introduction

Historically, the term “random walk” appears for the first time in 1905, in Pearson’s letter [81] to *Nature* entitled “The problem of random walk”. In the same year, due to Einstein [44], “random walks” were associated with Brownian movements. A brief summary of Einstein’s idea is that a particle in a fluid without friction after colliding with a molecule changes its velocity, and in particular the change of velocity can be quickly dissipated if the fluid is very viscous. The overall result of this impact is the change of the position of the particle, which can be interpreted as a random walk. Three years later Langevin [62] proposed the equation describing Brownian movements from Einstein’s idea; however at that time the mathematical apparatus was too poor to solve this equation or even associate it with known theory. Wiener’s [93] axiomatisation of random processes and his rigorous construction of Brownian motion which appear in 1920s, indicated the right direction for future understanding of the Langevin equation. Mathematical tools allowing approximation of Brownian Motion (according to Wiener’s definition) by normalised increasing sums (also called the invariance principle) are due to, *inter alia*, Donsker [39], Doob [40], Kac [60] and Kolmogorov [61]; the following statement of the invariance principle is presently used: on the space of all continuous functions defined on the unit interval with the supermum norm, the scaled random walk $(X_n)_{n\in\mathbb{N}}$ converges in distribution, as $n \to \infty$, to the standard Brownian motion $(B_t)_{t\in[0,1]}$, where

$$X_n(t) = \frac{1}{\sqrt{n}}(x_1 + \ldots + x_m + (nt - m)x_{m+1}) \quad \text{for all } m \in \mathbb{N},\ t \in \left[\frac{m}{n}, \frac{m+1}{n}\right),$$

and $(x_n)$ is a sequence of independent, identically distributed random variables, with zero expectation and unit variance.

In early 1950s, Itô [59] developed stochastic calculus, which finally allowed one to deal with equations in which coefficients are random processes, and so the solution of the Langevin equation was correctly interpreted. Since then, random walks and more broadly stochastic calculus have found applications in various scientific fields including: cancer research [23], computer science [43], finance [75], physics [92], physical biology [78].

The rudiments of the quantum analogue of stochastic calculus were initiated by Hudson and Parthasarathy [56] in 1970s. Early results such as the first formulation of a noncommutative central limit theorem [34] concern the situation when classical random variables are replaced by a canonical pair $(p, q)$ of quantum mechanical momentum and position observables (self-adjoint operators) on $L^2(\mathbb{R})$ satisfying the Heisenberg commutation relations. One of the first quantum versions of Brownian motion was introduced 6 years later in [32], where the authors refer to it as a canonical Wiener process. Due to the commutation relations satisfied by this process and its representation on a symmetric Fock space over $L^2(\mathbb{R}_+)$, the relevance to the theory of algebras of canonical commutation relations (CCR) [11, 12] was emphasised. Development of quantum stochastic integrals [58] together with corresponding quantum stochastic differential equations (QSDEs) gave a natural extension of the Itô stochastic calculus.

**Quantum random walks.** The first analogues of the Donsker invariance principle are due to Lindsay and Parthasarathy [80, 69], where they showed that certain solutions of QSDEs, which generalise classical diffusions, can be approximated by so-called spin random walks. Further attempts in random walks approximation bring more descriptions of physical systems. In [9] Attal and Pautrat consider the situation when a small quantum system interacts with a stream of identical particles (in a discrete time-setup) according to repeated quantum interactions, that is, the first particle interacts with a system for a short period of time $h$, then it stops, the next particle from an infinite chain repeats the procedure and so on. The unitary evolutions of the repeated interaction system generate the quantum random walk. When we embed it into Fock space and let $h \to 0^+$, the limit objects are the solutions of QSDEs (quantum stochastic cocycles). In [9] it was assumed that each particle from an infinite chain is in the pure state induced

by a unit vector; physically it might correspond to the ground (or vacuum) state. The situation when the particle state is taken to be an arbitrary faithful normal state (in particular thermal Gibbs state) was first considered by Attal and Joye in [5], but we would also like to indicate Gough's work [52]. Moreover, Attal and Joye observed that the driving noises form a gauge-invariant quasifree representation of the CCR algebra. These convergence results were generalised independently by Sahu [86] and Belton [17] for quantum flows. However, only Belton's results [18, 19] consider the situation where the particle state is different than a pure state. In contrast to [9], Belton's approach was coordinate-free and no assumption of Hilbert–Schmidt type on the cocycle generator coefficients was made. Furthermore, in [19] the result was established in the case when the particle state is not even faithful. Recently, Das and Lindsay [35] have obtained quantum random walks approximation results for Banach algebras. All previously discussed approximations concern cocycles with bounded generators. In 2008 Bouten and van Handel [25] have used semigroup methods and Trotter–Kato theorem to obtain quantum random walk approximation results for unitary cocycles with certain unbounded generators.
The theory of quantum random walks has been broadly applied, including to the dilation theory of quantum dynamical semigroups ([17], [86]), approximation of quantum Lévy processes ([48]), quantum feedback control and quantum filtering ([53], [26]) and repeated-interaction models for the atom-maser ([29, 28, 31]).

**The quasifree picture.** Quantum integrals are defined with respect to fundamental processes, which come from creation and annihilation operators associated with the Fock representation of the CCR algebra. Quantum stochastic calculus in which fundamental processes are obtained through different representations of the CCR (e.g. Araki–Woods representations [12]), was investigated by Lindsay in his dissertation entitled “A Quantum Stochastic Calculus” and in [57] together with Hudson. Stochastic Integration for those quasifree representation of the CCR was also investigated by Barnett Streater and Wildein in [13]. Recently, Lindsay and Margetts have established a complete theory of quasifree stochastic calculus in [67, 68].

Inspired by work of Attal and Joye and recent results of Lindsay and Margetts we provide random walk approximation to cocycles which are strictly quasifree.

## Description of the contents

The contents of this thesis are presented in 5 chapters. In Chapter 1 we discuss the rudiments of quantum stochastic calculus. We start by recalling the definition of the symmetric Fock space and its basic properties. Next, we review the basic class of operators on the symmetric Fock space, including an abstract gradient and divergence introduced by Lindsay in [64]. In Section 1.2 we start with the definition of operator processes followed by the example of fundamental ones. By exploiting the abstract gradient and divergence we adapt the Lindsay definition of abstract Hitsuda–Skorohod and Itô integrals and thanks to them we introduce quantum stochastic integrals. Further, the Fundamental Formulae and Estimate are also discussed. After discussing quantum stochastic differential equations (QSDEs) and quantum stochastic cocycles, we focus on the semigroup representations of such cocycles. Those play an important role in our quantum random walk approximation theorem. For the reader's convenience and the completeness of the chapter we provide proofs of some theorems which are usually omitted in the literature. We end the chapter by quoting the characterisation theorem [71, 50] of isometric quantum stochastic cocycles in terms of the cocycle's generator.

Chapter 2 presents the recent development of Lindsay and Margetts [67, 68] on quasifree stochastic calculus. We start with introducing the partial (matrix) transpose, which plays an important technical function in quasifree stochastic analysis. The definition that we suggest is based on the one from [68], however we restrict it to the case of bounded operators. Next, we explore the theory of CCR algebras, focusing on the representations which are most relevant to a quasifree stochastic calculus; these are representations which induce gauge-invariant and squeezed states. Starting from different subalgebras of the CCR and by employing duality theorems we obtain von Neumann algebras and their commutants which will play the role of noise algebras. Next we discuss quasifree integrals and Fundamental Formulea. The chapter ends with a brief summary of the results on quasifree stochastic differential equations (Qf-SDEs) and associated cocycles obtained in [68].

After two preliminary chapters, quantum random walks, the central topic of this thesis, is contained in Chapter 3 and 4. We start by recalling the definition of toy Fock space, which is a discrete version of the symmetric Fock space. Next we define quantum random walks and in Theorem 3.1.12 we show their convergence to quantum stochastic cocycles, by employing the semigroup decomposition of quantum stochastic cocycles and the notion of associated semigroups, introduced by Lindsay and Wills (see [74]). Theorem 3.1.12 is a special case of the quantum Donsker invariance principle proved in [17], however our proof is independent and uses more elementary tools such as Euler's exponential formula (Theorem A.0.9). Next, we give an example of random walk generators which allow the associated random walks to approximate every isometric quantum stochastic cocycle. The construction of these generators is inspired by [80]. Afterwards, we investigate the products of random walks; the corresponding approximation theorem, that is, Theorem 3.1.16, is a consequence of Theorem 3.1.12 and a simple algebraic trick. We also give a sufficient condition implying that the product of two quantum stochastic cocycles forms a cocycle. Section 3.2 is devoted to the application of the results obtained at the beginning of Chapter 3. We discuss the repeated quantum interactions model, and show that our convergence results can be applied in this context. In Examples 3.2.5, 3.2.6 and 3.2.7 we generalise the results obtained by Attal and Pautrat (see [9]) and moreover we give necessary and sufficient conditions on the interaction Hamiltonian to obtain the desired convergence. Example 3.2.9 shows that the recent result by Attal, Deschamps and Pellegrini [10] that concerns the bipartite models in repeated quantum interactions can be viewed as a special case of Theorem 3.1.16.

Chapter 4 is inspired by Attal and Joye's work [5]. We investigate quantum random walks which arise from the repeated quantum interaction model in which the incoming particles are in a faithful normal state. We employ a concrete GNS representation to present the state in a vector state form and then we construct a rotation (Example 4.1.1) that maps the corresponding cyclic vector to the ampliation of the unit vector that was exploited in the standard case. This allows us to apply Theorem 3.1.12 to give a proof of the relevant convergence theorems. The results obtained correspond to Examples 3.2.5, 3.2.6; however the driving noises are influenced by a faithful normal state rather than a pure one. Further, we employ techniques involving conditional expectation, which were initiated by Belton in [18, 19], to obtain a simple condition, at the level of generators, which suffices for the convergence of the associated random walks, including the product case. The last section of this chapter is devoted to random walks which are embedded in a smaller space then in the previous case, when the incoming particles were in faithful normal state. We refer to them as compressed random walks. The results obtained are analogous to the previous investigation, however as pointed out, such cocycles have 'smaller' noise space.

The last chapter is devoted to quasifree cocycles and so it exploits all the previous results. We start with a technical lemma involving the partial transpose investigated at the beginning of Chapter 2. Due to the form of operators considered we explain in detail that the assumptions for the existence of the partial transpose are automatically satisfied. Further, in Theorem 5.2.2 we give sufficient conditions for the cocycle obtained by random walk approximation, whose driving noises depend on the particle state, to be a gauge-invariant quasifree cocycle. Corollary 5.2.3 gives moreover necessary and sufficient conditions, but it is restricted to a special case of the obtained cocycle. Theorem 5.2.5 establishes a similar result to Theorem 5.2.2, however it concerns squeezed quasifree cocycles rather than only gauge-invariant ones. Similarly, Corollary 5.2.6 is analogous to Corollary 5.2.3, but it concerns the squeezed case.

Appendix A contains results concerning Euler's exponential formula which was employed in Theorem 3.1.12 and Theorem 4.4.3. Due to the specific form of these results, it is hard to find them in the literature; however they are elementary.

## Notation and conventions

The symbol := is to be read as 'is defined to equal' (or similarly).
The sets of non-negative integers and non-negative real numbers are denoted by $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and $\mathbb{R}_+ := [0, \infty)$ .

The indicator function of a measurable set $A$ of a measure space $(X, \Sigma, \mu)$ is denoted by $\mathbb{1}_A$. For a vector valued function $f : \mathbb{R}_+ \to V$ and an interval $I \subset \mathbb{R}_+$, we denote the product of $f$ and $\mathbb{1}_I$ by $f_I$.

We will be using complex and separable Hilbert spaces unless otherwise stated and inner products are antilinear in the first and linear in the second argument.

The algebraic tensor product between two vector spaces $V_1$ and $V_2$ is denoted by $V_1 \underline{\otimes} V_2$. For Hilbert spaces $\mathsf{H}_1$ and $\mathsf{H}_2$, $\mathsf{H}_1 \otimes \mathsf{H}_2$ denotes their Hilbert-space tensor product. For von Neumann algebras $\mathsf{M}$ and $\mathsf{N}$, the ultraweak closure of the algebraic tensor product $\mathsf{M} \underline{\otimes} \mathsf{N}$ is denoted by $\mathsf{M} \overline{\otimes} \mathsf{N}$.

Let $\mathsf{H}$ be a Hilbert space. For $u \in \mathsf{H}$ we define the bounded operators

$$\langle u| : \mathsf{H} \to \mathbb{C} \text{ by } \langle u|\, v = \langle u, v \rangle \,,$$

and

$$|u\rangle : \mathbb{C} \to \mathsf{H} \text{ by } |u\rangle\, \alpha = \alpha u.$$

The operators $\langle u|$ and $|u\rangle$ are mutually adjoint and $\langle u|\,|v\rangle = \langle u, v \rangle$ . For another Hilbert space $\mathrm{K}$ and $u \in \mathsf{H}$, we define $E_u : \mathrm{K} \to \mathrm{K} \otimes \mathsf{H}$ to be the ampliation of $|u\rangle$, that is, $E_u = I_{\mathrm{K}} \otimes |u\rangle$, and denote its adjoint by $E^u$.

# Chapter 1

# Quantum stochastic calculus

This chapter collects the rudiments of quantum stochastic calculus on the symmetric Fock space. All the sections contain material which is briefly discussed without presenting the proofs, but while referring to the appropriate references. For more details we recommend the reader consult the books [79], [77] and [63].

## 1.1 Symmetric Fock Space

Let $\mathsf{H}$ be a Hilbert space. *Symmetric Fock space* over $\mathsf{H}$ is defined by

$$\Gamma(\mathsf{H}) = \bigoplus_{n\geqslant 0} \mathsf{H}^{\vee n},$$

where $\mathsf{H}^{\vee n}$ is the symmetric $n$-fold tensor product of $\mathsf{H}$, that is, the closed subspace of $\mathsf{H}^{\otimes n}$ generated by $\{u^{\otimes n}: u \in \mathsf{H}\}$, and $\mathsf{H}^{\vee 0} := \mathbb{C}$.

The symmetric Fock space was introduced by the physicist Fock in [47]; a mathematical interpretation of this article can be found in [33].

The *exponential vector* of $u \in \mathsf{H}$ is

$$\varepsilon(u) := \left(\left(\sqrt{n!}\right)^{-1} u^{\otimes n}\right)_{n\geqslant 0} = \left(1, u, \frac{u\otimes u}{\sqrt{2}}, \frac{u\otimes u\otimes u}{\sqrt{3!}}, \dots\right) \in \Gamma(\mathsf{H}).$$

Exponential vectors form a linearly independent and total set in $\Gamma(\mathsf{H})$ ([79, Proposition 19.4 p. 126], [63, Proposition 1.32]). It is easy to see that

$$\langle \varepsilon(u), \varepsilon(v)\rangle = e^{\langle u,v\rangle} \text{ for all } u,\ v \in \mathsf{H}.$$

For $S \subseteq \mathsf{H}$ we denote

$$\mathcal{E}(S) := \operatorname{span}\{\varepsilon(u) \colon u \in S\}.$$

The inequality ([63, equation (1.24)])

$$\|\varepsilon(u) - \varepsilon(v)\| \leqslant \|u - v\| e^{\frac{1}{2}(\|u\|+\|v\|)^2} \quad \text{for all } u,\ v \in \mathsf{H} \tag{1.1}$$

shows that the map $u \mapsto \varepsilon(u)$ is continuous.
The continuity of $u \mapsto \varepsilon(u)$ yields that if $S$ is a dense subset of $\mathsf{H}$, then $\mathcal{E}(S)$ is a dense subspace of $\Gamma(\mathsf{H})$ ([79, Corollary 19.5, p. 127], [63, Corollary 1.30]).

Furthermore, Fock space possesses the *exponential property* ([79, Proposition 19.6 p. 127], [63, Proposition 1.31.]), that is, for any Hilbert spaces $\mathsf{H}_1$, $\mathsf{H}_2$ the following holds

$$\Gamma(\mathsf{H}_1 \oplus \mathsf{H}_2) = \Gamma(\mathsf{H}_1) \otimes \Gamma(\mathsf{H}_2),$$

where the equality sign "$=$" means that left and right-hand side of the equality are isometrically isomorphic via the map

$$\varepsilon(u \oplus v) \mapsto \varepsilon(u) \otimes \varepsilon(v)$$

for all $u \in \mathsf{H}_1$, $v \in \mathsf{H}_2$.

## Fock operators

In this section we present some important Fock space linear operators.

The *number operator* on $\Gamma(\mathsf{H})$ is a positive self-adjoint operator a domain

$$\operatorname{Dom} \mathcal{N} := \left\{ \xi \in \Gamma(\mathsf{H}) \colon \sum_{n \geqslant 0} n^2 \|\xi_n\|^2 < \infty \right\}$$

and action

$$\mathcal{N}\xi = (n\xi_n)_{n\geqslant 0}.$$

Since $\mathcal{N}$ is positive and self-adjoint, we are provided with the operator $\sqrt{\mathcal{N}}$. For more details we refer the reader, to [63, p. 202] and [15, p. 16].

For $T \in \mathcal{B}(\mathsf{H})$, we define $\Gamma_0(T)$ by the prescription

$$\Gamma_0(T)\varepsilon(u) = \varepsilon(Tu).$$

In particular, $\Gamma_0(T^*) \subset \Gamma_0(T)^*$ and $\Gamma_0(T)^*$ is a closed extension of $\Gamma_0(T^*)$.

We define the *second quantisation* on $\mathit{\Gamma}(\mathsf{H})$ as the closed extension of $\Gamma_0(T)$ and denote it by $\Gamma(T)$. Moreover, $\Gamma(T)$ has the core $\mathcal{E}(\mathsf{H})$, and it is unitary, isometric, contractive if $T$ is. See also [79, p. 135], [63, Second quantisation, p. 208] and [77, p. 63]. *Annihilation* and *creation* operators are closed, mutually-adjoint operators defined on the core $\mathcal{E}(\mathsf{H})$ by

$$\begin{aligned} a(u)\varepsilon(v) &= \langle u, v\rangle\, \varepsilon(v), \\ a^\dagger(u)\varepsilon(v) &= \frac{d}{dt}\varepsilon(v+tu)\Big|_{t=0}, \quad \text{for all } u,\ v \in \mathsf{H}, \end{aligned}$$

respectively.

They both have common domain and satisfy the *canonical commutation relations*

$$[a(u), a^\dagger(v)] = \langle u, v\rangle\, I \quad \text{for all } u, v \in \mathsf{H}.$$

For more details the reader is recommended to consult [79, Proposition 20.12, p. 144; Proposition 20.14, p. 146] and [77, creation and annihilation operators, p. 61].

Let $(U_t)_{t\in\mathbb{R}}$ be a one parameter unitary group in $\mathsf{H}$. By Stone's theorem ([95, Theorem 1, p. 345]) there exists a unique self-adjoint (not necessarily bounded) operator $H$ on $\mathsf{H}$ such that for all $t \in \mathbb{R}$ we have

$$U_t = e^{-\mathrm{i}tH}.$$

Then for a one parameter unitary group $(\Gamma(U_t))_{t\in\mathbb{R}}$ there is a unique self-adjoint operator $\lambda(H)$ on $\Gamma(\mathsf{H})$ such that

$$\Gamma(U_t) = e^{-\mathrm{i}t\lambda(H)}, \quad t \in \mathbb{R}.$$

The operator $\lambda(H)$ is called the *differential second quantisation* of $H$. We can extend this definition to an arbitrary $L \in \mathcal{B}(\mathsf{H})$ as follows let $\lambda_0(L)$ be defined by the prescription

$$\lambda_0(L)\varepsilon(u) = \frac{\mathrm{d}}{\mathrm{d}t}\, \varepsilon(e^{tL}u)\Big|_{t=0}.$$

In particular, $\lambda_0(L^*) \subset \lambda_0(L)^*$ and $\lambda_0(L)^*$ is a closed extension of $\lambda_0(L^*)$. The second quantisation of $L$ on $\Gamma(\mathsf{H})$ is the closed extension of $\lambda_0(L)$, it equals to

$$\lambda\left(\frac{L+L^*}{2}\right) + \mathrm{i}\lambda\left(\frac{L-L^*}{2\mathrm{i}}\right).$$

We denote the second quantisation of $L$ on $\Gamma(\mathsf{H})$ as $\lambda(L)$. Moreover, $\mathcal{E}(\mathsf{H})$ is the core for $\lambda(L)$, and $\lambda(I) = \mathcal{N}$. For more details we refer the reader to [79, Proposiiton 20.7, p. 140; Proposition 20.12, p. 144; Proposition 20.13, p. 145] and [63, Differential second quantisation, p. 208].

*Momentum* and *position* positions operators are self-adjoint operators defined by

$$\begin{aligned} p(u) &:= \mathrm{i}\,\overline{a^\dagger(u) - a(u)}, \\ q(u) &:= \overline{a^\dagger(u) + a(u)}, \end{aligned}$$

respectively. In particular, $q(u) = -p(\mathrm{i}u)$ and $\mathcal{E}(\mathsf{H})$ is a core for $p(u)$ and $q(u)$.

By virtue of Stone's Theorem $p(u)$ generates strongly continuous one-parameter unitary group $(W_0(tu))_{t\in\mathbb{R}}$, where

$$W_0(tu) = e^{-\mathrm{i}tp(u)}.$$

A unitary operator $W_0(u)$ is called a *Fock–Weyl* operator, and its action on the exponential vectors is

$$W_0(u)\varepsilon(v) = e^{-\frac{1}{2}\|u\|^2 - \langle u,v\rangle}\varepsilon(u+v). \tag{1.2}$$

Moreover, Fock–Weyl operators satisfy the following relations

$$W_0(u)W_0(v) = e^{-\mathrm{i}\,\mathrm{Im}\langle u,v\rangle} W_0(u+v), \tag{1.3}$$
$$W_0(u)W_0(v) = e^{-2\mathrm{i}\mathrm{Im}\langle u,v\rangle} W_0(v)W_0(u), \tag{1.4}$$
$$W_0(u)^* = W_0(-u) \tag{1.5}$$

for all $u$, $v \in \mathsf{H}$.
For more details concerning Fock–Weyl operators we refer the reader to [79, Chapter II: 20 The Weyl Representation] and [77, Weyl operators, p. 65]. For a different approach see [63, Fock-space operators p. 207].

The abstract *gradient* and *divergence* operators

$$\nabla\colon \mathrm{Dom}\nabla \subset \Gamma(\mathsf{H}) \to \mathsf{H} \otimes \Gamma(\mathsf{H}) \quad \text{and} \quad \mathcal{S}\colon \mathrm{Dom}\mathcal{S} \subset \mathsf{H} \otimes \Gamma(\mathsf{H}) \to \Gamma(\mathsf{H}) \tag{1.6}$$

are densely defined, closed, mutually-adjoint operators satisfying

$$\nabla\varepsilon(u) = u \otimes \varepsilon(u),$$
$$\mathcal{S}(u \otimes \varepsilon(v)) = \frac{d}{dt}\varepsilon(v+tu)\Big|_{t=0}.$$

In particular, $\mathcal{E}(\mathsf{H})$ is a core for $\nabla$, and $\mathsf{H} \underline{\otimes} \mathcal{E}(\mathsf{H})$ is a core for $\mathcal{S}$.

Those two operators were originally introduced by Lindsay in [64, p. 69 after Proposition 1.2] in the context of Fock–Guichardet space [54]. Then they were extended to symmetric Fock space over an arbitrary Hilbert space $\mathsf{H}$ in [63, Section 3.1]. The abstract gradient and divergence were also studied in [2], where the author calls them the universal annihilation and creation operators, which is due to the following observation ([2, Proposition 3.3]);

**Proposition 1.1.1.** *For each* $u \in \mathsf{H}$

- $(\langle u| \otimes I_{\Gamma(\mathsf{H})})\nabla = a(u)$,
- $\mathcal{S}(|u\rangle \otimes I_{\Gamma(\mathsf{H})}) = a^\dagger(u)$

*on* $\mathcal{E}(\mathsf{H})$.

Gradient $\nabla$ and divergence $\mathcal{S}$ are also related to the number operator, and for $\mathcal{S}$ we have an isometric-type equality with a correction term ([63, Proposition 3.1], [2, Theorem 3.2 and Theorem 3.4]);.

**Proposition 1.1.2.** *The following relations hold*

1. $\mathrm{Dom}\nabla = \mathrm{Dom}\sqrt{\mathcal{N}}$,
2. $\mathcal{S}\nabla = \mathcal{N}$,
3. *For* $z_1, z_2 \in \mathrm{Dom}(I_{\mathrm{H}} \otimes \sqrt{\mathcal{N}+1})$,

$$\langle \mathcal{S}z_1, \mathcal{S}z_2 \rangle = \langle z_1, z_2 \rangle + \big\langle (I_{\mathrm{H}} \otimes \nabla)z_1, (\Pi \otimes I_{\Gamma(\mathsf{H})})(I_{\mathrm{H}} \otimes \nabla)z_2 \big\rangle,$$

*where* $\Pi$ *denotes the tensor flip on* $\mathrm{H} \otimes \mathrm{H}$*, that is,* $\Pi$ *is an isometric isomorphism such that* $x \otimes y \mapsto y \otimes x$*, for all* $x$*,* $y \in \mathsf{H}$*.*

## 1.2 Quantum stochastic analysis

Quantum stochastic analysis arises from the natural filtration of the symmetric Fock space over $L^2(\mathbb{R}_+;\mathsf{k})$, where $\mathsf{k}$ is a separable Hilbert space, called the *noise dimension space.*

Now and for the rest of the thesis fix a Hilbert space $\mathfrak{h}$. We refer to $\mathfrak{h}$ as the *initial space.*

A piece of notation is required; let $I \subset \mathbb{R}_+$ be an interval, and denote

$$\mathcal{F}_I^{\mathsf{k}} := \Gamma(L^2(I,\mathsf{k})) \quad \text{and} \quad \Omega_{I,\mathsf{k}} = \varepsilon(0) \in \mathcal{F}_I^{\mathsf{k}}.$$

We skip the subscript $I$ whenever $I = \mathbb{R}_+$.

The exponential property yields the tensor decomposition

$$\mathcal{F}^{\mathsf{k}} = \mathcal{F}^{\mathsf{k}}_{[0,s)} \otimes \mathcal{F}^{\mathsf{k}}_{[s,t)} \otimes \mathcal{F}^{\mathsf{k}}_{[t,\infty)}$$

for all $0 \leqslant s \leqslant t$.

**Definition 1.2.1.** We call a subset $S \subset L^2(\mathbb{R}_+;\mathsf{k})$ *admissible* if

- $\mathcal{E}(S)$ is dense in $\mathcal{F}^{\mathsf{k}}$,
- $f_{[0,t)} \in S$ for all $f \in S$, $t \in \mathbb{R}_+$.

Denote by $\mathbb{S} := \operatorname{span}\{c_{[0,t)} : c \in \mathsf{k}, t \geqslant 0\} \subset L^2(\mathbb{R}_+;\mathsf{k})$ the set of all step functions in $L^2(\mathbb{R}_+;\mathsf{k})$. Given any $A \subset \mathsf{k}$, let

$$\mathbb{S}_A := \{f \in \mathbb{S} : f \text{ is } A-\text{valued}\}.$$

Note that $\mathbb{S}_A$ is admissible.

**Proposition 1.2.2.** *If $A$ is a total subset of $\mathsf{k}$ and $0 \in A$, then $\mathcal{E}(\mathbb{S}_A)$ is dense in $\mathcal{F}^{\mathsf{k}}$.*

For the proof we refer the reader to [63, Proposition 2.1] in which the proof is an adaptation of the main result in [87].

## Operator processes

For further analysis we will require the quantum analogue of stochastic processes.

**Definition 1.2.3.** Let $\mathcal{D}$ be a dense subspace of $\mathfrak{h}$ and let $S \subset L^2(\mathbb{R}_+;\mathsf{k})$ be admissible. An *operator process* on $\mathfrak{h}$ with noise dimension space $\mathsf{k}$ is a family $X = (X_t)_{t \geqslant 0}$ of linear operators

$$X_t : \mathcal{D} \underline{\otimes} \mathcal{E}(S) \to \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}}$$

such that

- the map $t \mapsto \langle x, X_t y\rangle$ is measurable for all $x \in \mathsf{h} \otimes \mathcal{F}$, $y \in \mathcal{D} \underline{\otimes} \mathcal{E}(S)$,
- the process is *adapted*, that is

  $$\begin{aligned}&\langle u \otimes \varepsilon(f), X_t(v \otimes \varepsilon(g))\rangle \\ &= \big\langle u \otimes \varepsilon(f_{[0,t)}), X_t(v \otimes \varepsilon(g_{[0,t)}))\big\rangle \big\langle \varepsilon(f_{[t,\infty)}), \varepsilon(g_{[t,\infty)})\big\rangle\end{aligned}$$

  for all $u, v \in \mathfrak{h}$, $f \in L^2(\mathbb{R}_+;\mathsf{k})$, $g \in S$ and $t \in \mathbb{R}_+$.

Therefore, if $X$ is an operator process we can write it as

$$X_t = X_{t)} \otimes I_{\mathcal{E}(S|_{[t,\infty)})}, \tag{1.7}$$

where $X_{t)}: \mathcal{D} \underline{\otimes} \mathcal{E}(S|_{[0,t)}) \to \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}}_{[0,t)}$.

The operator process $X$ is called *measurable* (*continuous*) if the function

$$t \mapsto X_t x$$

is strongly measurable (respectively, continuous) for all $x \in \mathcal{D} \underline{\otimes} \mathcal{E}(S)$. It is called *bounded* (*contractive*, *isometric*, *unitary* etc.) if all the operators $X_t$ have this property.

**Remark 1.2.4.** Every continuous operator process is measurable. Pettis' theorem ([82, Theorem 1.1.]) yields that the operator process $X = (X_t)_{t \geqslant 0}$ is measurable, if and only if, the function $t \mapsto X_t \eta$ is a.e. separably valued (some subset of $\mathbb{R}_+$ with full measure is carried to a separable subset of the range of $X_t$).

Using the notion of creation, annihilation and preservation operators defined earlier we give an example of operator processes.

**Example 1.2.5** (Fundamental operator processes)**.** Let $f, g \in L^2_{\text{loc}}(\mathbb{R}_+; \mathsf{k})$ and $R \in L^\infty_{\text{loc}}(\mathbb{R}_+; \mathcal{B}(\mathsf{k}))$, where this last space is the Banach space of locally bounded, strongly measurable functions from $\mathbb{R}_+$ with values in Banach space $\mathcal{B}(\mathsf{k})$. *Creation*, *annihilation* and *preservation* (or *gauge*) processes $A^\dagger_{|f\rangle}$, $A_{\langle g|}$ and $N_R$ are defined by

$$\begin{aligned} A^\dagger_{|f\rangle} &:= (a^\dagger(f_{[0,t)}))_{t \geqslant 0}, \\ A_{\langle g|} &:= (a(g_{[0,t)}))_{t \geqslant 0}, \\ N_R &:= (\lambda(R_{[0,t)}))_{t \geqslant 0}, \end{aligned}$$

respectively. These are continuous operator processes ([58, Proposition 4.1. p. 306]).

The next example of an operator process will tell us how to reverse the time of a given operator process $X = (X_t)_{t \geqslant 0}$.

**Example 1.2.6** (Time reversal)**.** For each $t \in \mathbb{R}_+$ the operator $r_t \in \mathcal{B}(L^2(\mathbb{R}_+; \mathsf{k}))$ is given by

$$(r_t f)(r) := \begin{cases} f(t-r) & , \text{ if } r \leqslant t \\ f(r) & , \text{ if } r > t. \end{cases} \tag{1.8}$$

Let $R_t$ be the second quantisation of $r_t$, that is, it is the bounded operator on $\mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}}$ such that

$$R_t(u \otimes \varepsilon(f)) = u \otimes \varepsilon(r_t f). \tag{1.9}$$

We call $R = (R_t)_{t \geqslant 0}$ the *time reflection process*. Note that each $R_t$ is a self-adjoint and unitary operator.
For any bounded operator process $X = (X_t)_{t \geqslant 0}$, the *time reversed process* of $X$ is given by $(R_t X_t R_t)_{t \geqslant 0}$.

## Hitsuda–Skorohod and Itô integrals

Exploiting the abstract gradient and divergence of Malliavin calculus from (1.6), we will define the abstract Hitsuda–Shorohod and Itô integrals, which we require to produce quantum integrals. These techniques were first introduced independently by Lindsay in [63] and Belavkin in [14].

We ampliate the abstract gradient and divergence operators, with for $\mathsf{H} = L^2(\mathbb{R}_+; \mathsf{k})$, in the following manner

$$\nabla \colon \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}} \to L^2(\mathbb{R}_+; \mathsf{k}) \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}}, \quad \mathcal{S} \colon L^2(\mathbb{R}_+; \mathsf{k}) \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}} \to \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}},$$

and

$$\nabla(u \otimes \varepsilon(f)) = f \otimes u \otimes \varepsilon(f),$$
$$\mathcal{S}(g \otimes u \otimes \varepsilon(f)) = u \otimes \frac{d}{dt}\varepsilon(f + tg)\Big|_{t=0}.$$

We call $\mathcal{S}$ the *Hitsuda–Skorohod integral* and $\nabla$ is called the *gradient operator of Malliavin calculus.*

**Notation 1.2.7.** For $z \in \mathrm{Dom}\mathcal{S}$, $\xi \in \mathrm{Dom}\nabla$, and $t > 0$ we set

$$\mathcal{S}_t z := \mathcal{S} z_{[0,t)} \quad \text{and} \quad \nabla_t \xi := (\nabla \xi)_t \in \mathsf{k} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}}.$$

Apart from the Hitsuda–Skorohod integral we also use Bochner integrals, which are vector-valued integrals with respect to time. For $z \in L^1(\mathbb{R}_+; \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$, set

$$\mathcal{T} z := \int_0^\infty z(s)\, \mathrm{d}s,$$

and for $z \in L^1_{\mathrm{loc}}(\mathbb{R}_+; \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$, and $t \geqslant 0$, set

$$\mathcal{T}_t z := \int_0^t z(s)\, \mathrm{d}s.$$

Note that $\mathbb{R}_+ \ni t \mapsto \mathcal{T}_t z \in \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}}$ is a continuous function.

**Definition 1.2.8.** For all $f \in L^2(\mathbb{R}_+; \mathsf{k})$, a function $z \in L^p(\mathbb{R}_+; \mathsf{k} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$, where $1 \leqslant p \leqslant \infty$, is called $\varepsilon(f)$-*adapted* if, for each $t \in \mathbb{R}_+$,

$$z_t = z_{t)} \otimes \varepsilon(f|_{[t,\infty)}), \text{ where } z_{t)} \in \mathsf{k} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}}_{[0,t)}. \tag{1.10}$$

We denote by $L^p_{\varepsilon(f)}(\mathbb{R}_+; \mathsf{k} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$ the subspace of such functions, and the orthogonal projection in $L^2(\mathbb{R}_+; \mathsf{k} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$ with range $L^2_{\varepsilon(f)}(\mathbb{R}_+; \mathsf{k} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$ is denoted by $P^{\varepsilon(f)}$. In particular, in case of $\varepsilon(0)$-adapted functions we exchange $\varepsilon(0)$-script for the one with $\Omega$, that is, we use $L^p_\Omega(\mathbb{R}_+; \mathsf{k} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$ and $P^\Omega$, respectively.

Definition of $\varepsilon(f)$-adaptedness also makes sense for locally integrable functions. We denote the space of all locally $p$-integrable $\varepsilon(f)$-adapted functions by $L^p_{\varepsilon(f),\mathrm{loc}}(\mathbb{R}_+; \mathsf{k} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$.

The Hitsuda-Skorohod integral possesses the following properties.

**Proposition 1.2.9.** *We have*

- $L^2_{\varepsilon(f)}(\mathbb{R}_+; \mathsf{k} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}}) \subset \mathrm{Dom}\mathcal{S}$, *and for all* $z \in L^2_{\varepsilon(f)}(\mathbb{R}_+; \mathsf{k} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$,

$$\|\mathcal{S}z\| \leqslant \left(\|f\| + \sqrt{1 + \|f\|^2}\right) \|z\|.$$

- *For all* $z \in L^2_{\varepsilon(f)}(\mathbb{R}_+; \mathsf{k} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$*, the function*

$$\mathbb{R}_+ \ni t \mapsto \mathcal{S}_t z \in \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}}$$

*is continuous.*

- *For two triples* $(f, z^{(0)}, z^{(1)})$ *and* $(g, w^{(0)}, w^{(1)})$*, where* $f$*,* $g \in L^2(\mathbb{R}_+; \mathsf{k})$*,* $z^{(0)}$*,* $w^{(0)} \in L^1_{\varepsilon(f)}(\mathbb{R}_+; \mathsf{k} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$*, and* $z^{(1)}$*,* $w^{(1)} \in L^2_{\varepsilon(f)}(\mathbb{R}_+; \mathsf{k} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$*, we have*

$$\left\langle \widehat{\mathcal{S}} z, \widehat{\mathcal{S}} w \right\rangle = \int \left\{ \left\langle z_t^{(1)}, w_t^{(1)} \right\rangle + \left\langle z_t, \left( \begin{smallmatrix} 1 \\ g(t) \end{smallmatrix} \right) \otimes \widehat{\mathcal{S}}_t w \right\rangle + \left\langle \left( \begin{smallmatrix} 1 \\ f(t) \end{smallmatrix} \right) \otimes \widehat{\mathcal{S}}_t z, w_t \right\rangle \right\} \, \mathrm{d}t,$$

*where* $z = \begin{pmatrix} z^{(0)} \\ z^{(1)} \end{pmatrix}$, $w = \begin{pmatrix} w^{(0)} \\ w^{(1)} \end{pmatrix}$ *and* $\widehat{\mathcal{S}} z = \mathcal{T} z^{(0)} + \mathcal{S} z^{(1)}$.

For the proof we refer the reader to [63, Theorem 3.3, p. 225; Corollary 3.4, p. 226; Theorem 3.5, p. 227].

**Definition 1.2.10** (Itô integral)**.** The *abstract Itô integral* is defined by

$$\mathcal{I} := \mathcal{S}|_{L^2_\Omega(\mathbb{R}_+; \mathsf{k} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})};$$

its adjoint $D$ is said to be the *adapted gradient operator*.

Denote by $V_\Omega$ the inclusion map from $L^2_\Omega(\mathbb{R}_+; \mathsf{k} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$ to $L^2(\mathbb{R}_+; \mathsf{k} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$. Then the adjoint $V_\Omega^*$ may be viewed as $P^\Omega$ if we think about it as an operator from $L^2(\mathbb{R}_+; \mathsf{k} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$ onto the subspace $L^2_\Omega(\mathbb{R}_+; \mathsf{k} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$ of $L^2(\mathbb{R}_+; \mathsf{k} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$.

Then we can write explicitly

$$\mathcal{I} = \mathcal{S} V_\Omega.$$

In particular, $\mathcal{I}$ and $D$ are bounded operators with $\mathcal{I}$ being an isometry.
Set $\mathcal{I}_t z := \mathcal{I} z_{[0,t)}$ for each $z \in L^2_{\Omega,\mathrm{loc}}(\mathbb{R}_+; \mathsf{k} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}})$ and $t \in \mathbb{R}_+$.

For more details on abstract Itô calculus on Fock space we refer the reader to [3] and [7] (Fock–Guichardet approach).

## Quantum stochastic integrals

Integrals with respect to the fundamental operator processes (Example 1.2.5) are defined in this section.

Let $\widehat{\mathsf{k}} := \mathbb{C} \oplus \mathsf{k}$, for $c \in \mathsf{k}$ set $\widehat{c} := \binom{1}{c} \in \widehat{\mathsf{k}}$ and for $f \in L^2(\mathbb{R}_+; \mathsf{k})$ denote $\widehat{f}(t) := \widehat{f(t)}$ for each $t \in \mathbb{R}_+$.

The *Itô projection* is given by

$$\Delta := \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathsf{k}} \end{bmatrix} \otimes I_{\mathfrak{h}} \in \mathcal{B}(\widehat{\mathsf{k}} \otimes \mathfrak{h}) \text{ and } \Delta^{\perp} := I_{\widehat{\mathsf{k}} \otimes \mathfrak{h}} - \Delta. \tag{1.11}$$

Let $\mathcal{D}$ be a dense subspace of $\mathfrak{h}$ and let $S \subset L^2(\mathbb{R}_+; \mathsf{k})$ be admissible.

**Definition 1.2.11** (Quantum stochastic integrand)**.** A measurable operator process $F = (F_t)_{t \geqslant 0}$ on $\widehat{\mathsf{k}} \otimes \mathfrak{h}$ such that $F_t \colon \widehat{\mathsf{k}} \underline{\otimes} \mathcal{D} \underline{\otimes} \mathcal{E}(S) \to \widehat{\mathsf{k}} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}}$ for each $t \in \mathbb{R}_+$, is called a *QS integrand on* $\mathbb{R}_+$ if

- the function $t \mapsto (\Delta^{\perp} \otimes I_{\mathcal{F}^{\mathsf{k}}}) F_t(\widehat{f}(t) \otimes u \otimes \varepsilon(f))$ is Bochner integrable,
- $t \mapsto (\Delta \otimes I_{\mathcal{F}^{\mathsf{k}}}) F_t(\widehat{f}(t) \otimes u \otimes \varepsilon(f))$ is square integrable.

If the above functions are only locally integrable, then $F$ is said to be a *QS integrand*.

**Definition 1.2.12** (QS integral)**.** Let $F$ be a QS integrand on $\mathbb{R}_+$.
The *QS integral* of $F$ is defined by

$$\Lambda(F) \colon \mathcal{D} \underline{\otimes} \mathcal{E} \to \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}}, \quad \Lambda(F)(u \otimes \varepsilon(f)) = \widehat{\mathcal{S}} z,$$

where $z_t = F_t(\widehat{f}(t) \otimes u \otimes \varepsilon(f))$.

The definition makes sense also in the locally integrable case.
If $F$ is a QS integrand then $F_{[0,t)}$, for each $t \geqslant 0$, is a QS integrand on $\mathbb{R}_+$. Set

$$\Lambda_t(F) := \Lambda(F_{[0,t)}).$$

**Remark 1.2.13.** Since $\widehat{\mathsf{k}} = \mathbb{C} \oplus \mathsf{k}$, we can write each QS integrand $F$ in a matrix form

$$F_t = \begin{bmatrix} K_t & M_t \\ L_t & P_t \end{bmatrix}.$$

In this case we will be using the following notation

$$F = \begin{bmatrix} K & M \\ L & P \end{bmatrix}.$$

**Definition 1.2.14** (Time, creation, annihilation and preservation integrals)**.** Let $F$ be a QS integrand on $\mathbb{R}_+$.

- If
$$F = \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix},$$
then $T(K) := \Lambda\left(\left[\begin{smallmatrix} K & 0 \\ 0 & 0 \end{smallmatrix}\right]\right)$ is said to be a *time integral.*
- If
$$F = \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix},$$
then $A(M) := \Lambda\left(\left[\begin{smallmatrix} 0 & M \\ 0 & 0 \end{smallmatrix}\right]\right)$ is said to be a *annihilation integral.*
- If
$$F = \begin{bmatrix} 0 & 0 \\ L & 0 \end{bmatrix},$$
then $A^*(L) := \Lambda\left(\left[\begin{smallmatrix} 0 & 0 \\ L & 0 \end{smallmatrix}\right]\right)$ is said to be a *creation integral.*
- If
$$F = \begin{bmatrix} 0 & 0 \\ 0 & P \end{bmatrix},$$
then $N(P) := \Lambda\left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & P \end{smallmatrix}\right]\right)$ is said to be a *preservation integral.*

All above integrals are in particular continuous operator processes.

Similarly, as for the QS integral we denote

- $T_t(K) := T(K_{[0,t)})$,
- $A_t(M) := A(M_{[0,t)})$,
- $A^*_t(L) := A^*(L_{[0,t)})$,
- $N_t(P) := N(P_{[0,t)})$.

**Remark 1.2.15.** Quantum stochastic integrals were introduced by Hudson and Parathasarathy in [58]. Since then, there have been many different reformulations and generalisations, the development of which, until the 1990s, can be found in books, e.g., [79] and [77]. Here, the definition of quantum stochastic integrals is taken from [63], where the author exploits the gradient and divergence of Malliavin calculus. A different, implicit approach based on the classical Itô calculus can be found in [8]. The latter two approaches are extended in articles [3] and [7]. Finaly there is now an explicit approach based on the abstract Itô integral [66].

Now, we present the fundamental formulae of quantum stochastic calculus.

**Theorem 1.2.16** (First Fundamental Formula)**.** *Let $F$ be a QS integrand on $\mathbb{R}_+$. Then*

$$\langle u \otimes \varepsilon(f), \Lambda(F)(v \otimes \varepsilon(g))\rangle = \int \left\langle u \otimes \varepsilon(f), E^{\widehat{f}(s)} F_s E_{\widehat{g}(s)}(v \otimes \varepsilon(g))\right\rangle \mathrm{d}s$$

*for all $u \in \mathfrak{h}$, $v \in \mathcal{D}$, $f \in L^2(\mathbb{R}_+; \mathsf{k})$ and $g \in \mathcal{S}$.*

For the proof we refer the reader to [63, Theorem 3.13 p. 232].

The estimate below is a consequence of condition (1) in Proposition 1.2.9.

**Theorem 1.2.17** (Fundamental Estimate)**.** *Let $F$ be a QS integrand on $\mathbb{R}_+$. Then*

$$\begin{aligned} &\|\Lambda(F)(u \otimes \varepsilon(f))\| \\ &\leqslant \int \left\|(\Delta^{\perp} \otimes I_{\mathcal{F}^{\mathsf{k}}}) F_s(\widehat{f}(t) \otimes u \otimes \varepsilon(f))\right\| \mathrm{d}s \\ &+ C_f \left(\int \left\|(\Delta \otimes I_{\mathcal{F}^{\mathsf{k}}}) F_s(\widehat{f}(t) \otimes u \otimes \varepsilon(f))\right\|^2 \mathrm{d}s\right)^{\frac{1}{2}} \end{aligned}$$

*for all $u \in \mathcal{D}$, $f \in \mathcal{S}$, where $C_f = \|f\| + \sqrt{1 + \|f\|^2}$.*

This estimate is improved in [66]. The Skorohod isometry, that is, condition (3) in Proposition 1.2.9, yields the following.

**Theorem 1.2.18** (Second Fundamental Formula)**.** *Let $F$ and $G$ be QS integrands on $\mathbb{R}_+$. Then*

$$\begin{aligned}
&\langle \Lambda(G)(u \otimes \varepsilon(f)), \Lambda(F)(v \otimes \varepsilon(g))\rangle \\
&= \int \left\langle \Lambda_s(G)(u \otimes \varepsilon(f)), E^{\widehat{f}(s)} F_s E_{\widehat{g}(s)}(v \otimes \varepsilon(g))\right\rangle \mathrm{d}s \\
&+ \int \left\langle E^{\widehat{g}(s)} G_s E_{\widehat{f}(s)}(u \otimes \varepsilon(f)), \Lambda_s(F)(v \otimes \varepsilon(g))\right\rangle \mathrm{d}s \\
&+ \int \left\langle G_s E_{\widehat{f}(s)}(u \otimes \varepsilon(f)), (\Delta \otimes I_{\mathcal{F}^{\mathsf{k}}}) F_s E_{\widehat{g}(s)}(v \otimes \varepsilon(g))\right\rangle \mathrm{d}s
\end{aligned}$$

*for all $u$, $v \in \mathcal{D}$, $f$ and $g \in S$.*

For the proof we refer the reader to [63, Theorem 3.15, p. 234].

The QS integral consists of time, annihilation, creation and preservation integrals. Thus according to the Second Fundamental Formula, the correction term (the third term of the sum in the Second Fundamental Formula) may vary for different combinations of coefficients of the integrand.

**Example 1.2.19.** Let $F$ and $G$ be quantum integrands such that

$$F_t = \begin{bmatrix} 0 & 0 \\ L_t & 0 \end{bmatrix} \text{ and } G_t = \begin{bmatrix} 0 & 0 \\ 0 & P \end{bmatrix}$$

for each $t \in \mathbb{R}_+$. By applying the Second Fundamental Formula to

$$\langle A_t^*(L)u \otimes \varepsilon(f), N_t(P)v \otimes \varepsilon(g)\rangle$$

we obtain that the correction term equals to

$$\begin{aligned}
&\int_0^t \left\langle E_{\widehat{f}(s)}(u \otimes \varepsilon(f)), \begin{bmatrix} 0 & L_s^* P_s \\ 0 & 0 \end{bmatrix} E_{\widehat{g}(s)}(v \otimes \varepsilon(g))\right\rangle \mathrm{d}s \\
&= \langle u \otimes \varepsilon(f), A_t(M)v \otimes \varepsilon(g)\rangle
\end{aligned}$$

for all $u$, $v \in \mathcal{D}$, $f$ and $g \in S$, where $M_t = L_t^* P_t$. For simplicity we denote $\langle \mathrm{d}A_t^*, \mathrm{d}N_t\rangle := \langle A_t^*(L)u \otimes \varepsilon(f), N_t(P)v \otimes \varepsilon(g)\rangle$ and we say that it has the correction term of type $\mathrm{d}A_t$.

We have sixteen different possibilities, but only four of them are non-zero. According to the procedure and notation in Example 1.2.19, we present all non-zero correction terms in the quantum Itô table underneath.

| | the correction term |
|---|---|
| $\langle \mathrm{d}A_t^*, \mathrm{d}A_t^* \rangle$ | $\mathrm{d}t$ |
| $\langle \mathrm{d}A_t^*, \mathrm{d}N_t \rangle$ | $\mathrm{d}A_t$ |
| $\langle \mathrm{d}N_t, \mathrm{d}A_t^* \rangle$ | $\mathrm{d}A_t^*$ |
| $\langle \mathrm{d}N_t, \mathrm{d}N_t \rangle$ | $\mathrm{d}N_t$ |

Table 1.1: Quantum Itô table.

**Remark 1.2.20.** Since we haven't defined adjoint operator processes, our quantum Itô table has a slightly different form than the one which usually appears in the literature. Therefore, we refer the reader to [63, Example 3.17 p. 235].

## QSDEs and QS cocycles

In this section we investigate the connections between the solutions of quantum stochastic differential equations and quantum stochastic cocycles. Quantum stochastic differential equations and quantum stochastic cocycles have not only mathematical beauty, but also important applications in physics, e.g., they describe the interaction between atoms and the electromagnetic field in the weak coupling limit [24]. For applications in quantum optics we refer the reader to [49].

Let $F = (F_t)_{t \geqslant 0}$ be a bounded operator process on $\widehat{\mathsf{k}} \otimes \mathfrak{h}$ and let $T \in \mathcal{B}(\mathfrak{h})$. We say that an operator process $X = (X_t)_{t \geqslant 0}$ on $\mathfrak{h}$ is a *weak solution* of the left quantum stochastic differential equation

$$\mathrm{d}X_t = \widehat{X}_t F_t \, \mathrm{d}\Lambda_t, \quad X_0 = T \otimes I_{\mathcal{F}^{\mathsf{k}}}, \tag{1.12}$$

where $\widehat{X}_t = I_{\widehat{\mathsf{k}}} \otimes X_t$, if it satisfies

$$\begin{aligned}&\langle u \otimes \varepsilon(f), (X_t - T \otimes I_{\mathcal{F}^{\mathsf{k}}})(v \otimes \varepsilon(g))\rangle \\ &= \int_0^t \Big\langle u \otimes \varepsilon(f), X_s E^{\widehat{f}(s)} F_s E_{\widehat{g}(s)}(v \otimes \varepsilon(g))\Big\rangle \, \mathrm{d}s,\end{aligned} \tag{1.13}$$

for all $u \in \mathfrak{h}$, $v \in \mathcal{D}$, $f \in L^2(\mathbb{R}_+; \mathsf{k})$ and $g \in S$.

**Definition 1.2.21.** We call an operator process $X = (X_t)_{t\geqslant 0}$ on $\mathfrak{h}$ a *strong solution* of (1.12) if it is a weak solution of (1.12) and $t \mapsto \widehat{X}_t F_t$ is a QS integrand.

**Remark 1.2.22.** Since QS integrals are continuous operator processes so are the strong solutions.

If, instead of (1.12) an operator process $X = (X_t)_{t\geqslant 0}$ satisfies

$$\begin{aligned}&\langle u \otimes \varepsilon(f), (X_t - T \otimes I_{\mathcal{F}^{\mathsf{k}}})(v \otimes \varepsilon(g))\rangle \\ &= \int_0^t \Big\langle u \otimes \varepsilon(f), E^{\widehat{f}(s)} F_s E_{\widehat{g}(s)} X_s (v \otimes \varepsilon(g))\Big\rangle \, \mathrm{d}s,\end{aligned}$$

for all $u \in \mathfrak{h}$, $v \in \mathcal{D}$, $f \in L^2(\mathbb{R}_+; \mathsf{k})$ and $g \in S$. We call it a weak solution of the right quantum stochastic differential equation

$$\mathrm{d}X_t = F_t \widehat{X}_t \, \mathrm{d}\Lambda_t, \quad X_0 = T \otimes I_{\mathcal{F}^{\mathsf{k}}}. \tag{1.14}$$

**Remark 1.2.23.** Picard iteration assures the existence and uniqueness of a weak solution of the above quantum stochastic differential equations, whenever $F_t = F \otimes I_{\mathcal{F}^{\mathsf{k}}}$ for some $F \in \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ ([63, Theorem 4.2, p. 240], [88, Theorem 5.3.1, p. 129]).

Further we abbreviate a quantum stochastic differential equation to *QSDE*.

Let $S \subset L^2(\mathbb{R}_+; \mathsf{k})$ be an admissible set (Definition 1.2.1). For an operator $T$ on $\mathcal{F}^{\mathsf{k}}$ with domain $\mathcal{E}(S)$ its shifts are defined by

$$\sigma_t(T) = I_{\mathcal{F}_{[0,t)}} \underline{\otimes} S_t T S_t^* \big|_{\mathcal{E}(S)}, \tag{1.15}$$

where the right shift $S_t\colon \mathcal{F}^{\mathsf{k}} \to \mathcal{F}^{\mathsf{k}}_{[t,\infty)}$ is given by

$$S_t\varepsilon(f) = \varepsilon(s_t f), \quad s_t f(r) = f(r-t) \text{ for all } f \in L^2(\mathbb{R}_+, \mathsf{k}),\ t \in \mathbb{R}_+.$$

**Definition 1.2.24** (Quantum stochastic cocycles)**.** Let $X = (X_t)_{t\geqslant 0}$ be an operator process on $\mathfrak{h}$ (Definition 1.2.3). We call $X$ a left *QS cocycle* on $\mathfrak{h}$ if it satisfies

$$X_{s+t} = X_s\widetilde{\sigma}_s(X_t) \text{ for all } s,t \geqslant 0, \tag{1.16}$$

where

$$\widetilde{\sigma}_s(X_t) := (\mathrm{id}_{\mathcal{B}(\mathfrak{h})} \underline{\otimes}\, \sigma_s)(X_t)\big|_{\mathcal{D}\,\underline{\otimes}\,\mathcal{E}(S)}.$$

If instead of condition (1.16) we assume that $X$ satisfies

$$X_{s+t} = \widetilde{\sigma}_s(X_t)X_s \text{ for all } s,t \geqslant 0,$$

then we call it a right *QS cocycle* on $\mathfrak{h}$.

We will be using left QS cocycles associated with left QSDEs. All the theorems which are true for left QS cocycles associated with left QSDE are also valid for right QS cocycles associated with right QSDEs. We also show the connection between left and right cocycles, which involves the time reflection process (Example 1.2.6) and allows us to use only left cocycles and obtain the results for the right ones for free. Henceforth, the term "QS cocycle" will refer to a left QS cocycle.

**Definition 1.2.25.** If $X$ is a QS cocycle and it satisfies strongly the QSDE

$$\mathrm{d}X_t = \widehat{X}_t(F \otimes I_{\mathcal{F}^{\mathsf{k}}})\,\mathrm{d}\Lambda_t, \quad X_0 = I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}}}, \tag{1.17}$$

where $F \in \mathcal{B}(\widehat{\mathsf{k}} \otimes \mathfrak{h})$, then we call $F$ the *stochastic generator* of $X$, and denote $X$ by $X^F$.

**Proposition 1.2.26.** *Let $X = (X_t)_{t \geqslant 0}$ be the unique strong solution of the QSDE*

$$dX_t = \widehat{X}_t(F \otimes I_{\mathcal{F}^{\mathsf{k}}})\, d\Lambda_t, \quad X_0 = I_{\mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}}}, \tag{1.18}$$

*where $F \in \mathcal{B}(\widehat{\mathsf{k}} \otimes \mathfrak{h})$. Then $X$ is a QS cocycle with generator $F$, simply $X = X^F$.*

For the proof we refer the reader to [74, Theorem 2.3 b.]. For an operator process $X = (X_t)_{t \geqslant 0}$ on $\mathfrak{h}$ define the following operators on $\mathfrak{h}$;

$$P_t^{c,d} := E^{\varepsilon(c_{[0,t)})} X_t E_{\varepsilon(d_{[0,t)})} \text{ for all } c,\ d \in \mathsf{k},\ t \geqslant 0. \tag{1.19}$$

**Proposition 1.2.27** (Proposition 3.2 in [72])**.** *The following statements are equivalent*

- *$X$ is a QS cocycle on $\mathfrak{h}$ such that $E^{\varepsilon(f)} X_t E_{\varepsilon(g)} \in \mathcal{B}(\mathfrak{h})$ for all $f,\ g \in S$.*
- *For each $c, d \in \mathsf{k}$, $(P_t^{c,d})_{t \geqslant 0}$ is a semigroup on $\mathfrak{h}$, and for all right-continuous step functions $f$ and $g \in S$,*

  $$E^{\varepsilon(f_{[0,t)})} X_t E_{\varepsilon(g_{[0,t)})} = P_{t_1 - t_0}^{f(t_0), g(t_0)} \dots P_{t - t_n}^{f(t_n), g(t_n)}, \tag{1.20}$$

  *where $\{0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant t\}$ contains the discontinuities of $f_{[0,t)}$ and $g_{[0,t)}$.*

*The same holds for right QS cocycles except that the product in (1.20) is in the reverse order.*

*Proof.* Since in the literature the proof of the above proposition is usually omitted or left as an exercise, we present it here. Observe that

$$E^{\varepsilon(c_{[s,s+t)})} \widetilde{\sigma}_s(X_t) E_{\varepsilon(d_{[s,s+t]})} = E^{\varepsilon(s_s^* c_{[s,s+t)})} X_t E_{\varepsilon(s_s^* d_{[s,s+t]})} = E^{\varepsilon(c_{[0,t)})} X_t E_{\varepsilon(d_{[0,t]})}$$

for all $c,\ d \in \mathsf{k}$, $0 \leqslant s < t < \infty$. Thus, $X$ is a QS cocycle if and only if $(P_t^{c,d})_{t \geqslant 0}$ is a semigroup on $\mathfrak{h}$ for each $c, d \in \mathsf{k}$.

Fix $t \in \mathbb{R}_+$. To manifest that (1.20) holds if and only if $X$ is a QS cocycle consider right-continuous step functions $f$ and $g \in S \subseteq L^2(\mathbb{R}_+; \mathsf{k})$ such that the set $\{0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n < t\}$ for $n \in \mathbb{N}$ contains all the discontinuities of $f_{[0,t)}$ and $g_{[0,t)}$. For simplicity set $t_{n+1} := t$, we arrive at

$$\begin{aligned}
&E^{\varepsilon(f_{[0,t)})} X_t E_{\varepsilon(g_{[0,t)})} \\
=&E^{\varepsilon\left(\sum_{i=0}^{n} f(t_i)_{[t_i,t_{i+1})}\right)} X_t E_{\varepsilon\left(\sum_{i=0}^{n} g(t_i)_{[t_i,t_{i+1})}\right)} \\
=&E^{\varepsilon(f(t_0)_{[0,t_1)})} X_{t_1} E_{\varepsilon(g(t_0)_{[0,t_1)})} E^{\varepsilon\left(\sum_{i=1}^{n} f(t_i)_{[t_i,t_{i+1})}\right)} \sigma_{t_1}(X_{t-t_1}) E_{\varepsilon\left(\sum_{i=1}^{n} g(t_i)_{[t_i,t_{i+1})}\right)} \\
=&E^{\varepsilon(f(t_0)_{[0,t_1)})} X_{t_1} E_{\varepsilon(g(t_0)_{[0,t_1)})} E^{\varepsilon\left(\sum_{i=1}^{n} f(t_i)_{[t_i-t_1,t_{i+1}-t_1)}\right)} X_{t-t_1} E_{\varepsilon\left(\sum_{i=1}^{n} g(t_i)_{[t_i-t_1,t_{i+1}-t_1)}\right)}.
\end{aligned}$$

Therefore, by repeating the above procedure we obtain

$$\begin{aligned}
&E^{\varepsilon(f_{[0,t)})} X_t E_{\varepsilon(g_{[0,t]})} \\
=&E^{\varepsilon\left(\sum_{i=0}^{n} f(t_i)_{[t_i,t_{i+1})}\right)} X_t E_{\varepsilon\left(\sum_{i=0}^{n} g(t_i)_{[t_i,t_{i+1})}\right)} \\
=&E^{\varepsilon(f(t_0)_{[0,t_1)})} X_{t_1} E_{\varepsilon(g(t_0)_{[0,t_1)})} \ldots E^{\varepsilon(f(t_n)_{[t_n,t)})} \sigma_{t_n}(X_{t-t_n}) E_{\varepsilon(g(t_n)_{[t_n,t)})} \\
=&E^{\varepsilon(f(t_0)_{[0,t_1-t_0)})} X_{t_1-t_0} E_{\varepsilon(g(t_0)_{[0,t_1-t_0)})} \ldots E^{\varepsilon(f(t_n)_{[0,t-t_n)})} X_{t-t_n} E_{\varepsilon(g(t_n)_{[0,t-t_n)})} \\
=&P_{t-t_0}^{f(t_0),g(t_0)} \ldots P_{t-t_n}^{f(t_n),g(t_n)}.
\end{aligned}$$

□

We refer to $\{(P_t^{c,d})_{t\geqslant 0}: c, d \in \mathsf{k}\}$ as the family of semigroups associated to the cocycle $X$ and we denote the generator of the semigroup $(P_t^{c,d})_{t\geqslant 0}$ by $H_{c,d}$.

**Definition 1.2.28.** A QS cocycle $X$ is called *Markov regular* if each of the semigroup in $\{(P_t^{c,d})_{t\geqslant 0}: c, d \in \mathsf{k}\}$ is norm continuous.

The next proposition is an easy consequence of the semigroup representation of the cocycle;

**Proposition 1.2.29.** *Let* $F \in \mathcal{B}(\widehat{\mathsf{k}} \otimes \mathfrak{h})$ *and let* $(F_m)$ *be a sequence in* $\mathcal{B}(\widehat{\mathsf{k}} \otimes \mathfrak{h})$ *such that* $F_m \to F$ *in norm as* $m \to \infty$.

*If $f, g \in L^2(\mathbb{R}_+; \mathsf{k})$ are right-continuous step functions and $T \in \mathbb{R}_+$ then*

$$\lim_{m\to\infty} \sup_{t\in[0,T]} \left\| E^{\varepsilon(f)} \left( X_t^{F_m} - X_t^F \right) E_{\varepsilon(g)} \right\| = 0,$$

*where $X^F = (X_t^F)$, $X^{F_m} = (X_t^{F_m})$ are Markov-regular QS cocycles.*

*Proof.* Let $f, g$ be right-continuous step functions and let the set $\{0 = t_0 < t_1 < \dots < t_k < t\}$ contains the discontinuities of $f_{[0,t)}$ and $g_{[0,t)}$, where $k \in \mathbb{N}$. Then, by Proposition 1.2.27, we get that

$$E^{\varepsilon(f_{[0,t)})} X_t^{F_m} E_{\varepsilon(g_{[0,t)})} = P(m)_{t_1-t_0}^{f(t_0),g(t_0)} \dots P(m)_{t-t_n}^{f(t_n),g(t_n)},$$

where $P(m)_t^{c,d} := E^{\varepsilon(c_{[0,t)})} X_t^{F_m} E_{\varepsilon(d_{[0,t]})}$.

Moreover, by Lemma A.0.7 each semigroup $P(m)_{t_{i+1}-t_i}^{f(t_i),g(t_i),}$ converges in norm, locally uniformly in time, to $P_{t_{i+1}-t_i}^{f(t_i),g(t_i)} = E^{\varepsilon(f(t_i)_{[0,t_{i+1}-t_i)})} X_{t_{i+1}-t_i}^F E_{\varepsilon(g(t_i)_{[0,t_{i+1}-t_i)})}$ for $i \in \{0, \dots, n\}$ and $t_{n+1} = t$. We end the proof by applying Proposition 1.2.27 again. □

**Proposition 1.2.30.** *Let $X^F$ be a QS cocycle with generator $F \in \mathcal{B}(\widehat{\mathsf{k}} \otimes \mathfrak{h})$. Then $X^F$ is Markov regular and for each $c, d \in \mathsf{k}$ the generator of a semigroup $(P_t^{c,d})_{t\geqslant 0}$ is given by*

$$H_{c,d} = E^{\widehat{c}} F E_{\widehat{d}} + \langle c, d \rangle I_{\mathfrak{h}}. \tag{1.21}$$

The proof can be found in [74], the formula (1.20) (below Corollary 1.4 ) in [74] corresponds to our (1.21).

**Proposition 1.2.31.** *Let $X = (X_t)_{t\geqslant 0}$ be a Markov-regular contractive QS cocycle, then $X$ is the unique strong solution of the QSDE (1.18) for some $F \in \mathcal{B}(\widehat{\mathsf{k}} \otimes \mathfrak{h})$.*

The proof can be found in [74, Theorem 4.1].

To summarise we state the following theorem.

**Theorem 1.2.32.** *Let $X$ be a contractive operator process on $\mathfrak{h}$. Then the following statements are equivalent:*

- *$X$ is a Markov-regular QS cocycle.*

- *$X$ is the unique strong solution of the QSDE (1.17).*

*Proof.* If $X$ is a Markov-regular QS cocycle then Proposition 1.2.31 delivers the QSDE to which $X$ is the unique strong solution. The converse holds by Proposition 1.2.26 and Proposition 1.2.30. □

**Proposition 1.2.33.** *Let $X^F = (X_t^F)_{t\geqslant 0}$ be a bounded QS cocycle with the stochastic generator $F \in \mathcal{B}(\widehat{\mathsf{k}} \otimes \mathfrak{h})$. Then its time reflection process $(Y_t = R_t X_t^F R_t)_{t\geqslant 0}$ is the right QS cocycle with generator $F$.*

The proposition above is a part of [71, Theorem 7.2].

**Theorem 1.2.34.** *Let $X^F = (X_t^F)_{t\geqslant 0}$ be a QS cocycle with the stochastic generator $F \in \mathcal{B}(\widehat{\mathsf{k}} \otimes \mathfrak{h})$. Then the following are equivalent:*

*1). $X^F$ is a family of isometries.*

*2). $F + F^* + F^*\Delta F = 0$.*

*3). $F$ has a block matrix form*

$$F = \begin{bmatrix} iH - \frac{1}{2}L^*L & -L^*W \\ L & W - I \end{bmatrix}, \tag{1.22}$$

*where $H \in \mathcal{B}(\mathfrak{h})$ is self-adjoint, $L \in \mathcal{B}(\mathfrak{h}; \mathsf{k} \otimes \mathfrak{h})$, and $W \in \mathcal{B}(\mathsf{k} \otimes \mathfrak{h})$ an isometry.*

*Furthermore, $X^F$ is a unitary cocycle, if and only if, $W$ in (1.22) is a unitary operator, which holds if and only if*

$$F + F^* + F\Delta F^* = F + F^* + F^*\Delta F = 0. \tag{1.23}$$

For more general case, where $X^F$ is a contraction QS cocycle we refer the reader to the original article [71, Propositions 7.5 and 7.6] and for the matrix form of the generator to [50, Theorem 6.2].

# Chapter 2

# Quasifree stochastic calculus

The theory of quasifree stochastic calculus was initiated in 1985 by Lindsay in his thesis entitled "A Quantum Stochastic Calculus". Recently, in [67, 68] Lindsay and Margetts has extended the previous theory by investigating various Araki-Woods representation of the CCR algebras ([11]). Here we present some of the results which they obtained, however we restrict their theory to the bounded setting.

## 2.1 Conjugate space – partial transpose

In this section we discuss the partial transpose of bounded linear operators. For more details, including the case for unbounded linear operators, we refer the reader to the original articles [67], [68].

**Definition 2.1.1.** If $\mathsf{k}$ is a Hilbert space then $\overline{\mathsf{k}}$ denotes the Hilbert space conjugate to $\mathsf{k}$, i.e. $\overline{\mathsf{k}} = \{\overline{u} : u \in \mathsf{k}\}$, with

$$\overline{u} + \overline{v} = \overline{u + v}, \ \alpha\overline{u} = \overline{\overline{\alpha}u} \text{ and } \langle \overline{u}, \overline{v} \rangle = \langle v, u \rangle \text{ for all } u, \ v \in \mathsf{k} \text{ and } \alpha \in \mathbb{C}.$$

The bijective map $j : \mathsf{k} \to \overline{\mathsf{k}}$ given by $c \mapsto \overline{c}$ is called the *conjugation*.

**Definition 2.1.2.** Let $\mathsf{k}_1$, $\mathsf{k}_2$ be Hilbert spaces with conjugations $j_1$, $j_2$, respectively, that is, $j_i : \mathsf{k}_i \to \overline{\mathsf{k}_i}$; $c \mapsto \overline{c}$ for all $c \in \mathsf{k}_i$ $(i = 1, 2)$. Let $\mathcal{D}$ be a dense subspace of $\mathsf{k}_1$ and let $\overline{\mathcal{D}} := \{\overline{c} : c \in \mathcal{D}\}$. For a linear operator $T : \mathcal{D} \subset \mathsf{k}_1 \to \mathsf{k}_2$, its *conjugate* operator $\overline{T}\colon \overline{\mathcal{D}} \subset \overline{\mathsf{k}_1} \to \overline{\mathsf{k}_2}$ is defined by

$$\overline{T} := j_2 T j_1^{-1}, \quad \overline{c} \mapsto \overline{Tc}.$$

Given $B \in \mathcal{B}(\mathsf{k}_1, \mathsf{k}_2)$, the transpose of $B$ is defined by setting

$$B^{\mathrm{t}} := \overline{B}^* \in \mathcal{B}(\overline{\mathsf{k}_2}, \overline{\mathsf{k}_1}).$$

In [67, 68] to build quasifree stochastic calculus the authors exploited the notion of the partial (matrix) transpose. Before we introduce it let us make the following observation.

Fix a Hilbert space $\mathsf{H}$. Let $\mathcal{B}(\mathsf{k}_1; \mathsf{k}_2)$ or $\mathcal{B}(\mathsf{H})$ be infinite-dimensional, we can map $A \otimes X$ to $A^{\mathrm{t}} \otimes X$, where $A \in \mathcal{B}(\mathsf{k}_1; \mathsf{k}_2)$ and $X \in \mathcal{B}(\mathsf{H})$, however due to the lack of completely boundedness of the transpose ([42, Proposition 2.2.7, p. 27]) this definition cannot be extended to an arbitrary element of $\mathcal{B}(\mathsf{k}_1; \mathsf{k}_2) \overline{\otimes} \mathcal{B}(\mathsf{H})$ by continuity. To solve this problem, we will consider some special classes of operators.

**Lemma 2.1.3.** *Transpose operation restricted to the Hilbert–Schmidt class of operators is a unitary operator.*

*Proof.* We can identify the class of Hilbert–Schmidt operators $A \in \mathrm{HS}(\mathsf{k}_1; \mathsf{k}_2)$ with $\mathsf{k}_2 \otimes \overline{\mathsf{k}_1}$ via isometric isomorphism given by a prescription

$$\psi_{\mathsf{k}_1, \mathsf{k}_2} : |c\rangle \langle d| \mapsto c \otimes \overline{d} \tag{2.1}$$

for all $c \in \mathsf{k}_2$, $d \in \mathsf{k}_1$. Now, observe that $(|c\rangle \langle d|)^{\mathrm{t}} = |\overline{d}\rangle\langle\overline{c}|$. Moreover, $\psi^{-1}_{\overline{\mathsf{k}_2}, \overline{\mathsf{k}_1}}(|\overline{d}\rangle\langle\overline{c}|) = \overline{d} \otimes c$ and the tensor flip map

$$\Pi\colon \mathsf{k}_2 \otimes \overline{\mathsf{k}_1} \to \overline{\mathsf{k}_1} \otimes \mathsf{k}_2, \quad c \otimes \overline{d} \mapsto \overline{d} \otimes c$$

is unitary. Therefore transpose restricted to the Hilbert–Schmidt class is unitary as a composition of three unitary maps $\psi_{\overline{\mathsf{k}_2}, \overline{\mathsf{k}_1}} \circ \Pi \circ \psi_{\mathsf{k}_1, \mathsf{k}_2}$. □

To define the partial transpose we will need a few technicalities.

**Proposition 2.1.4.** *The map* $\mathrm{E}_{\mathsf{H};\mathsf{k}_1,\mathsf{k}_2}\colon \mathcal{B}(\mathsf{k}_1 \otimes \mathsf{H}; \mathsf{k}_2 \otimes \mathsf{H}) \to \mathcal{B}(\mathsf{H}; \mathcal{B}(\mathsf{k}_1; \mathsf{k}_2 \otimes \mathsf{H}))$ *such that*

$$\mathrm{E}_{\mathsf{H};\mathsf{k}_1,\mathsf{k}_2}(T)u := TE_u \tag{2.2}$$

*for all* $T \in \mathcal{B}(\mathsf{k}_1 \otimes \mathsf{H}; \mathsf{k}_2 \otimes \mathsf{H})$*,* $u \in \mathsf{H}$*, is a continuous bijective linear map, with the inverse defined by the prescription*

$$\mathrm{E}^{-1}_{\mathsf{H};\mathsf{k}_1,\mathsf{k}_2}(S)(c \otimes u) = S(u)c, \tag{2.3}$$

*for all* $S \in \mathcal{B}(\mathsf{H}; \mathcal{B}(\mathsf{k}_1; \mathsf{k}_2 \otimes \mathsf{H}))$*,* $u \in \mathsf{H}$ *and* $c \in \mathsf{k}_1$*.*

*Proof.* Linearity is immediate, for the continuity note that

$$\|E_{\mathsf{H};\mathsf{k}_1,\mathsf{k}_2}(T) - E_{\mathsf{H};\mathsf{k}_1,\mathsf{k}_2}(S)\| = \sup_{u \in \mathsf{H}:\|u\|=1} \|(T-S)E_u\| \leqslant \|T - S\|$$

for any $T$, $S \in \mathcal{B}(\mathsf{k}_1 \otimes \mathsf{H}; \mathsf{k}_2 \otimes \mathsf{H})$. It is easy to check that (2.3) defines the inverse of $E_{\mathsf{H};\mathsf{k}_1,\mathsf{k}_2}$. □

Now, let $\mathsf{M}$ be a von Neumann algebra acting on $\mathsf{H}$.

Observe that if $\mathrm{E}^{-1}_{\mathsf{H};\mathsf{k}_1,\mathsf{k}_2}(S) \in \mathcal{B}(\mathsf{k}_1; \mathsf{k}_2) \overline{\otimes} \mathsf{M}$ for some $S \in \mathcal{B}(\mathsf{H}; \mathcal{B}(\mathsf{k}_1; \mathsf{k}_2 \otimes \mathsf{H}))$, then for all $c \in \mathsf{k}_1$, $d \in \mathsf{k}_2$ the operator $u \mapsto E^d S(u)c = E^d \mathrm{E}^{-1}_{\mathsf{H};\mathsf{k}_1,\mathsf{k}_2}(S)E_c u$ is an element of $\mathsf{M}$.

Set

$$\mathcal{B}(\mathsf{H}; \mathrm{HS}(\mathsf{k}_1; \mathsf{k}_2 \otimes \mathsf{H}))_{\mathsf{M}} := \{S \in \mathcal{B}(\mathsf{H}; \mathrm{HS}(\mathsf{k}_1; \mathsf{k}_2 \otimes \mathsf{H}))\colon E^d S(\cdot)c \in \mathsf{M}\ \forall_{c \in \mathsf{k}_2, d \in \mathsf{k}_2}\}.$$

We identify $\mathrm{HS}(\mathsf{k}_1; \mathsf{k}_2 \otimes \mathsf{H})$ with $\mathrm{HS}(\mathsf{k}_1; \mathsf{k}_2) \otimes \mathsf{H}$ via isometric isomorphism given by the following prescription

$$|d \otimes u\rangle \langle c| \mapsto |d\rangle \langle c| \otimes u$$

for all $u \in \mathsf{H}$, $c \in \mathsf{k}_1$ and $d \in \mathsf{k}_2$.
For $S \in \mathcal{B}(\mathsf{H}; \mathrm{HS}(\mathsf{k}_1; \mathsf{k}_2 \otimes \mathsf{H}))_{\mathsf{M}}$ define an operator $S_{\mathsf{T}} \in \mathcal{B}(\mathsf{H}; \mathrm{HS}(\overline{\mathsf{k}_2}; \overline{\mathsf{k}_1} \otimes \mathsf{H}))_{\mathsf{M}}$ by

$$S_{\mathsf{T}} u = (U \otimes I_{\mathsf{H}})(Su) \quad \text{for all } u \in \mathsf{H},$$

where $U\colon \mathrm{HS}(\mathsf{k}_1;\mathsf{k}_2) \to \mathrm{HS}(\overline{\mathsf{k}_2};\overline{\mathsf{k}_1})$ is the transpose, that is, $U\colon A \mapsto A^{\mathrm{t}}$.

For simplicity denote

$$\mathrm{Mat}(\mathsf{M},\mathsf{H})_{\mathsf{k}_1,\mathsf{k}_2} := \left\{T \in \mathcal{B}(\mathsf{k}_1;\mathsf{k}_2)\,\overline{\otimes}\,\mathsf{M}\colon \forall_{u\in\mathsf{H}}\ TE_u \in \mathrm{HS}(\mathsf{k}_1;\mathsf{k}_2\otimes\mathsf{H})\right\}. \tag{2.4}$$

**Definition 2.1.5.** The *partial transpose* is the (unique) assignment ${}^{\mathsf{T}}\colon T \mapsto T^{\mathsf{T}}$ which makes the following diagram commutative:

$$\begin{array}{ccc}
\mathrm{Mat}(\mathsf{M},\mathsf{H})_{\overline{\mathsf{k}_2},\overline{\mathsf{k}_1}} & \xleftarrow{\ \mathrm{E}^{-1}_{\mathsf{H};\overline{\mathsf{k}_2},\overline{\mathsf{k}_1}}\ } & \mathcal{B}(\mathsf{H};\mathrm{HS}(\overline{\mathsf{k}_2};\overline{\mathsf{k}_1}\otimes\mathsf{H}))_{\mathsf{M}} \\
{}^{\mathsf{T}}\colon T\mapsto T^{\mathsf{T}}\ \Big\uparrow & & \Big\uparrow\ {}_{\mathsf{T}}\colon S\mapsto S_{\mathsf{T}} \\
\mathrm{Mat}(\mathsf{M},\mathsf{H})_{\mathsf{k}_1,\mathsf{k}_2} & \xrightarrow[\ \mathrm{E}_{\mathsf{H};\mathsf{k}_1,\mathsf{k}_2}\ ]{} & \mathcal{B}(\mathsf{H};\mathrm{HS}(\mathsf{k}_1;\mathsf{k}_2\otimes\mathsf{H}))_{\mathsf{M}}
\end{array}$$

where $\mathrm{E}_{\mathsf{H};\mathsf{k}_1,\mathsf{k}_2}\colon \mathrm{Mat}(\mathsf{M},\mathsf{H})_{\mathsf{k}_1,\mathsf{k}_2} \to \mathcal{B}(\mathsf{H};\mathrm{HS}(\mathsf{k}_1;\mathsf{k}_2\otimes\mathsf{H}))_{\mathsf{M}}$ is the bijective linear map defined by (2.2) with the inverse satisfying (2.3).

The operator $T^{\mathsf{T}}$ is then called the *partial transpose* of $T$.
The corresponding *conjugation operation* for $T \in \mathrm{Mat}(\mathsf{M},\mathsf{H})_{\mathsf{k}_1,\mathsf{k}_2}$ is defined as follows

$$T^{\mathsf{c}} := \left(T^{\mathsf{T}}\right)^{*} \in \mathcal{B}(\overline{\mathsf{k}_1};\overline{\mathsf{k}_2})\,\overline{\otimes}\,\mathsf{M}. \tag{2.5}$$

**Remark 2.1.6.** Note that the transpose of the adjoint might be not well-defined, but if for $T \in \mathrm{Mat}(\mathsf{M},\mathsf{H})_{\mathsf{k}_1,\mathsf{k}_2}$ its adjoint $T^* \in \mathrm{Mat}(\mathsf{M},\mathsf{H})_{\mathsf{k}_2,\mathsf{k}_1}$ then $T^{\mathsf{c}} = (T^*)^{\mathsf{T}}$.

It is easy to see that $(H\otimes X)^{\mathsf{T}} = H^{\mathrm{t}}\otimes X$ for $X \in \mathsf{M}$ and $H \in \mathrm{HS}(\mathsf{k}_1;\mathsf{k}_2)$. In particular, for all $c \in \mathsf{k}$ we have $|c\rangle^{\mathsf{T}} = |c\rangle^{\mathrm{t}} = \langle\overline{c}|$ and so $\langle c|^{\mathsf{T}} = |\overline{c}\rangle$.

**Lemma 2.1.7.** *Let $T \in \mathrm{Mat}(\mathsf{M},\mathsf{H})_{\mathsf{k},\mathbb{C}}$. Then, for all $c \in \mathsf{k}$,*

$$TE_c = E^{\overline{c}}T^{\mathsf{T}}. \tag{2.6}$$

*Proof.* Now, for all $u$, $v \in \mathsf{H}$ we have that

$$\begin{aligned}
&\left\langle u, E^{\overline{c}} T^{\mathsf{T}}(v)\right\rangle \\
&= \left\langle \overline{c} \otimes u, T^{\mathsf{T}}(v)\right\rangle \\
&= \left\langle \overline{c} \otimes u, \mathrm{E}^{-1}_{\mathsf{H};\mathbb{C},\overline{\mathsf{k}}}((\mathrm{E}_{\mathsf{H};\mathsf{k},\mathbb{C}}(T))_{\mathsf{T}})(v)\right\rangle \\
&= \langle \overline{c} \otimes u, ((\mathrm{E}_{\mathsf{H};\mathsf{k},\mathbb{C}}(T))_{\mathsf{T}})(v)1\rangle \\
&= \langle \overline{c} \otimes u, (U \otimes I_{\mathsf{H}})(TE_v)1\rangle .
\end{aligned}$$

By applying the definition of the transpose we arrive at

$$\begin{aligned}
&\langle \overline{c} \otimes u, (U \otimes I_{\mathsf{H}})(TE_v)1\rangle \\
&= \langle 1 \otimes u, TE_v c\rangle \\
&= \langle 1 \otimes u, T(c \otimes v)\rangle \\
&= \langle u, TE_c(v)\rangle .
\end{aligned}$$

□

**Lemma 2.1.8.** *If $T \in \mathcal{B}(\mathfrak{h}; \mathsf{k} \otimes \mathfrak{h})$ then*

$$T^* E_c = E^{\overline{c}} T^{\mathsf{c}}. \tag{2.7}$$

*for all $c \in \mathsf{k}$.*

*Proof.* First observe that if $T \in \mathcal{B}(\mathfrak{h}; \mathsf{k} \otimes \mathfrak{h})$ then $T \in \mathrm{Mat}(\mathsf{M}, \mathsf{H})_{\mathbb{C},\mathsf{k}}$. Therefore,

$$\begin{aligned}
&\left\langle u, E^{\overline{c}} (T^{\mathsf{T}})^* v\right\rangle \\
&= \left\langle \overline{c} \otimes u, (T^{\mathsf{T}})^* v\right\rangle \\
&= \left\langle T^{\mathsf{T}}(\overline{c} \otimes u), v\right\rangle \\
&= \langle Tu, v \otimes c\rangle \\
&= \langle u, T^*(v \otimes c)\rangle \\
&= \langle u, TE_c v\rangle .
\end{aligned}$$

□

## 2.2 The CCR algebra

In this section we briefly discuss the theory of the CCR algebra. For more details we recommend the reader consult the books [27], [83].

Let $V$ be a real vector space equipped with a *non-degenerate symplectic* form $\sigma$, that is, $\sigma\colon V \times V \to \mathbb{R}$ is a bilinear form which is skew symmetric and non-degenerate:

- $\sigma(u,v) = -\sigma(v,u)$, for all $u,\ v \in V$,
- if $\sigma(u,v) = 0$ for all $u \in V$ then $v = 0$.

From now we will refer to the pair $(V,\sigma)$ as a *symplectic space*. Let $(V_1,\sigma_1)$ and $(V_2,\sigma_2)$ be symplectic spaces, we call a map $T\colon V_1 \to V_2$ *symplectic* if it is real linear and

$$\sigma_2(Tu,Tv) = \sigma_1(u,v) \text{ for all } u,v \in V_1. \tag{2.8}$$

Relation (2.8) implies the injectivity of $T$. Thus, if $T$ is a surjective symplecitc map then it is bijective. We call a surjective symplectic map $T\colon V \to V$ a *symplectic automorphism*.

**Definition 2.2.1.** The *CCR algebra* $\mathrm{CCR}(V,\sigma)$ is the C*-algebra generated by elements $\{w_u\colon u \in V\}$ satisfying

- $w_u w_v = e^{-i\sigma(u,v)} w_{u+v}$ for all $u,\ v \in V$,
- $w_u^* = w_{-u}$, for all $u \in V$.

**Theorem 2.2.2** (Slawny)**.** *For any symplectic space $(V,\sigma)$ the C*-algebra $\mathrm{CCR}(V,\sigma)$ exists and it is unique up to isomorphism.*

For the proof we refer the reader to [83, Theorem 2.1, p. 10].

If $V = \mathsf{H}$, where $\mathsf{H}$ is a complex Hilbert space, and $\sigma = \operatorname{Im}\langle\cdot,\cdot\rangle$ then the Weyl operators on $\Gamma(\mathsf{H})$ defined in (1.2) give us a representation of $\mathrm{CCR}(\mathsf{H},\operatorname{Im}\langle\cdot,\cdot\rangle)$ and moreover the vacuum vector which is cyclic for this representation induces the state $\varphi$ on the algebra such that $\varphi(w_u) = e^{-\frac{1}{2}\|u\|^2}$ for all $u \in \mathsf{H}$. This representation is called the *Fock representation* and the state is called the *Fock state*.

We abbreviate $\mathrm{CCR}(V,\sigma)$ to $\mathrm{CCR}(V)$ if $V$ is a real subspace of the complex Hilbert space $\mathsf{H}$ and $\sigma$ is the imaginary part of the inner product on $\mathsf{H}$.

A state $\varphi$ on $\mathrm{CCR}(V,\sigma)$ is called a *quasifree* state if there exists a symmetric positive bilinear form $\alpha\colon V \times V \to \mathbb{R}$ such that

$$\varphi(w_u) = e^{-\frac{1}{2}\alpha(u,u)} \quad \text{for all } u \in V.$$

If $V$ is a complex subspace of $\mathsf{H}$ then $\varphi$ is called *gauge invariant* if

$$\alpha(zu,zu) = \alpha(u,u)$$

for all $u \in V$ and $z \in \{w \in \mathbb{C}\colon |w| = 1\}$.

**Theorem 2.2.3.** *Let $(V',\sigma')$ be a symplectic space, where $V'$ is a real dense subspace of a complex Hilbert space $\mathsf{H}'$. The* Bogoliubov transformation *induced by a symplectic map $Q\colon V \to V'$ is a *-monomorphism*

$$\Phi_Q\colon \mathrm{CCR}(V,\sigma) \to \mathrm{CCR}(V',\sigma'), \quad w_u \mapsto w_{Qu}$$

*for all $u \in V$. It induces a representation $\pi_Q$ of $\mathrm{CCR}(V,\sigma)$ on $\Gamma(\mathsf{H}')$ satisfying*

$$\pi_Q(w_u) = W_0(Qu)$$

*and a quasifree state*

$$\varphi(w_u) = e^{-\frac{1}{2}\|Qu\|^2}$$

*for all $u \in V$.*

For the proof we refer the reader to [27, Theorem 5.2.8, p. 94].

## Quasifree states via representation

Set $\mathsf{H} := L^2(\mathbb{R}_+;\mathsf{k})$, where $\mathsf{k}$ is a separable complex Hilbert space. The conjugation $j$ induces the conjugation $J\colon L^2(\mathbb{R}_+;\mathsf{k}) \to L^2(\mathbb{R}_+;\overline{\mathsf{k}})$ such that

$$(Jf)(t) = j(f(t)) = \overline{f(t)} \quad \text{for all } f \in L^2(\mathbb{R}_+;\mathsf{k}).$$

We use the following real-linear "doubling map" $\iota\colon L^2(\mathbb{R}_+;\mathsf{k}) \to L^2(\mathbb{R}_+;\mathsf{k}\oplus\overline{\mathsf{k}})$ defined by setting

$$\iota(f) = \begin{pmatrix} f \\ -Jf \end{pmatrix} \tag{2.9}$$

for all $f \in L^2(\mathbb{R}_+;\mathsf{k})$.

**Example 2.2.4** (Gauge-invariant quasifree states)**.** Let $T$ be a positive self-adjoint operator on $L^2(\mathbb{R}_+;\mathsf{k})$. Set $V = \mathrm{Dom}(T)$ and note that

$$\Sigma_T := \begin{bmatrix} \sqrt{I+T^2} & 0 \\ 0 & JTJ^* \end{bmatrix} \tag{2.10}$$

is also a positive self-adjoint operator with dense domain

$$\mathrm{Dom}\Sigma_T := \mathrm{span}_{\mathbb{C}}\iota(V) \subset L^2(\mathbb{R}_+;\mathsf{k}\oplus\overline{\mathsf{k}}).$$

Note that, for all $f, g \in V$

$$\begin{aligned} &\langle \Sigma_T\iota(f), \Sigma_T\iota(g)\rangle \\ =&\left\langle \begin{pmatrix} \sqrt{I+T^2}f \\ -JTf \end{pmatrix}, \begin{pmatrix} \sqrt{I+T^2}g \\ -JTg \end{pmatrix}\right\rangle \\ =&\left\langle (I+T^2)f, g\right\rangle + \langle Tg, Tf\rangle \\ =&\langle f, g\rangle + 2\mathrm{Re}\,\langle Tf, Tg\rangle\,. \end{aligned}$$

Thus if $\sigma = \mathrm{Im}\,\langle\cdot,\cdot\rangle$ then $\Sigma_T\iota$ is symplectic and, furthermore, the map

$$\pi_{\Sigma,T}\colon \mathrm{CCR}(V) \to \mathcal{B}(\mathcal{F}^{\mathsf{k}\oplus\overline{\mathsf{k}}}), \quad w_f \mapsto W_0(\Sigma_T\iota(f)) \tag{2.11}$$

defines a representation of $\mathrm{CCR}(V)$ on $\mathcal{F}^{\mathsf{k}\oplus\overline{\mathsf{k}}}$. The corresponding vacuum vector $\Omega$ induces the gauge-invariant quasifree state $\varphi$ such that

$$\varphi(w_f) := \langle \Omega, W_0(\Sigma_T\iota(f)\Omega\rangle = e^{-\frac{1}{2}\|\Sigma_T\iota(f)\|^2} = e^{-\frac{1}{2}\left\|\sqrt{I+2T^2}f\right\|^2}$$

for all $f \in V$.

**Example 2.2.5** (Squeezed states)**.** Let $\Phi_Q$ be a Boguliubov transformation induced by a symplectic automorphism $Q$ of $V := \mathrm{Dom}(T)$. Since in our case $L^2(\mathbb{R}_+;\mathsf{k})$ is separable, Thereom 2.5, Corollary 2.6 and Corollary 2.8 in [55] give the polar decomposition of $Q$, that is

$$Q = U\cosh(D)|_V + UK\sinh(D)|_V, \tag{2.12}$$

where $U$ is a unitary operator on $L^2(\mathbb{R}_+;\mathsf{k})$, $K$ is an antilinear involution on $L^2(\mathbb{R}_+;\mathsf{k})$ (that is, $K = K^* = K^{-1}$) and $D$ is self-adjoint and positive. Moreover, this decomposition is unique in the sense of [55, Theorem 2.5 (e), p. 4296]. To see more relations between $U$, $D$, $K$ and $Q$ we refer the reader to [55, Theorem 2.5].

Now note that $\Sigma_T \circ \iota \circ Q = \Sigma_{T,Q} \circ \iota$, where

$$\Sigma_{T,Q} = \begin{bmatrix} \sqrt{I+T^2}U\cosh(D) & -\sqrt{I+T^2}UK\sinh(D)J^* \\ -JTUK\sinh(D) & JTU\cosh(D)J^* \end{bmatrix}. \tag{2.13}$$

The map $\Sigma_{T,Q}\iota$ is symplectic as a composition of two symplectic maps, and the map

$$\pi_{\Sigma,T,Q}\colon \mathrm{CCR}(V) \to \mathcal{B}(\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}), \quad w_f \mapsto W_0(\Sigma_{T,Q}\iota(f)) \tag{2.14}$$

defines a representation of $\mathrm{CCR}(V)$ on $\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}$.

The corresponding 'squeezed' quasifree state on $\mathrm{CCR}(V)$ has the form

$$\varphi(w_f) := \langle \Omega, W_0(\Sigma_{T,Q}\iota(f))\Omega\rangle = e^{-\frac{1}{2}\|\Sigma_{T,Q}\iota(f)\|^2} = e^{-\frac{1}{2}\|\sqrt{I+2T^2}Qf\|^2}$$

for all $f \in V$.

## The duality theorem

Now we briefly discuss the duality theorem for the CCR algebra. We generate a von Neumann algebra and its commutant starting from different subalgebras of the CCR algebra which acts on the symmetric Fock space. Tomita–Takesaki modular theory shows how to establish a connection between the von Neumann algebra $\mathsf{M}$ and its commutant $\mathsf{M}'$, through the modular conjugation $J_\xi$, that is,

$J_\xi \mathsf{M} J_\xi = \mathsf{M}'$, provided that $\mathsf{M}$ admits a cyclic and separating vector $\xi$ ([90, Theorem 1.19, p. 13]).

Let $\mathsf{K}$ be a complex Hilbert space, and let the non-degenerate symplectic form $\sigma$ be the imaginary part of the inner product on $\mathsf{K}$. We consider $\mathrm{CCR}(\mathsf{K})$ which is generated by the elements $\{W_0(u): u \in \mathsf{K}\}$, in particular it acts on the Fock space $\Gamma(\mathsf{K})$. For a subspace $\mathsf{K}_0$ of $\mathsf{K}$, we write $\mathrm{CCR}(\mathsf{K}_0)$ for the subalgebra of $\mathrm{CCR}(\mathsf{K})$ generated by $\{W_0(u): u \in \mathsf{K}_0\}$. If $\mathsf{K}_0$ is a real subspace of $\mathsf{K}$ such that $\mathsf{K}_0 + \mathrm{i}\mathsf{K}_0$ is dense in $\mathsf{K}$, then the vacuum vector $\Omega$ is cyclic for $\mathrm{CCR}(\mathsf{K}_0)$ ([83, Proposition 7.7 p. 65]).

We denote the von Neumann algebra $(\mathrm{CCR}(\mathsf{K}_0))''$ by $\mathsf{M}_{\mathsf{K}_0}$. The vacuum vector $\Omega$ is cyclic for $\mathsf{M}_{\mathsf{K}_0}$, as long as, we assume that $\mathsf{K}_0 + \mathrm{i}\mathsf{K}_0$ is dense in $\mathsf{K}$. Set $\mathsf{K}_1$ to be the symplectic complement of $\mathsf{H}_0$, that is,

$$\mathsf{K}_1 := \{u \in \mathsf{K}: \forall_{v \in \mathsf{K}_0} \ \ \sigma(v,u) = 0\}.$$

It is easy to see that $\mathsf{K}_1 = \mathrm{i}\mathsf{K}_0^{\perp}$, where $\perp$ is the orthogonal complement with respect to the real part of the inner product on $\mathsf{K}$ ([83, p. 66]). Using the property (1.4) we observe that $W_0(u)W_0(v) = W_0(v)W_0(u)$ for all $u \in \mathsf{K}_0$, $v \in \mathsf{K}_1$, and so

$$\mathsf{M}_{\mathsf{K}_1} \subset \mathsf{M}'_{\mathsf{K}_0}.$$

To obtain the equality we have to assume that $\mathsf{K}_0 \cap \mathrm{i}\mathsf{K}_0 = \{0\}$ and use Tomita–Takesaki modular theory. Therefore, let us state the duality theorem;

**Theorem 2.2.6.** *[83, Theorem 7.8, p. 66]. Let $\mathsf{K}_0$ be a closed real subspace of $\mathsf{K}$ such that $\mathsf{K}_0 + \mathrm{i}\mathsf{K}_0$ is dense in $\mathsf{K}$ and $\mathsf{K}_0 \cap \mathrm{i}\mathsf{K}_0 = \{0\}$. Then $\mathsf{M}'_{\mathsf{K}_0} = \mathsf{M}_{\mathrm{i}\mathsf{K}_0^{\perp}}$. Moreover, the vacuum vector $\Omega$ is cyclic and separating for both $\mathsf{M}_{\mathsf{K}_0}$ and $\mathsf{M}_{\mathrm{i}\mathsf{K}_0^{\perp}}$.*

We also refer the reader to the original article [11], the proof of [83, Theorem 7.8, p. 66] comes from [41].

For next two examples let us fix $\mathsf{K} := L^2(\mathbb{R}_+; \mathsf{k} \oplus \overline{\mathsf{k}})$.

**Example 2.2.7** (Gauge-invariant case)**.** Let $\iota': L^2(\mathbb{R}_+; \overline{\mathsf{k}}) \to L^2(\mathbb{R}_+; \overline{\mathsf{k}} \oplus \mathsf{k})$ be the map

$$\iota' := \begin{bmatrix} I \\ -J^* \end{bmatrix}. \tag{2.15}$$

According to Example 2.2.4, define a closed operator

$$\Sigma'_T := \begin{bmatrix} 0 & T \\ J\sqrt{1+T^2}J^* & 0 \end{bmatrix} \tag{2.16}$$

with dense domain

$$\mathrm{Dom}\,\Sigma'_T := \mathrm{span}_{\mathbb{C}}\iota'(JV) \subset \overline{\mathsf{K}}.$$

One can show that

- $\mathrm{Ran}\,\iota + \mathrm{i}\mathrm{Ran}\,\iota = \mathsf{K}$, $\mathrm{Ran}\,\iota \cap \mathrm{i}\mathrm{Ran}\,\iota = \{0\}$, and
- $\mathrm{Ran}\,\iota' + \mathrm{i}\mathrm{Ran}\,\iota' = \overline{\mathsf{K}}$, $\mathrm{Ran}\,\iota' \cap \mathrm{i}\mathrm{Ran}\,\iota' = \{0\}$.

For the proof we refer the reader to [67, Lemma 4.4, p. 15]).

Set

- $\mathsf{K}_0 := \mathrm{cl}\,\Sigma_T\iota(V)$, which is the closure of the range of the operator

$$\begin{bmatrix} \sqrt{I+T^2} \\ -JT \end{bmatrix},$$

- $\mathsf{K}_1 := \mathrm{cl}\,\Sigma'_T\iota'(JV)$, which is the closure of the range of the operator

$$\begin{bmatrix} -T \\ J\sqrt{I+T^2} \end{bmatrix},$$

If $T$ is injective then $T$ has dense range ([76, p. 231]) and so both $\Sigma_T$ and $\Sigma'_T$ have dense ranges, moreover $\mathsf{K}_0 + \mathrm{i}\mathsf{K}_0$ is dense in $\mathsf{K}$, and $\mathsf{K}_0 \cap \mathrm{i}\mathsf{K}_0 = \{0\}$. In particular, we get that

$$\mathsf{K}_1 = \mathrm{i}\mathsf{K}_0^{\perp}.$$

Hence we can apply Theorem 2.2.6 and obtain a von Neumann algebra

$$\mathrm{N}_\Sigma := (\mathrm{CCR}(\mathsf{K}_0))'' = (\pi_{\Sigma,T}(\mathrm{CCR}(V)))'' \tag{2.17}$$

which acts on the Hilbert space $\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}$, and its commutant

$$\mathrm{N}'_\Sigma = (\mathrm{CCR}(\mathsf{K}_1))'' = (\pi_{\Sigma',T}(\mathrm{CCR}(JV)))''$$

both having $\Omega$ as a cyclic and separating vector.

**Example 2.2.8** (Squeezed case)**.** Now, we use the duality theorem for the case where the input is the operators from Example 2.2.5. Similarly to the preceding example, we define a closed operator

$$\Sigma'_{T,Q} = \begin{bmatrix} -TUK\sinh(D)J^* & TU\cosh(D) \\ J\sqrt{I+T^2}U\cosh(D)J^* & -J\sqrt{I+T^2}UK\sinh(D) \end{bmatrix} \tag{2.18}$$

on the dense domain

$$\mathrm{Dom}\,\Sigma'_{T,Q} := \mathrm{span}_{\mathbb{C}}\iota'(JV) \subset \overline{\mathsf{K}},$$

where $U$, $K$, $D$ are obtained from the polar decomposition (2.12). Set

- $\mathsf{K}_0 := \mathrm{cl}\,\Sigma_{T,Q}\iota(V)$, which is the closure of the range of the operator

$$\begin{bmatrix} \sqrt{I+T^2}U(\cosh(D)-K\sinh(D)) \\ JTU(\cosh(D)-K\sinh(D)) \end{bmatrix},$$

- $\mathsf{K}_1 := \mathrm{cl}\,\Sigma'_{T,Q}\iota'(JV)$, which is the closure of the range of the operator

$$\begin{bmatrix} -TU(\cosh(D)+K\sinh(D))J^* \\ J\sqrt{I+T^2}U(\cosh(D)+K\sinh(D))J^* \end{bmatrix},$$

If $\Sigma_{T,Q}$ has a dense range (e.g., when $D$ is bounded [67, Example (Squeezed states)]) then $\mathsf{K}_0 + \mathrm{i}\mathsf{K}_0$ is dense in $\mathsf{K}$, and $\mathsf{K}_0 \cap \mathrm{i}\mathsf{K}_0 = \{0\}$. Moreover, we get

$$\mathsf{K}_1 = \mathrm{i}\mathsf{K}_0^\perp.$$

Hence, we can apply Theorem 2.2.6 to obtain a von Neumann algebra

$$\mathrm{N}_\Sigma := (\mathrm{CCR}(\mathsf{K}_0))'' = \big(\pi_{\Sigma,T,Q}(\mathrm{CCR}(V))\big)'' \tag{2.19}$$

which acts on the Hilbert space $\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}$ and its commutant

$$\mathrm{N}'_\Sigma = (\mathrm{CCR}(\mathsf{K}_1))'' = (\pi_{\Sigma',T}(\mathrm{CCR}(JV)))''$$

both having $\Omega$ as a cyclic and separating vector.

In [67, p. 12-19], [68, p. 3-5] authors has established the general conditions for the operators $\Sigma$ and $\Sigma'$ to generate von Neumann algebras

$$N_\Sigma := (W_0(\Sigma\iota(f)) \colon f \in V)'', \quad N'_\Sigma = (W_0(\Sigma'\iota'(Jf)) \colon f \in V)''.$$

For further calculations the following lemma will be helpful;

**Lemma 2.2.9.** *Let $f \in V$ be such that $\iota'(Jf) \in \mathrm{Dom}(\Sigma^*\Sigma')$, where $\Sigma$ and $\Sigma'$ are either, $\Sigma_T$ and $\Sigma'_T$ or $\Sigma_{T,Q}$ and $\Sigma'_{T,Q}$, respectively defined in Examples 2.2.4, 2.2.5, 2.2.8 and 2.2.7. Then*

$$J^{\bar{\mathsf{k}}\oplus\mathsf{k}}\Pi(\Sigma^*\Sigma'\iota'(Jf)) = -\Sigma^*\Sigma'\iota'(Jf),$$

*where*

- $J^{\bar{\mathsf{k}}\oplus\mathsf{k}} \colon L^2(\mathbb{R}_+; \bar{\mathsf{k}} \oplus \mathsf{k}) \to L^2(\mathbb{R}_+; \mathsf{k} \oplus \bar{\mathsf{k}})$ *is the conjugation, that is,*

$$(J^{\bar{\mathsf{k}}\oplus\mathsf{k}} f)(t) = \overline{f(t)} \quad \textit{for all } f \in L^2(\mathbb{R}_+; \bar{\mathsf{k}} \oplus \mathsf{k})),$$

- $\Pi \colon L^2(\mathbb{R}_+; \mathsf{k} \oplus \bar{\mathsf{k}}) \to L^2(\mathbb{R}_+; \bar{\mathsf{k}} \oplus \mathsf{k})$ *is a direct-sum flip, i.e.*

$$\Pi(g \oplus \bar{h}) = \bar{h} \oplus g \quad \textit{for all } g,\ h \in L^2(\mathbb{R}_+; \mathsf{k}).$$

For the proof we refer the reader to [68, Lemma 1.2].

## 2.3 Quasifree stochastic analysis

Before we start to discuss quasifree stochastic analysis, we have to add a piece of notation and make a few assumptions for the operators $\Sigma_T$ defined in (2.10) and $\Sigma_{T,Q}$ given by (2.13).

**Notation 2.3.1.** For simplicity let us use the following notation.

- Henceforth, $\Sigma$ is of the form $\Sigma_T$ and $\Sigma_{T,Q}$ as in Examples 2.2.4 and 2.2.5.
- Similarly, $\Sigma'$ is of the form $\Sigma'_T$ and $\Sigma'_{T,Q}$ as in Examples 2.2.8 and 2.2.7.
- We will consider only the case when $\Sigma$ and $\Sigma'$ are bounded operators, in other words $T$, $Q$ and $D \in \mathcal{B}(L^2(\mathbb{R}_+; \mathsf{k}))$.
- Let $W(f) := W_0(\Sigma \circ \iota(f))$ and $W'(f) := W_0(\Sigma' \circ \iota'(Jf))$ for every function $f \in L^2(\mathbb{R}_+; \mathsf{k})$.
- Instead $\iota'(Jf)$ we will write simply $\iota(f)$, when this expression follows the operator $\Sigma'$, that is, we will write $\Sigma'\iota(f)$ instead of $\Sigma'\iota'(Jf)$.

We would like to be able to write

$$(\Sigma f)(t) = \Sigma_t(f(t)) \tag{2.20}$$

for all $f \in L^2(\mathbb{R}_+; \mathsf{k})$, for some family of operators $(\Sigma_t)_{t\geqslant 0}$ on $\mathsf{k} \oplus \overline{\mathsf{k}}$. For this to hold we will need some conditions on the operators in Example 2.2.4 and 2.2.5. The natural assumption is delivered by the following theorem.

**Theorem 2.3.2.** *Let $A \in \mathcal{B}(L^2(\mathbb{R}_+; \mathsf{k})$. There exists a strongly measurable function $B\colon \mathbb{R}_+ \to \mathcal{B}(\mathsf{k});\ t \mapsto B_t$ such that*

$$(Af)(t) = B_t(f(t)) \tag{2.21}$$

*for each $t \geqslant 0$, if and only if $A$ commutes with $M_g \in \mathcal{B}(L^2(\mathbb{R}_+; \mathsf{k}))$ for all $g \in L^\infty(\mathbb{R}_+)$, where $(M_g f)(t) = g(t)f(t)$.*

For the proof we refer the reader to [89, Theorem 7.10, p. 259].

**Remark 2.3.3.** The condition which says that $A$ commutes with $M_g \in \mathcal{B}(L^2(\mathbb{R}_+;\mathsf{k}))$ for all $g \in L^\infty(\mathbb{R}_+)$, could be written more simply as $A \in \left(\mathsf{D} \overline{\otimes} \mathbb{C} I_{\mathsf{k}}\right)'$, where $\mathsf{D}$ (often called the diagonal algebra) stands for the von Neumann algebra $L^\infty(\mathbb{R}_+)$, elements of which act as multiplication operators on $L^2(\mathbb{R}_+)$.

Theorem 2.3.2 let us changed the technical conditions assumed in [67, p. 17] or [68, Proposition 1.3] to obtain $\Sigma$ of the form (2.20). However, it was possible since we assumed that $\Sigma$ is a bounded operator.

Further in this chapter we will be abusing notation by writing $\Sigma_t^*$ for $(\Sigma^*)_t$, and $\Sigma_t \otimes I_{\mathsf{H}}$ instead of $(\Sigma \otimes I_{\mathsf{H}})_t$ for each $t \in \mathbb{R}_+$, where $\mathsf{H}$ is a Hilbert space.

## Quasifree integrals

Stochastic integrals for the representation of CCR algebras were studied in [57, 13] and recently in [67, 68]. Here, we discuss quasifree integrals defined as in [68].

Let $\mathsf{A}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{h}$, and let us denote $\mathsf{M} := \mathsf{A} \overline{\otimes} \mathsf{N}$, where $\mathsf{N}$ is $\{W_0(\Sigma_T \iota(f)) : f \in L^2(\mathbb{R}_+;\mathsf{k})\}''$, as defined in (2.17), or $\{W_0(\Sigma_{T,Q} \iota(f)) : f \in L^2(\mathbb{R}_+;\mathsf{k})\}''$, as in (2.19). .

For each $t \in \mathbb{R}_+$, let $\mathsf{M}_t := \mathsf{A} \overline{\otimes} \mathsf{N}_t$, where

$$\mathsf{N}_t := \{W(f) : f \in p_t(L^2(\mathbb{R}_+;\mathsf{k}))\}''. \tag{2.22}$$

and $p_t \in \mathcal{B}(L^2(\mathbb{R}_+;\mathsf{k}))$ is the projection given by $f \mapsto f_{[0,t)}$.

**Definition 2.3.4.** A (bounded) $\Sigma$-*quasifree process* on $\mathfrak{h}$ is a family of bounded linear operators $X = (X_t)_{t \geqslant 0}$ such that $X_t \in \mathsf{M}_t$ for all $t \in \mathbb{R}_+$.

**Definition 2.3.5** (Quasifree integrand)**.** A *quasifree integrand* on $\mathfrak{h}$ is a family of linear operators $F = (F_t)_{t \geqslant 0}$, such that

$$F_t = \begin{bmatrix} K_t & M_t \\ L_t & 0 \end{bmatrix}, \tag{2.23}$$

where, for each $t \in \mathbb{R}_+$,

- $K_t \in \mathsf{M}$,

- $M_t \in \mathcal{M}(\mathsf{M}, \mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}})_{\mathsf{k},\mathbb{C}}$, that is, $M_t \in \mathcal{B}(\mathsf{k};\mathbb{C}) \overline{\otimes} \mathsf{M}$ and

$$M_t E_x \in \mathrm{HS}(\mathsf{k}; \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}) \text{ for all } x \in \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}},$$

- $L_t \in \mathcal{B}(\mathbb{C}; \mathsf{k}) \otimes_{\mathsf{m}} \mathsf{M}$,

and

- $t \mapsto K_t(u \otimes \Omega) \in L^1_{\Omega,\mathrm{loc}}(\mathbb{R}_+; \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}})$ for all $u \in \mathfrak{h}$,
- $z : t \mapsto \begin{pmatrix} L_t(u\otimes\Omega) \\ M_t^{\mathsf{T}}(u\otimes\Omega) \end{pmatrix} \in L^2_{\Omega,\mathrm{loc}}(\mathbb{R}_+; (\mathsf{k} \oplus \bar{\mathsf{k}}) \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}})$ for all $u \in \mathfrak{h}$.

**Remark 2.3.6.** Let $\mathcal{I}\colon L^2_\Omega(\mathbb{R}_+; (\mathsf{k}\oplus\bar{\mathsf{k}})\otimes\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}) \to \mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}$ denote the abstract Itô integral (Definition 1.2.10). The inclusion map from the space of all vacuum adapted square-integrable functions (Definition 1.2.8) $L^2_\Omega(\mathbb{R}_+; (\mathsf{k}\oplus\bar{\mathsf{k}})\otimes\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}})$ to $L^2(\mathbb{R}_+; (\mathsf{k} \oplus \bar{\mathsf{k}}) \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}})$ is denoted by $V_\Omega$.

**Definition 2.3.7** (Quasifree integral)**.** Let $F = (\left[\begin{smallmatrix} K_t & M_t \\ L_t & 0 \end{smallmatrix}\right])_{t\geqslant 0}$ be a quasifree integrand on $\mathfrak{h}$, the $\Sigma$-*quasifree integral* $\Lambda^\Sigma_t(F)\colon \mathfrak{h} \underline{\otimes} \mathsf{N}'\Omega \to \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}$ is defined by

$$\Lambda^\Sigma_t(F)(u\otimes X'\Omega) = (I_\mathfrak{h}\otimes X')\left(\mathcal{T}_t(z^{(0)}) + \mathcal{I}_t(V^*_\Omega(\Sigma \otimes I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}})V_\Omega)(z^{(1)})\right), \quad (2.24)$$

where $z^{(0)}_t = K_t(u \otimes \Omega)$, $z^{(1)}_t = \begin{pmatrix} L_t(u \otimes \Omega) \\ M_t^{\mathsf{T}}(u \otimes \Omega) \end{pmatrix}$.

Note that $\Lambda^\Sigma_t(F)$ has a bounded extension.

**Remark 2.3.8.** For the quasifree integrand $F$ from the preceding definition we define *quasifree time*, *annihilation* and *creation* integrals by

- $T^\Sigma_t(K) := \Lambda^\Sigma_t\left(\left[\begin{smallmatrix} K_t & 0 \\ 0 & 0 \end{smallmatrix}\right]\right)$, where

$$F_t = \begin{bmatrix} K_t & 0 \\ 0 & 0 \end{bmatrix},$$

- $A_t^{\Sigma}(M) := \Lambda_t^{\Sigma}\left(\left[\begin{smallmatrix} 0 & M_t \\ 0 & 0 \end{smallmatrix}\right]\right)$, where

$$F_t = \begin{bmatrix} 0 & M_t \\ 0 & 0 \end{bmatrix},$$

- $A_t^{*,\Sigma}(L) := \Lambda_t^{\Sigma}\left(\left[\begin{smallmatrix} 0 & 0 \\ L_t & 0 \end{smallmatrix}\right]\right)$, where

$$F_t = \begin{bmatrix} 0 & 0 \\ L_t & 0 \end{bmatrix},$$

respectively.
Caution: In quasifree stochastic calculus there is no number integral. For more details we refer the reader to [68, Remark and Lemma 7.5, p. 21].

To draw the analogy between quantum stochastic analysis and quasifree stochastic analysis we state three fundamental statemants.

**Theorem 2.3.9** (First Fundamental Formula). *Let $F$ be a quasifree integrand on $\mathfrak{h}$ of the form (2.23). Then*

$$\begin{aligned}
&\langle u \otimes W'(f)\Omega, \Lambda_t^{\Sigma}(F)(v \otimes W'(g)\Omega)\rangle \\
&= \int_0^t \left\langle \widehat{\Sigma'\iota(h)}(s) \otimes u \otimes W'(f)\Omega, (\widehat{\Sigma}_s \otimes I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}}) \left( \begin{bmatrix} K_s \\ L_s \\ M_s^{\mathsf{T}} \end{bmatrix} (v \otimes W'(g)\Omega) \right) \right\rangle \mathrm{d}s
\end{aligned}$$

*for all $u, v \in \mathfrak{h}$, $f$ and $g \in L^2(\mathbb{R}_+; \mathsf{k})$, where $h = f - g$ and $\widehat{\Sigma}_s := I_{\mathbb{C}} \oplus \Sigma_s$.*

*Proof.* Let $u, v \in \mathfrak{h}$, $f$ and $g \in L^2(\mathbb{R}_+; \mathsf{k})$. Then

$$\begin{aligned}
&\langle u \otimes W'(f)\Omega, \Lambda_t^\Sigma(F)(v \otimes W'(g)\Omega)\rangle \\
=&\langle u \otimes W'(-g)W'(f)\Omega, \mathcal{T}_t\big(K.v \otimes \Omega\big)\rangle \\
+&\left\langle u \otimes W'(-g)W'(f)\Omega, \mathcal{I}(V_\Omega^*(\Sigma \otimes I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}})V_\Omega)\begin{pmatrix} L.(u \otimes \Omega) \\ M.^\top(u \otimes \Omega)\end{pmatrix}_{[0,t)}\right\rangle \\
=&\int_0^t \langle u \otimes W'(f)\Omega, K_s(v \otimes W'(g)\Omega)\rangle\, \mathrm{d}s \\
+&\int_0^t \left\langle \Sigma'\iota(f-g)(s) \otimes W'(-g)W'(f)\Omega, (\Sigma_s \otimes I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}})\begin{pmatrix} L_s(u \otimes \Omega) \\ M_s^\top(u \otimes \Omega)\end{pmatrix}\right\rangle \mathrm{d}s \\
=&\int_0^t \left\langle \widehat{\Sigma'\iota(f-g)}(s) \otimes u \otimes W'(f)\Omega, (\widehat{\Sigma}_s \otimes I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}})\left(\begin{bmatrix} K_s \\ L_s \\ M_s^\top \end{bmatrix}(v \otimes W'(g)\Omega)\right)\right\rangle \mathrm{d}s.
\end{aligned}$$

The outline of the proof can be found in [68, Proposition 4.1]; here we give a more detailed version for reader's convenience. □

The bridge between quantum stochastic and quasifree calculus is the following corollary of the First Fundamental Formula.

**Corollary 2.3.10.** *Let $F$ be a quasifree integrand on $\mathfrak{h}$ of the form (2.23). Then*

$$\Lambda_t^\Sigma(F) = \Lambda_t(F^\square) \text{ on } \mathfrak{h} \underline{\otimes} \mathsf{N}'\Omega,$$

*where*

$$F_t^\square = \begin{bmatrix} K_t & \begin{bmatrix} M_t & L_t^\top \end{bmatrix}(\Sigma_t^* \otimes I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}}) \\ (\Sigma_t \otimes I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}})\begin{bmatrix} L_t \\ M_t^\top \end{bmatrix} & 0 \end{bmatrix} \tag{2.25}$$

*is defined on $\widehat{\mathsf{k} \oplus \bar{\mathsf{k}}} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}$.*

*Proof.* In [68] this corollary is a consequence of Explicit Formulae [68, Proposition 4.3], we derive it directly from the First Fundamental Formula.

First note that for any $f \in L^2(\mathbb{R}_+; \mathsf{k})$ we have

$$W'(f)\Omega = e^{-\frac{1}{2}\|\Sigma'\iota(f)\|^2}\varepsilon(\Sigma'\iota(f)).$$

For simplicity let $c(\Sigma', f) := e^{-\frac{1}{2}\|\Sigma'\iota(f)\|^2}$.

If $u, v \in \mathfrak{h}$, and $f, g \in L^2(\mathbb{R}_+; \mathsf{k})$ then we obtain

$$\begin{aligned}
&\langle u \otimes W'(f)\Omega, \Lambda_t(F^\square)(v \otimes W'(g)\Omega)\rangle \\
=&c(\Sigma', f)c(\Sigma', g)\int_0^t \left\langle u \otimes \varepsilon(\Sigma'\iota(f)), E^{\widehat{\Sigma'\iota(f)}(s)} F^\square E_{\widehat{\Sigma'\iota(g)}(s)}(v \otimes \varepsilon(\widehat{\Sigma'}\iota(g))\right\rangle \, \mathrm{d}s \\
=&\int_0^t \left\langle u \otimes W'(f)\Omega, E^{\widehat{\Sigma'\iota(f)}(s)} F^\square E_{\widehat{\Sigma'\iota(g)}(s)}(v \otimes W'(g)\Omega))\right\rangle \, \mathrm{d}s \\
=&\int_0^t \langle u \otimes W'(f)\Omega, K_s(v \otimes W'(g)\Omega))\rangle \, \mathrm{d}s \\
&+ \int_0^t \left\langle u \otimes W'(f)\Omega, E^{\Sigma'\iota(f)(s)}(\Sigma_s \otimes I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}}) \begin{bmatrix} L_s \\ M_s^\top \end{bmatrix} (v \otimes W'(g)\Omega))\right\rangle \, \mathrm{d}s \\
&+ \int_0^t \left\langle u \otimes W'(f)\Omega, \begin{bmatrix} M_s & L_s^\top \end{bmatrix}(\Sigma_s^* \otimes I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}}) E_{\Sigma'\iota(g)(s)}(v \otimes W'(g)\Omega))\right\rangle \, \mathrm{d}s.
\end{aligned}$$

Lemmas 2.1.7 and 2.2.9 yield that

$$\begin{aligned}
&\begin{bmatrix} M_s & L_s^\top \end{bmatrix}(\Sigma_s^* \otimes I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}}) E_{\Sigma'\iota(g)(s)} \\
=&\begin{bmatrix} L_s^\top & M_s \end{bmatrix} E_{\Pi((\Sigma^*\Sigma'\iota(g))(s))} \\
=&E^{J^{\bar{\mathsf{k}}\oplus\mathsf{k}}\Pi((\Sigma^*\Sigma'\iota(g))(s))} \begin{bmatrix} L_s \\ M_s^\top \end{bmatrix} \\
=&E^{\Sigma'\iota(-g)(s)}(\Sigma_s \otimes I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}}) \begin{bmatrix} L_s \\ M_s^\top \end{bmatrix},
\end{aligned}$$

where $\Pi\colon L^2(\mathbb{R}_+; \mathsf{k} \oplus \bar{\mathsf{k}}) \to L^2(\mathbb{R}_+; \bar{\mathsf{k}} \oplus \mathsf{k})$ is a direct-sum flip.

Therefore

$$\begin{aligned}
&\int_0^t \langle u \otimes W'(f)\Omega, K_s(v \otimes W'(g)\Omega))\rangle \, \mathrm{d}s \\
&+ \int_0^t \left\langle u \otimes W'(f)\Omega, E^{\Sigma'\iota(f)(s)}(\Sigma_s \otimes I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}}) \begin{bmatrix} L_s \\ M_s^\top \end{bmatrix} (v \otimes W'(g)\Omega))\right\rangle \, \mathrm{d}s \\
&+ \int_0^t \left\langle u \otimes W'(f)\Omega, \begin{bmatrix} M_s & L_s^\top \end{bmatrix}(\Sigma_s^* \otimes I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}}) E_{\Sigma'\iota(g)(s)}(v \otimes W'(g)\Omega))\right\rangle \, \mathrm{d}s \\
=&\int_0^t \left\langle \widehat{\Sigma'\iota(h)}(s) \otimes u \otimes W'(f)\Omega, (\widehat{\Sigma}_s \otimes I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}}) \left( \begin{bmatrix} K_s \\ L_s \\ M_s^\top \end{bmatrix} (v \otimes W'(g)\Omega) \right) \right\rangle \, \mathrm{d}s \\
=&\langle u \otimes W'(f)\Omega, \Lambda_t^\Sigma(F)(v \otimes W'(g)\Omega)\rangle.
\end{aligned}$$

□

An immediate consequence of $\mathcal{I}$ being an isometry and the triangle inequality is the following fundamental estimate for quasifree calculus ([68, Proposition 4.1]).

**Proposition 2.3.11** (Fundamental Estimate)**.** *For all $u \in \mathfrak{h}$ we have*

$$\|\Lambda_t^\Sigma(F)(u \otimes \Omega)\| \leqslant \|\mathcal{T}_t(z^{(0)})\| + \left\|(\Sigma \otimes I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}})(z^{(1)}_{[0,t)})\right\|,$$

*where* $z_t^{(0)} = K_t(u \otimes \Omega)$, $z_t^{(1)} = \begin{pmatrix} L_t(u \otimes \Omega) \\ M_t^{\mathsf{T}}(u \otimes \Omega) \end{pmatrix}$.

Corollary 2.3.10 and Second Fundamental Formula for QS (Theorem 1.2.18) yields the following result.

**Theorem 2.3.12** (Second Fundamental Formula)**.** *Let* $F = \left(\left[\begin{smallmatrix} K_t & M_t \\ L_t & 0 \end{smallmatrix}\right]\right)_{t\geqslant 0}$ *and* $G = \left(\left[\begin{smallmatrix} R_t & N_t \\ P_t & 0 \end{smallmatrix}\right]\right)_{t\geqslant 0}$ *be quasifree integrands on $\mathfrak{h}$. Then*

$$\begin{aligned}
&\left\langle \Lambda_t^\Sigma(F)(u \otimes W'(f)\Omega), \Lambda_t^\Sigma(G)(v \otimes W'(g)\Omega)\right\rangle \\
&= \int_0^t \left\langle \widehat{\Sigma' \iota(h)}(s) \otimes \Lambda_s^\Sigma(F)(u \otimes W_t'(f)\Omega), (\widehat{\Sigma}_s \otimes I)\left(\begin{bmatrix} R_s \\ P_s \\ N_s^{\mathsf{T}} \end{bmatrix}(v \otimes W'(g)\Omega)\right)\right\rangle \mathrm{d}s \\
&- \int_0^t \left\langle (\widehat{\Sigma}_s \otimes I)\left(\begin{bmatrix} K_s \\ L_s \\ M_s^{\mathsf{T}} \end{bmatrix}(v \otimes W'(f)\Omega)\right), \widehat{\Sigma' \iota(h)}(s) \otimes \Lambda_s^\Sigma(G)(u \otimes W'(g)\Omega)\right\rangle \mathrm{d}s \\
&+ \int_0^t \left\langle (\widehat{\Sigma}_s \otimes I)\left(\begin{bmatrix} K_s \\ L_s \\ M_s^{\mathsf{T}} \end{bmatrix}(v \otimes W'(f)\Omega)\right), \Delta_{\mathsf{k}\oplus\bar{\mathsf{k}}}(\widehat{\Sigma}_s \otimes I)\left(\begin{bmatrix} R_s \\ P_s \\ N_s^{\mathsf{T}} \end{bmatrix}(v \otimes W'(g)\Omega)\right)\right\rangle \mathrm{d}s
\end{aligned}$$

*for all $u$, $v \in \mathfrak{h}$, $f$ and $g \in L^2(\mathbb{R}_+; \mathsf{k})$, where $h = f - g$, $I = I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}}$ and* $\Delta_{\mathsf{k}\oplus\bar{\mathsf{k}}} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & I_{\mathsf{k}\oplus\bar{\mathsf{k}}} \end{smallmatrix}\right] \otimes I \in \mathcal{B}(\widehat{\mathsf{k}\oplus\bar{\mathsf{k}}} \otimes \mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}})$.

For more details including the proof we refer the reader to [68, Theorem 4.5].

The QF integral consists of time, annihilation and creation integrals, thus according to the Second Fundamental Formula, the correction term may vary for different combinations of coefficients of the integrand.

**Example 2.3.13.** Let $F = \left(\left[\begin{smallmatrix} K_t & M_t \\ L_t & 0 \end{smallmatrix}\right]\right)_{t\geqslant 0}$ and $G = \left(\left[\begin{smallmatrix} R_t & N_t \\ P_t & 0 \end{smallmatrix}\right]\right)_{t\geqslant 0}$ be quasifree integrands on $\mathfrak{h}$. By applying the Second Fundamental Formula to

$$\left\langle A_t^{\Sigma}(M)u \otimes \Omega, A_t^{\Sigma}(N)v \otimes \Omega \right\rangle$$

we obtain that the correction term equals to

$$\begin{aligned} &\int_0^t \left\langle v \otimes \Omega, \left[\begin{smallmatrix} 0 & 0 & M_s^{*,\top} \end{smallmatrix}\right] (\widehat{\Sigma}_s \otimes I)^* \Delta_{\mathsf{k}\oplus\bar{\mathsf{k}}} (\widehat{\Sigma}_s \otimes I) \left( \left[\begin{smallmatrix} 0 \\ 0 \\ N_s^{\top} \end{smallmatrix}\right] (v \otimes \Omega) \right) \right\rangle \mathrm{d}s \\ = &\int_0^t \left\langle v \otimes \Omega, \left[\begin{smallmatrix} 0 & M_s^{*,\top} \end{smallmatrix}\right] (\Sigma_s^* \Sigma_s \otimes I) \left( \left[\begin{smallmatrix} 0 \\ N_s^{\top} \end{smallmatrix}\right] (v \otimes \Omega) \right) \right\rangle \mathrm{d}s. \end{aligned}$$

Let $\Sigma$ be the covariance from Example 2.2.4 then we obtain

$$\begin{aligned} &\int_0^t \left\langle v \otimes \Omega, \left[\begin{smallmatrix} 0 & M_s^{*,\top} \end{smallmatrix}\right] (\Sigma_s^* \Sigma_s \otimes I) \left( \left[\begin{smallmatrix} 0 \\ N_s^{\top} \end{smallmatrix}\right] (v \otimes \Omega) \right) \right\rangle \mathrm{d}s. \\ = &\int_0^t \left\langle v \otimes \Omega, M_s^{*,\top} (jT^2(s)j^* \otimes I) N_s^{\top} (v \otimes \Omega) \right\rangle \mathrm{d}s. \end{aligned}$$

for all $u, v \in \mathfrak{h}$.

For simplicity we denote $\left\langle dA_t^{\Sigma}, dA_t^{\Sigma} \right\rangle := \left\langle A_t^{\Sigma}(M)u \otimes W'(f), A_t^{\Sigma}(N)v \otimes W'(g) \right\rangle$. When $\Sigma$ is taken to be the one from Example 2.2.4 then the correction term is of type $jT^2(t)j^*\mathrm{d}t$.

We have nine different possibilities, four of them might be non-zero, depending on the covariance operator $\Sigma$. Let $\Sigma_t^* \Sigma_t$ be of the form

$$\Sigma_t^* \Sigma_t = |\Sigma_t|^2 = \begin{bmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{bmatrix},$$

for some $\alpha(t) \in \mathcal{B}(\mathsf{k})$, $\beta(t) \in \mathcal{B}(\mathsf{k}; \bar{\mathsf{k}})$, $\gamma(t) \in \mathcal{B}(\bar{\mathsf{k}}, \mathsf{k})$, $\delta(t) \in \mathcal{B}(\bar{\mathsf{k}})$, $t \in \mathbb{R}_+$.

Analogous to the procedure in Example 1.2.19, we present all non-zero correction terms in the quantum Itô table underneath.

| | the correction term |
|---|---|
| $\left\langle \mathrm{d}A_t^{\Sigma}, \mathrm{d}A_t^{\Sigma} \right\rangle$ | $\delta(t)\mathrm{d}t$ |
| $\left\langle \mathrm{d}A_t^{\Sigma}, \mathrm{d}A_t^{*,\Sigma} \right\rangle$ | $\gamma(t)\mathrm{d}t$ |
| $\left\langle \mathrm{d}A_t^{*,\Sigma}, \mathrm{d}A_t^{*,\Sigma} \right\rangle$ | $\alpha(t)\mathrm{d}t$ |
| $\left\langle \mathrm{d}A_t^{*,\Sigma}, \mathrm{d}A_t^{\Sigma} \right\rangle$ | $\beta(t)\mathrm{d}t$ |

Table 2.1: Quantum Itô table.

In particular, if $\Sigma = \Sigma_T$, where $\Sigma_T$ is as in Examples 2.2.4 then:

| | the correction term |
|---|---|
| $\left\langle \mathrm{d}A_t^{\Sigma}, \mathrm{d}A_t^{\Sigma} \right\rangle$ | $jT^2(t)j^*\mathrm{d}t$ |
| $\left\langle \mathrm{d}A_t^{\Sigma}, \mathrm{d}A_t^{*,\Sigma} \right\rangle$ | $0$ |
| $\left\langle \mathrm{d}A_t^{*,\Sigma}, \mathrm{d}A_t^{*,\Sigma} \right\rangle$ | $(I+T^2)(t)\mathrm{d}t$ |
| $\left\langle \mathrm{d}A_t^{*,\Sigma}, \mathrm{d}A_t^{\Sigma} \right\rangle$ | $0$ |

For more details we recommend the reader consult [68, Corollary 4.6 and Example 1, p. 13].

## Quasifree-SDEs and Σ-quasifree cocycles

This section is based on [68], a recent development by Lindsay and Margetts .

Let $F$ be a quasifree integrand on $\mathfrak{h}$ of the form (2.23) and let $T \in \mathcal{B}(\mathfrak{h})$. We say that a bounded $\Sigma$-quasifree process $X = (X_t)_{t\geqslant 0}$ on $\mathfrak{h}$ is a *weak solution* of the (left) quasifree stochastic differential equation

$$\mathrm{d}X_t = \widehat{X}_t F_t \,\mathrm{d}\Lambda_t^{\Sigma}, \quad X_0 = T \otimes I_{\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}}, \tag{2.26}$$

where $\widehat{X}_t = I_{\widehat{\mathsf{k}}} \otimes X_t$, if it satisfies

$$\begin{aligned}&\langle u \otimes W'(f)\Omega, (X_t - X_0)(v \otimes \Omega)\rangle \\ &= \int_0^t \left\langle \widehat{\Sigma' \iota(f)}(s) \otimes u \otimes W'(f), \widetilde{X_s}\left((\widehat{\Sigma}_s \otimes I)\left(\begin{bmatrix} K_s \\ L_s \\ M_s^{\mathsf{T}} \end{bmatrix}(v \otimes \Omega)\right)\right)\right\rangle \mathrm{d}s,\end{aligned}$$

for all, $u$, $v \in \mathfrak{h}$ and $f \in L^2(\mathbb{R}_+; \mathsf{k})$, where $\widetilde{X_s} = I_{\widehat{\mathsf{k} \oplus \bar{\mathsf{k}}}} \otimes X_s$.

We call the solution *strong* if $\widehat{X}F = (\widehat{X}_t F_t)_{t \geqslant 0}$ is a quasifree integrand on $\mathfrak{h}$. We abbreviate a quasifree stochastic differential equation to *Qf-SDE*.

**Proposition 2.3.14.** *Let $F$ be a quasifree integrand on $\mathfrak{h}$ of the form (2.23). If a bounded operator process $X = (X_t)_{t \geqslant 0}$ such that each $X_t \in \mathsf{M}$ is a strong solution (Definition 1.2.21) of the QSDE*

$$\mathrm{d}X_t = \widetilde{X_t} F_t^{\square}\, \mathrm{d}\Lambda_t, \quad X_0 = T \otimes I_{\mathcal{F}^{\mathsf{k} \oplus \bar{\mathsf{k}}}}, \tag{2.27}$$

*with $F_t^{\square}$ given by (2.25), then $X$ is a strong solution the Qf-SDE (2.26).*

*Proof.* First note that $\widehat{X}F$ is a quasifree integrand on $\mathfrak{h}$. Next, by Corollary 2.3.10 we obtain that $X$ satisfies weakly the Qf-SDE (2.26). □

The existence and the uniqueness of the strong solution of the Qf-SDE (2.26) is obtained in a similar way as it was done for QSDEs, Remark 1.2.23. For more details we refer the reader to [68, Theorem 6.1].

**Definition 2.3.15** (Σ-quasifree cocycles)**.** A left Σ-*quasifree cocycle* on a Hilbert space $\mathfrak{h}$ is a QS cocycle (Definition 1.2.24) $X = (X_t)_{t \geqslant 0}$, where $X_t \in \mathsf{M}_t$ for each $t \in \mathbb{R}_+$.

**Assumption.** *From now until the end of the thesis we make the following assumption on Σ.*

**A**. *Covariance Σ is time-constant, that is,*

$$\Sigma = I_{L^2(\mathbb{R}_+)} \otimes Z, \tag{2.28}$$

*where $Z \in \mathcal{B}(\mathsf{k} \oplus \bar{\mathsf{k}})$.*

**Proposition 2.3.16.** *Let $X$ be a unique strong solution of the Qf-SDE*

$$\mathrm{d}X_t = \widehat{X}_t(F \otimes I_{\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}})\,\mathrm{d}\Lambda_t^{\Sigma}, \quad X_0 = I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}},$$

*then $X$ is a $\Sigma$-quasifree cocycle.*

For the proof we refer the reader to [68, Lemma 7.1].

**Theorem 2.3.17.** *Let $X$ be a Markov-regular contraction $\Sigma$-quasifree cocycle. Then $X$ is a strong solution of the Qf-SDE*

$$\mathrm{d}X_t = \widetilde{X}_t(F \otimes I_{\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}})\,\mathrm{d}\Lambda_t^{\Sigma}, \quad X_0 = I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}},$$

*where $F = \left[\begin{smallmatrix} K & -L^* \\ L & 0 \end{smallmatrix}\right] \in \mathcal{B}(\widehat{\mathsf{k}\oplus\bar{\mathsf{k}}} \otimes \mathfrak{h})$. Furthermore, the following statements are equivalent.*

- *$X$ is a unitary cocycle.*
- *$X$ is an isometry cocycle.*
- *There exists a self-adjoint operator $H \in \mathcal{B}(\mathfrak{h})$ such that*

$$K = -\mathrm{i}H - \frac{1}{2}\begin{bmatrix} L^* & -L^{\intercal} \end{bmatrix}(I_{\mathfrak{h}} \otimes Z^*Z)\left[\begin{smallmatrix} L \\ -L^{\mathsf{c}} \end{smallmatrix}\right].$$

For more details including the proof we refer the reader to [68, Theorem 7.9]. An immediate consequence of Proposition 2.3.14, Proposition 2.3.16 and the preceding theorem is the following.

**Corollary 2.3.18.** *Let $F = \left[\begin{smallmatrix} K & -L^* \\ L & 0 \end{smallmatrix}\right]$ for some self-adjoint operator $K \in \mathcal{B}(\mathfrak{h})$ and $L \in \mathcal{B}(\mathfrak{h}; \mathfrak{h}\otimes\mathsf{k})$. If a bounded operator process $X = (X_t)_{t\geqslant 0}$ satisfies strongly the QSDE*

$$\mathrm{d}X_t = \widetilde{X_t}(F^{\square} \otimes I_{\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}})\,\mathrm{d}\Lambda_t, \quad X_0 = I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}}, \tag{2.29}$$

*with*

$$F^{\square} = \begin{bmatrix} K & -\begin{bmatrix} L^* & -L^{\intercal} \end{bmatrix}(Z^* \otimes I_{\mathfrak{h}}) \\ (Z \otimes I_{\mathfrak{h}})\left[\begin{smallmatrix} L \\ -L^{\mathsf{c}} \end{smallmatrix}\right] & 0 \end{bmatrix} \tag{2.30}$$

*and*

$$K = -iH - \frac{1}{2}\begin{bmatrix} L^* & -L^\top \end{bmatrix}(I_{\mathfrak{h}} \otimes Z^* Z)\begin{bmatrix} L \\ -L^c \end{bmatrix}$$

*for some self-adjoint operator* $H \in \mathcal{B}(\mathfrak{h})$*, then* $X$ *is a unitary* $\Sigma$*-quasifree cocycle.*

# Chapter 3

# Quantum random walk approximation

Quantum random-walk approximation to Markov regular QS cocycles is discussed in this chapter. As well as the mathematical description, we provide the reader with some physical interpretations, based on repeated quantum interactions introduced in [9] and studied by many others, e.g. [5], [6], [28], [38], [10] and [29]. Our convention here is slightly different than the one in Chapter 2; the generators of cocycles are operators on $\mathfrak{h} \otimes \widehat{\mathsf{k}}$ rather than on $\widehat{\mathsf{k}} \otimes \mathfrak{h}$. However this does not make a difference since in Chapter 2 we could employ tensor flips to deal with the space $\mathfrak{h} \otimes \widehat{\mathsf{k}}$, but, it is more elegant not to use too many tensor flips. We start by introducing the discrete version of Fock space $\mathcal{F}^{\mathsf{k}}$ on which random walks will be defined.

## 3.1 Random walk convergence

### Toy Fock space

The toy Fock space, introduced by J. L. Journé ([77, p. 18]), is a discrete model for quantum stochastic calculus on $\mathcal{F}^{\mathsf{k}}$. For more details on the topic we refer the reader to [16] (Fock space setting) and [4] (Fock–Guichardet space setting).

Let $\widehat{\mathsf{k}}$ be a separable Hilbert space with distinguished unit vector $e_0$, and let

$$\mathsf{k} := \widehat{\mathsf{k}} \ominus \mathbb{C}e_0 = \{x \in \widehat{\mathsf{k}} : \langle y, x \rangle = 0 \text{ for all } y \in \mathbb{C}e_0\}.$$

**Definition 3.1.1.** *Toy Fock space* over $\mathsf{k}$ is the countable tensor product

$$\Upsilon^{\mathsf{k}} := \bigotimes_{n \geqslant 0} \widehat{\mathsf{k}}_{(n)}$$

with respect to the stabilising sequence $(\epsilon_{(n)} := e_0)_{n \geqslant 0}$, where $\widehat{\mathsf{k}}_{(n)} = \widehat{\mathsf{k}}$ for each $n$; the subscript $(n)$ indicates the relevant copy.

For all $n \in \mathbb{N}$ let

$$\Upsilon^{\mathsf{k}}_{n)} := \bigotimes_{m=0}^{n-1} \widehat{\mathsf{k}}_{(m)} \quad \text{and} \quad \Upsilon^{\mathsf{k}}_{[n} := \bigotimes_{m=n}^{\infty} \widehat{\mathsf{k}}_{(m)}, \tag{3.1}$$

where $\Upsilon^{\mathsf{k}}_{0)} := \mathbb{C}$. For natural numbers $n$, $k$ such that $n \geqslant k$ let

$$\Upsilon^{\mathsf{k}}_{[k,n)} = \bigotimes_{m=k}^{n-1} \widehat{\mathsf{k}}_{(n)} \quad \text{and} \quad \Upsilon^{\mathsf{k}}_{(k,n]} = \bigotimes_{m=k+1}^{n} \widehat{\mathsf{k}}_{(n)}. \tag{3.2}$$

The identity $\Upsilon^{\mathsf{k}}_{k)} \otimes \Upsilon^{\mathsf{k}}_{[k,n)} \otimes \Upsilon_{[n} = \Upsilon^{\mathsf{k}}$ is the analogue of the continuous tensor-product structure of the symmetric Fock space.

**Remark 3.1.2.** Let $h > 0$ and set $I_{h,n} := [hn, h(n+1))$ for all $n \in \mathbb{N}$. We identify Fock space $\mathcal{F}^{\mathsf{k}}$ and $\mathcal{F}^{\mathsf{k},h} := \bigotimes_{n \geqslant 0} \mathcal{F}^{\mathsf{k}}_{I_{h,n}}$ by the following isometric isomorphism:

$$\mathcal{F}^{\mathsf{k}} \to \mathcal{F}^{\mathsf{k},h}; \quad \varepsilon(f) \mapsto \bigotimes_{n \geqslant 0} \varepsilon(f|_{I_{h,n}}), \tag{3.3}$$

where the tensor product is taken with respect to the stabilising sequence $(\Omega_{I_{h,n}})_{n \geqslant 0}$.

To embed toy Fock space over $\mathsf{k}$ into $\mathcal{F}^{\mathsf{k}}$ let us define the following map.

**Definition 3.1.3.** Let $h > 0$. For all $n \in \mathbb{N}$ define the natural isometry

$$J^n_h : \widehat{\mathsf{k}} \to \mathcal{F}^{\mathsf{k}}_{[hn,h(n+1))}; \begin{pmatrix} \alpha \\ c \end{pmatrix} \mapsto \alpha \Omega_{[hn,h(n+1))} + \frac{1}{\sqrt{h}} c \mathbf{1}_{[hn,h(n+1))}. \tag{3.4}$$

This gives an isometric embedding

$$J_h := \bigotimes_{n \geqslant 0} J_h^n \colon \Upsilon^{\mathsf{k}} \to \mathcal{F}_h^{\mathsf{k}}. \tag{3.5}$$

Note that $J_h^* \varepsilon(f) = \bigotimes_{m \geqslant 0} \widehat{f_h(m)}$, where $f_h(m) = \frac{1}{\sqrt{h}} \int_{hm}^{h(m+1)} f(t)dt$.

**Proposition 3.1.4.** *As $h \to 0^+$ the projection $J_h J_h{}^*$ converges strongly to $I_{\mathcal{F}^{\mathsf{k}}}$.*

For the proof we refer the reader to [16, Theorem 2.1].

## Approximation theorem

The main aim of this section is to give a simple proof of the random walk approximation theorem for the Markov regular QS cocycles. Similar approximation theorem for mapping cocycles (quantum flows) can be found in the literature, e.g. [86, Theorem 3.3] or [17, Theorem 7.6, p. 431].

Similarly to the preceding section, we will be using a separable Hilbert space $\widehat{\mathsf{k}}$ with a distinguished unit vector $e_0$, and $\mathsf{k}$ will stand for $\mathsf{k} := \widehat{\mathsf{k}} \ominus \mathbb{C}e_0$.

**Definition 3.1.5.** Let $G \in \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}})$, $n \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$. The ampliation of $G$ to $\mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}}^{\otimes n})$ in the $k$-th place is the operator $G_k^{(n)} \in \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}}^{\otimes n})$ that

$$\left\langle u \otimes \bigotimes_{i=1}^{n} x_i, G_k^{(n)}(v \otimes \bigotimes_{i=1}^{n} y_i) \right\rangle = \langle u \otimes x_k, G(v \otimes y_k)\rangle \prod_{i \neq k} \langle x_i, y_i\rangle . \tag{3.6}$$

That is, $G_k^{(n)}$ acts as $G$ on the tensor product of $\mathfrak{h}$ with the $k$-th copy of $\widehat{\mathsf{k}}$ and $G_k^{(n)}$ acts as the identity on the other $n-1$ components of $\widehat{\mathsf{k}}^{\otimes n}$.

**Definition 3.1.6.** If $h > 0$ and $G \in \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ then the *embedded quantum random walk* with *generator* $G$ and *step size* $h$ is the operator process $(X_t^{G,h})_{t \geqslant 0}$ such that

$$X_t^{G,h} := \begin{cases} I_{\mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}}_{[0,\infty)}} & \text{for } n = 0, \\ \left(I_{\mathfrak{h}} \otimes \bigotimes_{k=0}^{n-1} J_h^k\right) G_1^{(n)} \cdots G_n^{(n)} \left(I_{\mathfrak{h}} \otimes \bigotimes_{k=0}^{n-1} J_h^k\right)^* \otimes I_{\mathcal{F}^{\mathsf{k}}_{[nh,\infty)}} & \text{for } n \geqslant 1 \end{cases} \tag{3.7}$$

if $t \in [nh, (n+1)h)$, for all $t \in \mathbb{R}_+$.

Before we can write a statement of a random-walk approximation theorem we will need a few technicalities.

**Definition 3.1.7.** If $h > 0$ and $G \in \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ then the *modification* $m(G,h)$ is defined by setting

$$m(G,h) := \left(\frac{1}{\sqrt{h}}\Delta^{\perp} + \Delta\right)(G - I_{\mathfrak{h}\otimes\widehat{\mathsf{k}}})\left(\frac{1}{\sqrt{h}}\Delta^{\perp} + \Delta\right),$$

where $\Delta^{\perp} = I_{\mathfrak{h}} \otimes |e_0\rangle\langle e_0|$ and $\Delta := I_{\mathfrak{h}\otimes\widehat{\mathsf{k}}} - \Delta^{\perp}$.

**Remark 3.1.8.** Note that $\Delta$ defined above coincides with the one defined in (1.11). In particular, if $G = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ for $A \in \mathcal{B}(\mathfrak{h})$, $B \in \mathcal{B}(\mathfrak{h}\otimes\mathsf{k};\mathfrak{h})$, $C \in \mathcal{B}(\mathfrak{h};\mathfrak{h}\otimes\mathsf{k})$ and $D \in \mathcal{B}(\mathfrak{h}\otimes\mathsf{k})$ then

$$m(G,h) = \begin{bmatrix} \frac{1}{h}(A - I_{\mathfrak{h}}) & \frac{1}{\sqrt{h}}B \\ \frac{1}{\sqrt{h}}C & D - I_{\mathfrak{h}\otimes\mathsf{k}} \end{bmatrix}.$$

**Definition 3.1.9.** Let $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in \widehat{\mathsf{k}}$. We define a map

$$E_{x_1\otimes x_2\otimes\ldots\otimes x_n}: \mathfrak{h} \to \mathfrak{h} \otimes \widehat{\mathsf{k}}^{\otimes n} \quad \text{by} \quad E_{x_1\otimes x_2\otimes\ldots\otimes x_n}u = u \otimes x_1 \otimes x_2 \otimes \ldots \otimes x_n$$

and denote its adjoint by $E^{x_1\otimes x_2\otimes\ldots\otimes x_n}$.

**Lemma 3.1.10.** *If* $G, H \in \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ *then*

$$E^{x_1\otimes x_2}G_1^{(2)}H_2^{(2)}E_{y_1\otimes y_2} = E^{x_1}GE_{y_1}E^{x_2}HE_{y_2} \tag{3.8}$$

*and*

$$E^{x_1\otimes x_2}G_2^{(2)}H_1^{(2)}E_{y_1\otimes y_2} = E^{x_2}GE_{y_2}E^{x_1}HE_{y_1} \tag{3.9}$$

*for all* $x_1,\ x_2,\ y_1,\ y_2 \in \widehat{\mathsf{k}}$.

*Proof.* First note that

$$\begin{aligned} E_{x_1\otimes x_2}u &= u \otimes x_1 \otimes x_2 \\ &= E_{x_1}u \otimes x_2 \\ &= (E_{x_1} \otimes I_{\widehat{\mathsf{k}}})E_{x_2}u. \end{aligned}$$

and so

$$E_{x_1 \otimes x_2} = (E_{x_1} \otimes I_{\hat{\mathsf{k}}}) E_{x_2},$$
$$E^{x_1 \otimes x_2} = E^{x_2} (E^{x_1} \otimes I_{\hat{\mathsf{k}}}). \tag{3.10}$$

By applying (3.10) we obtain

$$\begin{aligned} &E^{x_1 \otimes x_2} G_1^{(2)} H_2^{(2)} E_{y_1 \otimes y_2} \\ &= E^{x_2} (E^{x_1} \otimes I_{\hat{\mathsf{k}}}) G_1^{(2)} H_2^{(2)} (E_{y_1} \otimes I_{\hat{\mathsf{k}}}) E_{y_2}. \end{aligned}$$

Then

$$\begin{aligned} &E^{x_2} (E^{x_1} \otimes I_{\hat{\mathsf{k}}}) G_1^{(2)} H_2^{(2)} (E_{y_1} \otimes I_{\hat{\mathsf{k}}}) E_{y_2} \\ &= E^{x_2} (E^{x_1} G \otimes I_{\hat{\mathsf{k}}}) H_2^{(2)} (E_{y_1} \otimes I_{\hat{\mathsf{k}}}) E_{y_2} \end{aligned}$$

and since

$$H_2^{(2)} (E_{y_1} \otimes I_{\hat{\mathsf{k}}}) = (E_{y_1} \otimes I_{\hat{\mathsf{k}}}) H_2^{(2)}$$

we obtain

$$\begin{aligned} &E^{x_2} (E^{x_1} G \otimes I_{\hat{\mathsf{k}}}) H_2^{(2)} (E_{y_1} \otimes I_{\hat{\mathsf{k}}}) E_{y_2} \\ &= E^{x_2} (E^{x_1} G E_{y_1} \otimes I_{\hat{\mathsf{k}}}) H E_{y_2}. \end{aligned}$$

Now note that for any $A \in \mathcal{B}(\mathfrak{h})$ the following holds

$$E^x \left( A \otimes I_{\hat{\mathsf{k}}} \right) = A E^x,$$

therefore

$$\begin{aligned} &E^{x_2} (E^{x_1} G E_{y_1} \otimes I_{\hat{\mathsf{k}}}) H E_{y_2} \\ &= E^{x_1} G E_{y_1} E^{x_2} H E_{y_2}, \end{aligned}$$

and hence,

$$E^{x_1 \otimes x_2} G_1^{(2)} H_2^{(2)} E_{y_1 \otimes y_2} = E^{x_1} G E_{y_1} E^{x_2} H E_{y_2}.$$

To show that (3.9) holds we start by applying (3.10)

$$\begin{aligned} & E^{x_1 \otimes x_2} G_2^{(2)} H_1^{(2)} E_{y_1 \otimes y_2} \\ =& E^{x_2} (E^{x_1} \otimes I_{\hat{\mathsf{k}}}) G_2^{(2)} H_1^{(2)} (E_{y_1} \otimes I_{\hat{\mathsf{k}}}) E_{y_2} \end{aligned}$$

Now observe that $(E^{x_1} \otimes I_{\hat{\mathsf{k}}}) G_2^{(2)} = G(E^{x_2} \otimes I_{\hat{\mathsf{k}}})$, because $G_2^{(2)}$ acts as an identity on the first copy of $\hat{\mathsf{k}}$, and so

$$\begin{aligned} & E^{x_2} (E^{x_1} \otimes I_{\hat{\mathsf{k}}}) G_2^{(2)} H_1^{(2)} (E_{y_1} \otimes I_{\hat{\mathsf{k}}}) E_{y_2} \\ =& E^{x_2} G (E^{x_1} \otimes I_{\hat{\mathsf{k}}}) H_1^{(2)} (E_{y_1} \otimes I_{\hat{\mathsf{k}}}) E_{y_2}. \end{aligned}$$

Furthermore,

$$H_1^{(2)} (E_{y_1} \otimes I_{\hat{\mathsf{k}}}) E_{y_2} = \Pi (E_{y_2} \otimes I_{\hat{\mathsf{k}}}) H E_{y_1}$$

where $\Pi \in \mathcal{B}(\mathfrak{h} \otimes \hat{\mathsf{k}} \otimes \hat{\mathsf{k}})$ is given by

$$\Pi(u \otimes y \otimes x) = u \otimes x \otimes y$$

for all $x, y \in \hat{\mathsf{k}}$. Therefore,

$$\begin{aligned} & E^{x_2} G (E^{x_1} \otimes I_{\hat{\mathsf{k}}}) H_1^{(2)} (E_{y_1} \otimes I_{\hat{\mathsf{k}}}) E_{y_2} \\ =& E^{x_2} G (E^{x_1} \otimes I_{\hat{\mathsf{k}}}) \Pi (E_{y_2} \otimes I_{\hat{\mathsf{k}}}) H E_{y_1}, \end{aligned}$$

Note that $(E^{x_1} \otimes I_{\hat{\mathsf{k}}}) \Pi (E_{y_2} \otimes I_{\hat{\mathsf{k}}}) = E_{y_2} E^{x_1}$ and hence,

$$E^{x_2} G (E^{x_1} \otimes I_{\hat{\mathsf{k}}}) \pi (E_{y_2} \otimes I_{\hat{\mathsf{k}}}) H E_{y_1} = E^{x_2} G E_{y_2} E^{x_1} H E_{y_1}$$

□

**Lemma 3.1.11.** *If* $G \in \mathcal{B}(\mathfrak{h} \otimes \hat{\mathsf{k}})$ *then*

$$E^{x_1 \otimes \cdots \otimes x_n} G_1^{(n)} \cdots G_n^{(n)} E_{y_1 \otimes \cdots \otimes y_n} = E^{x_1} G E_{y_1} \cdots E^{x_n} G E_{y_n} \tag{3.11}$$

*for all* $x_1, \ldots, x_n, y_1, \ldots, y_n \in \hat{\mathsf{k}}$.

*Proof.* We proceed by induction on $n$. For $n = 2$ the condition is satisfied by the preceding lemma. Assume (3.11) holds for $n$. Now we show the inductive step

$$
\begin{aligned}
&E^{x_1\otimes\cdots\otimes x_n\otimes x_{n+1}}G_1^{(n+1)}\cdots G_n^{(n+1)}G_{n+1}^{(n+1)}E_{y_1\otimes\cdots\otimes y_n\otimes y_{n+1}}\\
&= E^{x_{n+1}}(E^{x_1\otimes\cdots\otimes x_n}\otimes I_{\widehat{\mathsf{k}}})(G_1^{(n)}\otimes I_{\widehat{\mathsf{k}}})\cdots(G_n^{(n)}\otimes I_{\widehat{\mathsf{k}}})G_{n+1}^{(n+1)}(E_{y_1\otimes\cdots\otimes y_n}\otimes I_{\widehat{\mathsf{k}}})\,E_{y_{n+1}}\\
&= E^{x_{n+1}}\left(E^{x_1\otimes\cdots\otimes x_n}G_1^{(n)}\cdots G_n^{(n)}\otimes I_{\widehat{\mathsf{k}}}\right)G_{n+1}^{(n+1)}\left(E_{y_1\otimes\cdots\otimes y_n}\otimes I_{\widehat{\mathsf{k}}}\right)E_{y_{n+1}}\\
&= E^{x_{n+1}}\left(E^{x_1\otimes\cdots\otimes x_n}G_1^{(n)}\cdots G_n^{(n)}E_{y_1\otimes\cdots\otimes y_n}\otimes I_{\widehat{\mathsf{k}}}\right)GE_{y_{n+1}}\\
&= E^{x_1\otimes\cdots\otimes x_n}G_1^{(n)}\cdots G_n^{(n)}E_{y_1\otimes\cdots\otimes y_n}E^{x_{n+1}}GE_{y_{n+1}}\\
&= E^{x_1}GE_{y_1}\cdots E^{x_n}GE_{y_n}E^{x_{n+1}}GE_{y_{n+1}}.
\end{aligned}
$$

□

The next theorem is a main result of this section. It is a special case of a quantum analogue of Donsker's invariant principle proved in [17, Theorem 7.6, p. 431], however we give a new more elementary proof.

**Theorem 3.1.12.** *Let $G\colon(0,\infty)\to\mathcal{B}(\mathfrak{h}\otimes\widehat{\mathsf{k}})$, and let $F\in\mathcal{B}(\mathfrak{h}\otimes\widehat{\mathsf{k}})$ be such that*

$$m(G(h),h)\overset{h\to 0^+}{\to}F \tag{3.12}$$

*in norm.*

*If $f,\ g\in L^2(\mathbb{R}_+;\mathsf{k})$ are right-continuous step functions and $T\in\mathbb{R}_+$ then*

$$\lim_{h\to 0^+}\sup_{t\in[0,T]}\left\|E^{\varepsilon(f)}\left(X_t^{G(h),h}-X_t^F\right)E_{\varepsilon(g)}\right\|=0, \tag{3.13}$$

*where $X^F=(X_t^F)$ is the Markov-regular QS cocycle with generator $F$.*

*Proof.* Observe that

$$E^{\widehat{\alpha x}}=E^{\hat{x}}\left(\Delta^{\perp}+\alpha\Delta\right)\quad\text{and}\quad E_{\widehat{\alpha x}}=\left(\Delta^{\perp}+\alpha\Delta\right)E_{\hat{x}}$$

for each $\alpha\in\mathbb{C}$ and $x\in\mathsf{k}$.
For each $x,\ y\in\mathsf{k}$ denote by $(P_t^{x,y})_{t\geqslant 0}$ the associated semigroup of the cocycle $X^F$.

Now, let $t \in \mathbb{R}_+$, let $c, d \in \mathsf{k}$ and let $h > 0$. Then $t \in [nh, (n+1)h)$ for some $n \in \mathbb{N}$ and

$$
\begin{aligned}
&\left(E^{\widehat{\sqrt{h}c}} G(h) E_{\widehat{\sqrt{h}d}}\right)^n \\
&= \left(E^{\widehat{\sqrt{h}c}} \left(G(h) - I_{\mathfrak{h}\otimes\widehat{\mathsf{k}}}\right) E_{\widehat{\sqrt{h}d}} + I_{\mathfrak{h}} + h\langle c, d\rangle I_{\mathfrak{h}}\right)^n \\
&= \left(E^{\widehat{\sqrt{h}c}} (\sqrt{h}\Delta^\perp + \Delta) m(G(h), h)(\sqrt{h}\Delta^\perp + \Delta) E_{\widehat{\sqrt{h}d}} + I_{\mathfrak{h}} + h\langle c, d\rangle I_{\mathfrak{h}}\right)^n \\
&= \left(E^{\widehat{c}} (\sqrt{h}\Delta^\perp + \sqrt{h}\Delta)(F + o(1))(\sqrt{h}\Delta^\perp + \sqrt{h}\Delta) E_{\widehat{d}} + I_{\mathfrak{h}} + h\langle c, d\rangle I_{\mathfrak{h}}\right)^n \\
&= \left(I_{\mathfrak{h}} + h\left(E^{\widehat{c}} F E_{\widehat{d}} + \langle c, d\rangle I_{\mathfrak{h}}\right) + o(h)\right)^n \\
&\overset{h\to 0^+}{\to} \exp\left\{t(E^{\widehat{c}} F E_{\widehat{d}} + \langle c, d\rangle I_{\mathfrak{h}})\right\},
\end{aligned}
$$

where the convergence holds in norm due to Proposition A.0.9 (Euler's exponential formula).

According to Proposition 1.2.30 we arrive at

$$\exp\left\{t(E^{\widehat{c}} F E_{\widehat{d}} + \langle c, d\rangle I_{\mathfrak{h}})\right\} = P_t^{c,d}$$

and hence,

$$\left(E^{\widehat{\sqrt{h}c}} G(h) E_{\widehat{\sqrt{h}d}}\right)^n \to P_t^{c,d} \quad \text{as } h \to 0^+. \tag{3.14}$$

Let $f$, $g$ be right-continuous step functions and let $\{0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = t\}$ for some $k \in \mathbb{N}$, contain the discontinuities of $f_{[0,t)}$ and $g_{[0,t)}$. Let $l_i$ be such that

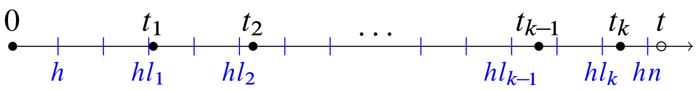
$$hl_i \leqslant t_i < hl_i + h \quad \text{for } i = 0, \ldots, k+1.$$

In particular,

$$hn = hl_{k+1} \leqslant t = t_{k+1} < hl_{k+1} + h$$

so $l_{k+1} = n$.

See the figure below:




.

By applying Lemma 3.1.11 we obtain

$$
\begin{aligned}
&E^{\varepsilon(f)} X_t^{G(h),h} E_{\varepsilon(g)} \\
&= \langle \varepsilon(f_{[hn,\infty)}), \varepsilon(g_{[hn,\infty)}) \rangle \, E^{\bigotimes_{m=0}^{n-1} \widehat{f_h(m)}} G(h)_1^{(n)} \cdots G(h)_n^{(n)} E_{\bigotimes_{m=0}^{n-1} \widehat{g_h(m)}} \\
&= \langle \varepsilon(f_{[hn,\infty)}), \varepsilon(g_{[hn,\infty)}) \rangle \, E^{\widehat{f_h(0)}} G(h) E_{\widehat{g_h(0)}} \cdots E^{\widehat{f_h(n-1)}} G(h) E_{\widehat{f_h(n-1)}}.
\end{aligned}
$$

The functions $f$ and $g$ are constant on

$$
(hl_i + h, hl_{i+1}) \subset (t_i, t_{i+1}) \quad \text{for } i = 0, \ldots, k,
$$

so

$$
\prod_{j=0}^{n-1} E^{\widehat{f_h(j)}} G(h) E_{\widehat{g_h(j)}} = \prod_{i=0}^{k} E^{\widehat{f_h(l_i)}} G(h) E_{\widehat{g_h(l_i)}} \left( E^{\widehat{\sqrt{h} f(t_i)}} G(h) E_{\widehat{\sqrt{h} g(t_i)}} \right)^{l_{i+1}-l_i-1}.
$$

Now, for all i=0, . . . , k

$$
f_h(l_i) = \frac{1}{\sqrt{h}} \int_{hl_i}^{h(l_i+1)} f(s) \, \mathrm{d}s \to 0 \quad \text{as } h \to 0^+,
$$

because $f$ is a step function, so bounded and similarly for $g$.
Moreover, observe that

$$
\begin{aligned}
E^{\widehat{0}} G(h) E_{\widehat{0}} &= E^{\widehat{0}} (G(h) - I_{\mathfrak{h} \otimes \widehat{\mathsf{k}}}) E_{\widehat{0}} + I_{\mathfrak{h}} \\
&= h E^{\widehat{0}} m(G(h), h) E_{\widehat{0}} + I_{\mathfrak{h}} \\
&\to I_{\mathfrak{h}} \quad \text{as } h \to 0^+.
\end{aligned}
$$

Hence,

$$
E^{\widehat{f_h(l_i)}} G(h) E_{\widehat{g_h(l_i)}} \to I_{\mathfrak{h}} \quad \text{as } h \to 0^+.
$$

Furthermore, by (3.14),

$$
\left( E^{\widehat{\sqrt{h} f(t_i)}} G(h) E_{\widehat{\sqrt{h} g(t_i)}} \right)^{l_{i+1}-l_i-1} \to P_{\tau_i}^{f(t_1), g(t_i)},
$$

where $\tau_i \sim h(l_{i+1} - l_i - 1)$, but

$$h(l_{i+1} - l_i - 1) = hl_{i+1} - hl_i - h \to t_{i+1} - t_i \quad \text{as } h \to 0^+$$

and so $\tau_i = t_{i+1} - t_i$.
Hence,

$$\begin{aligned} & E^{\varepsilon(f)} X_t^{G(h),h} E_{\varepsilon(g)} \\ \xrightarrow{h \to 0^+} & \left\langle \varepsilon(f_{[t,\infty)}), \varepsilon(g_{[t,\infty)}) \right\rangle P_{t_1-t_0}^{f(t_0),g(t_0)} \dots P_{t-t_k}^{f(t_k),g(t_k)} \\ = & \left\langle \varepsilon(f_{[t,\infty)}), \varepsilon(g_{[t,\infty)}) \right\rangle E^{\varepsilon(f_{[0,t)})} X_t^F E_{\varepsilon(g_{[0,t)})} \\ = & E^{\varepsilon(f)} X_t^F E_{\varepsilon(g)}. \end{aligned}$$

To see that this convergence is uniform for all $t \in [0, T]$, where $T > 0$ is arbitrary, suppose first that $t \in [T_1, T_2]$ for some $T_1,\ T_2 \in [0, T]$, and $f, g$ are constant on $(T_1, T_2]$. From the previous working

$$\begin{aligned} & E^{\varepsilon(f)} X_t^{G(h),h} E_{\varepsilon(g)} \\ & = \left\langle \varepsilon(f_{[hn,\infty)}), \varepsilon(g_{[hn,\infty)}) \right\rangle \prod_{i=0}^{k} E^{\widehat{f_h(l_i)}} G(h) E_{\widehat{g_h(l_i)}} \left( E^{\widehat{\sqrt{h} f(t_i)}} G(h) E_{\widehat{\sqrt{h} g(t_i)}} \right)^{l_{i+1}-l_i-1}, \end{aligned} \tag{3.15}$$

where $nh \leqslant t < (n+1)h$ and the $l_i$ are as we previously used in the proof. As $t$ varies in $(T_1, T_2]$, since $t > t_k$, the last point of discontinuity, only two terms in (3.15) vary:

- $\left\langle \varepsilon(f_{[hn,\infty)}), \varepsilon(g_{[hn,\infty)}) \right\rangle = \exp\left(\int_{nh}^{T_2} \langle f(s), g(s) \rangle \ \mathrm{d}s\right) \exp\left(\int_{T_2}^{\infty} \langle f(s), g(s) \rangle \ \mathrm{d}s\right)$ and
- $\left( E^{\widehat{\sqrt{h} f(t_k)}} G(h) E_{\widehat{\sqrt{h} g(t_k)}} \right)^{n-l_k-1}$.

Note that, the integers $l_0, \dots, l_k$ do not depend on $t$, but $l_{k+1} = n$ does. Thus convergence is uniform on $(T_1, T_2]$ if

$$\exp\left(\int_{nh}^{T_2} \langle f(s), g(s) \rangle \ \mathrm{d}s\right) \to \exp\left(\int_{t}^{T_2} \langle f(s), g(s) \rangle \ \mathrm{d}s\right) \tag{3.16}$$

and

$$\left(E^{\widehat{\sqrt{h}f(t_k)}}G(h)E_{\widehat{\sqrt{h}g(t_k)}}\right)^n \to P_t^{f(t_k),g(t_k)} \tag{3.17}$$

uniformly in $t$.

Condition (3.17) holds by Proposition A.0.9 (Euler's exponential formula).

To show (3.16) first note that

$$\begin{aligned} &\left|\int_{nh}^{T_2} \langle f(s), g(s)\rangle \, \mathrm{d}s - \int_t^{T_2} \langle f(s), g(s)\rangle \, \mathrm{d}s\right| \\ \leqslant \; & (t-nh)\|f\|_\infty \|g\|_\infty \leqslant h\|f\|_\infty \|g\|_\infty, \end{aligned} \tag{3.18}$$

where $\|f\|_\infty = \sup_{t\in\mathbb{R}_+} |f(t)|$.

Now let $z \in \mathbb{C}$ and $\{z_h : h \in (0,\infty)\} \subset \mathbb{C}$ be such that $z_h \to z$ as $h \to 0^+$. We take $\delta > 0$ such that $|z_h| < |z| + 1$ whenever $0 < h < \delta$. By using a similar argument to the one in the proof of Lemma A.0.7 we obtain that if $0 < h < \delta$ then

$$|\exp(z_h) - \exp(z)| \leqslant |z_h - z| e^{|z|+1}. \tag{3.19}$$

Hence, we show that (3.16) holds by combining (3.18) and (3.19).

Finally, we can write $[0, T]$ as

$$\{0\} \cup (0, T_1] \cup (T_1, T_2] \cup \ldots \cup (T_{n-1}, T_n]$$

where $f$ and $g$ are constant on $(T_i, T_{i+1}]$ for $i = 0, \ldots, n-1$. Since convergence is uniform on each of the interval, it is uniform on the whole $[0, T]$. □

Next two lemmas will tell us when we can have a stronger convergence than the one in Theorem 3.1.12, that is, the convergence in equation (3.13).

**Lemma 3.1.13.** *Let $\mathsf{H}_1$ and $\mathsf{H}_2$ be Hilbert spaces, let $\mathcal{D}$ be a total subset of $\mathsf{H}_2$ and let $T \geqslant 0$. If $\{X_t : t \in [0, T]\}$ and $\{Y_t^{(h)} : h > 0, t \in [0, T]\}$ are bounded subsets of $\mathcal{B}(\mathsf{H}_1 \otimes \mathsf{H}_2)$ such that*

$$\lim_{h\to 0^+} \sup_{t\in[0,T]} \|E^c(Y_t^{(h)} - X_t)E_d\| = 0$$

*for all* $c, d \in \mathcal{D}$ *then*

$$\lim_{h\to 0^+} \sup_{t\in[0,T]} \left|\left\langle u, (Y_t^{(h)} - X_t)v\right\rangle\right| = 0$$

*for all* $u, v \in \mathsf{H}_1 \otimes \mathsf{H}_2$

*Proof.* If $x$, $y \in \mathsf{H}_1$ and $c$, $d \in \mathcal{D}$ then

$$\begin{aligned}
&\sup_{t\in[0,T]} \left|\left\langle x\otimes c, (Y_t^{(h)} - X_t)(y\otimes d)\right\rangle\right| \\
\leqslant &\|x\|\|y\| \sup_{t\in[0,T]} \|E^c(Y_t^{(h)} - X_t)E_d\| \\
\to &0 \quad \text{as } h\to 0^+.
\end{aligned}$$

It follows immediately that

$$\lim_{h\to 0} \sup_{t\in[0,T]} \left|\left\langle \theta_1, (Y_t^{(h)} - X_t)\theta_2\right\rangle\right| = 0$$

for all $\theta_1$, $\theta_2 \in \mathsf{H}_1 \underline{\otimes} \mathcal{D}$.

Let $u$ and $v \in \mathsf{H}_1 \otimes \mathsf{H}_2$ and let $(u_n)$ and $(v_n)$ be sequences in $\mathsf{H}_1 \underline{\otimes} \mathcal{D}$ such that

$$\|u_n - u\| \to 0 \quad \text{and} \quad \|v_n - v\| \to 0 \text{ as } n\to\infty.$$

Since $\sup_{t\in[0,T]} (\|X_t\| + \|Y_t^{(h)})\| < \infty$ we obtain

$$\begin{aligned}
&\sup_{t\in[0,T]} \left|\left\langle u, (Y_t^{(h)} - X_t)v\right\rangle\right| \\
\leqslant &\sup_{t\in[0,T]} \left\{\left|\left\langle u-u_n, (Y_t^{(h)} - X_t)v - v_n\right\rangle\right| + \left|\left\langle u_n, (Y_t^{(h)} - X_t)v - v_n\right\rangle\right|\right. \\
&\left. + \left|\left\langle u-u_n, (Y_t^{(h)} - X_t)v_n\right\rangle\right| + \left|\left\langle u_n, (Y_t^{(h)} - X_t)v_n\right\rangle\right|\right\} \\
\leqslant &\sup_{t\in[0,T]} \left\{\|u-u_n\|\|v-v_n\|(\|Y_t^{(h)}\| + \|X_t\|) + \|u_n\|\|v-v_n\|(\|Y_t^{(h)}\| + \|X_t\|)\right. \\
&\left. + \|u-u_n\|\|v-v_n\|(\|Y_t^{(h)}\| + \|X_t\|)\left|\left\langle u_n, (Y_t^{(h)} - X_t)v_n\right\rangle\right|\right\} \\
\to &0 \quad \text{as } h\to 0^+ \quad \text{and} \quad n\to\infty.
\end{aligned}$$

□

**Lemma 3.1.14.** *Let $T \geqslant 0$. If $(Y_t^h)_{t\in[0,T]}$ is a family of contractions in $\mathcal{B}(\mathsf{H})$ for all $h > 0$ and $(X_t)_{t\in[0,T]}$ is a family of isometries in $\mathcal{B}(\mathsf{H})$ such that $t \mapsto X_t u$ is continuous for all $u \in \mathsf{H}$, and*

$$\lim_{h\to 0^+} \sup_{t\in[0,T]} \left|\left\langle u, (Y_t^{(h)} - X_t)v\right\rangle\right| \to 0, \text{ for all } u, v \in \mathsf{H}, \tag{3.20}$$

*then*

$$\lim_{h\to 0^+} \sup_{t\in[0,T]} \left\|(Y_t^{(h)} - X_t)u\right\| \to 0, \text{ for all } u \in \mathsf{H}.$$

*Proof.* Fix $u \in \mathsf{H}$ and $T > 0$.
If $X$ is is an isometry and $Y$ is an contraction then

$$\begin{aligned}
&\|(X-Y)u\|^2 \\
=&\langle (X-Y)u, (X-Y)u\rangle \\
=&\|u\|^2 + \|Yu\|^2 - \langle Xu, Yu\rangle - \langle Yu, Xu\rangle \\
=&\|u\|^2 + \|Yu\|^2 - \langle Xu, (Y-X)u\rangle - \langle Xu, Xu\rangle - \langle (Y-X)u, Xu\rangle - \langle Xu, Xu\rangle \\
=&\|Yu\|^2 - \|u\|^2 + 2\mathrm{Re}\,\langle (X-Y)u, Xu\rangle \\
\leqslant& 2\mathrm{Re}\,\langle (X-Y)u, Xu\rangle \\
\leqslant& 2|\,\langle (X-Y)u, Xu\rangle\,|.
\end{aligned}$$

Then

$$\sup_{t\in[0,T]} \left\|(Y_t^{(h)} - X_t)u\right\| \leqslant 2 \sup_{t\in[0,T]} \left|\left\langle (X_t - Y_t^{(h)})u, X_t u\right\rangle\right|.$$

As $t \mapsto X_t u$ is continuous on $\mathbb{R}_+$, it is uniformly continuous on $[0, T]$.
If $s$, $t \in [0, T]$ then

$$\begin{aligned}
&\left|\left\langle (X_t - Y_t^{(h)})u, X_t u\right\rangle\right| \\
\leqslant& \left|\left\langle (X_t - Y_t^{(h)})u, (X_t - X_s)u\right\rangle\right| + \left|\left\langle (X_t - Y_t^{(h)})u, X_s u\right\rangle\right| \\
\leqslant& 2\|u\|\|(X_t - X_s)u\| + \left|\left\langle (X_t - Y_t^{(h)})u, X_s u\right\rangle\right|.
\end{aligned}$$

Given $\varepsilon > 0$ let $0 = t_0 < t_2 < \ldots < t_n = T$ be such that

$$\|(X_t - X_{t_i})u\| < \frac{\varepsilon}{2\|u\| + 1}$$

if $t \in [t_i, t_{i+1}]$, for $i = 0, \ldots, n-1$. Then

$$\begin{aligned}
&\sup_{t\in[0,T]} \left|\left\langle (X_t - Y_t^{(h)})u, X_t u\right\rangle\right| \\
\leqslant &\frac{2\|u\|\varepsilon}{2\|u\| + 1} + \max_{0\leqslant i\leqslant n-1} \left|\left\langle (X_t - Y_t^{(h)})u, X_{t_i} u\right\rangle\right| \\
< &\varepsilon + \max_{0\leqslant i\leqslant n-1} \left|\left\langle (X_t - Y_t^{(h)})u, X_{t_i} u\right\rangle\right| \\
\to &\varepsilon \quad \text{as } h \to 0.
\end{aligned}$$

As $\varepsilon$ is arbitrary, the result follows. □

In [80, Theorem 4.1] Parthasarathy showed that the Markov-regular unitary QS cocycle can be obtained as a limit (in the sense of Definition on p. 156 in [80]) of quantum random walks (defined in [80, equation (4.3), p. 161]). However, the theorem was proved under the assumption that the initial space $\mathfrak{h} = \mathbb{C}$. The random walk generator used in the next example is taken to be of a similar form to the one considered on p. 163 in [80], but $\mathfrak{h}$ is an arbitrary Hilbert space finite or infinte-dimensional.

**Example 3.1.15** (Toy Weyl generator)**.** Take

$$X(H, L, W) := \begin{bmatrix} e^{-\mathrm{i}H} & 0 \\ 0 & e^{-\mathrm{i}H} \otimes I \end{bmatrix} \begin{bmatrix} (I - L^*L)^{\frac{1}{2}} & -L^*W \\ L & (I - LL^*)^{\frac{1}{2}} W \end{bmatrix}, \tag{3.21}$$

where $L \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$ has norm less or equal 1, $W \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{k})$ is an isometry and $H \in \mathcal{B}(\mathfrak{h})$ is self-adjoint. Note that $X(H, L, W)$ is an isometry.

For $h > 0$ set $G(h) := X(hH, \sqrt{h}L, W)$ and denote

$$F := \begin{bmatrix} -\mathrm{i}H - \frac{1}{2}L^*L & -L^*W \\ L & W - I \end{bmatrix}. \tag{3.22}$$

By Theorem 1.2.34 we know that the Markov-regular QS cocycle generated by

$F$, that is, $X^F$ is a family of isometries.
Observe that

$$\begin{aligned}
&m(G(h),h)\\
=&\begin{bmatrix} \frac{1}{\sqrt{h}}I & 0 \\ 0 & I \end{bmatrix}\left(\begin{bmatrix} I-\mathrm{i}hH+o(h) & 0 \\ 0 & I-\mathrm{i}hH\otimes I+o(h) \end{bmatrix}\right.\\
&\left.\begin{bmatrix} I-\frac{1}{2}hL^*L+o(h) & -\sqrt{h}L^*W \\ \sqrt{h}L & W-\frac{1}{2}hLL^*W+o(h) \end{bmatrix}-\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\right)\begin{bmatrix} \frac{1}{\sqrt{h}}I & 0 \\ 0 & I \end{bmatrix}\\
=&\begin{bmatrix} I-\mathrm{i}H-\frac{1}{2}hL^*L+o(1) & -L^*W+o(1) \\ L+o(1) & W-I+o(1) \end{bmatrix}\\
=&F+o(1)\\
\to&F \quad \text{as } h\to 0^+.
\end{aligned}$$

Therefore Theorem 3.1.12 together with Lemma 3.1.13 and Lemma 3.1.14 imply that, if $\theta \in \mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}}$ and $T\in\mathbb{R}_+$ then

$$\lim_{h\to 0^+}\sup_{t\in[0,T]}\left\|\left(X_t^{G(h),h}-X_t^F\right)\theta\right\|=0.$$

## Random walk products

The results presented in this section concerns the product of random walks.

**Theorem 3.1.16.** *Let*

$$G_1\colon(0,\infty)\to\mathcal{B}(\mathfrak{h}\otimes\widehat{\mathsf{k}}),\quad G_2\colon(0,\infty)\to\mathcal{B}(\mathfrak{h}\otimes\widehat{\mathsf{k}}),$$

*and let* $F_1\in\mathcal{B}(\mathfrak{h}\otimes\widehat{\mathsf{k}})$, $F_2\in\mathcal{B}(\mathfrak{h}\otimes\widehat{\mathsf{k}})$ *be such that*

$$m(G_1(h),h)\overset{h\to0^+}{\to}F_1 \quad \textit{and} \quad m(G_2(h),h)\overset{h\to0^+}{\to}F_2$$

*in norm.*

*If* $f,g\in L^2(\mathbb{R}_+;\mathsf{k})$ *are right-continuous step functions and* $T\in\mathbb{R}_+$ *then*

$$\lim_{h\to 0^+} \sup_{t\in[0,T]} \left\| E^{\varepsilon(f)} \left( X_t^{G_1(h)G_2(h),h} - X_t^F \right) E_{\varepsilon(g)} \right\| = 0,$$

*where* $X^F = (X_t^F)$ *is the Markov-regular QS cocycle with generator*

$$F := F_1 + F_2 + F_1 \Delta F_2.$$

*Proof.* We claim that $m(G_1(h)G_2(h), h) \to F_1 + F_2 + F_1\Delta F_2$.

The key step of the proof is the following observation

$$ab - 1 = (a-1)(b-1) + a - 1 + b - 1,$$

for any elements $a, b$ of a unital Banach algebra.

Thus,

$$\begin{aligned}
&m(G_1(h)G_2(h), h)\\
=&\left(\frac{1}{\sqrt{h}}\Delta^\perp + \Delta\right)(G_1(h)G_2(h) - I)\left(\frac{1}{\sqrt{h}}\Delta^\perp + \Delta\right)\\
=&\left(\frac{1}{\sqrt{h}}\Delta^\perp + \Delta\right)(G_1(h) - I)(G_2(h) - I)\left(\frac{1}{\sqrt{h}}\Delta^\perp + \Delta\right) + m(G_1(h), h)\\
&+ m(G_2(h), h).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\left(\frac{1}{\sqrt{h}}\Delta^\perp + \Delta\right)(G_1(h) - I)(G_2(h) - I)\left(\frac{1}{\sqrt{h}}\Delta^\perp + \Delta\right)\\
=&m(G_1(h), h)(h\Delta^\perp + \Delta)m(G_2(h), h)\\
=&m(G_1(h), h)\Delta m(G_2(h), h) + h\, m(G_1(h), h)\Delta^\perp m(G_2(h), h).
\end{aligned}$$

Therefore,

$$m(G_1(h)G_2(h), h) \overset{h\to 0^+}{\to} F_1 + F_2 + F_1\Delta F_2.$$

We finish the proof by applying Theorem 3.1.12. □

**Definition 3.1.17.** We say that the operators $F_1,\ F_2 \in \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ *commute on the initial space* $\mathfrak{h}$ if and only if for all $x_1,\ x_2,\ y_1,\ y_2 \in \widehat{\mathsf{k}}$ the following holds

$$E^{x_1} F_1 E_{y_1} E^{x_2} F_2 E_{y_2} = E^{x_2} F_2 E_{y_2} E^{x_1} F_1 E_{y_1}. \tag{3.23}$$

**Corollary 3.1.18.** *Let*

$$G_1\colon (0,\infty) \to \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}}), \quad G_2\colon (0,\infty) \to \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}}),$$

*and let* $F_1,\ F_2 \in \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ *be such that*

$$m(G_1(h),h) \overset{h\to 0^+}{\to} F_1 \quad \text{and} \quad m(G_2(h),h) \overset{h\to 0^+}{\to} F_2$$

*in norm.*

*If* $G_1(h)$ *and* $G_2(h)$ *commute on the initial space* $\mathfrak{h}$ *for all* $h > 0$ *then*

$$\lim_{h\to 0^+} \sup_{t\in[0,T]} \left\| E^{\varepsilon(f)} \left( X_t^{G_1(h),h} X_t^{G_2(h),h} - X_t^F \right) E_{\varepsilon(g)} \right\| = 0,$$

*for all right-continuous step functions* $f,\ g \in L^2(\mathbb{R}_+;\mathsf{k})$ *and* $T \in \mathbb{R}_+$*,where* $X^F = (X_t^F)$ *is the Markov-regular QS cocycle with generator*

$$F := F_1 + F_2 + F_1 \Delta F_2.$$

*Proof.* First we show that $X_t^{G_1(h),h} X_t^{G_2(h),h} = X_t^{G_1(h)G_2(h),h}$ holds if and only if $G_1(h)$ and $G_2(h)$ commute on the initial space. Since $J_h^k$ is an isometry for any $k \in \mathbb{N}$ it is sufficient to show that $G_1(h)$ and $G_2(h)$ commute on the initial space if and only if

$$G_{1,1}^{(n)}(h) \dots G_{1,n}^{(n)}(h) G_{2,1}^{(n)}(h) \dots G_{2,n}^{(n)}(h) = G_{1,1}^{(n)}(h) G_{2,1}^{(n)}(h) \dots G_{1,n}^{(n)}(h) G_{2,n}^{(n)}(h) \tag{3.24}$$

for all $n \in \mathbb{N}$, where $G_{l,j}^{(n)}(h)$, for $l \in \{1,2\}$, is the ampliation of $G_l(h)$ to $\mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}}^{\otimes n})$ in the $j$-th place. Now we show that the equation (3.24) implies that $G_1(h)$ and

$G_2(h)$ commute on the initial space. Note that (3.24) yields the equality

$$G^{(2)}_{2,1}G^{(2)}_{1,2} = G^{(2)}_{1,2}G^{(2)}_{2,1}.$$

By applying Lemma 3.1.10 we obtain that

$$E^{x_1\otimes x_2}G^{(2)}_{2,1}(h)G^{(2)}_{1,2}(h)E_{y_1\otimes y_2} = E^{x_1}G_2(h)E_{y_1}E^{x_2}G_1(h)E_{y_2}.$$

On the other hand

$$E^{x_1\otimes x_2}G^{(2)}_{1,2}(h)G^{(2)}_{2,1}(h)E_{y_1\otimes y_2} = E^{x_2}G_1(h)E_{y_2}E^{x_1}G_2(h)E_{y_1}$$

due to the equation (3.9). Therefore, $G_1(h)$ and $G_2(h)$ commute on the initial space. Application of Lemma 3.1.10 and the induction on $n$ delivers that the equation (3.24) holds if $G_1(h)$ and $G_2(h)$ commute on the initial space.

Finally, to finish the proof we apply Theorem 3.1.16. □

**Proposition 3.1.19.** *Let $X^{F_1}$, $X^{F_2}$ be Markov-regular QS cocycles generated by $F_1$, $F_2 \in \mathcal{B}(\mathfrak{h}\otimes\widehat{\mathsf{k}})$, respectively. If $F_1$ and $F_2$ commute on the initial space $\mathfrak{h}$ then $(X^{F_1}_t X^{F_2}_t)_{t\geqslant 0}$ is a QS cocycle with generator $F_1 + F_2 + F_1\Delta F_2$.*

*Proof.* Assume that $F_1$ and $F_2$ commute on the initial space $\mathfrak{h}$. Let $h > 0$.

Define

$$G_1(h) := h\Delta^\perp F_1\Delta^\perp + \sqrt{h}\Delta^\perp F_1\Delta + \sqrt{h}\Delta F_1\Delta^\perp + \Delta^\perp F_1\Delta^\perp + I,$$
$$G_2(h) := h\Delta^\perp F_2\Delta^\perp + \sqrt{h}\Delta^\perp F_2\Delta + \sqrt{h}\Delta F_2\Delta^\perp + \Delta^\perp F_2\Delta^\perp + I.$$

Then $m(G_1(h), h) \to F_1$ and $m(G_2(h), h) \to F_2$ in norm as $h \to 0^+$. It is easy to see that $G_1(h)$ and $G_2(h)$ commute on the initial space $\mathfrak{h}$. Thus,

$$X^{G_1(h),h}_t X^{G_2(h),h}_t = X^{G_1(h)G_2(h),h}_t.$$

By applying, e.g., quantum Donsker's invariance principle, that is, Theorem 7.6

p. 27 in [16], we obtain that

$$\lim_{h\to 0^+}\left\|\left(X_t^{G_1(h),h}-X_t^{F_1}\right)(u\otimes\varepsilon(f))\right\|=0 \text{ and}$$
$$\lim_{h\to 0^+}\left\|\left(X_t^{G_2(h),h}-X_t^{F_2}\right)(u\otimes\varepsilon(f))\right\|=0$$

for all $u\in\mathfrak{h}$ and $f\in L^2(\mathbb{R}_+;\mathsf{k})$. Observe that

$$\begin{aligned}&\left\langle u\otimes\varepsilon(f), X_t^{G_1(h)G_2(h),h}(v\otimes\varepsilon(g))\right\rangle\\ =&\left\langle u\otimes\varepsilon(f), X_t^{G_1(h),h}X_t^{G_2(h),h}(v\otimes\varepsilon(g))\right\rangle\\ =&\left\langle (X_t^{G_1(h),h})^*(u\otimes\varepsilon(f)), X_t^{G_2(h),h}(v\otimes\varepsilon(g))\right\rangle\\ \overset{h\to 0^+}{\to}&\left\langle u\otimes\varepsilon(f), X_t^{F_1}X_t^{F_2}(v\otimes\varepsilon(g))\right\rangle\end{aligned}$$

for all $u, v\in\mathfrak{h}$ and $f, g\in L^2(\mathbb{R}_+;\mathsf{k})$. On the other hand, by Corollary 3.1.18,

$$\left\langle u\otimes\varepsilon(f), X_t^{G_1(h)G_2(h),h}(v\otimes\varepsilon(g))\right\rangle\to\left\langle u\otimes\varepsilon(f), X_t^{F}(v\otimes\varepsilon(g))\right\rangle,$$

where $F = F_1+F_2+F_1\Delta F_2$. Therefore, due to the uniqueness of the limit $(X_t^{F_1}X_t^{F_2})_{t\geqslant 0}$ is the Markov-regular QS cocycle with generator $F_1+F_2+F_1\Delta F_2$. □

**Remark 3.1.20.** Instead of using the quantum Donsker's invariance principle in the proof of the preceding proposition, we could use our convergence results if we assume that the cocycles $X^{F_1}$, $X^{F_2}$ consist of isometries.

Let $X^{F_1}$, $X^{F_2}$ be isometric Markov-regular QS cocycles and assume that $F_1$ and $F_2$ commute on the initial space $\mathfrak{h}$. By Theorem 1.2.34 we know that $F_1$ and $F_2$ are of the form (3.22) for some $L_1, L_2\in\mathcal{B}(\mathfrak{h};\mathfrak{h}\otimes\mathsf{k})$, isometries $W_1, W_2\in\mathcal{B}(\mathfrak{h}\otimes\mathsf{k})$ and self-adjoint $H_1, H_2\in\mathcal{B}(\mathfrak{h})$. Now, set

$$G_1(h) := X(hH_1,\sqrt{h}L_1,W_1)\quad\text{and}\quad G_2(h) := X(hH_2,\sqrt{h}L_2,W_2)$$

as in Example 3.1.15. Thus, $G_1(h)$ and $G_2(h)$ commute on the initial space $\mathfrak{h}$.

Example 3.1.15 yields that $X_t^{G_1(h),h}\to X_t^{F_1}$ and $X_t^{G_2(h),h}\to X_t^{F_2}$ in strong operator topology as $h\to 0^+$. Analogously to the proof of the preceding proposition we obtain that $(X_t^{F_1} X_t^{F_2})_{t \geqslant 0}$ is a Markov-regular QS cocycle with generator $F_1 + F_2 + F_1 \Delta F_2$.

We may also consider infinite products. The next lemma is a simple tool which will be used to obtain a non-trivial limit when we take an arbitrary large number of generators.

**Lemma 3.1.21.** *For any $d \in \mathbb{C}$ the following holds*

$$\lim_{n\to\infty} \sum_{k=2}^{n} \binom{n}{k} \left(\frac{d}{n}\right)^k = e^d - 1 - d.$$

*Proof.* By virtue of the binomial theorem we obtain

$$\begin{aligned} \lim_{n\to\infty} \sum_{k=2}^{n} \binom{n}{k} \left(\frac{d}{n}\right)^k &= \lim_{n\to\infty} \left( \sum_{k=0}^{n} \binom{n}{k} \left(\frac{d}{n}\right)^k - 1 - d \right) \\ &= \lim_{n\to\infty} \left(1 + \frac{d}{n}\right)^n - 1 - d = e^d - 1 - d. \end{aligned}$$

□

**Proposition 3.1.22.** *Let $G = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ for some $A \in \mathcal{B}(\mathfrak{h})$, $B \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{k}; \mathfrak{h})$, $C \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$, $D \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{k})$. For all $n \in \mathbb{N}$ let $G_n \colon (0, \infty) \to \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ satisfy*

$$m(G_n(h), h)) \to \frac{G}{n}$$

*in norm as $h \to 0^+$. If $f$, $g \in L^2(\mathbb{R}_+; \mathsf{k})$ are (right-continuous) step functions in $L^2(\mathbb{R}_+; \mathsf{k})$ and $T \in \mathbb{R}_+$ then*

$$\lim_{n\to\infty} \lim_{h\to 0^+} \sup_{t\in[0,T]} \left\| E^{\varepsilon(f)} \left( X_t^{G_n(h)^n, h} - X_t^F \right) E_{\varepsilon(g)} \right\| = 0,$$

*where $X^F = (X_t^F)$ is the Markov-regular QS cocycle with generator*

$$F = \begin{bmatrix} A + B \exp_2(D) C & B \exp_1(D) \\ \exp_1(D) C & \exp(D) - I \end{bmatrix},$$

*where* $\exp_1(D) := \sum_{n \geqslant 1} \frac{D^{n-1}}{n!}$, $\exp_2(D) := \sum_{n \geqslant 2} \frac{D^{n-2}}{n!}$.

*Proof.* First note that induction yields

$$m(G_n(h)^n, h) \to \sum_{k=1}^{n} \binom{n}{k} \frac{G}{n} \left( \Delta \frac{G}{n} \right)^{k-1}$$

in norm as $h \to 0^+$. Now, observe that for any $k \in \mathbb{N}$

$$\underbrace{G\Delta G \dots \Delta G}_{G \text{ appears } k\text{-times}} = \begin{bmatrix} BD^{k-2}C & BD^{k-1} \\ D^{k-1}C & D^k \end{bmatrix}.$$

Therefore,

$$\begin{aligned}
&\sum_{k=1}^{n} \binom{n}{k} \frac{G}{n} \left( \Delta \frac{G}{n} \right)^{k-1} \\
=&G + \sum_{k=2}^{n} \binom{n}{k} \frac{1}{n^k} \begin{bmatrix} BD^{k-2}C & BD^{k-1} \\ D^{k-1}C & D^k \end{bmatrix} \\
=&\begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} \left( \sum_{k=2}^{n} \binom{n}{k} \frac{1}{n^k} \begin{bmatrix} D^{k-2} & D^{k-1} \\ D^{k-1} & D^k \end{bmatrix} \right) \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}
\end{aligned}$$

in norm.

The previous lemma gives us the following norm convergence

- $\sum_{k=2}^{n} \binom{n}{k} \frac{1}{n^k} D^k \overset{n\to\infty}{\to} \exp(D) - I - D$,
- $\sum_{k=2}^{n} \binom{n}{k} \frac{1}{n^k} D^{k-1} \overset{n\to\infty}{\to} \exp_1(D) - I$,
- $\sum_{k=2}^{n} \binom{n}{k} \frac{1}{n^k} D^{k-2} \overset{n\to\infty}{\to} \exp_2(D)$.

We arrive at

$$\begin{aligned}
&\begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} \left( \sum_{k=2}^{n} \binom{n}{k} \frac{1}{n^k} \begin{bmatrix} D^{k-2} & D^{k-1} \\ D^{k-1} & D^k \end{bmatrix} \right) \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \\
\overset{n\to\infty}{\to}& \begin{bmatrix} B \exp_2(D) C & B(\exp_1(D) - I) \\ (\exp_1(D) - I)C & \exp(D) - I - D \end{bmatrix}.
\end{aligned}$$

Hence,

$$m(G_n(h)^n, h) \to \begin{bmatrix} A + B\exp_2(D)C & B\exp_1(D) \\ \exp_1(D)C & \exp(D) - I \end{bmatrix}$$

in norm, as $h \to 0^+$ and $n \to \infty$.
We end the proof by applying Proposition 1.2.29.

□

## 3.2 Open quantum systems

Let $\mathsf{H}_{\mathsf{sys}}$ be a Hilbert space and let $H_{\mathsf{sys}} \in \mathcal{B}(\mathsf{H}_{\mathsf{sys}})$ be self-adjoint. We refer to the pair $S = (\mathsf{H}_{\mathsf{sys}}, H_{\mathsf{sys}})$ as a *quantum system*; we call $\mathsf{H}_{\mathsf{sys}}$ a *state space* for $S$ and $H_{\mathsf{sys}}$ a *Hamiltonian* for $S$ (it describes the dynamics of the system $S$). We say that $S$ is an *open quantum system* if it is coupled to another quantum system $B = (\mathsf{H}_{\mathsf{env}}, H_{\mathsf{env}})$, which we call *the environment*, and interacts with it. We may view $S$ as a triple $(\mathsf{H}_{\mathsf{sys}}, H_{\mathsf{sys}}, \omega_{\mathsf{sys}})$, if the quantum system $S$ is considered to be in the normal state $\omega_{\mathsf{sys}}\colon \mathcal{B}(\mathsf{H}_{\mathsf{sys}}) \to \mathbb{C}$ (an ultraweakly continuous positive linear functional of norm 1). If the open quantum system $S$ interacts with the environment $B$, then the Hilbert space of the total system $S + B$ is given by $\mathsf{H}_{\mathsf{sys}} \otimes \mathsf{H}_{\mathsf{env}}$. The Hamiltonian $H_{\mathsf{tot}}(t)$ of $S + B$ may be taken to be of the form

$$H_{\mathsf{tot}}(t) := H_{\mathsf{sys}} \otimes I_{\mathsf{H}_{\mathsf{env}}} + I_{\mathsf{H}_{\mathsf{sys}}} \otimes H_{\mathsf{env}} + H_{\mathsf{int}}(t),$$

where $H_{\mathsf{int}}(t) \in \mathcal{B}(\mathsf{H}_{\mathsf{sys}} \otimes \mathsf{H}_{\mathsf{env}})$ is the Hamiltonian describing the interaction between the system $S$ and the environment $B$ at time $t$. The evolution of the total system is given by a differential equation

$$\frac{\mathrm{d}U_t}{\mathrm{d}t} = -\mathrm{i}H_{\mathsf{tot}}(t)U(t).$$

For more details we refer the reader to [30, 3.1.3 Dynamics of open systems, p. 115].

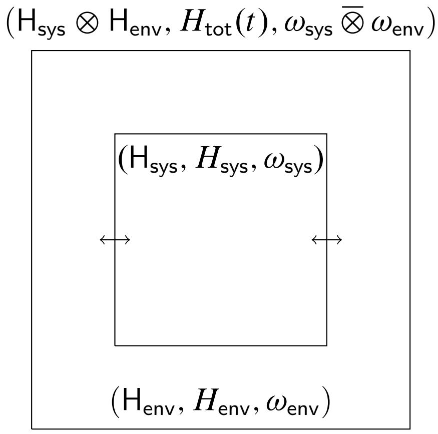



Figure 3.1: Interaction of the open quantum system $S$ with the environment $B$ at time $t$.

We will be exploiting the above model further in this chapter, but before let us formulate some important technicalities.

**Lemma 3.2.1.** *If $a$, $b$ and $c$ are elements of a Banach algebra and $h > 0$ then*

$$\exp\left(a + \sqrt{h}b + hc\right) = \exp(a) + \sqrt{h} f(a,b) + h(f(a,c) + g(a,b)) + o(h) \text{ as } h \to 0^+,$$

*where*

$$f(x,y) := \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{j=0}^{n-1} x^j y x^{n-1-j}$$

*and*

$$g(x,y) := \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{j=0}^{n-2} \sum_{k=0}^{n-2-j} x^j y x^k y x^{n-3-j-k}.$$

For the proof use a definition of the exponential function in Banach algebras (it is defined via power series) and then group the terms according to $h$.

**Lemma 3.2.2.** *Let*

$$G(h) := \exp(A + \sqrt{h}B + hC), \tag{3.25}$$

*where $A$, $B$, $C \in \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}})$. We obtain that*

$$\begin{aligned} m(G(h),h) \overset{h\to 0^+}{\to} & \Delta^\perp(f(A,C)+g(A,B))\Delta^\perp + \Delta^\perp f(A,B)\Delta + \Delta f(A,B)\Delta^\perp \\ & + \Delta(\exp A - I)\Delta \\ =: & F \end{aligned} \tag{3.26}$$

*in norm, if and only if*

$$\Delta^\perp(\exp(A)-I) = 0 = (\exp(A)-I)\Delta^\perp \quad \text{and} \quad \Delta^\perp f(A,B)\Delta^\perp = 0. \tag{3.27}$$

*Proof.* By Lemma 3.2.1, the modification $m(G(h),h)$ equals to the sum of $F$ and the reminder term

$$\begin{aligned} & \frac{1}{h}\Delta^\perp(\exp(A)-I)\Delta^\perp + \frac{1}{\sqrt{h}}\Delta^\perp(\exp(A)-I)\Delta \\ + & \frac{1}{\sqrt{h}}\Delta(\exp(A)-I)\Delta^\perp + \frac{1}{\sqrt{h}}\Delta^\perp f(A,B)\Delta^\perp + o(h). \end{aligned}$$

Hence, $m(G(h),h)$ converges as claimed if and only if $\Delta^\perp(\exp(A)-I)\Delta^\perp = 0$ and

$$\Delta^\perp(\exp(A)-I)\Delta + \Delta(\exp(A)-I)\Delta^\perp + \Delta^\perp f(A,B)\Delta^\perp = 0,$$

by considering the associated block matrices, the second identity holds if and only if

$$\Delta^\perp(\exp(A)-I)\Delta = \Delta(\exp(A)-I)\Delta^\perp = \Delta^\perp f(A,B)\Delta^\perp = 0.$$

□

**Lemma 3.2.3.** *Let $G(h)$ be defined as in Lemma 3.2.2. If $\Delta^\perp A = 0 = A\Delta^\perp$ and $\Delta^\perp B\Delta^\perp = 0$ then*

$$\begin{aligned} m(G(h),h) \overset{h\to 0^+}{\to} & \Delta^\perp(C + B\exp_2(A)B)\Delta^\perp + \Delta^\perp B\exp_1(A)\Delta \\ + & \ \Delta\exp_1(A)B\Delta^\perp + \Delta(\exp A - I)\Delta \\ := & \ F \end{aligned}$$

*in norm, where*

$$\exp_1(A) := \sum_{n\geqslant 1} \frac{1}{n!} A^{n-1} \quad \textit{and} \quad \exp_2(A) := \sum_{n\geqslant 2} \frac{1}{n!} A^{n-2}.$$

*Proof.* Assume that $\Delta^\perp A = 0 = A\Delta^\perp$. Then the first condition in (3.27) holds and the second one becomes

$$\Delta^\perp B \Delta^\perp = 0.$$

Furthermore,

- $\Delta^\perp f(A, B) = \Delta^\perp B \exp_1(A)$,
- $f(A, B)\Delta^\perp = \exp_1(A) B \Delta^\perp$,
- $\Delta^\perp g(A, B)\Delta^\perp = \Delta^\perp B \exp_2(A) B \Delta^\perp$,
- $\Delta^\perp f(A, C)\Delta^\perp = \Delta^\perp C \Delta^\perp$,

Hence,

$$F = \Delta^\perp(C + B\exp_2(A)B)\Delta^\perp + \Delta^\perp B\exp_1(A)\Delta + \Delta\exp_1(A)B\Delta^\perp + \Delta(\exp A - I)\Delta. \tag{3.28}$$

□

An immediate consequence from the preceding lemma is the following corollary:

**Corollary 3.2.4.** *Assume that $G(h)$ is as in Lemma 3.2.2. If $A = 0$ then we obtain*

$$F = \Delta^\perp \left( C + \frac{1}{2} B^2 \right) \Delta^\perp + \Delta^\perp B \Delta + \Delta B \Delta^\perp.$$

*If instead we assume that $B = 0$ and $\Delta^\perp A = 0 = A\Delta^\perp$ then*

$$F = \Delta^\perp C \Delta^\perp + \Delta(\exp A - I)\Delta.$$

## Repeated interactions

Let $\widehat{\mathsf{k}}$ be a separable Hilbert space with a distinguished unit vector $e_0$. In this section we discuss the repeated quantum interactions, during short time intervals of length $h$, between the open quantum system $S = (\mathfrak{h}, H_{\mathsf{sys}})$ with the environment $B$, which is given by an infinite chain of identical quantum systems. The coupled system $S + B$ is called a *repeated interaction quantum system.* The model of repeated quantum interactions was introduced by Attal and Pautrat in [9] and investigated in many others articles, e.g. [5], [6], [5], [38], [28], [29] and [10]. Let $P = (\widehat{\mathsf{k}}, H_{\mathsf{par}})$ be a typical element of the infinite chain $B$. Thus the Toy Fock space $\Upsilon(\widehat{\mathsf{k}})$ is the state space for $B$. The system $S$ interacts first with $P$ during the time interval $[0, h)$ for some small $h > 0$, according to the total Hamiltonian

$$H_{\mathsf{tot}}(h) := H_{\mathsf{sys}} \otimes I_{\widehat{\mathsf{k}}} + I_{\mathfrak{h}} \otimes H_{\mathsf{par}} + \lambda_h H_{\mathsf{int}}, \tag{3.29}$$

where $H_{\mathsf{int}} \in \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ is the interaction Hamiltonian and $\lambda_h \geqslant 0$ represents the strength of interaction depending on $h$. Then the system $S$ stops interacting with $P$ and starts with the next particle, again for a period of length $h$. This procedure is continued for one particle after another, and so on.

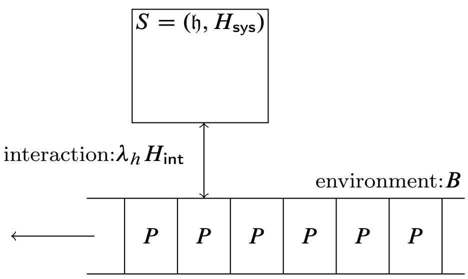



Figure 3.2: A repeated interaction system.

The unitary operator representing the evolution during the time of length $h$ is given by

$$U(h) := e^{-\mathrm{i}hH_{\mathsf{tot}}(h)}.$$

Let $n \in \mathbb{N}$. To describe the sequence of interactions, we consider the unitary operator $U(h)_k^{(n)}$, which is the ampliation of $U(h)$ to $\mathfrak{h} \otimes \widehat{\mathsf{k}}^{\otimes n}$ with the notation according to Definition 3.1.5. The coupled evolution during the time $[0, hn)$ is given by

$$V(h)_n := U(h)_1^{(n)} \cdots U(h)_n^{(n)}.$$

Hence, the whole evolution is described by the family $(V(h)_n)_{n\in\mathbb{N}}$. We will be investigating the behaviour of the evolutions as $h \to 0^+$. To obtain a non-trivial limit we have to scale the total Hamiltonian in terms of $h$ in a way that strengthens the interaction. We consider three different scalings, one of order $\sqrt{h}$, another of $h$, and the third combined. When $\widehat{\mathsf{k}}$ is finite dimensional, it was shown in [9] that those scalings lead to the limits which are Markov-regular unitary QS cocycles. Later this was generalised to infinite dimensional case, e.g. in [17]. Similarly to [9, IV.2 Typical Hamiltonian: weak coupling and low density ], below we present all possible limits while different scaling is chosen. In contrast to the results in [9] and [17], we also give the necessary and sufficient conditions on the interaction Hamiltonian to obtain the limit. In contrast to [9], the continuous-time limit of unitary evolutions are left QS cocycles, although due to Proposition 1.2.33 no generality is lost.

**Example 3.2.5** (Scaling of order $\sqrt{h}$)**.** First let us consider

$$H_{\mathsf{tot}}(h) := H_{\mathsf{sys}} \otimes I_{\widehat{\mathsf{k}}} + I_{\mathfrak{h}} \otimes H_{\mathsf{par}} + \frac{1}{\sqrt{h}} H_{\mathsf{int}}. \tag{3.30}$$

Then the associated unitary evolution can be expressed as follows

$$U(h) := \exp\left(-\mathrm{i}hH_{\mathsf{tot}}(h)\right) = \exp\Big(-\mathrm{i}\sqrt{h}H_{\mathsf{int}} + h\left(-\mathrm{i}H_{\mathsf{sys}} \otimes I_{\widehat{\mathsf{k}}} - \mathrm{i}I_{\mathfrak{h}} \otimes H_{\mathsf{par}}\right)\Big).$$

Lemma 3.2.3 and Corollary 3.2.4 imply that

$$\begin{aligned} m(U(h), h) \to &- \Delta^{\perp}\left(\mathrm{i}H_{\mathsf{sys}} \otimes I_{\widehat{\mathsf{k}}} + \mathrm{i}I_{\mathfrak{h}} \otimes H_{\mathsf{par}} + \frac{1}{2}H_{\mathsf{int}}^2\right)\Delta^{\perp} \\ &- \mathrm{i}\Delta^{\perp}H_{\mathsf{int}}\Delta - \mathrm{i}\Delta H_{\mathsf{int}}\Delta^{\perp} =: F \end{aligned} \tag{3.31}$$

in norm as $h \to 0^+$, if and only if $\Delta^{\perp}H_{\mathsf{int}}\Delta^{\perp} = 0$.

Set $\mathsf{k} := \widehat{\mathsf{k}} \ominus \mathbb{C}e_0$, where $e_0$ is a distinguished unit vector in $\widehat{\mathsf{k}}$. The condition

$\Delta^{\perp} H_{\text{int}} \Delta^{\perp} = 0$ holds if and only if

$$H_{\text{int}} = \begin{bmatrix} 0 & L^* \\ L & D \end{bmatrix} \tag{3.32}$$

for some $L \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$ and self-adjoint $D \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{k})$.

Thus, if $H_{\text{int}}$ has the matrix form (3.32) then

$$F = \begin{bmatrix} -\mathrm{i}\left(H_{\text{sys}} + \langle e_0, H_{\text{par}} e_0 \rangle I_{\mathfrak{h}}\right) - \frac{1}{2} L^* L & -\mathrm{i} L^* \\ -\mathrm{i} L & 0 \end{bmatrix}. \tag{3.33}$$

Hence, to obtain the most general form of (3.33) it is sufficient to consider interaction such that

$$H_{\text{int}} = \begin{bmatrix} 0 & L^* \\ L & 0 \end{bmatrix} \tag{3.34}$$

for some $L \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$. Observe that $F$ satisfies the conditions in (1.23).

Therefore, Theorem 3.1.12 together with Lemma 3.1.13 and Lemma 3.1.14 yield that $X_t^{U(h),h}$ converges in the strong operator topology to the Markov-regular unitary QS cocycle $(X_t^F)_{t \geqslant 0}$.

The interaction Hamiltonian given by (3.34) together with the scaling of order $\sqrt{h}$ correspond to a so-called typical renormalised dipole Hamiltonian which is often considered in the weak-coupling limit, also called the van Hove limit. Further in the thesis we will give more details and references concerning this type of the limit.

**Example 3.2.6** (Scaling of order $h$)**.** To exploit another possibility of scaling set

$$H_{\text{tot}}(h) := H_{\text{sys}} \otimes I_{\widehat{\mathsf{k}}} + I_{\mathfrak{h}} \otimes H_{\text{par}} + \frac{1}{h} H_{\text{int}} \quad \text{for all } h > 0. \tag{3.35}$$

Furthermore,

$$U(h) := \exp\left(-\mathrm{i} h H_{\text{tot}}(h)\right) = \exp\left(-\mathrm{i} H_{\text{int}} + h\left(-\mathrm{i} H_{\text{sys}} \otimes I_{\widehat{\mathsf{k}}} - \mathrm{i} I_{\mathfrak{h}} \otimes H_{\text{par}}\right)\right).$$

Lemma 3.2.3 and Corollary 3.2.4 imply that if $\Delta^{\perp}H_{\mathsf{int}} = H_{\mathsf{int}}\Delta^{\perp} = 0$ then

$$\begin{aligned} m(U(h), h) \to &- \Delta^{\perp}\left(\mathrm{i}H_{\mathsf{sys}} \otimes I_{\widehat{\mathsf{k}}} + \mathrm{i}I_{\mathfrak{h}} \otimes H_{\mathsf{par}}\right)\Delta^{\perp} + \Delta(\exp\left(-\mathrm{i}H_{\mathsf{int}}\right) - I)\Delta \\ &=: F \quad \text{as } h \to 0^{+}, \end{aligned}$$

The condition $\Delta^{\perp}H_{\mathsf{int}} = H_{\mathsf{int}}\Delta^{\perp} = 0$ holds if and only if

$$H_{\mathsf{int}} = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \tag{3.36}$$

for some self-adjoint $D \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{k})$. Therefore, if $H_{\mathsf{int}}$ has the matrix form (3.36) then

$$F = \begin{bmatrix} -\mathrm{i}\left(H_{\mathsf{sys}} + \langle e_0, H_{\mathsf{par}}e_0\rangle I_{\mathfrak{h}}\right) & 0 \\ 0 & e^{-\mathrm{i}D} - I \end{bmatrix},$$

and such $F$ satisfies the conditions in (1.23).

We finish by applying Theorem 3.1.12, Lemma 3.1.13 and Lemma 3.1.14 to obtain that $X_t^{U(h),h}$ converges in the strong operator topology to the Markov-regular unitary QS cocycle $(X_t^F)_{t\geqslant 0}$.

The next example is a generalisation of [9, Theorem 19, p. 35], however we also give the necessary and sufficient conditions on the interaction Hamiltonian to obtain the limit.

**Example 3.2.7** (Combined limits)**.** For the combined scaling let us consider self-adjoint operators $H_{\mathsf{sys}} \in \mathcal{B}(\mathfrak{h})$, $H_{\mathsf{par}} \in \mathcal{B}(\widehat{\mathsf{k}})$,

$$H_{\mathsf{tot}}(h) := H_{\mathsf{sys}} \otimes I_{\widehat{\mathsf{k}}} + I_{\mathfrak{h}} \otimes H_{\mathsf{par}} + \frac{1}{\sqrt{h}}H'_{\mathsf{int}} + \frac{1}{h}H''_{\mathsf{int}} \quad \text{for all } h > 0, \tag{3.37}$$

where $H'_{\mathsf{int}} \in \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ and $H''_{\mathsf{int}} \in \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}})$.

The associated unitary evolution is given by

$$\begin{aligned} U(h) :&= \exp\left(-\mathrm{i}hH_{\mathsf{tot}}(h)\right) \\ &= \exp\Big(-\mathrm{i}H''_{\mathsf{int}} - \mathrm{i}\sqrt{h}H'_{\mathsf{int}} + h\left(-\mathrm{i}H_{\mathsf{sys}} \otimes I_{\widehat{\mathsf{k}}} - \mathrm{i}I_{\mathfrak{h}} \otimes H_{\mathsf{par}}\right)\Big). \end{aligned}$$

Hence, by Lemma 3.2.3 if

$$\Delta^{\perp} H'_{\mathrm{int}} \Delta^{\perp} = 0 \text{ and } \Delta^{\perp} H''_{\mathrm{int}} = H''_{\mathrm{int}} \Delta^{\perp} = 0 \tag{3.38}$$

then we have

$$\begin{aligned} m(U(h), h) \to & - \Delta^{\perp}(\mathrm{i}\left(H_{\mathsf{sys}} \otimes I_{\widehat{\mathsf{k}}} + I_{\mathfrak{h}} \otimes H_{\mathsf{par}}\right) + H'_{\mathrm{int}} \exp_2\left(-\mathrm{i} H''_{\mathrm{int}}\right) H'_{\mathrm{int}}) \Delta^{\perp} \\ & - \mathrm{i} \Delta^{\perp} H'_{\mathrm{int}} \exp_1\left(-\mathrm{i} H''_{\mathrm{int}}\right) \Delta - \mathrm{i} \Delta \exp_1\left(-\mathrm{i} H''_{\mathrm{int}}\right) H'_{\mathrm{int}} \Delta^{\perp} \\ & + \Delta(\exp\left(-\mathrm{i} H''_{\mathrm{int}}\right) - I) \Delta \\ =: & \; F \quad \text{as } h \to 0^{+}. \end{aligned}$$

Note that conditions (3.38) hold if and only if

$$H'_{\mathrm{int}} = \begin{bmatrix} 0 & V^* \\ V & C \end{bmatrix} \text{ and } H''_{\mathrm{int}} = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \tag{3.39}$$

for some $V \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$ and self-adjoint operators $C, D \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{k})$. Therefore, if $H'_{\mathrm{int}}$ and $H''_{\mathrm{int}}$ have the matrix form (3.39) then

$$F = \begin{bmatrix} -\mathrm{i}\left(H_{\mathsf{sys}} + \langle e_0, H_{\mathsf{par}} e_0 \rangle I_{\mathfrak{h}}\right) - V^* \exp_2(-\mathrm{i} D) V & -\mathrm{i} V^* \exp_1(-\mathrm{i} D) \\ -\mathrm{i} \exp_1(-\mathrm{i} D) V & \exp(-\mathrm{i} D) - I \end{bmatrix}. \tag{3.40}$$

Hence, to obtain (3.40) it is sufficient to consider

$$H'_{\mathrm{int}} = \begin{bmatrix} 0 & V^* \\ V & 0 \end{bmatrix}$$

for some $V \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$ and $H''_{\mathrm{int}}$ of the form as in (3.39).

To show that $F$ given by (3.40) generates a unitary Markov-regular QS cocycle, according to Theorem 1.2.34 it is sufficient to show that it has a form

$$F = \begin{bmatrix} -iH - \frac{1}{2} L^* L & -L^* W \\ L & W - I \end{bmatrix} \tag{3.41}$$

for some $L \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$, self-adjoint $H \in \mathcal{B}(\mathfrak{h})$, and unitary $W \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{k})$.

To do that, first note that $L = -\mathrm{i}\exp_1(-\mathrm{i}D)V$, $W = \exp(-\mathrm{i}D)$.
Now, for each $0 \neq z \in \mathbb{C}$ we have $\exp_1(z) = \frac{1}{z}(e^z - 1)$ which implies that $\exp_1(z)e^{-z} = \frac{1}{z}(e^z - 1)e^{-z} = \exp_1(-z)$. Thus, the holomorphic calculus yields that

$$-L^*W = -\mathrm{i}V^* \exp_1(\mathrm{i}D)\exp(-\mathrm{i}D) = -\mathrm{i}V^* \exp_1(-\mathrm{i}D).$$

We have also

$$-\frac{1}{2}L^*L = -\frac{1}{2}V^* \exp_1(\mathrm{i}D)\exp_1(-\mathrm{i}D)V,$$

while the top-left entry of the matrix (3.40) containing $V$ equals

$$-V^* \exp_2(-\mathrm{i}D)V.$$

The last operator from the matrix (3.41) to determine is $H$. Note that simple algebraic operations give us the following equalities

- $-\frac{1}{2}\exp_1(\mathrm{i}x)\exp_1(-\mathrm{i}x) = \frac{1}{x^2}(\frac{e^{\mathrm{i}x}+e^{-\mathrm{i}x}}{2} - 1) = \frac{1}{x^2}(\cos x - 1)$,
- $-\exp_2(-\mathrm{i}x) = \frac{1}{x^2}(e^{-\mathrm{i}x} + \mathrm{i}x - 1)$ for each $x \in \mathbb{R} \setminus \{0\}$.

Now, observe that

$$-\exp_2(-\mathrm{i}x) = \frac{1}{x^2}(e^{-\mathrm{i}x} + \mathrm{i}x - 1) = \frac{1}{x^2}(\cos x - 1) - \mathrm{i}\frac{1}{x^2}(\sin x - x),$$

and set $\sin_1(x) := \sum_{n\geqslant 1}(-1)^n \frac{x^{2n-1}}{(2n+1)!}$. Therefore,

$$H = H_{\mathsf{sys}} + \langle e_0, H_{\mathsf{par}}e_0\rangle I_{\mathfrak{h}} - \mathrm{i}V^* \sin_1(D)V.$$

Hence, Theorem 3.1.12 together with Lemma 3.1.13 and Lemma 3.1.14 imply that $X_t^{U(h),h}$ converges in strong operator topology to the Markov-regular unitary QS cocycle $(X_t^F)_{t\geqslant 0}$.

**Remark 3.2.8.** Note that the above example can be also obtained by Proposition

3.1.22 if we take $G_n(h)$ such that its modification converges in norm to

$$G = \begin{bmatrix} -\mathrm{i}\left(H_{\mathsf{sys}} + \langle e_0, H_{\mathsf{par}} e_0\rangle I_{\mathfrak{h}}\right) & -\mathrm{i}V^* \\ -\mathrm{i}V & -\mathrm{i}D \end{bmatrix}.$$

Then

$$\lim_{n\to\infty}\lim_{h\to 0+} m(U(h)^n, h) = F$$

in norm, where $F$ is of the form (3.40).

The next example is a generalisation of [10, Theorem 3.1]. The physical description which we discuss below comes from [10, 2 Description of the Bipartite Model], however in our case we don't have to assume that the space of a single particle is finite dimensional. In contrast to [10], to obtain a limit cocycle we present a completely different technique than the one used in the proof of [10, Theorem 3.1]. We show that [10, Theorem 3.1] can be obtained as a special case of Proposition 3.1.16.

**Example 3.2.9** (Bipartite model)**.** Let $S_1 = (\mathfrak{h}_1, H^{(1)}_{\mathsf{sys}})$ and $S_2 = (\mathfrak{h}_2, H^{(2)}_{\mathsf{sys}})$ be two quantum systems, and assume that they do not interact together. The evolution of the quantum system $S := S_1 + S_2$ with the state space $\mathfrak{h}_1 \otimes \mathfrak{h}_2$ is therefore expressed by

$$H_{\mathsf{sys}} = H^{(1)}_{\mathsf{sys}} \otimes I_{\mathfrak{h}_2} + I_{\mathfrak{h}_1} \otimes H^{(2)}_{\mathsf{sys}}.$$

Let $\mathfrak{h} := \mathfrak{h}_1 \otimes \mathfrak{h}_2$. Let $S$ be coupled to the environment made of an infinite chain of identical systems $P = (\widehat{\mathsf{k}}, H_{\mathsf{par}})$, which is represented by a state space $\Upsilon(\widehat{\mathsf{k}})$. To describe the interaction between $S$ and $B$, we use the quantum repeated interactions model. Thus $P$ interacts with $S$ for a short time then stops, letting the next to repeat the procedure, and so on. A single interaction between $P$ and $S$ is described as follows; $P$ interacts first with $S_1$ during time $h$ without interacting with $S_2$, then it stops and start interacting with system $S_2$ without interacting with $S_1$ it happens for the length of time $h$ and then stops. The dynamics of the coupled system with the state space $\mathfrak{h} \otimes \widehat{\mathsf{k}}$ is described by the following Hamiltonians, for the first interaction we have

$$H_{\text{tot}}^{(1)}(h) := H_{\text{sys}}^{(1)} \otimes I_{\mathfrak{h}_2} \otimes I_{\widehat{\mathsf{k}}} + I_{\mathfrak{h}} \otimes H_{\text{par}} + H_{\text{int}}^{(1)}(h),$$

for the second one

$$H_{\text{tot}}^{(2)}(h) := I_{\mathfrak{h}_1} \otimes H_{\text{sys}}^{(2)} \otimes I_{\widehat{\mathsf{k}}} + I_{\mathfrak{h}_1} \otimes H_{\text{par}} + H_{\text{int}}^{(2)}(h),$$

where $H_{\text{par}} \in \mathcal{B}(\widehat{\mathsf{k}})$, $H_{\text{int}}^{(1)}(h) \in \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ acts on $\mathfrak{h}_2$ as an identity operator $I_{\mathfrak{h}_2}$, $H_{\text{int}}^{(2)}(h) \in \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ acts as $I_{\mathfrak{h}_1}$ on $\mathfrak{h}_1$. Each total Hamiltonian yield the unitary evolutions

$$U_1(h) := e^{-\mathrm{i}hH_{\text{tot}}^{(1)}(h)} \quad \text{and} \quad U_2(h) := e^{-\mathrm{i}hH_{\text{tot}}^{(2)}(h)}.$$

The combined evolution of the single interaction between $S$ and a particle is given by

$$V(h) := U_2(h)U_1(h).$$

The sequence of interactions is described as before in quantum repreated interactions, that is, we consider the sequence of unitary operators $V(h)_k^{(n)}$, which is the ampliation of $V(h)$ to $\mathfrak{h} \otimes \widehat{\mathsf{k}}^{\otimes n}$ with the notation according to Definition 3.1.5. The coupled evolution during the time $[0, 2hn)$ is given by

$$W(h)_n := V(h)_1^{(n)} \cdots V(h)_n^{(n)},$$

and so the whole evolution is described by the family $(W(h)_n)_{n\geqslant 1}$. To investigate the behaviour of the evolutions as $h \to 0^+$, let us consider the following interaction Hamiltonians:

$$H_{\text{int}}^{(1)}(h) := \frac{1}{\sqrt{h}} H_{\text{int}}^{(1)} := \frac{1}{\sqrt{h}} \begin{bmatrix} 0 & L^* \\ L & 0 \end{bmatrix},$$
$$H_{\text{int}}^{(2)}(h) := \frac{1}{\sqrt{h}} H_{\text{int}}^{(2)} := \frac{1}{\sqrt{h}} \begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix}$$

for some $L \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$ which acts on $\mathfrak{h}_2$ as $I_{\mathfrak{h}_2}$ and $M \in \mathcal{B}(\mathfrak{h}_2; \mathfrak{h}_2 \otimes \mathsf{k})$ which acts on $\mathfrak{h}_1$ as $I_{\mathfrak{h}_1}$. Therefore, by (3.31) we obtain that

$$\begin{aligned} m(U_1(h),h) \to &- \Delta^{\perp}\left(\mathrm{i}H^{(1)}_{\mathsf{sys}} \otimes I_{\mathfrak{h}_1} \otimes I_{\widehat{\mathsf{k}}} + \mathrm{i}I_{\mathfrak{h}} \otimes H_{\mathsf{par}} + \frac{1}{2}\left(H^{(1)}_{\mathsf{int}}\right)^2\right)\Delta^{\perp} \\ &- \mathrm{i}\Delta^{\perp}H^{(1)}_{\mathsf{int}}\Delta - \mathrm{i}\Delta H^{(1)}_{\mathsf{int}}\Delta^{\perp} =: F_1, \\ m(U_2(h),h) \to &- \Delta^{\perp}\left(\mathrm{i}I_{\mathfrak{h}_0} \otimes H''_{\mathsf{sys}} \otimes I_{\widehat{\mathsf{k}}} + \mathrm{i}I_{\mathfrak{h}} \otimes H_{\mathsf{par}} + \frac{1}{2}\left(H^{(2)}_{\mathsf{int}}\right)^2\right)\Delta^{\perp} \\ &- \mathrm{i}\Delta^{\perp}H^{(2)}_{\mathsf{int}}\Delta - \mathrm{i}\Delta H^{(2)}_{\mathsf{int}}\Delta^{\perp} =: F_2 \end{aligned}$$

in norm as $h \to 0^+$. Note that

$$\begin{aligned} F :&= F_1 + F_2 + F_2\Delta F_1 \\ &= \begin{bmatrix} -\mathrm{i}(H^{(1)}_{\mathsf{sys}} \otimes I_{\mathfrak{h}_2} + \langle e_0, H_{\mathsf{par}}(e_0)\rangle I_{\mathfrak{h}}) - \frac{1}{2}L^*L & -L^* \\ L & 0 \end{bmatrix} \\ &+ \begin{bmatrix} -\mathrm{i}(I_{\mathfrak{h}_1} \otimes H^{(2)}_{\mathsf{sys}} + \langle e_0, H_{\mathsf{par}}(e_0)\rangle I_{\mathfrak{h}}) - \frac{1}{2}M^*M & -M^* \\ M & 0 \end{bmatrix} \\ &+ \begin{bmatrix} -M^*L & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{3.42}$$

satisfies conditions (1.23).

Hence, Proposition 3.1.16 together with Lemma 3.1.13 and Lemma 3.1.14 yield that $X_t^{U_2(h)U_1(h),h}$ converges in the strong operator topology to the Markov-regular unitary QS cocycle $(X_t^F)_{t\geqslant 0}$.

Now let us establish the result similar to [10, Theorem 3.2], that is, we would like to find a Hamiltonian on $\mathfrak{h}\otimes\widehat{\mathsf{k}}$ according to which the usual repeated quantum interactions (described at the beginning of Section 3.2) lead us to the unitary cocycle generated by $F$ given in (3.42). To do that we compare the generator $F$ with the one which is usually obtained during repeated quantum interactions (Example 3.2.5), see (3.33) and (3.31). We will also be able to identity the interaction Hamiltonian which has been created by the environment between $S_1$ and $S_2$.

Observe that

$$-M^*L - \frac{1}{2}(M^*M + L^*L) = \frac{1}{2}(L^*M - M^*L) - \frac{1}{2}(M+L)^*(L+M).$$

Therefore we could obtain $F$ by using the usual repeated quantum interaction with the total Hamiltonian

$$H_{\mathsf{tot}}^{(1),(2)}(h) = H_{\mathsf{sys}}^{(1),(2)} \otimes I_{\widehat{\mathfrak{k}}} + I_{\mathfrak{h}} \otimes 2H_{\mathsf{par}} + \frac{1}{\sqrt{h}}\left(H_{\mathsf{int}}^{(1)} + H_{\mathsf{int}}^{(2)}\right),$$

where

$$H_{\mathsf{sys}}^{(1),(2)} = H_{\mathsf{sys}}^{(1)} \otimes I_{\mathfrak{h}_2} + I_{\mathfrak{h}_1} \otimes H_{\mathsf{sys}}^{(2)} + \frac{\mathrm{i}}{2}(L^*M - M^*L).$$

In particular,

$$\frac{\mathrm{i}}{2}(L^*M - M^*L)$$

represents the interaction by the environment between $S_1$ and $S_2$.

# Chapter 4

# Random walks and thermalisation

In this chapter section we will again be using the quantum repeated interactions model. However we assume that an infinite chain of identical systems is such that each system in the chain is in a normal faithful state $\omega$. An interesting example is to consider the thermal Gibbs state at inverse temperature $\beta$ given by the density matrix

$$\rho_\beta := \frac{1}{\mathrm{Tr}(e^{-\beta H_{\mathsf{par}}})} e^{-\beta H_{\mathsf{par}}},$$

however for $\rho_\beta$ to be well-defined we have to assume that $e^{-\beta H_{\mathsf{par}}}$ is a trace class operator. The zero-temperature case, refers to the repeated interactions which we considered before. By employing the Gelfand-Naimark theorem we will investigate the limits of unitary evolutions, which, in contrast to the zero temperature case, depend on the given faithful normal state. The first results regarding such limits are due to Attal and Joye [5], where the noise space is assumed to be finite dimensional and the approach is coordinate dependent. It was generalised to infinite dimensional noise space by Belton in [18], where his approach was coordinate free and mapping cocycles were mainly investigated. Later, in [19] it was shown that the convergence result can be obtained even with particles being in an arbitrary normal state. In our case the noise space is infinite dimensional and the particle state is faithful and normal. We investigate only operator processes. We obtain necessary and sufficient conditions on the interaction Hamiltonians for the model to have a limit.

## 4.1 Thermal states

### Concrete GNS representation

Let $\rho$ be a density matrix which acts on a separable Hilbert space $\mathsf{K}$, that is,

$$\rho \geqslant 0,\ \rho^{\frac{1}{2}} \in \mathrm{HS}(\mathsf{K}),\ \mathrm{Tr}\rho = 1$$

and assume that the corresponding normal state $\omega\colon A \to \mathrm{Tr}(\rho A)$ is faithful. Fix an orthonormal basis $\{e_n\}_{n=0}^N$ of $\mathsf{K}$, where $N \in \mathbb{N}$ or $N = \infty$, such that

$$\rho = \sum_{n=0}^{N} \gamma_n \left|e_n\right\rangle \left\langle e_n\right|, \tag{4.1}$$

where

$$\sum_{n=0}^{N} \gamma_n = 1, \quad \gamma_0 \geqslant \gamma_1 \geqslant \ldots > 0.$$

Assume that the eigenvalue $\gamma_0$ is nondegenerate, so $\gamma_0 > \gamma_1$.
The injective normal unital *-homomorphism

$$\pi\colon \mathcal{B}(\mathsf{K}) \to \mathcal{B}(\mathsf{K} \otimes \overline{\mathsf{K}}),\ \pi(T) = T \otimes I_{\overline{\mathsf{K}}}, \tag{4.2}$$

together with the vector $\xi = \sum_{n=0}^{N} \sqrt{\gamma_n}\ e_n \otimes \overline{e_n} \in \mathsf{K} \otimes \overline{\mathsf{K}}$ give a GNS representation of $(\mathcal{B}(\mathsf{K}), \omega)$; that is, $\omega(A) = \langle \xi, \pi(A)\xi \rangle$ for all $A \in \mathcal{B}(\mathsf{K})$.

### Rotation

Here, we start with some important technicalities.

Let $R \in \mathcal{B}(\mathsf{K} \otimes \overline{\mathsf{K}})$ be a unitary operator such that $R\xi = e_0 \otimes \overline{e_0}$ and let $\widetilde{R} := I_{\mathfrak{h}} \otimes R$ be its ampliation.
Below we present an example of such a unitary operator.

**Example 4.1.1** (Rotation $R$). Let

$$\tilde{\xi} := \frac{\sum_{i=0}^{N} \sqrt{\gamma_i}\; e_i \otimes \overline{e_i}}{\sqrt{1-\gamma_0}}. \tag{4.3}$$

Note that $\{e_0 \otimes \overline{e_0}, \tilde{\xi}\}$ is an orthonormal set and $\xi = \alpha e_0 \otimes \overline{e_0} + \beta \tilde{\xi}$, where $\alpha = \sqrt{\gamma_0}$ and $\beta = \sqrt{1-\gamma_0}$. According to the orthogonal decomposition

$$\mathsf{K} \otimes \overline{\mathsf{K}} = \mathbb{C}e_0 \otimes \overline{e_0} \oplus \mathbb{C}\tilde{\xi} \oplus ((\mathsf{k} \otimes \mathbb{C}\overline{e_0}) \oplus (\mathbb{C}e_0 \otimes \overline{\mathsf{k}})) \oplus \left((\mathsf{k} \otimes \overline{\mathsf{k}}) \ominus \mathbb{C}\tilde{\xi}\right), \tag{4.4}$$

where $\mathsf{k} := \mathsf{K} \ominus \mathbb{C}e_0$, let

$$R := \begin{bmatrix} \alpha & \beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \tag{4.5}$$

where $I$ represents the appropriate identity operator.
It is easy to see that $R$ is unitary and

$$R\xi = R \begin{pmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = e_0 \otimes \overline{e_0}.$$

We will always emphasize whenever we will be using the operator $R$ from Example 4.1.1 instead of a general $R$ which satisfies the condition

$$R\xi = e_0 \otimes \overline{e_0}.$$

Unless otherwise specified, we will be using a general $R$.

**Definition 4.1.2.** If $h > 0$ and $G \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{K} \otimes \overline{\mathsf{K}})$ then the *R-modification*

$m_R(G,h)$ is defined by setting

$$m_R(G,h) := \left(\frac{1}{\sqrt{h}}\Delta^\perp + \Delta\right)\widetilde{R}(G - I_{\mathfrak{h}\otimes\mathsf{K}\otimes\overline{\mathsf{K}}})\widetilde{R}^*\left(\frac{1}{\sqrt{h}}\Delta^\perp + \Delta\right).$$

where $\Delta^\perp = I_{\mathfrak{h}} \otimes |e_0 \otimes \overline{e_0}\rangle\langle e_0 \otimes \overline{e_0}|$ and $\Delta := I_{\mathfrak{h}\otimes\widehat{\mathsf{k}}} - \Delta^\perp$.

**Remark 4.1.3.** Note that if $\widehat{\mathsf{k}} = \mathsf{K} \otimes \overline{\mathsf{K}}$ and its distinguished unit vector $\eta = e_0 \otimes \overline{e_0}$ then $\Delta$ defined above coincides with the one defined in (1.11) and

$$m_R(G,h) = m(\widetilde{R}G\widetilde{R}^*, h),$$

**Lemma 4.1.4.** *Let*

$$G(h) := \exp\left(A + \sqrt{h}B + hC\right), \tag{4.6}$$

*where* $A, B, C \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{K} \otimes \overline{\mathsf{K}})$. *We obtain that*

$$\begin{aligned} m_R(G(h),h) \overset{h\to 0^+}{\to} & \Delta^\perp\widetilde{R}(f(A,C) + g(A,B))\widetilde{R}^*\Delta^\perp \\ & +\Delta^\perp\widetilde{R}f(A,B)\widetilde{R}^*\Delta + \Delta\widetilde{R}f(A,B)\widetilde{R}^*\Delta^\perp \\ & +\Delta(\widetilde{R}\exp(A)\widetilde{R}^* - I)\Delta \\ =: & \; F \end{aligned}$$

*in norm, if and only if*

$$E^\xi(\exp(A) - I)\widetilde{R}^* = 0 = \widetilde{R}(\exp(A) - I)E_\xi \quad \text{and} \quad E^\xi f(A,B)E_\xi = 0, \tag{4.7}$$

*where*

$$f(X,Y) := \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{j=0}^{n-1} X^j Y X^{n-1-j}$$

*and*

$$g(X,Y) := \sum_{n=2}^{\infty}\frac{1}{n!}\sum_{j=0}^{n-2}\sum_{k=0}^{n-2-j} X^j Y X^k Y X^{n-3-j-k}.$$

*for all* $X, Y \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{K} \otimes \overline{\mathsf{K}})$.

*Proof.* Lemma 3.2.2 implies that

$$\begin{aligned} m_R(G(h),h) &\overset{h\to 0^+}{\to} \Delta^\perp\widetilde{R}(f(A,C)+g(A,B))\widetilde{R}^*\Delta^\perp + \Delta^\perp\widetilde{R}f(A,B)\widetilde{R}^*\Delta \\ &\quad +\Delta\widetilde{R}f(A,B)\widetilde{R}^*\Delta^\perp + \Delta\widetilde{R}(\exp A - I)\widetilde{R}^*\Delta \\ &=: F \end{aligned}$$

in norm, if and only if

$$\Delta^\perp\widetilde{R}(\exp(A)-I)\widetilde{R}^* = 0 = \widetilde{R}(\exp(A)-I)\widetilde{R}^*\Delta^\perp \quad \text{and} \quad \Delta^\perp\widetilde{R}f(A,B)\widetilde{R}^*\Delta^\perp = 0. \tag{4.8}$$

Now, note that the following holds:

- $\Delta^\perp\widetilde{R} = \left(I_\mathfrak{h}\otimes|e_0\otimes\overline{e_0}\rangle\right)\left(I_\mathfrak{h}\otimes\langle\xi|\right) = E_{e_0\otimes\overline{e_0}}E^\xi,$
- $\widetilde{R}^*\Delta^\perp = \left(I_\mathfrak{h}\otimes|\xi\rangle\right)\left(I_\mathfrak{h}\otimes\langle e_0\otimes\overline{e_0}|\right) = E_\xi E^{e_0\otimes\overline{e_0}}.$

Hence, the condition (4.8) if and only if (4.7) is satisfied. □

**Lemma 4.1.5.** *Let $G(h)$ be defined as in the preceding lemma.*
*If $E^\xi A = AE_\xi = 0$ and $E^\xi BE_\xi = 0$ then*

$$\begin{aligned} m_R(G(h),h) &\overset{h\to 0^+}{\to} \Delta^\perp\widetilde{R}(C + B\exp_2(A)B)\widetilde{R}^*\Delta^\perp \\ &\quad + \Delta^\perp\widetilde{R}B\exp_1(A)\widetilde{R}^*\Delta + \Delta\widetilde{R}\exp_1(A)B\widetilde{R}^*\Delta^\perp \\ &\quad + \Delta(\widetilde{R}\exp(A)\widetilde{R}^* - I)\Delta. \\ &:= F \end{aligned}$$

*in norm, where*

$$\exp_1(A) := \sum_{n\geqslant 1}\frac{1}{n!}A^{n-1} \quad \textit{and} \quad \exp_2(A) := \sum_{n\geqslant 2}\frac{1}{n!}A^{n-2}.$$

*Proof.* Assume that $E^\xi A = AE_\xi = 0$. Then the first condition in (4.7) holds and the second one becomes

$$E^\xi BE_\xi = 0.$$

Furthermore,

- $\Delta^\perp \widetilde{R} f(A,B) = \Delta^\perp \widetilde{R} B \exp_1(A)$,
- $f(A,B)\widetilde{R}^* \Delta^\perp = \exp_1(A) B \widetilde{R}^* \Delta^\perp$,
- $\Delta^\perp \widetilde{R} g(A,B)\widetilde{R}^* \Delta^\perp = \Delta^\perp \widetilde{R} B \exp_2(A) B \widetilde{R}^* \Delta^\perp$,
- $\Delta^\perp \widetilde{R} f(A,C)\widetilde{R}^* \Delta^\perp = \Delta^\perp \widetilde{R} C \widetilde{R}^* \Delta^\perp$,

where

$$\exp_1(A) := \sum_{n\geqslant 1} \frac{1}{n!} A^{n-1} \quad \text{and} \quad \exp_2(A) := \sum_{n\geqslant 2} \frac{1}{n!} A^{n-2}.$$

Hence,

$$\begin{aligned} F &= \Delta^\perp \widetilde{R}(C + B \exp_2(A) B)\widetilde{R}^* \Delta^\perp \\ &\quad + \Delta^\perp \widetilde{R} B \exp_1(A)\widetilde{R}^* \Delta + \Delta \widetilde{R} \exp_1(A) B \widetilde{R}^* \Delta^\perp \\ &\quad + \Delta(\widetilde{R} \exp(A)\widetilde{R}^* - I)\Delta. \end{aligned}$$

□

An immediate consequence from the preceding lemma is the following corollary:

**Corollary 4.1.6.** *Let $G(h)$ be defined as in Lemma 4.1.4. If $A = 0$ then we obtain*

$$F = \Delta^\perp \widetilde{R}\left(C + \frac{1}{2}B^2\right)\widetilde{R}^* \Delta^\perp + \Delta^\perp \widetilde{R} B \widetilde{R}^* \Delta + \Delta \widetilde{R} B \widetilde{R}^* \Delta^\perp.$$

*If instead we assume that $B = 0$ and $E^\xi A = A E_\xi = 0$ then*

$$F = \Delta^\perp \widetilde{R} C \widetilde{R}^* \Delta^\perp + \Delta(\widetilde{R} \exp(A)\widetilde{R}^* - I)\Delta.$$

## 4.2 Limit cocycles

In this section we investigate repeated quantum interaction when each particle from the infinite chain is in a faithful normal state $\omega$. We consider different scaling of the interaction Hamiltonians and so the corresponding random walks will converge to unitary QS cocycles, whose noises depend on the state $\omega$. Example 4.2.6 is a generalisation of the main result obtained by Attal and Joye in [5], Example 4.2.12 generalises the work by Dhahri [38].

## Weak coupling limits

As we mentioned by the end of Example 3.2.5, the interaction Hamiltonian together with the scaling, which are exploited there, correspond to a so-called typical renormalised dipole Hamiltonian which is often considered in the weak coupling limit (van Hove limit). The notion of weak coupling limit was first studied by van Hove [91]. The mathematical picture comes from the series of articles by Davies [36], [37]. Briefly, in the weak coupling limit we assume that the interaction between a quantum system and the environment is such that the influence of the system on the environment is small [30, 3.3.1 Weak - coupling Limit]. In this section we scale the interaction Hamiltonian in the same way as in Example 3.2.5, therefore we expect that it will correspond to a dipole Hamiltonian.

To obtain a more general picture we start with 'particle' operators defined on $\mathsf{K}\otimes\overline{\mathsf{K}}$, rather than ampliating the ones defined on $\mathsf{K}$ to $\mathsf{K}\otimes\overline{\mathsf{K}}$. However, having the ampliation, which appears naturally in repeated quantum interactions, when each particle from the infinite chain is in a faithful normal state $\omega$, will be discussed in detail in 4.2.6.
According to (3.29) we take

$$H_{\mathrm{tot}}(h) := H_{\mathrm{sys}} \otimes I_{\mathsf{K}\otimes\overline{\mathsf{K}}} + I_{\mathfrak{h}} \otimes H_{\mathrm{par}} + \frac{1}{\sqrt{h}} H_{\mathrm{int}} \tag{4.9}$$

where $\widehat{\mathsf{k}}$ is replaced by $\mathsf{K}\otimes\overline{\mathsf{K}}$, and so $H_{\mathrm{par}} \in \mathcal{B}(\mathsf{K}\otimes\overline{\mathsf{K}})$ and $H_{\mathrm{int}} \in \mathcal{B}(\mathfrak{h}\otimes\mathsf{K}\otimes\overline{\mathsf{K}})$.

The associated unitary evolution is naturally given by

$$U(h) := \exp\left(-\mathrm{i}hH_{\mathrm{tot}}(h)\right). \tag{4.10}$$

Lemma 4.1.6 yields

$$m_R(U(h), h) \to -\Delta^\perp \widetilde{R}\left(\mathrm{i}H_{\mathrm{sys}} \otimes I_{\mathsf{K}\otimes\overline{\mathsf{K}}} + \mathrm{i}I_{\mathfrak{h}} \otimes H_{\mathrm{par}} + \frac{1}{2}H_{\mathrm{int}}^2\right)\widetilde{R}^*\Delta^\perp \\ - \mathrm{i}\left(\Delta^\perp \widetilde{R} H_{\mathrm{int}} \widetilde{R}^* \Delta + \Delta \widetilde{R} H_{\mathrm{int}} \widetilde{R}^* \Delta^\perp\right) \tag{4.11}$$

in norm as $h \to 0^+$ if and only if

$$E^\xi H_{\mathrm{int}} E_\xi = 0. \tag{4.12}$$

Set

$$F := -\Delta^\perp \widetilde{R}\left(\mathrm{i}H_{\mathrm{sys}} \otimes I_{\mathsf{K}\otimes\overline{\mathsf{K}}} + \mathrm{i}I_{\mathfrak{h}} \otimes H_{\mathrm{par}} + \frac{1}{2}H_{\mathrm{int}}^2\right)\widetilde{R}^*\Delta^\perp \\ - \mathrm{i}\Delta^\perp \widetilde{R} H_{\mathrm{int}} \widetilde{R}^* \Delta - \mathrm{i}\Delta \widetilde{R} H_{\mathrm{int}} \widetilde{R}^* \Delta^\perp \tag{4.13}$$

**Lemma 4.2.1.** *The Markov-regular QS cocycle with stochastic generator $F$ defined in (4.13) is unitary.*

*Proof.* We want to show that the condition (1.23) is satisfied, that is,

$$F + F^* + F\Delta F^* = F + F^* + F^*\Delta F = 0. \tag{4.14}$$

Observe that

$$F + F^* = -\Delta^\perp \widetilde{R} H_{\mathrm{int}}^2 \widetilde{R}^* \Delta^\perp.$$

Now we obtain

$$\begin{aligned} F\Delta F^* &= \Delta^\perp \widetilde{R} H_{\mathrm{int}} \widetilde{R}^* \Delta \widetilde{R} H_{\mathrm{int}} \widetilde{R}^* \Delta^\perp \\ &= \Delta^\perp \widetilde{R}\left(H_{\mathrm{int}}^2 - H_{\mathrm{int}} \widetilde{R}^* \Delta^\perp \widetilde{R} H_{\mathrm{int}}\right)\widetilde{R}^*\Delta^\perp \\ &= \Delta^\perp \widetilde{R} H_{\mathrm{int}}^2 \widetilde{R}^* \Delta^\perp - \Delta^\perp \widetilde{R} H_{\mathrm{int}} E_\xi E^\xi H_{\mathrm{int}} \widetilde{R}^* \Delta^\perp, \end{aligned}$$

and hence

$$\begin{aligned} F\Delta F^* =&\Delta^\perp \widetilde{R} H_{\text{int}}^2 \widetilde{R}^* \Delta^\perp - \underbrace{E^\xi H_{\text{int}} E_\xi}_{=0} E^\xi H_{\text{int}} E_\xi \otimes |e_0 \otimes \overline{e_0}\rangle \langle e_0 \otimes \overline{e_0}| \\ =&\Delta^\perp \widetilde{R} H_{\text{int}}^2 \widetilde{R}^* \Delta^\perp. \end{aligned}$$

Similarly, $F^*\Delta F = \Delta^\perp \widetilde{R} H_{\text{int}}^2 \widetilde{R}^* \Delta^\perp$ and thus (4.14) holds. □

**Remark 4.2.2.** Since the modification $m_R(U(h), h)$, where $U(h)$ is given by (4.10), converges in norm to $F$ defined as in (4.13), as $h \to 0^+$, we will still be able to apply the random walk approximation theorem, that is, Theorem 3.1.12 (bearing in mind that $\widehat{\mathsf{k}} = \mathsf{K} \otimes \overline{\mathsf{K}}$ and the distinguished unit vector $\eta = e_0 \otimes \overline{e_0}$) together with Lemma 3.1.13 and Lemma 3.1.14 to obtain that

$$\lim_{h \to 0^+} \sup_{t \in [0,T]} \|(X_t^{\widetilde{R}U(h)\widetilde{R}^*, h} - X_t^F)x\| = 0$$

for all $x \in \mathfrak{h} \otimes \mathcal{F}^{(\mathsf{K}\otimes\overline{\mathsf{K}})\ominus\mathbb{C}e_0\otimes\overline{e_0}}$ and $T \in \mathbb{R}_+$, where $(X_t^F)_{t\geqslant 0}$ is the unitary Markov-regular QS cocycle with generator $F$.

In [5, Proposition 8], Attal and Joye show that the limit cocycle obtained via their convergence theorem ([5, Theorem 7]) is quasifree (Definition 2.3.15), thus the driving noises form a representation of the relevant CCR algebra.
However, in our case the limit cocycles are defined on $\mathfrak{h} \otimes \mathcal{F}^{(\mathsf{K}\otimes\overline{\mathsf{K}})\ominus\mathbb{C}e_0\otimes\overline{e_0}}$ rather than on $\mathfrak{h} \otimes \mathcal{F}^{(\mathsf{k}\otimes\mathbb{C}\overline{e_0})\oplus(\mathbb{C}e_0\otimes\overline{\mathsf{k}})}$, where $\mathsf{k} := \mathsf{K} \ominus \mathbb{C}e_0$, how it was done in the quasifree case. To solve this problem let us do the following; first observe that we can decompose

$$\mathsf{K} \otimes \overline{\mathsf{K}} \cong \mathbb{C} \oplus (\mathsf{k} \oplus \overline{\mathsf{k}}) \oplus \mathsf{k} \otimes \overline{\mathsf{k}}, \quad \text{where } \mathsf{k} := \mathsf{K} \ominus \mathbb{C}e_0. \tag{4.15}$$

Secondly, if the generator of the limit cocycle has the form

$$F = \begin{bmatrix} F_0^0 & F_+^0 & 0 \\ F_0^+ & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \begin{bmatrix} \mathcal{B}(\mathfrak{h}) & \mathcal{B}(\mathfrak{h}\otimes(\mathsf{k}\oplus\overline{\mathsf{k}});\mathfrak{h}) & \mathcal{B}(\mathfrak{h}\otimes\mathsf{k}\otimes\overline{\mathsf{k}};\mathfrak{h}) \\ \mathcal{B}(\mathfrak{h};\mathfrak{h}\otimes(\mathsf{k}\oplus\overline{\mathsf{k}})) & \mathcal{B}(\mathfrak{h}\otimes(\mathsf{k}\oplus\overline{\mathsf{k}})) & \mathcal{B}(\mathfrak{h}\otimes\mathsf{k}\otimes\overline{\mathsf{k}};\mathfrak{h}\otimes(\mathsf{k}\oplus\overline{\mathsf{k}})) \\ \mathcal{B}(\mathfrak{h};\mathfrak{h}\otimes\mathsf{k}\otimes\overline{\mathsf{k}}) & \mathcal{B}(\mathfrak{h}\otimes(\mathsf{k}\oplus\overline{\mathsf{k}});\mathfrak{h}\otimes\mathsf{k}\otimes\overline{\mathsf{k}}) & \mathcal{B}(\mathfrak{h}\otimes\mathsf{k}\otimes\overline{\mathsf{k}}) \end{bmatrix} \tag{4.16}$$

according to this decomposition, then we can think about $F$ as an element of

$\mathcal{B}\big(\mathfrak{h} \otimes \big(\mathbb{C} \oplus (\mathsf{k} \oplus \overline{\mathsf{k}})\big)\big)$ with the following matrix form:

$$F = \begin{bmatrix} F_0^0 & F_+^0 \\ F_0^+ & 0 \end{bmatrix} \in \begin{bmatrix} \mathcal{B}(\mathfrak{h}) & \mathcal{B}(\mathfrak{h} \otimes (\mathsf{k} \oplus \overline{\mathsf{k}}); \mathfrak{h}) \\ \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes (\mathsf{k} \oplus \overline{\mathsf{k}})) & \mathcal{B}(\mathfrak{h} \otimes (\mathsf{k} \oplus \overline{\mathsf{k}})) \end{bmatrix}.$$

Hence, we can see that the QS cocycle generated by such $F$ will be defined on the desired space.

Furthermore, let us state the following necessary and sufficient condition to obtain the cocycle's generator to be of the form (4.16);

**Lemma 4.2.3.** *The operator $F$ given by (4.13) has the form (4.16) if and only if $E^{c \otimes \overline{d}} \widetilde{R} H_{\text{int}} E_\xi = 0$ for all $c, d \in \mathsf{k}$.*

*Proof.* First note that $F$ has the form (4.16) if and only if

$$\text{im}\left(\Delta \widetilde{R} H_{\text{int}} \widetilde{R}^* \Delta^\perp\right) \subset \mathfrak{h} \otimes (\mathsf{k} \oplus \overline{\mathsf{k}}).$$

Now, observe that $\text{im}\left(\Delta \widetilde{R} H_{\text{int}} \widetilde{R}^* \Delta^\perp\right) \subset \mathfrak{h} \otimes (\mathsf{k} \oplus \overline{\mathsf{k}})$ if and only if

$$(\widetilde{R} H_{\text{int}})(\mathfrak{h} \otimes \mathbb{C}\xi) \subset \mathsf{h} \otimes \left(\mathbb{C}e_0 \otimes \overline{e_0} \oplus (\mathsf{k} \oplus \overline{\mathsf{k}})\right),$$

because

$$\text{im}(\widetilde{R}\Delta^\perp) = \mathfrak{h} \otimes \mathbb{C}\xi \quad \text{and} \quad \Delta(\mathfrak{h} \otimes \mathbb{C}e_0 \otimes \overline{e_0}) = \{0\}.$$

Moreover, $(\widetilde{R} H_{\text{int}})(\mathfrak{h} \otimes \mathbb{C}\xi) \subset \mathsf{h} \otimes \left(\mathbb{C}e_0 \otimes \overline{e_0} \oplus (\mathsf{k} \oplus \overline{\mathsf{k}})\right)$ if and only if

$$\text{im}(\widetilde{R} H_{\text{int}} E_\xi) \perp \mathsf{h} \otimes (\mathsf{k} \otimes \overline{\mathsf{k}}).$$

Hence the required necessary and sufficient condition is that

$$E^{c \otimes \overline{d}} \widetilde{R} H_{\text{int}} E_\xi = 0$$

for all $c, d \in \mathsf{k}$. □

**Lemma 4.2.4.** *With $R$ defined as in Example 4.1.1 and $H_{\text{int}}$ self-adjoint, it holds* *that* $E^{\xi} H_{\text{int}} E_{\xi} = 0$ *and* $E^{c\otimes\overline{d}} \widetilde{R} H_{\text{int}} E_{\xi} = 0$ *for all* $c, d \in \mathsf{k}$ *if and only if*

$$H_{\text{int}} = \begin{bmatrix} A & -\frac{\alpha}{\beta}A & C^* & D^* \\ -\frac{\alpha}{\beta}A & \frac{\alpha^2}{\beta^2}A & G^* & -\frac{\alpha}{\beta}D^* \\ C & G & M & N^* \\ D & -\frac{\alpha}{\beta}D & N & P \end{bmatrix}, \tag{4.17}$$

where $\alpha = \sqrt{\gamma_0}$, $\beta = \sqrt{1-\gamma_0}$, $A$, $M$, $P$ are self-adjoint operators and $B$, $C$, $D$, $G$, $N$ are bounded operators defined according to the decomposition (4.4).

*Proof.* First assume that $H_{\text{int}}$ is of the form (4.17). Clearly,

$$E^{\xi} H_{\text{int}} E_{\xi} = E^{\xi} \begin{pmatrix} 0 \\ 0 \\ \alpha C + \beta G \\ 0 \end{pmatrix} = 0 \quad \text{and}$$

$$E^{c\otimes\overline{d}} \widetilde{R} H_{\text{int}} E_{\xi} = E^{c\otimes\overline{d}} \begin{pmatrix} 0 \\ 0 \\ \alpha C + \beta G \\ 0 \end{pmatrix} = 0.$$

Let

$$H_{\text{int}} = \begin{bmatrix} A & B^* & C^* & D^* \\ B & E & G^* & H^* \\ C & G & M & N^* \\ D & H & N & P \end{bmatrix}, \tag{4.18}$$

where $A$, $E$, $M$, $P$ are self-adjoint operators and $B$, $C$, $D$, $G$, $H$, $N$ are bounded operators defined according to the decomposition (4.4).

Now, if $E^{c\otimes\overline{d}} \widetilde{R} H_{\text{int}} E_{\xi} = 0$ for all $c, d \in \mathsf{k}$ then we obtain the following system of equations to solve

$$\begin{aligned} -\alpha\beta A + \alpha^2 B - \beta^2 B^* + \alpha\beta E &= 0, \\ \alpha D + \beta H &= 0. \end{aligned}$$

By regrouping the terms and using the fact that $E$ is self-adjoint we get

$$\begin{aligned} B &= B^*, \\ E &= A + \left(\frac{\beta}{\alpha} - \frac{\alpha}{\beta}\right) B, \\ H &= -\frac{\alpha}{\beta} D. \end{aligned}$$

Therefore, we can write the interaction Hamiltonian as

$$H_{\text{int}} = \begin{bmatrix} A & B & C^* & D^* \\ B & A + \left(\frac{\beta}{\alpha} - \frac{\alpha}{\beta}\right) B & G^* & -\frac{\alpha}{\beta} D^* \\ C & G & M & N^* \\ D & -\frac{\alpha}{\beta} D & N & P \end{bmatrix},$$

where $A$, $B$, $M$, $P$ are self-adjoint operators.
Moreover, if $E^{\xi} H_{\text{int}} E_{\xi} = 0$ holds then we obtain that

$$\alpha^2 A + \alpha\beta B + \beta^2 A + \frac{\beta^3}{\alpha} B = 0$$

and since $\alpha^2 + \beta^2 = 1$ then

$$\begin{aligned} A + \beta\Big(\alpha + \frac{\beta^2}{\alpha}\Big) B &= 0 \text{ and so} \\ B &= -\frac{\alpha}{\beta} A. \end{aligned}$$

Hence, $H_{\text{int}}$ has the form (4.17). □

Let us summarise the results in this section by stating the following proposition:

**Proposition 4.2.5.** *Let $U(h)$ be a unitary operator defined as in (4.10) for all*

*$h > 0$ and let $R$ be the rotation from Example 4.1.1. It is sufficient to consider*

$$H_{\mathrm{int}} = \begin{bmatrix} 0 & 0 & C^* & 0 \\ 0 & 0 & G^* & 0 \\ C & G & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.19}$$

*where $C, G \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes (\mathsf{k} \oplus \overline{\mathsf{k}}))$, instead of $H_{\mathrm{int}}$ defined in (4.17), to obtain that*

$$\lim_{h \to 0^+} \sup_{t \in [0,T]} \|(X_t^{\widetilde{R}U(h)\widetilde{R}^*, h} - X_t^F)x\| = 0$$

*for all $x \in \mathfrak{h} \otimes \mathcal{F}^{(\mathsf{K} \otimes \overline{\mathsf{K}}) \ominus \mathbb{C}e_0 \otimes \overline{e_0}}$ and $T \in \mathbb{R}_+$, where $(X_t^F)_{t \geqslant 0}$ is the unitary Markov-regular QS cocycle with generator $F$ of the form*

$$F = \begin{bmatrix} F_0^0 & F_+^0 & 0 \\ F_0^+ & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{4.20}$$

*where*

$$\begin{aligned} F_0^0 &= -\mathrm{i} H_{\mathrm{sys}} - \mathrm{i} \langle \xi, H_{\mathrm{par}} \xi \rangle I_{\mathfrak{h}} \\ &\quad - \frac{1}{2} \Big( \gamma_0 C^* C + \sqrt{\gamma_0(1-\gamma_0)} (C^* G + G^* C) + (1-\gamma_0) G^* G \Big), \\ F_+^0 &= -\mathrm{i} \Big( \sqrt{\gamma_0} C^* + \sqrt{1-\gamma_0} G^* \Big) \quad \textit{and} \\ F_0^+ &= -\mathrm{i} \Big( \sqrt{\gamma_0} C + \sqrt{1-\gamma_0} G \Big), \end{aligned}$$

*Proof.* According to (4.11)

$$m_R(U(h), h) \to F$$

in norm, as $h \to 0^+$, where $F$ is given by (4.13), if and only if

$$E^{\xi} H_{\mathrm{int}} E_{\xi} = 0.$$

Thus, if $E^{\xi} H_{\mathrm{int}} E_{\xi} = 0$ then Theorem 3.1.12 together with Lemma 3.1.13 and

Lemma 3.1.14 guarantee that

$$\lim_{h\to 0+}\sup_{t\in[0,T]}\|(X_t^{\widetilde{R}U(h)\widetilde{R}^*,h}-X_t^F)x\|=0$$

for all $x\in\mathfrak{h}\otimes\mathcal{F}^{(\mathsf{K}\otimes\overline{\mathsf{K}})\ominus\mathbb{C}e_0\otimes\overline{e_0}}$ and $T\in\mathbb{R}_+$.
By Lemma 4.2.3 we know that $F$ is of the form (4.16) if and only if

$$E^{c\otimes\overline{d}}\widetilde{R}H_{\mathrm{int}}E_\xi=0$$

for all $c,d\in\mathsf{k}$. By applying Lemma 4.2.4 we can write $H_{\mathrm{int}}$ explicitly in the matrix form (4.17). Now, let us calculate all non-zero entries of $F$ which as we know has the matrix form

$$F=\begin{bmatrix}F_0^0 & F_+^0 & 0\\ F_0^+ & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

for some $F_0^0\in\mathcal{B}(\mathfrak{h})$, $F_+^0\in\mathcal{B}(\mathfrak{h}\otimes(\mathsf{k}\oplus\overline{\mathsf{k}});\mathfrak{h})$ and $F_0^+\in\mathcal{B}(\mathfrak{h};\mathfrak{h}\otimes(\mathsf{k}\oplus\overline{\mathsf{k}}))$.
Top left corner of the matrix is obtained as follows

$$\begin{aligned}F_0^0 =&-E^{e_0\otimes\overline{e_0}}\Delta^\perp\widetilde{R}\left(\mathrm{i}H_{\mathrm{sys}}\otimes I_{\mathsf{K}\otimes\overline{\mathsf{K}}}+\mathrm{i}I_{\mathfrak{h}}\otimes H_{\mathrm{par}}+\frac{1}{2}H_{\mathrm{int}}^2\right)\widetilde{R}^*\Delta^\perp E_{e_0\otimes\overline{e_0}}\\
=&-\mathrm{i}H_{\mathrm{sys}}-\mathrm{i}\langle\xi,H_{\mathrm{par}}\xi\rangle I_{\mathfrak{h}}-\frac{1}{2}E^\xi H_{\mathrm{int}}^2E_\xi\\
=&-\mathrm{i}H_{\mathrm{sys}}-\mathrm{i}\langle\xi,H_{\mathrm{par}}\xi\rangle I_{\mathfrak{h}}\\
&-\frac{1}{2}\left(\gamma_0C^*C+\sqrt{\gamma_0(1-\gamma_0)}(C^*G+G^*C)+(1-\gamma_0)G^*G\right).\end{aligned}$$

Now let $c,d\in\mathsf{k}$, we arrive at

$$\begin{aligned}&F_+^0E_{c+\overline{d}}\\
&=-\mathrm{i}E^{e_0\otimes\overline{e_0}}\Delta^\perp\widetilde{R}H_{\mathrm{int}}\widetilde{R}^*\Delta E_{c+\overline{d}}\\
&=-\mathrm{i}\left(\sqrt{\gamma_0}C^*+\sqrt{1-\gamma_0}G^*\right)E_{c+\overline{d}}.\end{aligned}$$

Therefore,

$$F_+^0 = -\mathrm{i}\left(\sqrt{\gamma_0}C^* + \sqrt{1-\gamma_0}G^*\right)$$

Similarly, we get that $F_0^+ = -\mathrm{i}\left(\sqrt{\gamma_0}C + \sqrt{1-\gamma_0}G\right)$.
Hence, it was sufficient to consider $H_{\text{int}}$ of the form

$$H_{\text{int}} = \begin{bmatrix} 0 & 0 & C^* & 0 \\ 0 & 0 & G^* & 0 \\ C & G & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.21}$$

where $C, G \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes (\mathsf{k} \oplus \overline{\mathsf{k}}))$. □

The following example is the generalisation of [5, Theorem 7].

**Example 4.2.6.** Now, let us again consider a repeated quantum interactions model. As we mentioned earlier, we assume that all the particles from an infinite chain are in a normal faithful state $\omega$, induced by a density matrix $\rho$ given by (4.1). Therefore, we consider the same total Hamiltonian as in Example 3.2.5, that is,

$$H_{\mathsf{tot}}(h) := H_{\mathsf{sys}} \otimes I_{\mathsf{K}} + I_{\mathfrak{h}} \otimes H_{\mathsf{par}} + \frac{1}{\sqrt{h}} H_{\mathsf{int}},$$

where $H_{\mathsf{sys}} \in \mathcal{B}(\mathfrak{h})$, $H_{\mathsf{par}} \in \mathcal{B}(\mathsf{K})$, and $H_{\mathsf{int}} \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{K})$. The associated unitary evolution is given by

$$U(h) := \exp\left(-\mathrm{i}hH_{\mathsf{tot}}(h)\right) = \exp\left(-\mathrm{i}\sqrt{h}H_{\mathsf{int}} + h\left(-\mathrm{i}H_{\mathsf{sys}} \otimes I_{\mathsf{K}} - \mathrm{i}I_{\mathfrak{h}} \otimes H_{\mathsf{par}}\right)\right).$$

However, to include the state of each particle in the interaction we will involve the GNS representation $\pi$ defined in (4.2) and the unitary operator $R \in \mathcal{B}(\mathsf{K} \otimes \overline{\mathsf{K}})$ which maps $\xi \mapsto e_0 \otimes \overline{e_0}$.
Now observe that

$$\widetilde{\pi}(U(h)) = \exp\left(-\mathrm{i}\sqrt{h}(H_{\mathsf{int}} \otimes I_{\overline{\mathsf{K}}} + h\left(-\mathrm{i}H_{\mathsf{sys}} \otimes I_{\mathsf{K}\otimes\overline{\mathsf{K}}} - \mathrm{i}I_{\mathfrak{h}} \otimes H_{\mathsf{par}} \otimes I_{\overline{\mathsf{K}}}\right)\right)$$

where $\widetilde{\pi} = \mathrm{id}_{\mathcal{B}(\mathfrak{h})} \,\overline{\otimes}\, \pi$ is the ampliation of the representation $\pi$.

Thus, by Lemma 4.1.5 the norm-limit of

$$\lim_{h\to 0^+} m_R(\widetilde{\pi}(U(h)), h)$$

exists if $E^{\xi}(H_{\mathsf{int}} \otimes I_{\overline{\mathsf{K}}})E_{\xi} = 0$ and in this case

$$\begin{aligned} m_R(\widetilde{\pi}(U(h)), h) \overset{h\to 0^+}{\to} & -\Delta^{\perp}\widetilde{R}\left(\mathrm{i}H_{\mathsf{sys}} \otimes I_{\mathsf{K}\otimes\overline{\mathsf{K}}} + \mathrm{i}I_{\mathfrak{h}} \otimes H_{\mathsf{par}} \otimes I_{\overline{\mathsf{K}}} + \frac{1}{2}H_{\mathsf{int}}^2 \otimes I_{\overline{\mathsf{K}}}\right)\widetilde{R}^*\Delta^{\perp} \\ & -\mathrm{i}\Delta^{\perp}\widetilde{R}\,(H_{\mathsf{int}} \otimes I_{\overline{\mathsf{K}}})\,\widetilde{R}^*\Delta - \mathrm{i}\Delta\widetilde{R}\,(H_{\mathsf{int}} \otimes I_{\overline{\mathsf{K}}})\,\widetilde{R}^*\Delta^{\perp} \\ =: \; & F. \end{aligned} \tag{4.22}$$

Now assume that $H_{\mathsf{int}} = \begin{bmatrix} A & V^* \\ V & B \end{bmatrix}$ for some $V \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$ and self-adjoint operators $A \in \mathcal{B}(\mathfrak{h})$, $B \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{k})$, where the matrix decomposition is due to the identification $\mathsf{K} = \mathbb{C}e_0 \oplus \mathsf{k}$.
Then, according to the decomposition (4.15),

$$H_{\mathsf{int}} \otimes I_{\overline{\mathsf{K}}} = \begin{bmatrix} A & [V^* \;\; 0] & 0 \\ \begin{bmatrix} V \\ 0 \end{bmatrix} & \begin{bmatrix} B & 0 \\ 0 & A\otimes I_{\overline{\mathsf{k}}} \end{bmatrix} & \begin{bmatrix} 0 \\ V^*\otimes I_{\overline{\mathsf{k}}} \end{bmatrix} \\ 0 & [0 \;\; V\otimes I_{\overline{\mathsf{k}}}] & B\otimes I_{\overline{\mathsf{k}}} \end{bmatrix}.$$

Caution: this is not the same $4\times 4$ decomposition as in (4.4).
The expression $E^{\xi}(H_{\mathsf{int}} \otimes I_{\overline{\mathsf{K}}})E_{\xi}$ can be written as

$$\begin{aligned} & E^{\alpha e_0\otimes\overline{e_0}+\beta\widetilde{\xi}} \begin{bmatrix} A & [V^* \;\; 0] & 0 \\ \begin{bmatrix} V \\ 0 \end{bmatrix} & \begin{bmatrix} B & 0 \\ 0 & A\otimes I_{\overline{\mathsf{k}}} \end{bmatrix} & \begin{bmatrix} 0 \\ V^*\otimes I_{\overline{\mathsf{k}}} \end{bmatrix} \\ 0 & [0 \;\; V\otimes I_{\overline{\mathsf{k}}}] & B\otimes I_{\overline{\mathsf{k}}} \end{bmatrix} E_{\alpha e_0\otimes\overline{e_0}+\beta\widetilde{\xi}} \\ = & \alpha^2 A + \beta^2 E_{\widetilde{\xi}}(B \otimes I_{\overline{\mathsf{k}}})E_{\widetilde{\xi}}, \end{aligned}$$

where $\alpha = \sqrt{\gamma_0}$ and $\beta = \sqrt{1-\gamma_0}$. Thus, $E^{\xi}(H_{\mathsf{int}} \otimes I_{\overline{\mathsf{K}}})E_{\xi} = 0$ if and only if

$$A = -\frac{\beta^2}{\alpha^2}E^{\widetilde{\xi}}(B \otimes I_{\mathsf{k}})E_{\widetilde{\xi}}$$

Let $R$ be as in Example 4.1.1; we can write it according to the decomposition

(4.15) as follows:

$$R = \begin{bmatrix} \alpha & 0 & \beta \left\langle \tilde{\xi} \right| \\ 0 & I_{\mathsf{k}\oplus\bar{\mathsf{k}}} & 0 \\ -\beta \left| \tilde{\xi} \right\rangle & 0 & \alpha I_{\mathbb{C}\tilde{\xi}} \oplus I_{(\mathsf{k}\otimes\bar{\mathsf{k}})\ominus\mathbb{C}\tilde{\xi}} \end{bmatrix}. \tag{4.23}$$

In particular, $I_{\mathbb{C}\tilde{\xi}} = \left|\tilde{\xi}\right\rangle\left\langle\tilde{\xi}\right|$, and so

$$\widetilde{R} = \begin{bmatrix} \alpha I_{\mathfrak{h}} & 0 & \beta E^{\tilde{\xi}} \\ 0 & I_{\mathfrak{h}\otimes(\mathsf{k}\oplus\bar{\mathsf{k}})} & 0 \\ -\beta E_{\tilde{\xi}} & 0 & \alpha I_{\mathfrak{h}\otimes\mathbb{C}\tilde{\xi}} \oplus I_{\mathfrak{h}\otimes((\mathsf{k}\otimes\bar{\mathsf{k}})\ominus\mathbb{C}\tilde{\xi})} \end{bmatrix}. \tag{4.24}$$

According to Lemma 4.2.3 the operator $F$ given by (4.22) has the form (4.16) if and only if $E^{c\otimes\bar{d}}\widetilde{R}(H_{\text{int}} \otimes I_{\overline{\mathsf{K}}})E_{\xi} = 0$ for all $c, d \in \mathsf{k}$.
Furthermore,

$$\begin{aligned} &\widetilde{R}(H_{\text{int}} \otimes I_{\overline{\mathsf{K}}})E_{\xi} \\ &= \begin{bmatrix} \alpha^2 A + \beta^2 E^{\tilde{\xi}}(B \otimes I_{\mathsf{k}})E_{\tilde{\xi}} \\ \alpha V \\ \beta(V^* \otimes I_{\bar{\mathsf{k}}})E_{\tilde{\xi}} \\ -\alpha\beta E_{\tilde{\xi}}A + \beta\left(\alpha I_{\mathfrak{h}\otimes\mathbb{C}\tilde{\xi}} \oplus I_{\mathfrak{h}\otimes(\mathsf{k}\otimes\bar{\mathsf{k}})\ominus\mathbb{C}\tilde{\xi}}\right)(B \otimes I_{\bar{\mathsf{k}}})E_{\tilde{\xi}} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \alpha V \\ \beta(V^* \otimes I_{\bar{\mathsf{k}}}) \\ -\alpha\beta E_{\tilde{\xi}}A + \beta\left(\alpha I_{\mathfrak{h}\otimes\mathbb{C}\tilde{\xi}} \oplus I_{\mathfrak{h}\otimes(\mathsf{k}\otimes\bar{\mathsf{k}})\ominus\mathbb{C}\tilde{\xi}}\right)(B \otimes I_{\bar{\mathsf{k}}})E_{\tilde{\xi}} \end{bmatrix}. \end{aligned}$$

To obtain the condition $E^{c\otimes\bar{d}}\widetilde{R}(H_{\text{int}} \otimes I_{\overline{\mathsf{K}}})E_{\xi} = 0$ we have to check when

$$-\alpha\beta E_{\tilde{\xi}}A + \beta\left(\alpha I_{\mathfrak{h}\otimes\mathbb{C}\tilde{\xi}} \oplus I_{\mathfrak{h}\otimes(\mathsf{k}\otimes\bar{\mathsf{k}})\ominus\mathbb{C}\tilde{\xi}}\right)(B \otimes I_{\bar{\mathsf{k}}})E_{\tilde{\xi}} = 0.$$

Observe that we can write $(B \otimes I_{\bar{\mathsf{k}}})E_{\tilde{\xi}}$ in the following way:

$$E^{\tilde{\xi}}(B \otimes I_{\bar{\mathsf{k}}})E_{\tilde{\xi}} + (B \otimes I_{\bar{\mathsf{k}}})E_{\tilde{\xi}} - E_{\tilde{\xi}}E^{\tilde{\xi}}(B \otimes I_{\bar{\mathsf{k}}})E_{\tilde{\xi}}, \tag{4.25}$$

which splits it according to the direct sum decomposition

$$\mathfrak{h} \otimes \mathbb{C}\tilde{\xi} \oplus \mathfrak{h} \otimes (\mathsf{k} \otimes \overline{\mathsf{k}}) \ominus \mathbb{C}\tilde{\xi}.$$

Since $-\alpha\beta E_{\tilde{\xi}} A$ has its values in $\mathfrak{h} \otimes \mathbb{C}\tilde{\xi}$ then, according to (4.25),

$$(B \otimes I_{\overline{\mathsf{k}}})E_{\tilde{\xi}} - E_{\tilde{\xi}} E^{\tilde{\xi}} (B \otimes I_{\overline{\mathsf{k}}})E_{\tilde{\xi}} = 0,$$

but this is true if $B = 0$ or the image of $(B \otimes I_{\overline{\mathsf{k}}})E_{\tilde{\xi}}$ lies in $\mathfrak{h} \otimes \mathbb{C}\tilde{\xi}$. If $B = 0$ then $A = 0$, thus we consider the second case, that is,

$$\operatorname{im}\big((B \otimes I_{\overline{\mathsf{k}}})E_{\tilde{\xi}}\big) \subset \mathfrak{h} \otimes \mathbb{C}\tilde{\xi}.$$

Then we have to check when

$$-\alpha\beta E_{\tilde{\xi}} A + \alpha\beta E_{\tilde{\xi}} E^{\tilde{\xi}} (B \otimes I_{\mathsf{k}})E_{\tilde{\xi}} = 0,$$

but this holds if and only if $A = E^{\tilde{\xi}}(B \otimes I_{\mathsf{k}})E_{\tilde{\xi}}$ and, since by (4.2.6) we know that $A = -\frac{\beta^2}{\alpha^2} E^{\tilde{\xi}}(B \otimes I_{\mathsf{k}})E_{\tilde{\xi}}$, then $A = B = 0$.

Now note that if $H_{\mathsf{int}} = \begin{bmatrix} 0 & V^* \\ V & 0 \end{bmatrix}$ for some $V \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$ then we can write $H_{\mathsf{int}} \otimes I_{\overline{\mathsf{K}}}$ with respect to the decomposition (4.4) as follows

$$H_{\mathsf{int}} \otimes I_{\overline{\mathsf{K}}} = \begin{bmatrix} 0 & 0 & C^* & 0 \\ 0 & 0 & G^* & 0 \\ C & G & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $C = \begin{bmatrix} V \\ 0 \end{bmatrix}$ and $G = \begin{bmatrix} 0 \\ (V^* \otimes I_{\overline{\mathsf{k}}})E_{\tilde{\xi}} \end{bmatrix}$.

Hence, Proposition 4.2.5 leads to the following conclusion

$$\lim_{h \to 0^+} \sup_{t \in [0,T]} \|(X_t^{\tilde{R}\tilde{\pi}(U(h))\tilde{R}^*, h} - X_t^F)x\| = 0$$

for all $x \in \mathfrak{h} \otimes \mathcal{F}^{(\mathsf{K}\otimes\overline{\mathsf{K}})\ominus\mathbb{C}e_0\otimes\overline{e_0}}$ and $T \in \mathbb{R}_+$, where $(X_t^F)_{t\geqslant 0}$ is the unitary Markov-regular QS cocycle with generator $F$ having the form (4.16), where

$$\begin{aligned}
F_0^0 &= -\mathrm{i}H_{\mathrm{sys}} - \mathrm{i}\omega(H_{\mathrm{par}})I_{\mathfrak{h}} \\
&\quad - \frac{1}{2}\Big(\gamma_0 V^*V + (1-\gamma_0)E_{\tilde{\xi}}(VV^*\otimes I_{\overline{\mathsf{k}}})E_{\tilde{\xi}}\Big), \\
F_+^0 &= -\mathrm{i}\Big(\sqrt{\gamma_0}\begin{bmatrix} V^* & 0 \end{bmatrix} + \sqrt{1-\gamma_0}\begin{bmatrix} 0 & E^{\tilde{\xi}}(V\otimes I_{\overline{\mathsf{k}}}) \end{bmatrix}\Big), \\
F_0^+ &= -\mathrm{i}\left(\sqrt{\gamma_0}\begin{bmatrix} V \\ 0 \end{bmatrix} + \sqrt{1-\gamma_0}\begin{bmatrix} 0 \\ (V^*\otimes I_{\overline{\mathsf{k}}})E_{\tilde{\xi}} \end{bmatrix}\right).
\end{aligned}$$

**Example 4.2.7** (Bipartite model)**.** In this example we consider a generalised bipartite model (recall Example 3.2.9). Thus, according to Example 3.2.9 let us consider

- $H_{\mathsf{tot}}^{(1)}(h) := H_{\mathsf{sys}}^{(1)} \otimes I_{\mathfrak{h}_2\otimes\mathsf{K}\otimes\overline{\mathsf{K}}} + I_{\mathfrak{h}} \otimes H_{\mathsf{par}} + \frac{1}{\sqrt{h}}H_{\mathsf{int}}^{(1)}$,
- $H_{\mathsf{tot}}^{(2)}(h) := I_{\mathfrak{h}_1} \otimes H_{\mathsf{sys}}^{(2)} \otimes I_{\mathsf{K}\otimes\overline{\mathsf{K}}} + I_{\mathfrak{h}} \otimes H_{\mathsf{par}} + \frac{1}{\sqrt{h}}H_{\mathsf{int}}^{(2)}$,

where

- $H_{\mathsf{sys}}^{(1)} \in \mathcal{B}(\mathfrak{h}_1)$, $H_{\mathsf{par}} \in \mathcal{B}(\mathsf{K}\otimes\overline{\mathsf{K}})$, and
$$H_{\mathrm{int}}^{(1)} = \begin{bmatrix} 0 & 0 & M^* & 0 \\ 0 & 0 & P^* & 0 \\ M & P & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
where $M, P \in \mathcal{B}(\mathfrak{h}; \mathfrak{h}\otimes(\mathsf{k}\oplus\overline{\mathsf{k}}))$ are such that on $\mathfrak{h}_2$ they act as $I_{\mathfrak{h}_2}$, and the matrix decomposition is with respect to (4.4),
- $H_{\mathsf{sys}}^{(2)} \in \mathcal{B}(\mathfrak{h}_2)$ and
$$H_{\mathrm{int}}^{(2)} = \begin{bmatrix} 0 & 0 & N^* & 0 \\ 0 & 0 & Q^* & 0 \\ N & Q & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
where $N, Q \in \mathcal{B}(\mathfrak{h}; \mathfrak{h}\otimes(\mathsf{k}\oplus\overline{\mathsf{k}}))$ are such that on $\mathfrak{h}_1$ they act as $I_{\mathfrak{h}_1}$.

The associated unitary evolutions are given by

$$U_1(h) := \exp\left(-\mathrm{i}hH^{(1)}_{\mathsf{tot}}(h)\right) \quad \text{and} \quad U_2(h) := \exp\left(-\mathrm{i}hH^{(2)}_{\mathsf{tot}}(h)\right).$$

By (4.11) we obtain

$$\begin{aligned} m_R(U_1(h),h) &\overset{h\to 0^+}{\to} -\Delta^\perp \widetilde{R}\left(\mathrm{i}H^{(1)}_{\mathsf{sys}}\otimes I_{\mathfrak{h}_2\otimes\mathsf{K}\otimes\overline{\mathsf{K}}} + \mathrm{i}I_{\mathfrak{h}}\otimes H_{\mathsf{par}} + \frac{1}{2}(H^{(1)}_{\mathsf{int}})^2\right)\widetilde{R}^*\Delta^\perp \\ &\quad -\mathrm{i}\Delta^\perp\widetilde{R}H^{(1)}_{\mathsf{int}}\widetilde{R}^*\Delta - \mathrm{i}\Delta\widetilde{R}H^{(1)}_{\mathsf{int}}\widetilde{R}^*\Delta^\perp \\ &=: F \\ m_R(U_2(h),h) &\overset{h\to 0^+}{\to} -\Delta^\perp \widetilde{R}\left(\mathrm{i}I_{\mathfrak{h}_1}\otimes H^{(2)}_{\mathsf{sys}}\otimes I_{\mathsf{K}\otimes\overline{\mathsf{K}}} + \mathrm{i}I_{\mathfrak{h}}\otimes H_{\mathsf{par}} + \frac{1}{2}(H^{(2)}_{\mathsf{int}})^2\right)\widetilde{R}^*\Delta^\perp \\ &\quad -\mathrm{i}\Delta^\perp\widetilde{R}H^{(2)}_{\mathsf{int}}\widetilde{R}^*\Delta - \mathrm{i}\Delta\widetilde{R}H^{(2)}_{\mathsf{int}}\widetilde{R}^*\Delta^\perp \\ &=: G. \end{aligned}$$

By Proposition 3.1.16 the random walk $X_t^{\widetilde{R}U_2(h)U_1(h)\widetilde{R}^*}$ converges (in the sense as in Proposition 3.1.16 ) to the Markov-regular QS cocycle with generator $F+G+F\Delta G$. It is easy to check that $F+G+G\Delta F$ satisfies the conditions (1.23) and therefore it generates a unitary cocycle. Lemma 3.1.13 and Lemma 3.1.14 yield that the convergence of the random walk $X_t^{\widetilde{R}U_2(h)U_1(h)\widetilde{R}^*}$ is in particular a strong convergence, locally uniform in time $t$. Moreover,

$$G\Delta F = -\Delta^\perp\widetilde{R}H^{(2)}_{\mathsf{int}}\widetilde{R}^*\Delta\widetilde{R}H^{(1)}_{\mathsf{int}}\widetilde{R}^*\Delta^\perp.$$

Now, let $R$ be the rotation from Example 4.1.1 then the generator is

$$F+G-\begin{bmatrix} \gamma_0 N^*M+\sqrt{\gamma_0(1-\gamma_0)}Q^*M+\sqrt{\gamma_0(1-\gamma_0)}N^*P+(1-\gamma_0)Q^*P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{4.26}$$

where the matrix form is taken with respect to (4.15).

According to [10, 5 Thermal Environment] it is natural consider the bipartite model, which we described in Example 3.2.9, in the setting when each of the particles in an infinite chain are in a normal faithful state $\omega$, induced by a density matrix $\rho$ given by (4.1). The working below gives an appropriate description;

Let

- $H_{\mathsf{tot}}^{(1)}(h) := H_{\mathsf{sys}}^{(1)} \otimes I_{\mathfrak{h}_2 \otimes \mathsf{K}} + I_{\mathfrak{h}} \otimes H_{\mathsf{par}} + \frac{1}{\sqrt{h}} H_{\mathsf{int}}^{(1)}$,
- $H_{\mathsf{tot}}^{(2)}(h) := I_{\mathfrak{h}_1} \otimes H_{\mathsf{sys}}^{(2)} \otimes I_{\mathsf{K}} + I_{\mathfrak{h}} \otimes H_{\mathsf{par}} + \frac{1}{\sqrt{h}} H_{\mathsf{int}}^{(2)}$,

where

- $H_{\mathsf{sys}}^{(1)} \in \mathcal{B}(\mathfrak{h}_1)$, $H_{\mathsf{par}} \in \mathcal{B}(\mathsf{K})$, and $H_{\mathsf{int}}^{(1)} = \begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix}$ for some $M \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$ such that $M$ acts on $\mathfrak{h}_2$ as $I_{\mathfrak{h}_2}$,
- $H_{\mathsf{sys}}^{(2)} \in \mathcal{B}(\mathfrak{h}_2)$, and $H_{\mathsf{int}}^{(2)} = \begin{bmatrix} 0 & N^* \\ N & 0 \end{bmatrix}$ for some $N \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$ such that $N$ acts on $\mathfrak{h}_1$ as $I_{\mathfrak{h}_1}$.

The associated unitary evolutions are given by

$$U_1(h) := \exp\left(-\mathrm{i}h H_{\mathsf{tot}}^{(1)}(h)\right) \quad \text{and} \quad U_2(h) := \exp\left(-\mathrm{i}h H_{\mathsf{tot}}^{(2)}(h)\right)$$

we obtain that the random walk $X_t^{\widetilde{R}\widetilde{\pi}(U_2(h))\widetilde{\pi}(U_1(h))\widetilde{R}^*}$ converges in the strong operator topology to the unitary cocycle with the generator given by

$$F + G - \Delta^{\perp} \widetilde{R}(H_{\mathsf{int}}^{(2)} \otimes I_{\overline{\mathsf{K}}}) \widetilde{R}^* \Delta \widetilde{R}(H_{\mathsf{int}}^{(1)} \otimes I_{\overline{\mathsf{K}}}) \widetilde{R}^* \Delta^{\perp},$$

where

$$\begin{aligned} F = &- \Delta^{\perp} \widetilde{R} \left( \mathrm{i} H_{\mathsf{sys}}^{(1)} \otimes I_{\mathfrak{h}_2 \otimes \mathsf{K} \otimes \overline{\mathsf{K}}} + \mathrm{i} I_{\mathfrak{h}} \otimes H_{\mathsf{par}} \otimes I_{\overline{\mathsf{K}}} + \frac{1}{2} (H_{\mathsf{int}}^{(1)} \otimes I_{\overline{\mathsf{K}}})^2 \right) \widetilde{R}^* \Delta^{\perp} \\ &- \mathrm{i} \Delta^{\perp} \widetilde{R}(H_{\mathsf{int}}^{(1)} \otimes I_{\overline{\mathsf{K}}}) \widetilde{R}^* \Delta - \mathrm{i} \Delta \widetilde{R}(H_{\mathsf{int}}^{(1)} \otimes I_{\overline{\mathsf{K}}}) \widetilde{R}^* \Delta^{\perp}, \\ G = &- \Delta^{\perp} \widetilde{R} \left( \mathrm{i} I_{\mathfrak{h}_1} \otimes H_{\mathsf{sys}}^{(2)} \otimes I_{\mathsf{K} \otimes \overline{\mathsf{K}}} + \mathrm{i} I_{\mathfrak{h}} \otimes H_{\mathsf{par}} \otimes I_{\overline{\mathsf{K}}} + \frac{1}{2} (H_{\mathsf{int}}^{(2)} \otimes I_{\overline{\mathsf{K}}})^2 \right) \widetilde{R}^* \Delta^{\perp} \\ &- \mathrm{i} \Delta^{\perp} \widetilde{R}(H_{\mathsf{int}}^{(2)} \otimes I_{\overline{\mathsf{K}}}) \widetilde{R}^* \Delta - \mathrm{i} \Delta \widetilde{R}(H_{\mathsf{int}}^{(2)} \otimes I_{\overline{\mathsf{K}}}) \widetilde{R}^* \Delta^{\perp}. \end{aligned}$$

When we choose $R$ to be the rotation from Example 4.1.1 then the generator becomes

$$F + G - \begin{bmatrix} \gamma_0 N^* M + (1-\gamma_0) E^{\widetilde{\xi}} N M^* \otimes I_{\overline{\mathsf{k}}} E_{\widetilde{\xi}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{4.27}$$

where the matrix form is with respect to the decomposition (4.15).

## Low density limits

Here, we again discuss repeated quantum interaction model when the incoming particles are in the same faithful normal state. However, in contrast to weak coupling limits we let this state depend on $h$.

The interaction Hamiltonian together with the scaling in Example 3.2.6 are typical for the density limit [1]. The density limit is usually considered when an interaction corresponds to the preservation term (generalised number operator). For more details on a density limit including the physical interpretation we refer the reader to [1]. Here, we scale the interaction Hamiltonian in the same way as in Example 3.2.5, therefore we expect that it will correspond to the density limit. Let $R$ be the rotation from Example 4.1.1, that is,

$$R := \begin{bmatrix} \alpha & \beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \in \mathcal{B}(\mathsf{K} \otimes \overline{\mathsf{K}}),$$

where $\alpha = \sqrt{\gamma_0}$ and $\beta = \sqrt{1-\gamma_0}$ and the matrix decomposition is taken with respect to (4.4). Denote $\widetilde{R} := I_{\mathfrak{h}} \otimes R$.

**Theorem 4.2.8.** *For $h > 0$, let*

$$G(h) = \exp(A + hC),$$

*where $A, C \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{K} \otimes \overline{\mathsf{K}})$. If*

$$\beta^2 = \sum_{j=1}^{N} \gamma_j = o(h) \quad as \quad h \to 0^+ \tag{4.28}$$

*and*

$$\Delta^{\perp} A = 0 = A \Delta^{\perp} \tag{4.29}$$

*then*

$$m_R(G(h),h) \to \Delta(e^A - I)\Delta + \Delta^\perp C \Delta^\perp \tag{4.30}$$

*in norm, as* $h \to 0^+$.

*Proof.* First note that Lemma 3.2.1 implies

$$\begin{aligned} m_R(G(h),h) =& \frac{1}{h}\Delta^\perp \widetilde{R}(e^A - I)\widetilde{R}^*\Delta^\perp \\ &+ \frac{1}{\sqrt{h}}\Delta^\perp \widetilde{R}(e^A - I)\widetilde{R}^*\Delta + \frac{1}{\sqrt{h}}\Delta \widetilde{R}(e^A - I)\widetilde{R}^*\Delta^\perp \\ &+ \Delta \widetilde{R}(e^A - I)\widetilde{R}\Delta + \Delta^\perp \widetilde{R} f(A,C)\widetilde{R}^*\Delta^\perp \\ &+ o(1) \quad \text{as } h \to 0^+, \end{aligned}$$

where $f(A,C) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{j=0}^{n-1} A^j C A^{n-1-j}$.

Now note that

$$\Delta^\perp \widetilde{R} = \Delta^\perp \left( \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \oplus I \right) = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} \oplus 0$$

and since $\beta^2 = o(h)$ as $h \to 0^+$ then

$$\frac{1}{\sqrt{h}}(1-\alpha) = \frac{1}{\sqrt{h}}(1 - \sqrt{1-\beta^2}) = \frac{1}{\sqrt{h}}(1 - \sqrt{1 - o(h)}) = o(\sqrt{h})$$

and $\frac{1}{\sqrt{h}}\beta = o(1)$ as $h \to 0^+$.
Hence

$$\frac{1}{\sqrt{h}}(\Delta^\perp - \Delta^\perp \widetilde{R}) \to 0$$

in norm, as $h \to 0^+$. Furthermore, since $\widetilde{R} - I$ is a normal operator, by the spectral radius formula, its norm equals

$$\begin{aligned}\|\widetilde{R}-I\| &= \max\{|\lambda|: \lambda \in \sigma(\widetilde{R}-I\} \\ &= \max\{|\lambda|: (\lambda-\alpha_1)(\lambda-\alpha+1)+\beta^2=0\} \\ &= \max|\alpha-1\pm \mathrm{i}\beta| \\ &= \sqrt{(\alpha-1)^2+\beta^2} \\ &= 2(1-\alpha).\end{aligned}$$

In particular, if (4.28) holds then $\widetilde{R} \to I$ in norm, as $h \to 0^+$.

Moreover, if (4.29) holds then

$$\Delta^\perp(e^A - I) = 0 = (e^A - I)\Delta^\perp$$

and, if (4.28) holds as well, we obtain

$$\lim_{h\to 0^+} \frac{1}{h}\Delta^\perp \widetilde{R}(e^A - I)\widetilde{R}^*\Delta^\perp = \lim_{h\to 0^+} \big(\frac{1}{\sqrt{h}}\Delta^\perp\big)(e^A-I)\big(\frac{1}{\sqrt{h}}\Delta^\perp\big) = 0;$$

similarly,

$$\lim_{h\to 0^+} \frac{1}{\sqrt{h}}\Delta^\perp \widetilde{R}(e^A - I)\widetilde{R}^*\Delta = \lim_{h\to 0^+} \frac{1}{\sqrt{h}}\Delta^\perp(e^A - I)\widetilde{R}^*\Delta = 0$$

and

$$\lim_{h\to 0^+} \frac{1}{\sqrt{h}}\Delta \widetilde{R}(e^A - I)\widetilde{R}^*\Delta^\perp = \lim_{h\to 0^+} \Delta(e^A - I)\widetilde{R}^*\big(\frac{1}{\sqrt{h}}\Delta^\perp\big) = 0.$$

Thus, under assumptions (4.28) and (4.29),

$$m_R(G(h),h) \to \Delta(e^A - I)\Delta + \Delta^\perp f(A,C)\Delta^\perp = \Delta(e^A - I)\Delta + \Delta^\perp C\Delta^\perp$$

in norm, as $h \to 0^+$. □

Define

$$H_{\mathrm{tot}}(h) := H_{\mathrm{sys}} \otimes I_{\mathsf{K}\otimes\overline{\mathsf{K}}} + I_{\mathfrak{h}} \otimes H_{\mathrm{par}} + \frac{1}{h}H_{\mathrm{int}}$$

for all $h > 0$, where $H_{\mathrm{sys}} \in \mathcal{B}(\mathfrak{h})$, $H_{\mathrm{par}} \in \mathcal{B}(\mathsf{K}\otimes\overline{\mathsf{K}})$ and $H_{\mathrm{int}} \in \mathcal{B}(\mathfrak{h}\otimes\mathsf{K}\otimes\overline{\mathsf{K}})$ are self-adjoint.

The associated unitary evolution is given by

$$U(h) := \exp\left(-\mathrm{i}hH_{\mathrm{tot}}(h)\right) = \exp\left(-\mathrm{i}h\left(H_{\mathrm{sys}} \otimes I_{\mathsf{K}\otimes\overline{\mathsf{K}}} + I_{\mathfrak{h}} \otimes H_{\mathrm{par}}\right) - \mathrm{i}H_{\mathrm{int}}\right).$$

**Corollary 4.2.9.** *We have*

$$m_R(U(h),h) \to -\mathrm{i}\Delta^{\perp}\left(H_{\mathrm{sys}} \otimes I_{\mathsf{K}\otimes\overline{\mathsf{K}}} + I_{\mathfrak{h}} \otimes H_{\mathrm{par}}\right)\Delta^{\perp} + \Delta(\exp(-\mathrm{i}H_{\mathrm{int}}) - I)\Delta$$

*in norm as* $h \to 0^+$ *if*

$$\beta^2 = \sum_{j=1}^{N} \gamma_j = o(h) \quad as \quad h \to 0^+ \tag{4.31}$$

*and*

$$\Delta^{\perp}H_{\mathrm{int}} = 0 = H_{\mathrm{int}}\Delta^{\perp}. \tag{4.32}$$

*Proof.* It is an immediate consequence from the preceding theorem with $A = -\mathrm{i}H_{\mathrm{int}}$ and $C = -\mathrm{i}\left(H_{\mathrm{sys}} \otimes I_{\mathsf{K}\otimes\overline{\mathsf{K}}} + I_{\mathfrak{h}} \otimes H_{\mathrm{par}}\right)$. □

**Lemma 4.2.10.** *Denote*

$$F := -\mathrm{i}\Delta^{\perp}\left(H_{\mathrm{sys}} \otimes I_{\mathsf{K}\otimes\overline{\mathsf{K}}} + I_{\mathfrak{h}} \otimes H_{\mathrm{par}}\right)\Delta^{\perp} + \Delta(\exp(-\mathrm{i}H_{\mathrm{int}}) - I)\Delta. \tag{4.33}$$

*Theen the QS cocycle with stochastic generator* $F$ *is unitary.*

*Proof.* Again, we will verify the conditions (1.23). First note that for an arbitrary unitary operator $U$ we have

$$U + U^* - 2I = (U - I)(I - U^*).$$

Now,

$$\begin{aligned} F + F^* + F\Delta F^* &= \Delta\left(\exp(-\mathrm{i}H_{\text{int}}) + \exp(\mathrm{i}H_{\text{int}}) - 2I\right)\Delta \\ &\quad + \Delta(\exp(-\mathrm{i}H_{\text{int}}) - I)\Delta(\exp(\mathrm{i}H_{\text{int}}) - I)\Delta \\ &= \Delta\left(\exp(-\mathrm{i}H_{\text{int}}) - I\right)\left(I - \exp(\mathrm{i}H_{\text{int}})\right)\Delta \\ &\quad + \Delta(\exp(-\mathrm{i}H_{\text{int}}) - I)(\exp(\mathrm{i}H_{\text{int}}) - I)\Delta = 0. \end{aligned}$$

Similarly, $F + F^* + F^*\Delta F = 0$. □

**Example 4.2.11.** According to Example 3.2.6 we consider

$$H_{\mathsf{tot}}(h) := H_{\mathsf{sys}} \otimes I_{\widehat{\mathsf{k}}} + I_{\mathfrak{h}} \otimes H_{\mathsf{par}} + \frac{1}{h} H_{\mathsf{int}},$$

where $H_{\mathsf{sys}} \in \mathcal{B}(\mathfrak{h})$, $H_{\mathsf{par}} \in \mathcal{B}(\mathsf{K})$, and $H_{\mathsf{int}} \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{K})$. The associated unitary evolution is defined by

$$U(h) := e^{-\mathrm{i}hH_{\mathsf{tot}}(h)}.$$

If

$$\Delta^{\perp}(H_{\text{int}} \otimes I_{\overline{\mathsf{K}}}) = (H_{\text{int}} \otimes I_{\overline{\mathsf{K}}})\Delta^{\perp} = 0 \tag{4.34}$$

then by Corollary 4.2.9 we obtain

$$\begin{aligned} m_R(\widetilde{\pi}(U(h)), h) \to &-\mathrm{i}\Delta^{\perp}\left(H_{\text{sys}} \otimes I_{\mathsf{K}\otimes\overline{\mathsf{K}}} + I_{\mathfrak{h}} \otimes H_{\text{par}} \otimes I_{\overline{\mathsf{K}}}\right)\Delta^{\perp} \\ &+ \Delta(\exp(-\mathrm{i}H_{\text{int}}) \otimes I_{\overline{\mathsf{K}}} - I_{\mathfrak{h}\otimes\mathsf{K}\otimes\overline{\mathsf{K}}})\Delta =: F \end{aligned}$$

in norm as $h \to 0^+$.

Now assume that $H_{\mathsf{int}} = \begin{bmatrix} A & V^* \\ V & D \end{bmatrix}$ for some $V \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$ and self-adjoint $A \in \mathcal{B}(\mathfrak{h})$, $D \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{k})$, where the matrix decomposition is due to the identification $\mathsf{K} = \mathbb{C}e_0 \oplus \mathsf{k}$. Then according to the decomposition (4.15)

$$H_{\mathsf{int}} \otimes I_{\overline{\mathsf{K}}} = \begin{bmatrix} A & [V^* \;\; 0] & 0 \\ \begin{bmatrix} V \\ 0 \end{bmatrix} & \begin{bmatrix} D & 0 \\ 0 & A \otimes I_{\overline{\mathsf{k}}} \end{bmatrix} & \begin{bmatrix} 0 \\ V^* \otimes I_{\overline{\mathsf{k}}} \end{bmatrix} \\ 0 & [0 \;\; V \otimes I_{\overline{\mathsf{k}}}] & D \otimes I_{\overline{\mathsf{k}}} \end{bmatrix}.$$

Thus $\Delta^{\perp}(H_{\text{int}} \otimes I_{\overline{\mathsf{K}}}) = (H_{\text{int}} \otimes I_{\overline{\mathsf{K}}})\Delta^{\perp} = 0$ if and only if $H_{\mathsf{int}} = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}$.

Hence,

$$F = \begin{bmatrix} -\mathrm{i}H_{\mathrm{sys}} - \mathrm{i}\langle e_0, H_{\mathrm{par}} e_0 \rangle I_{\mathfrak{h}} & 0 & 0 \\ 0 & \begin{bmatrix} e^{-\mathrm{i}D} - I_{\mathfrak{h}\otimes\mathsf{k}} & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 & (e^{-\mathrm{i}D} - I_{\mathfrak{h}\otimes\mathsf{k}}) \otimes I_{\overline{\mathsf{k}}} \end{bmatrix}.$$

**Example 4.2.12.** In [38] Dhahri considers the repeated quantum interactions model, such that the system $S = (\mathfrak{h}, H_{\mathrm{sys}})$ interacts with a chain of particles represented by the state space $\bigotimes_{n \geqslant 0} \mathbb{C}^{n+1}$, so that the Hamiltonian of each piece of the chain is $H_{\mathrm{par}} \in \mathcal{B}(\mathbb{C}^{n+1})$. The associated total Hamiltonian defined on $\mathfrak{h} \otimes \mathbb{C}^{n+1}$ is given by

$$H_{\mathsf{tot}}(h) := H_{\mathsf{sys}} \otimes I_{\mathbb{C}^{n+1}} + I_{\mathfrak{h}} \otimes H_{\mathsf{par}} + \frac{1}{h} H_{\mathsf{int}}$$

for all $h > 0$, where $H_{\mathsf{int}} = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}$ for some self-adjoint $D \in \mathcal{B}(\mathfrak{h} \otimes \mathbb{C}^n)$. The thermal state $\omega \colon \mathcal{B}(\mathbb{C}^{n+1}) \to \mathbb{C}$ of each particle is defined by density matrix

$$\rho = \frac{e^{-\beta_T (H_{\mathrm{par}} - \mu N)}}{\mathrm{Tr}\left(e^{-\beta_T (H_{\mathrm{par}} - \mu N)}\right)},$$

where $\beta_T > 0$, $\mu < 0$, $N = \sum_{i=0}^{n} i \, |e_i\rangle \langle e_i|$ and $H_{\mathrm{par}} = \sum_{i=0}^{n} \alpha_i \, |e_i\rangle \langle e_i|$ for some real numbers $\alpha_i$.

Let $h > 0$ to obtain the random walk convergence the assumption $e^{\beta_T \mu} = h^2$ is made. However, according to the condition (4.28) in Theorem 4.2.8 observe that

$$\gamma_0 = \frac{e^{-\beta_T \alpha_0}}{\sum_{j=0}^{N} e^{-\beta_T (\alpha_j - \mu j)}} = \left( 1 + \sum_{j=1}^{N} e^{\beta_T (\alpha_j - \alpha_0)} \kappa^j \right)^{-1},$$

where $\kappa = e^{\beta_t \mu}$. Since

$$\beta = 1 - \gamma_0 = \kappa + O(\kappa^2)$$

if $\mu$ is chosen such that $\kappa = o(h)$ then (4.28) holds and we can apply Theorem 4.2.8.

## 4.3 Conditional expectation

Random walk approximation involving conditional expectations was first introduced by Belton in [18] to obtain QS cocycles whose generators depend on a thermal state. One of the advantages of this approximation technique, which is related to work that we have done in the previous section, is the simplification of Example 4.2.6; to obtain the same result we will have to verify a condition at the level of $\mathfrak{h} \otimes \mathsf{K}$, rather than $\mathfrak{h} \otimes \mathsf{K} \otimes \overline{\mathsf{K}}$.

**Definition 4.3.1.** Let $\mathcal{A}$ be a C*-algebra and let $\mathcal{A}_0$ be a C*-subalgebra of $\mathcal{A}$. A *conditional expectation* is an idempotent map from $\mathcal{A}$ to $\mathcal{A}$ of norm one and range $\mathcal{A}_0$.
Although, it is equivalent (see [22, Definition I.6.10.1; II.6.10.2 Theorem p. 132]) to say that a conditional expectation $d\colon \mathcal{A} \to \mathcal{A}_0$ is a completely positive contraction such that

- $d(a_0) = a_0$ for all $a_0 \in \mathcal{A}_0$,
- $d(a_0 a) = a_0 d(a)$ for all $a_0 \in \mathcal{A}_0$ and all $a \in \mathcal{A}$,
- $d(a a_0) = d(a) a_0$ for all $a_0 \in \mathcal{A}_0$ and all $a \in \mathcal{A}$.

Let $(\mathcal{M}, \phi)$ be a quantum probability space, that is, $\mathcal{M}$ is a von Neumann algebra and $\phi$ is a normal faithful state. Let $d\colon \mathcal{M} \to \mathcal{M}_0$ be a conditional expectation, where $\mathcal{M}_0$ is a C*-subalgebra of $\mathcal{M}$. Assume that $d$ preserves the state $\phi$, that is, $\phi \circ d = \phi$, then $d$ is ultraweakly continuous and $\mathcal{M}_0$ is a von Neumann algebra ([84, p. 251]).

**Example 4.3.2.** Let $(\mathcal{M}, \phi)$ be a quantum probability space. The identity map $\mathrm{id}_{\mathcal{M}}$ is a conditional expectation that preserves $\phi$, and so is the map $a \mapsto \phi(a) 1_{\mathcal{M}}$.

**Notation 4.3.3.** Set $\delta := \mathrm{id}_{\mathcal{B}(\mathfrak{h})} \overline{\otimes} d$, where $d\colon \mathcal{B}(\mathsf{K}) \to \mathcal{M}_0 \subset \mathcal{B}(\mathsf{K})$ is a conditional expectation that preserves $\omega$. Then $\delta$ is a conditional expectation onto $\mathcal{B}(\mathsf{h}) \overline{\otimes} \mathcal{M}_0$ which preserves $\widetilde{\omega} := \mathrm{id}_{\mathcal{B}(\mathfrak{h})} \overline{\otimes} \omega$. Furthermore,

- $\delta(A \otimes I_{\mathsf{K}}) = A \otimes I_{\mathsf{K}}$,
- $\delta(T_1 \delta(T_2)) = \delta(\delta(T_1) T_2)$,

- $\widetilde{\omega}(\delta(T_1)T_2) = \widetilde{\omega}(T_1\delta(T_2))$,
- $\widetilde{\omega}((A \otimes I_{\mathsf{K}})T) = A\widetilde{\omega}(T)$, for all $A \in \mathcal{B}(\mathfrak{h})$, $T$, $T_1$ and $T_2 \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{K})$.

**Example 4.3.4.** Define the *diagonal map* $\delta_e : \mathcal{B}(\mathfrak{h} \otimes \mathsf{K}) \to \mathcal{B}(\mathfrak{h} \otimes \mathsf{K})$ given by

$$\delta_e(T) = (I_{\mathfrak{h}} \otimes S)^*(T \otimes I_{\mathsf{K}})(I_{\mathfrak{h}} \otimes S),$$

where $S$ is the *Schur isometry* $S \in \mathcal{B}(\mathsf{K}; \mathsf{K} \otimes \mathsf{K})$, that is, $Se_j = e_j \otimes e_j$ for all $j$. If $\mathcal{D}_e$ is the maximal abelian subalgebra of $\mathcal{B}(\mathsf{K})$ generated by $\{|e_j\rangle\langle e_j|\}_{j \geqslant 0}$ then $\delta_e$ is the unique lifting of the conditional expectation onto $\mathcal{D}_e$ that preserves $\omega$. Note that we can write $\delta_e$ as $\mathrm{id}_{\mathcal{B}(\mathfrak{h})} \overline{\otimes}\, d$, where

$$d(S) = \sum_{k=1}^{N} \langle e_k, Se_k\rangle \, |e_k\rangle \, \langle e_k|$$

for each $S \in \mathcal{B}(\mathsf{K})$.

**Definition 4.3.5.** If $h > 0$ and $G \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{K})$ then the *$d$-modification* $m_d(G, h)$ is defined by setting

$$m_d(G, h) := \left(\frac{1}{h}\delta + \frac{1}{\sqrt{h}}\delta^{\perp}\right)(G - I_{\mathfrak{h}\otimes\mathsf{K}}),$$

where $\delta^{\perp} := \mathrm{id}_{\mathcal{B}(\mathfrak{h}\otimes\mathsf{K})} - \delta$.

The next theorem is a special case of [18, Theorem 3 p. 324]. We present a slightly different approach.

**Theorem 4.3.6.** *Let $G : (0, \infty) \to \mathcal{B}(\mathfrak{h} \otimes \mathsf{K})$, and let $F \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{K})$ be such that*

$$m_d(G(h), h) \overset{h \to 0^+}{\to} F \tag{4.35}$$

*in norm.*

*If $f$, $g \in L^2(\mathbb{R}_+, (\mathsf{K} \otimes \overline{\mathsf{K}}) \ominus \mathbb{C}(e_0 \otimes \overline{e_0}))$ are right-continuous step functions and $T \in \mathbb{R}_+$ then*

$$\lim_{h \to 0^+} \sup_{t \in [0,T]} \left\| E^{\varepsilon(f)} \left( X_t^{\widetilde{R}\widetilde{\pi}(G(h))\widetilde{R}^*, h} - X_t^{\Psi(F)} \right) E_{\varepsilon(g)} \right\| = 0,$$

*where* $X^{\Psi(F)} = (X_t^{\Psi(F)})$ *is a Markov-regular QS cocycle with generator*

$$\Psi(F) := \Delta^{\perp}\widetilde{R}\widetilde{\pi}(F)\widetilde{R}^*\Delta^{\perp} + \Delta\widetilde{R}(\widetilde{\pi}\circ\delta^{\perp})(F)\widetilde{R}^*\Delta^{\perp} + \Delta^{\perp}\widetilde{R}(\widetilde{\pi}\circ\delta^{\perp})(F)\widetilde{R}^*\Delta. \tag{4.36}$$

*Proof.* First note that $G(h) - I_{\mathfrak{h}\otimes\mathsf{K}} = (h\delta + \sqrt{h}\delta^{\perp})m_d(G(h), h)$ for all $h > 0$.

The identity $\widetilde{\omega}\circ\delta = \widetilde{\omega}$ implies that

$$\begin{aligned}
&\frac{1}{h}\Delta^{\perp}\widetilde{R}(\widetilde{\pi}(G(h)) - I_{\mathfrak{h}\otimes\mathsf{K}\otimes\overline{\mathsf{K}}})\widetilde{R}^*\Delta^{\perp} \\
&= \frac{1}{h}E^{\xi}\widetilde{\pi}(G(h) - I_{\mathfrak{h}\otimes\mathsf{K}})E_{\xi}E_{e_0\otimes\overline{e_0}}E^{e_0\otimes\overline{e_0}} \\
&= \frac{1}{h}\widetilde{\omega}(G(h) - I_{\mathfrak{h}\otimes\mathsf{K}})E_{e_0\otimes\overline{e_0}}E^{e_0\otimes\overline{e_0}} \\
&= \widetilde{\omega}\left(\frac{1}{h}\delta(G(h) - I_{\mathfrak{h}\otimes\mathsf{K}}) + \frac{1}{\sqrt{h}}\delta^{\perp}(G(h) - I_{\mathfrak{h}\otimes\mathsf{K}})\right)E_{e_0\otimes\overline{e_0}}E^{e_0\otimes\overline{e_0}} \\
&= \widetilde{\omega}\left(m_d(G(h), h)\right)E_{e_0\otimes\overline{e_0}}E^{e_0\otimes\overline{e_0}} \\
&= \Delta^{\perp}\widetilde{R}\widetilde{\pi}(m_d(G(h), h))\widetilde{R}^*\Delta^{\perp},
\end{aligned}$$

$$\frac{1}{\sqrt{h}}\Delta\widetilde{R}(\widetilde{\pi}(G(h)) - I_{\mathfrak{h}\otimes\mathsf{K}\otimes\overline{\mathsf{K}}})\widetilde{R}^*\Delta^{\perp} = \Delta\widetilde{R}\widetilde{\pi}((\sqrt{h}\delta + \delta^{\perp})(m_d(G(h), h))\widetilde{R}^*\Delta^{\perp},$$

$$\frac{1}{\sqrt{h}}\Delta^{\perp}\widetilde{R}(\widetilde{\pi}(G(h)) - I_{\mathfrak{h}\otimes\mathsf{K}\otimes\overline{\mathsf{K}}})\widetilde{R}^*\Delta = \Delta^{\perp}\widetilde{R}\widetilde{\pi}((\sqrt{h}\delta + \delta^{\perp})(m_d(G(h), h))\widetilde{R}^*\Delta,$$

$$\Delta\widetilde{R}(\widetilde{\pi}(G(h)) - I_{\mathfrak{h}\otimes\mathsf{K}\otimes\overline{\mathsf{K}}})\widetilde{R}^*\Delta = \Delta\widetilde{R}\widetilde{\pi}((h\delta + \sqrt{h}\delta^{\perp})(m_d(G(h), h))\widetilde{R}^*\Delta.$$

Set $\Theta := m_d(G(h), h) - F$ then

$$\begin{aligned}
m(\widetilde{R}\widetilde{\pi}(G(h))\widetilde{R}^*, h) - \Psi(F) = \Delta^{\perp}\widetilde{R}\widetilde{\pi}(\Theta)\widetilde{R}^*\Delta^{\perp} &+ \Delta\widetilde{R}(\widetilde{\pi}\circ\delta^{\perp})(\Theta)\widetilde{R}^*\Delta^{\perp} \\
&+ \Delta^{\perp}\widetilde{R}(\widetilde{\pi}\circ\delta^{\perp})(\Theta)\widetilde{R}^*\Delta + \sqrt{h}\mathrm{r}(G(h)),
\end{aligned}$$

where

$$\begin{aligned}\mathrm{r}(G(h)) &:= \Delta\widetilde{R}(\widetilde{\pi}\circ\delta)(m_d(G(h),h))\widetilde{R}^*\Delta^\perp + \Delta^\perp\widetilde{R}(\widetilde{\pi}\circ\delta)(m_d(G(h),h))\widetilde{R}^*\Delta \\ &\quad + \Delta\widetilde{R}\widetilde{\pi}(\sqrt{h}\delta+\delta^\perp)(m_d(G(h),h))\widetilde{R}^*\Delta.\end{aligned}$$

Therefore, the result follows by Theorem 3.1.12. □

**Example 4.3.7.** According to Example 3.2.5 we take

$$H_{\mathsf{tot}}(h) := H_{\mathsf{sys}}\otimes I_{\mathsf{K}} + I_{\mathfrak{h}}\otimes H_{\mathsf{par}} + \frac{1}{\sqrt{h}}H_{\mathsf{int}},$$

where $H_{\mathsf{sys}}\in\mathcal{B}(\mathfrak{h})$, $H_{\mathsf{par}}\in\mathcal{B}(\mathsf{K})$, and $H_{\mathsf{int}}\in\mathcal{B}(\mathfrak{h}\otimes\mathsf{K})$. The associated unitary evolution is given by

$$U(h) := \exp\left(-\mathrm{i}hH_{\mathsf{tot}}(h)\right) = \exp\Big(-\mathrm{i}\sqrt{h}H_{\mathsf{int}} + h\left(-\mathrm{i}H_{\mathsf{sys}}\otimes I_{\mathsf{K}} - \mathrm{i}I_{\mathfrak{h}}\otimes H_{\mathsf{par}}\right)\Big).$$

Let $\delta_e$ be the diagonal map from Example 4.3.4 and let

$$H_{\mathsf{int}} = \begin{bmatrix} 0 & V^* \\ V & 0 \end{bmatrix} \tag{4.37}$$

for some $V\in\mathcal{B}(\mathfrak{h};\mathfrak{h}\otimes(\mathsf{K}\ominus\mathbb{C}e_0))$ then

$$m_{d_e}(U(h),h) \overset{h\to 0^+}{\to} -\delta_e\left(\mathrm{i}H_{\mathsf{sys}}\otimes I_{\mathsf{K}} + \mathrm{i}I_{\mathfrak{h}}\otimes H_{\mathsf{par}} + \frac{1}{2}H_{\mathsf{int}}^2\right) - \begin{bmatrix} 0 & \mathrm{i}V^* \\ \mathrm{i}V & 0 \end{bmatrix} =: F.$$

Denote

$$\Psi(F) := \Delta^\perp\widetilde{R}\widetilde{\pi}(F)\widetilde{R}^*\Delta^\perp + \Delta\widetilde{R}(\widetilde{\pi}\circ\delta_e^\perp)(F)\widetilde{R}^*\Delta^\perp + \Delta^\perp\widetilde{R}(\widetilde{\pi}\circ\delta_e^\perp)(F)\widetilde{R}^*\Delta.$$

We show that

$$\Psi(F)+\Psi(F)^*+\Psi(F)\Delta\Psi(F)^* = \Psi(F)+\Psi(F)^*+\Psi(F)^*\Delta\Psi(F) = 0$$

and so by Theorem 1.2.32 $\Psi(F)$ generates the Markov-regular unitary QS cocycle generated. First note that

$$\begin{aligned}&\Psi(F)+\Psi(F)^*+\Psi(F)\Delta\Psi(F)^*\\=&\Delta^\perp\widetilde{R}\widetilde{\pi}(F+F^*)\widetilde{R}^*\Delta^\perp+\Delta^\perp\widetilde{R}(H_{\mathsf{int}}\otimes I_{\overline{\mathsf{K}}})\widetilde{R}^*\Delta\widetilde{R}(H_{\mathsf{int}}\otimes I_{\overline{\mathsf{K}}})\widetilde{R}^*\Delta^\perp.\end{aligned}$$

Now since $\widetilde{\omega}(T)=E^\xi\widetilde{\pi}(T)E_\xi$ for each $T\in\mathcal{B}(\mathsf{K})$, and $\widetilde{\omega}\circ\delta_e=\widetilde{\omega}$ we obtain

$$\begin{aligned}&\Delta^\perp\widetilde{R}\widetilde{\pi}(F+F^*)\widetilde{R}^*\Delta^\perp+\Delta^\perp\widetilde{R}(H_{\mathsf{int}}\otimes I_{\overline{\mathsf{K}}})\widetilde{R}^*\Delta\widetilde{R}(H_{\mathsf{int}}\otimes I_{\overline{\mathsf{K}}})\widetilde{R}^*\Delta^\perp\\=&-\Delta^\perp\widetilde{R}(H_{\mathsf{int}}^2\otimes I_{\overline{\mathsf{K}}})\widetilde{R}^*\Delta^\perp+\Delta^\perp\widetilde{R}(H_{\mathsf{int}}\otimes I_{\overline{\mathsf{K}}})\widetilde{R}^*\Delta\widetilde{R}(H_{\mathsf{int}}\otimes I_{\overline{\mathsf{K}}})\widetilde{R}^*\Delta^\perp.\end{aligned}$$

We can write $\Delta$ as $I-\Delta^\perp$ so

$$\begin{aligned}&-\Delta^\perp\widetilde{R}(H_{\mathsf{int}}^2\otimes I_{\overline{\mathsf{K}}})\widetilde{R}^*\Delta^\perp+\Delta^\perp\widetilde{R}(H_{\mathsf{int}}\otimes I_{\overline{\mathsf{K}}})\widetilde{R}^*\Delta\widetilde{R}(H_{\mathsf{int}}\otimes I_{\overline{\mathsf{K}}})\widetilde{R}^*\Delta^\perp\\=&-\Delta^\perp\widetilde{R}(H_{\mathsf{int}}\otimes I_{\overline{\mathsf{K}}})\widetilde{R}^*\Delta^\perp\widetilde{R}(H_{\mathsf{int}}\otimes I_{\overline{\mathsf{K}}})\widetilde{R}^*\Delta^\perp,\end{aligned}$$

but

$$\Delta^\perp\widetilde{R}(H_{\mathsf{int}}\otimes I_{\overline{\mathsf{K}}})\widetilde{R}^*\Delta^\perp=E_{e_0\otimes\overline{e_0}}E^\xi(H_{\mathsf{int}}\otimes I_{\overline{\mathsf{K}}})E_\xi E^{e_0\otimes\overline{e_0}}$$

and as we shown in Example 4.2.6

$$E^\xi(H_{\mathsf{int}}\otimes I_{\overline{\mathsf{K}}})E_\xi=0$$

for $H_{\mathsf{int}}$ of the form (4.37). Thus, $\Psi(F)+\Psi(F)^*+\Psi(F)\Delta\Psi(F)^*=0$, and similarly by repeating the above arguments in the different order

$$\Psi(F)+\Psi(F)^*+\Psi(F)^*\Delta\Psi(F)=0.$$

Hence, Theorem 4.3.6 together with Lemma 3.1.13 and Lemma 3.1.14 imply that

$$X_t^{\widetilde{R}\widetilde{\pi}(U(h))\widetilde{R}^*,h}\to X_t^{\Psi(F)}$$

in the strong operator topology as $h\to 0$, uniformly in $t\in[0,T]$, where $X^{\Psi(F)}=(X_t^{\Psi(F)})$ is the unitary Markov-regular QS cocycle with generator $\Psi(F)$.
Now let $R$ be the rotation from Example 4.1.1. Again, by using the argument

that $\widetilde{\omega}(T) = E^{\xi}\widetilde{\pi}(T)E_{\xi}$ for each $T \in \mathcal{B}(\mathsf{K})$, and $\widetilde{\omega} \circ \delta_e = \widetilde{\omega}$ we obtain

$$\begin{aligned}
\Delta^{\perp}\widetilde{R}\widetilde{\pi}(F)\widetilde{R}^*\Delta^{\perp} &= -\,\mathrm{i}H_{\mathrm{sys}} - \mathrm{i}\omega(H_{\mathrm{par}})I_{\mathfrak{h}} \\
&\qquad - \frac{1}{2}\left(\gamma_0 V^*V + (1-\gamma_0)E_{\widetilde{\xi}}(VV^* \otimes I_{\overline{\mathsf{k}}})E_{\widetilde{\xi}}\right), \\
\Delta^{\perp}\widetilde{R}(\widetilde{\pi} \circ \delta_e^{\perp})(F)\widetilde{R}^*\Delta &= -\,\mathrm{i}\begin{bmatrix} \sqrt{\gamma_0}V^* & \sqrt{1-\gamma_0}E^{\widetilde{\xi}}(V \otimes I_{\overline{\mathsf{k}}}) \end{bmatrix}, \\
\Delta\widetilde{R}(\widetilde{\pi} \circ \delta_e^{\perp})(F)\widetilde{R}^*\Delta^{\perp} &= -\,\mathrm{i}\begin{bmatrix} \sqrt{\gamma_0}V \\ \sqrt{1-\gamma_0}(V^* \otimes I_{\overline{\mathsf{k}}})E_{\widetilde{\xi}} \end{bmatrix}.
\end{aligned}$$

Therefore, the limit cocycle coincides with the one in Example 4.2.6.

Similarly, to Proposition 3.1.16, we can consider the products of random walks.

**Theorem 4.3.8.** *Let*

$$G_1\colon (0,\infty) \to \mathcal{B}(\mathfrak{h} \otimes \mathsf{K}), \quad G_2\colon (0,\infty) \to \mathcal{B}(\mathfrak{h} \otimes \mathsf{K})$$

*and let $F_1 \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{K})$, $F_2 \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{K})$ be such that*

$$m_d(G_1(h), h) \overset{h \to 0^+}{\to} F_1 \quad \text{and} \quad m_d(G_2(h), h) \overset{h \to 0^+}{\to} F_2$$

*in norm.*

*If $f, g \in L^2(\mathbb{R}_+; (\mathsf{K} \otimes \overline{\mathsf{K}}) \ominus \mathbb{C}(e_0 \otimes \overline{e_0}))$ are right-continuous step functions and $T \in \mathbb{R}_+$ then*

$$\lim_{h \to 0^+} \sup_{t \in [0,T]} \left\| E^{\varepsilon(f)} \left( X_t^{\widetilde{R}\widetilde{\pi}(G_1(h)G_2(h))\widetilde{R}^*, h} - X_t^{\Psi(F)} \right) E_{\varepsilon(g)} \right\| = 0,$$

*where $X^{\Psi(F)} = (X_t^{\Psi(F)})$ is a Markov-regular QS cocycle with generator*

$$\begin{aligned}
\Psi(F) :=& \Delta^{\perp}\widetilde{R}\widetilde{\pi}(F)\widetilde{R}^*\Delta^{\perp} + \Delta\widetilde{R}(\widetilde{\pi} \circ \delta^{\perp})(F)\widetilde{R}^*\Delta^{\perp} \\
&+ \Delta^{\perp}\widetilde{R}(\widetilde{\pi} \circ \delta^{\perp})(F)\widetilde{R}^*\Delta \\
\text{and} \quad F :=& F_1 + F_2 + \delta(\delta^{\perp}(F_1)\delta^{\perp}(F_2)). \qquad (4.38)
\end{aligned}$$

*Proof.* It is sufficient to show that $m_d(G_1(h)G_2(h), h) \overset{h \to 0^+}{\to} F_1 + F_2 + \delta(\delta^{\perp}(F_1)\delta^{\perp}(F_2))$.

Working similarly to the proof of Theorem 3.1.16 observe that

$$\begin{aligned}
&m_d(G_1(h)G_2(h), h) \\
&= \left(\frac{1}{h}\delta + \frac{1}{\sqrt{h}}\delta^{\perp}\right)((G_1(h) - I)(G_2(h) - I)) \\
&+ m_d(G_1(h), h) + m_d(G(h)_2, h).
\end{aligned}$$

Next, note that

$$\begin{aligned}
&\frac{1}{h}\delta\,((G_1(h) - I)(G_2(h) - I)) \\
&= \frac{1}{h}\delta\left((\delta + \delta^{\perp})(G_1(h) - I)(\delta + \delta^{\perp})(G_2(h) - I)\right) \\
&= \frac{1}{h}\delta\left(\delta^{\perp}(G_1(h) - I)\delta^{\perp}(G_2(h) - I)\right) + \frac{1}{h}\delta\,(G_1(h) - I)\,\delta\,(G_2(h) - I) \\
&= \delta\left(\delta^{\perp}(m_d(G_1(h), h))\delta^{\perp}(m_d(G_2(h), h))\right) + h\delta\,(m_d(G_1(h, h))\,\delta\,(m_d(G_2(h, h)) \\
&\to \delta\left(\delta^{\perp}(F_1)\delta^{\perp}(F_2)\right) \text{ as } h \to 0^+,
\end{aligned}$$

whereas

$$\begin{aligned}
&\frac{1}{\sqrt{h}}\delta^{\perp}\,((G_1(h) - I)(G_2(h) - I)) \\
=&\frac{1}{\sqrt{h}}\delta^{\perp}\left((\delta + \delta^{\perp})(G_1(h) - I)(\delta + \delta^{\perp})(G_2(h) - I)\right) \\
=&\frac{1}{\sqrt{h}}\delta^{\perp}\left(\delta^{\perp}(G_1(h) - I)\delta^{\perp}(G_2(h) - I)\right) + \frac{1}{\sqrt{h}}\delta\,(G_1(h) - I)\,\delta^{\perp}\,(G_2(h) - I) \\
&+ \frac{1}{\sqrt{h}}\delta^{\perp}\,(G_1(h) - I)\,\delta\,(G_2(h) - I) \\
=&\sqrt{h}\delta^{\perp}\left(\delta^{\perp}(m_d(G_1(h), h))\delta^{\perp}(m_d(G_2(h), h))\right) \\
&+ h\left(\delta(m_d(G_1(h), h))\delta^{\perp}(m_d(G_2(h), h)) + \delta^{\perp}(m_d(G_1(h), h))\delta(m_d(G_2(h), h))\right) \\
\to& 0 \quad \text{as } h \to 0^+.
\end{aligned}$$

Therefore,

$$m_d(G_1(h)G_2(h), h) \to F_1 + F_2 + \delta\left(\delta^{\perp}(F_1)\delta^{\perp}(F_2)\right) \text{ as } h \to 0^+$$

and the result follows by Theorem 4.3.6. □

**Corollary 4.3.9.** *Let*

$$G_1\colon (0,\infty)\to\mathcal{B}(\mathfrak{h}\otimes\mathsf{K}),\quad G_2\colon (0,\infty)\to\mathcal{B}(\mathfrak{h}\otimes\mathsf{K})$$

*and let* $F_1,\ F_2\in\mathcal{B}(\mathfrak{h}\otimes\mathsf{K})$ *be such that*

$$m_d(G_1(h),h)\stackrel{h\to0^+}{\to}F_1\quad\text{and}\quad m_d(G_2(h),h)\stackrel{h\to0^+}{\to}F_2$$

*in norm.*

*Let* $h>0$*, if* $G_1(h)$ *and* $G_2(h)$ *commute on the initial space* $\mathfrak{h}$ *then*

$$\lim_{h\to0^+}\sup_{t\in[0,T]}\left\|E^{\varepsilon(f)}\left(X_t^{\widetilde{R}\widetilde{\pi}(G_1(h)\widetilde{R}^*,h}X_t^{\widetilde{R}\widetilde{\pi}(G_2(h)\widetilde{R}^*,h}-X_t^{\Psi(F)}\right)E_{\varepsilon(g)}\right\|=0,$$

*for all right-continuous step functions* $f,\ g\in L^2(\mathbb{R}_+;(\mathsf{K}\otimes\overline{\mathsf{K}})\ominus\mathbb{C}(e_0\otimes\overline{e_0}))$ *and* $T\in\mathbb{R}_+$*, where* $X^{\Psi(F)}=(X_t^{\Psi(F)})$ *is a Markov-regular QS cocycle with generator (4.38).*

*Proof.* The follows the same argument given for Corollary 3.1.18. □

**Example 4.3.10.** Similarly to Example 4.2.7 ,let

- $H_{\mathsf{tot}}^{(1)}(h):=H_{\mathsf{sys}}^{(1)}\otimes I_{\mathfrak{h}_2\otimes\mathsf{K}}+I_{\mathfrak{h}}\otimes H_{\mathsf{par}}+\frac{1}{\sqrt{h}}H_{\mathsf{int}}^{(1)}$,
- $H_{\mathsf{tot}}^{(2)}(h):=I_{\mathfrak{h}_1}\otimes H_{\mathsf{sys}}^{(2)}\otimes I_{\mathsf{K}}+I_{\mathfrak{h}}\otimes H_{\mathsf{par}}+\frac{1}{\sqrt{h}}H_{\mathsf{int}}^{(2)}$,

where

- $H_{\mathsf{sys}}^{(1)}\in\mathcal{B}(\mathfrak{h}_1)$, $H_{\mathsf{par}}\in\mathcal{B}(\mathsf{K})$, and $H_{\mathsf{int}}^{(1)}=\begin{bmatrix}0 & M^*\\ M & 0\end{bmatrix}$ for some $M\in\mathcal{B}(\mathfrak{h};\mathfrak{h}\otimes\mathsf{k})$ which acts on $\mathfrak{h}_2$ as $I_{\mathfrak{h}_2}$,
- $H_{\mathsf{sys}}^{(2)}\in\mathcal{B}(\mathfrak{h}_2)$, and $H_{\mathsf{int}}^{(2)}=\begin{bmatrix}0 & N^*\\ N & 0\end{bmatrix}$ for some $N\in\mathcal{B}(\mathfrak{h};\mathfrak{h}\otimes\mathsf{k})$ which acts on $\mathfrak{h}_1$ on $I_{\mathfrak{h}_1}$.

The associated unitary evolutions are given by

$$U_1(h) := \exp\left(-\mathrm{i}hH^{(1)}_{\mathsf{tot}}(h)\right) \quad \text{and} \quad U_2(h) := \exp\left(-\mathrm{i}hH^{(2)}_{\mathsf{tot}}(h)\right).$$

By Example 4.3.7 we obtain

$$\begin{aligned} m_{d_e}(U_1(h), h) \overset{h\to 0^+}{\to} F_1 :=& -\delta_e\left(\mathrm{i}H^{(1)}_{\mathsf{sys}} \otimes I_{\mathfrak{h}_2\otimes\mathsf{K}} + \mathrm{i}I_{\mathfrak{h}} \otimes H_{\mathsf{par}} + \frac{1}{2}(H^{(1)}_{\mathsf{int}})^2\right) \\ & - \begin{bmatrix} 0 & \mathrm{i}M^* \\ \mathrm{i}M & 0 \end{bmatrix}, \\ m_{d_e}(U_2(h), h) \overset{h\to 0^+}{\to} F_2 :=& -\delta_e\left(\mathrm{i}I_{\mathfrak{h}_1} \otimes H^{(2)}_{\mathsf{sys}} \otimes I_{\mathsf{K}} + \mathrm{i}I_{\mathfrak{h}} \otimes H_{\mathsf{par}} + \frac{1}{2}(H^{(2)}_{\mathsf{int}})^2\right) \\ & - \begin{bmatrix} 0 & \mathrm{i}N^* \\ \mathrm{i}N & 0 \end{bmatrix} \end{aligned}$$

Applying Theorem 4.3.8 we obtain that

$$X_t^{\widetilde{R}\widetilde{\pi}(U_1(h)U_2(h))\widetilde{R}^*, h} \to X_t^{\Psi(F)}$$

strongly as $h \to 0^+$, uniformly in $t \in [0, T]$, where $X^{\Psi(F)} = (X_t^{\Psi(F)})_{t\geqslant 0}$ is the Markov-regular unitary QS cocycle with generator

$$\begin{aligned} \Psi(F) :=& \Delta^\perp \widetilde{R}\widetilde{\pi}(F)\widetilde{R}^*\Delta^\perp + \Delta\widetilde{R}(\widetilde{\pi}\circ\delta_e^\perp)(F)\widetilde{R}^*\Delta^\perp \\ & + \Delta^\perp\widetilde{R}(\widetilde{\pi}\circ\delta_e^\perp)(F)\widetilde{R}^*\Delta \\ \text{and} \quad F :=& F_1 + F_2 + \delta_e(\delta_e^\perp(F_1)\delta_e^\perp(F_2)). \end{aligned}$$

Now, let $R$ be the rotation from Example 4.1.1. Observe that

$$\Psi(F) = \Psi(F_1) + \Psi(F_2) + \Psi(\delta_e(\delta_e^\perp(F_1)\delta_e^\perp(F_2))),$$

and for $i \in \{1, 2\}$ the form of $\Psi(F_i)$ is as in Example 4.3.7.

Since $\widetilde{\omega}(T) = E^\xi\widetilde{\pi}(T)E_\xi$ for each $T \in \mathcal{B}(\mathsf{K})$, and $\widetilde{\omega}\circ\delta_e = \widetilde{\omega}$ the correction term equals

$$\Psi(\delta_e(\delta_e^\perp(F_1)\delta_e^\perp(F_2))) = -\begin{bmatrix} \gamma_0 M^*N+(1-\gamma_0)E^{\widetilde{\xi}}MN^*\otimes I_{\overline{\mathsf{k}}}E_{\widetilde{\xi}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where the matrix form is with respect to the decomposition (4.15). Therefore as we would expect, the result agrees with Example 4.2.7.

**Proposition 4.3.11.** *Let $F \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{K})$ and let $G_n\colon (0,\infty) \to \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ be such that*

$$m_d(G_n(h), h)) \to \frac{F}{n}$$

*in norm, as $h \to 0^+$, for all $n \in \mathbb{N}$.*
*If $f, g \in L^2(\mathbb{R}_+, (\mathsf{K} \otimes \overline{\mathsf{K}}) \ominus \mathbb{C}(e_0 \otimes \overline{e_0}))$ are right-continuous step functions and $T \in \mathbb{R}_+$ then*

$$\lim_{n\to\infty} \lim_{h\to 0^+} \sup_{t\in[0,T]} \left\| E^{\varepsilon(f)} \left( X_t^{\widetilde{R}\widetilde{\pi}(G_n(h)^n)\widetilde{R}^*,h} - X_t^{\Psi(H)} \right) E_{\varepsilon(g)} \right\| = 0,$$

*where $X^{\Psi(H)} = (X_t^{\Psi(H)})$ is the Markov-regular QS cocycle with generator*

$$\begin{aligned} \Psi(H) :=& \Delta^\perp \widetilde{R}\widetilde{\pi}(H)\widetilde{R}^*\Delta^\perp + \Delta\widetilde{R}(\widetilde{\pi}\circ\delta^\perp)(H)\widetilde{R}^*\Delta^\perp \\ &+ \Delta^\perp\widetilde{R}(\widetilde{\pi}\circ\delta^\perp)(H)\widetilde{R}^*\Delta, \\ \textit{and} \quad H :=& F + \frac{1}{2}\delta\big(\delta^\perp(F)\delta^\perp(F)\big). \end{aligned}$$

*Proof.* Induction yields that

$$m_d(G_n(h)^n, h) \overset{h\to 0^+}{\to} H_n := F + \frac{n(n-1)}{2n}\delta\left(\delta^\perp\Big(\frac{F}{n}\Big)\delta^\perp\Big(\frac{F}{n}\Big)\right).$$

By applying Theorem 4.3.6 we obtain that if $f, g \in L^2(\mathbb{R}_+, (\mathsf{K}\otimes\overline{\mathsf{K}})\ominus\mathbb{C}(e_0\otimes\overline{e_0}))$ are right-continuous step functions and $T \in \mathbb{R}_+$ then

$$\lim_{h\to 0^+} \sup_{t\in[0,T]} \left\| E^{\varepsilon(f)} \left( X_t^{\widetilde{R}\widetilde{\pi}(G_n(h)^n)\widetilde{R}^*,h} - X_t^{\Psi(H_n)} \right) E_{\varepsilon(g)} \right\| = 0,$$

where

$$\Psi(H_n) := \Delta^\perp \widetilde{R}\widetilde{\pi}(H_n)\widetilde{R}^*\Delta^\perp + \Delta\widetilde{R}(\widetilde{\pi}\circ\delta^\perp)(H_n)\widetilde{R}^*\Delta^\perp + \Delta^\perp\widetilde{R}(\widetilde{\pi}\circ\delta^\perp)(H_n)\widetilde{R}^*\Delta.$$

Since $H_n$ converges in norm to $H$ as $n \to \infty$, we get that $\Psi(H_n)$ converges in norm to $\Psi(H)$ as $n \to \infty$. We end the proof by applying Proposition 1.2.29. □

## 4.4 Compressed walks

In this section we construct random walks such that no special condition on the interaction Hamiltonian (stated in Remark 4.2.2) will have to be assumed to obtain convergence to the cocycles defined on $\mathfrak{h} \otimes \mathcal{F}^{(\mathsf{k}\otimes\overline{\mathbb{C}e_0})\oplus(\mathbb{C}e_0\otimes\bar{\mathsf{k}})}$, where $\mathsf{k} := \mathsf{K} \ominus \mathbb{C}e_0$.

Let us recall that the embedding map $J_h^n : \mathsf{K} \to \mathcal{F}^{\mathsf{k}}_{[hn,h(n+1))}$ is given by

$$\alpha e_0 + c \mapsto \alpha\Omega_{[hn,h(n+1))} + \frac{1}{\sqrt{h}} c\mathbf{1}_{[hn,h(n+1))},$$

for all $n \in \mathbb{N}_0$, $\alpha \in \mathbb{C}$ and $c \in \mathsf{k}$.

We denote $\overline{J}_h^n$ the similar embedding which maps $\overline{\mathsf{K}}$ to $\mathcal{F}^{\bar{\mathsf{k}}}_{[hn,h(n+1))}$ for all $n \in \mathbb{N}_0$.

By modifying Definition 3.1.6 we define a quantum random walk which embeds into $\mathfrak{h} \otimes \mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}$.

**Definition 4.4.1.** If $h > 0$ and $G \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{K} \otimes \overline{\mathsf{K}})$ then the *compressed embedded quantum random walk* with *generator* $G$ and *step size* $h$ is the operator process $(Y_t^{G,h})_{t\geqslant 0}$ such that

$$\begin{cases} I_{\mathfrak{h}\otimes\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}_{[0,\infty)}} & \text{for } n = 0, \\ \left(I_{\mathfrak{h}} \otimes \bigotimes_{k=0}^{n-1} J_h^k \otimes \overline{J}_h^k\right) G_1^{(n)} \cdots G_n^{(n)} \left(I_{\mathfrak{h}} \otimes \bigotimes_{k=0}^{n-1} J_h^k \otimes \overline{J}_h^k\right)^* \otimes I_{\mathcal{F}^{\mathsf{k}\oplus\bar{\mathsf{k}}}_{[nh,\infty)}} & \text{for } n \geqslant 1 \end{cases}$$

if $t \in [nh, (n+1)h)$, for all $t \in \mathbb{R}_+$.

**Remark 4.4.2.** According to (4.15) we can decompose

$$\mathsf{K} \otimes \overline{\mathsf{K}} \cong \mathbb{C} \oplus (\mathsf{k} \oplus \overline{\mathsf{k}}) \oplus (\mathsf{k} \otimes \overline{\mathsf{k}}).$$

Let us define an isometry $Q: \mathfrak{h} \otimes (\mathbb{C} \oplus (\mathsf{k} \oplus \overline{\mathsf{k}})) \to \mathfrak{h} \otimes \mathsf{K} \otimes \overline{\mathsf{K}}$ by setting

$$Q\left(u \otimes \left(\begin{smallmatrix} \alpha \\ c+\overline{d} \end{smallmatrix}\right)\right) = u \otimes \left(\alpha e_0 \otimes \overline{e_0} + c \otimes \overline{e_0} + e_0 \otimes \overline{d}\right)$$

for all $\alpha \in \mathbb{C}$, $c$, $d \in \mathsf{k}$.

**Theorem 4.4.3.** *Let $G: (0, \infty) \to \mathcal{B}(\mathfrak{h} \otimes \mathsf{K} \otimes \overline{\mathsf{K}})$, and let $F \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{K} \otimes \overline{\mathsf{K}})$ be such that*

$$m_R(G(h), h) \overset{h \to 0^+}{\to} F \tag{4.39}$$

*in norm, where $m_R(G(h), h)$ is defined as in (4.1.2).*

*If $f$, $g \in L^2(\mathbb{R}_+; \mathsf{k} \oplus \overline{\mathsf{k}})$ are right-continuous step functions and $T \in \mathbb{R}_+$ then*

$$\lim_{h \to 0^+} \sup_{t \in [0,T]} \left\| E^{\varepsilon(f)} \left( Y_t^{\widetilde{R}G(h)\widetilde{R}^*, h} - X_t^{Q^*FQ} \right) E_{\varepsilon(g)} \right\| = 0,$$

*where $X^{Q^*FQ} = (X_t^{Q^*FQ})_{t \geqslant 0}$ is the Markov-regular QS cocycle with generator $Q^*FQ$.*

*Proof.* The proof of this theorem is analogous to Theorem 3.1.12. Therefore, we are only going to show the following convergence:

$$\left\| E^{\varepsilon(c_{[0,t)} + \overline{d_{[0,t)}})} \left( Y_t^{\widetilde{R}G(h)\widetilde{R}^*, h} - X_t^{Q^*FQ} \right) E_{\varepsilon(c'_{[0,t)} + \overline{d'_{[0,t)}})} \right\| \to 0 \tag{4.40}$$

for all $c$, $c'$, $d$ and $d' \in \mathsf{k}$.

For simplicity denote $\widehat{x} := e_0 + x$ and $\widehat{x + \overline{y}} := \left(\begin{smallmatrix} 1 \\ x+\overline{y} \end{smallmatrix}\right) \in \mathbb{C} \oplus (\mathsf{k} \oplus \overline{\mathsf{k}})$ for all $x, y \in \mathsf{k}$.

Now, let $t \in \mathbb{R}_+$, let $c$, $c'$, $d$ and $d' \in \mathsf{k}$ and let $h > 0$. Then $t \in [nh, (n+1)h)$

for some $n \in \mathbb{N}$ and

$$\begin{aligned}
&E^{\varepsilon(c_{[0,t)}+\overline{d_{[0,t)}})} Y_t^{\widetilde{R}G(h)\widetilde{R}^*,h} E_{\varepsilon(c'_{[0,t)}+\overline{d'_{[0,t)}})} \\
=&\left\langle \varepsilon(c_{[hn,t)} + \overline{d_{[hn,t)}}), \varepsilon(c'_{[hn,t)} + \overline{d'_{[hn,t)}}) \right\rangle \\
&E^{\left(\widehat{\sqrt{h}c} \otimes \widehat{\sqrt{h}\,\overline{d}}\right)^{\otimes n}} (\widetilde{R}G(h)\widetilde{R}^*)_1^{(n)} \cdots (\widetilde{R}G(h)\widetilde{R}^*)_n^{(n)} E_{\left(\widehat{\sqrt{h}c'} \otimes \widehat{\sqrt{h}\,\overline{d'}}\right)^{\otimes n}}.
\end{aligned}$$

Since $\left\langle \varepsilon(c_{[hn,t)} + \overline{d_{[hn,t)}}), \varepsilon(c'_{[hn,t)} + \overline{d'_{[hn,t)}}) \right\rangle \to 1$ as $h \to 0^+$ and by applying Lemma 3.1.11 we obtain

$$E^{\varepsilon(c_{[0,t)}+\overline{d_{[0,t)}})} Y_t^{\widetilde{R}G(h)\widetilde{R}^*,h} E_{\varepsilon(c'_{[0,t)}+\overline{d'_{[0,t)}})} \sim \left( E^{\widehat{\sqrt{h}c} \otimes \widehat{\sqrt{h}\,\overline{d}}} \widetilde{R}G(h)\widetilde{R}^* E_{\widehat{\sqrt{h}c'} \otimes \widehat{\sqrt{h}\,\overline{d'}}} \right)^n,$$

where $\sim$ means that both expressions on the left and right-hand side have the same limit.

Now observe that

$$E^{\widehat{\sqrt{h}c} \otimes \widehat{\sqrt{h}\,\overline{d}}} = E^{e_0 \otimes \overline{e_0} + c \otimes \overline{e_0} + e_0 \otimes \overline{d} + \sqrt{h} c \otimes \overline{d}} (\Delta^\perp + \sqrt{h}\Delta) \tag{4.41}$$

and by taking the adjoint we obtain

$$E_{\widehat{\sqrt{h}c} \otimes \widehat{\sqrt{h}\,\overline{d}}} = (\Delta^\perp + \sqrt{h}\Delta) E_{e_0 \otimes \overline{e_0} + c \otimes \overline{e_0} + e_0 \otimes \overline{d} + \sqrt{h} c \otimes \overline{d}}. \tag{4.42}$$

Similarly

$$\begin{aligned}
E^{e_0 \otimes \overline{e_0} + c \otimes \overline{e_0} + e_0 \otimes \overline{d}} &= E^{\widehat{c+\overline{d}}} Q^* \\
E_{e_0 \otimes \overline{e_0} + c \otimes \overline{e_0} + e_0 \otimes \overline{d}} &= Q E_{\widehat{c+\overline{d}}}.
\end{aligned} \tag{4.43}$$

Note that

$$\begin{aligned}
&E^{\widehat{\sqrt{h}c} \otimes \widehat{\sqrt{h}\,\overline{d}}} \widetilde{R}G(h)\widetilde{R}^* E_{\widehat{\sqrt{h}c'} \otimes \widehat{\sqrt{h}\,\overline{d'}}} \\
=&E^{\widehat{\sqrt{h}c} \otimes \widehat{\sqrt{h}\,\overline{d}}} \left( \widetilde{R}G(h)\widetilde{R}^* - I_{\mathfrak{h} \otimes \mathsf{K} \otimes \overline{\mathsf{K}}} \right) E_{\widehat{\sqrt{h}c'} \otimes \widehat{\sqrt{h}\,\overline{d'}}} + I_{\mathfrak{h}} + h \left\langle c + \overline{d}, c' + \overline{d'} \right\rangle I_{\mathfrak{h}} \\
&+ o(h) \quad \text{as } h \to 0^+.
\end{aligned}$$

Now consider the first term of the above sum:

$$\begin{aligned}
&= E^{\widehat{\sqrt{h}c} \otimes \widehat{\sqrt{h}\,\overline{d}}} \left( \widetilde{R} G(h) \widetilde{R}^* - I_{\mathfrak{h} \otimes \mathsf{K} \otimes \overline{\mathsf{K}}} \right) E_{\widehat{\sqrt{h}c'} \otimes \widehat{\sqrt{h}\,\overline{d'}}} \\
&= E^{\widehat{\sqrt{h}c} \otimes \widehat{\sqrt{h}\,\overline{d}}} (\sqrt{h}\Delta^{\perp} + \Delta) m_R(G(h), h) (\sqrt{h}\Delta^{\perp} + \Delta) E_{\widehat{\sqrt{h}c'} \otimes \widehat{\sqrt{h}\,\overline{d'}}} \\
&= E^{\widehat{\sqrt{h}c} \otimes \widehat{\sqrt{h}\,\overline{d}}} (\sqrt{h}\Delta^{\perp} + \Delta) (F + o(1)) (\sqrt{h}\Delta^{\perp} + \Delta) E_{\widehat{\sqrt{h}c'} \otimes \widehat{\sqrt{h}\,\overline{d'}}} \quad \text{as } h \to 0^+.
\end{aligned}$$

By applying (4.41) and (4.42) we obtain

$$\begin{aligned}
& E^{\widehat{\sqrt{h}c} \otimes \widehat{\sqrt{h}\,\overline{d}}} (\sqrt{h}\Delta^{\perp} + \Delta) (F + o(1)) (\sqrt{h}\Delta^{\perp} + \Delta) E_{\widehat{\sqrt{h}c'} \otimes \widehat{\sqrt{h}\,\overline{d'}}} \\
&= h \left( E^{e_0 \otimes \overline{e_0} + c \otimes \overline{e_0} + e_0 \otimes \overline{d} + \sqrt{h} c \otimes \overline{d'}} F E_{e_0 \otimes \overline{e_0} + c' \otimes \overline{e_0} + e_0 \otimes \overline{d'} + \sqrt{h} c' \otimes \overline{d'}} \right) + o(h) \\
&= h \left( E^{e_0 \otimes \overline{e_0} + c \otimes \overline{e_0} + e_0 \otimes \overline{d}} F E_{e_0 \otimes \overline{e_0} + c' \otimes \overline{e_0} + e_0 \otimes \overline{d'}} \right) + o(h) \quad \text{as } h \to 0^+.
\end{aligned}$$

Now, we use (4.43) to get

$$\begin{aligned}
& h \left( E^{e_0 \otimes \overline{e_0} + c \otimes \overline{e_0} + e_0 \otimes \overline{d}} F E_{e_0 \otimes \overline{e_0} + c' \otimes \overline{e_0} + e_0 \otimes \overline{d'}} \right) + o(h) \\
&= h \left( E^{\widehat{c + \overline{d}}} Q^* F Q E_{\widehat{c' + \overline{d'}}} \right) + o(h) \quad \text{as } h \to 0^+.
\end{aligned}$$

Hence,

$$\begin{aligned}
& E^{\widehat{\sqrt{h}c} \otimes \widehat{\sqrt{h}\,\overline{d}}} \widetilde{R} G(h) \widetilde{R}^* E_{\widehat{\sqrt{h}c'} \otimes \widehat{\sqrt{h}\,\overline{d'}}} \\
&= I_{\mathfrak{h}} + h \left( E^{\widehat{c + \overline{d}}} Q^* F Q E_{\widehat{c' + \overline{d'}}} + \left\langle c + \overline{d}, c' + \overline{d'} \right\rangle I_{\mathfrak{h}} \right) + o(h) \quad \text{as } h \to 0^+.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& E^{\varepsilon(c_{[0,t)} + \overline{d_{[0,t)}})} Y_t^{\widetilde{R} G(h) \widetilde{R}^*, h} E_{\varepsilon(c'_{[0,t)} + \overline{d'_{[0,t)}})} \\
& \overset{h \to 0^+}{\to} \exp \left\{ t (E^{\widehat{c + \overline{d}}} Q^* F Q E_{\widehat{c' + \overline{d'}}} + \left\langle c + \overline{d}, c' + \overline{d'} \right\rangle I_{\mathfrak{h}}) \right\}
\end{aligned}$$

and by Proposition 1.2.27 together with Proposition 1.2.30 we obtain

$$\exp\left\{t(E^{\widehat{c+\overline{d}}}Q^*FQE_{\widehat{c'+\overline{d'}}} + \left\langle c+\overline{d}, c'+\overline{d'}\right\rangle I_{\mathfrak{h}})\right\}$$
$$= E^{\varepsilon(c_{[0,t)}+\overline{d_{[0,t)}})}X_t^{Q^*FQ}E_{\varepsilon(c'_{[0,t)}+\overline{d'_{[0,t)}})}.$$

Convergence for arbitrary right-continuous step functions $f, g \in L^2(\mathbb{R}_+; \mathsf{k} \oplus \overline{\mathsf{k}})$ and its uniformity on the bounded intervals of $\mathbb{R}_+$ holds and can be verified analogous to the proof of Theorem 3.1.12. □

**Example 4.4.4** (Weak coupling limits)**.** According to Section 4.2 we define

$$U(h) := \exp(-\mathrm{i}hH_{\mathrm{tot}}(h)),$$

where

$$H_{\mathrm{tot}}(h) := H_{\mathrm{sys}} \otimes I_{\mathsf{K}\otimes\overline{\mathsf{K}}} + I_{\mathfrak{h}} \otimes H_{\mathrm{par}} + \frac{1}{\sqrt{h}}H_{\mathrm{int}}$$

and

$$H_{\mathrm{int}} = \begin{bmatrix} 0 & 0 & C^* & 0 \\ 0 & 0 & G^* & 0 \\ C & G & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for some $C, G \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes (\mathsf{k} \oplus \overline{\mathsf{k}}))$ and $\mathsf{k} := \mathsf{K} \ominus \mathbb{C}e_0$.

Then $m_R(U(h), h) \overset{h\to 0^+}{\to} F$ in norm, where

$$F := -\Delta^{\perp}\widetilde{R}\left(\mathrm{i}H_{\mathrm{sys}} \otimes I_{\mathsf{K}\otimes\overline{\mathsf{K}}} + \mathrm{i}I_{\mathfrak{h}} \otimes H_{\mathrm{par}} + \frac{1}{2}H_{\mathrm{int}}^2\right)\widetilde{R}^*\Delta^{\perp}$$
$$-\mathrm{i}\Delta^{\perp}\widetilde{R}H_{\mathrm{int}}\widetilde{R}^*\Delta - \mathrm{i}\Delta\widetilde{R}H_{\mathrm{int}}\widetilde{R}^*\Delta^{\perp}$$

Hence, by Theorem 4.4.3 together with ith Lemma 3.1.13 and Lemma 3.1.14

$$Y_t^{\widetilde{R}U(h)\widetilde{R}^*,h} \to X_t^{Q^*FQ}$$

in the strong operator topology, as $h \to 0^+$, locally uniformly in $t$.

Moreover, if we consider $R$ to be a rotation from Example 4.1.1 then we obtain

$$Q^*FQ = \begin{bmatrix} F_0^0 & F_+^0 \\ F_0^+ & 0 \end{bmatrix} \in \begin{bmatrix} \mathcal{B}(\mathfrak{h}) & \mathcal{B}(\mathfrak{h}\otimes(\mathsf{k}\oplus\bar{\mathsf{k}});\mathfrak{h}) \\ \mathcal{B}(\mathfrak{h};\mathfrak{h}\otimes(\mathsf{k}\oplus\bar{\mathsf{k}})) & \mathcal{B}(\mathfrak{h}\otimes(\mathsf{k}\oplus\bar{\mathsf{k}})) \end{bmatrix}, \tag{4.44}$$

where

$$\begin{aligned} F_0^0 =& -\mathrm{i}H_{\mathrm{sys}} - \mathrm{i}\langle\xi, H_{\mathrm{par}}\xi\rangle I_{\mathfrak{h}} \\ & -\frac{1}{2}\Big(\gamma_0 C^*C + \sqrt{\gamma_0(1-\gamma_0)}(C^*G + G^*C) + (1-\gamma_0)G^*G\Big), \\ F_+^0 =& -\mathrm{i}\Big(\sqrt{\gamma_0}C^* + \sqrt{1-\gamma_0}G^*\Big), \\ F_0^+ =& -\mathrm{i}\Big(\sqrt{\gamma_0}C + \sqrt{1-\gamma_0}G\Big). \end{aligned}$$

An interesting special case of the above approximation can be obtained analogously to Example 4.2.6, that is, instead of considering the 'particle' operators on $\mathsf{K}\otimes\overline{\mathsf{K}}$, we ampliate the ones defined on $\mathsf{K}$ to $\mathsf{K}\otimes\overline{\mathsf{K}}$. It delivers us the cocycle with generator

$$F = \begin{bmatrix} F_0^0 & F_+^0 \\ F_0^+ & 0 \end{bmatrix}, \tag{4.45}$$

where

$$\begin{aligned} F_0^0 =& -\mathrm{i}H_{\mathrm{sys}} - \mathrm{i}\omega(H_{\mathrm{par}})I_{\mathfrak{h}} \\ & -\frac{1}{2}\Big(\gamma_0 V^*V + (1-\gamma_0)E_{\tilde{\xi}}(VV^*\otimes I_{\bar{\mathsf{k}}})E_{\tilde{\xi}}\Big), \\ F_+^0 =& -\mathrm{i}\Big(\sqrt{\gamma_0}\begin{bmatrix} V^* & 0 \end{bmatrix} + \sqrt{1-\gamma_0}\begin{bmatrix} 0 & E^{\tilde{\xi}}(V\otimes I_{\bar{\mathsf{k}}}) \end{bmatrix}\Big), \\ F_0^+ =& -\mathrm{i}\left(\sqrt{\gamma_0}\begin{bmatrix} V \\ 0 \end{bmatrix} + \sqrt{1-\gamma_0}\begin{bmatrix} 0 \\ (V^*\otimes I_{\bar{\mathsf{k}}})E_{\tilde{\xi}} \end{bmatrix}\right). \end{aligned}$$

for some $V \in \mathcal{B}(\mathfrak{h};\mathfrak{h}\otimes\mathsf{k})$ and self-adjoint $H_{\mathrm{sys}} \in \mathcal{B}(\mathfrak{h})$, $H_{\mathrm{par}} \in \mathcal{B}(\mathsf{k})$.

Combining the preceding theorem and Theorem 4.3.6 we obtain

**Corollary 4.4.5.** *Let $G\colon (0,\infty) \to \mathcal{B}(\mathfrak{h} \otimes \mathsf{K})$, and let $F \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{K})$ be such that*

$$m_d(G(h), h) \overset{h\to 0^+}{\to} F \tag{4.46}$$

*in norm, where $m_d(G(h), h)$ is defined as in (4.3.5).*

*If $f, g \in L^2(\mathbb{R}_+; \mathsf{k} \oplus \bar{\mathsf{k}})$ are right-continuous step functions and $T \in \mathbb{R}_+$ then*

$$\lim_{h\to 0^+} \sup_{t\in[0,T]} \left\| E^{\varepsilon(f)} \left( Y_t^{\widetilde{R}\widetilde{\pi}(G(h))\widetilde{R}^*, h} - X_t^{Q^*\Psi(F)Q} \right) E_{\varepsilon(g)} \right\| = 0,$$

*where $X^{Q^*\Psi(F)Q} = (X_t^{Q^*\Psi(F)Q})_{t\geqslant 0}$ is the Markov-regular QS cocycle with generator $Q^*\Psi(F)Q$ and $\Psi(F)$ is given by (4.36).*

Equipped with Theorem 4.4.3 we can proof the analogous result to Theorem 4.3.8 and Proposition 4.3.11:

**Proposition 4.4.6.** *Let*

$$G_1\colon (0,\infty) \to \mathcal{B}(\mathfrak{h} \otimes \mathsf{K}) \quad \textit{and} \quad G_2\colon (0,\infty) \to \mathcal{B}(\mathfrak{h} \otimes \mathsf{K})$$

*and let $F_1 \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{K})$, $F_2 \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{K})$ be such that*

$$m_d(G_1(h), h) \overset{h\to 0^+}{\to} F_1 \quad \textit{and} \quad m_d(G_2(h), h) \overset{h\to 0^+}{\to} F_2$$

*in norm.*

*If $f, g \in L^2(\mathbb{R}_+; \mathsf{k} \oplus \bar{\mathsf{k}})$ are right-continuous step functions and $T \in \mathbb{R}_+$ then*

$$\lim_{h\to 0^+} \sup_{t\in[0,T]} \left\| E^{\varepsilon(f)} \left( Y_t^{\widetilde{R}\widetilde{\pi}(G_1(h)G_2(h)\widetilde{R}^*, h} - X_t^{Q^*\Psi(F)Q} \right) E_{\varepsilon(g)} \right\| = 0,$$

*where $X^{Q^*\Psi(F)Q} = (X_t^{Q^*\Psi(F)Q})_{t\geqslant 0}$ is the Markov-regular QS cocycle with $\Psi(F)$ given by (4.38).*

**Proposition 4.4.7.** *Let $F \in \mathcal{B}(\mathfrak{h} \otimes \mathsf{K})$ and let $G_n\colon (0,\infty) \to \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ be such that*

$$m_d(G_n(h), h)) \to \frac{F}{n}$$

*in norm, as $h \to 0^+$, for all $n \in \mathbb{N}$.*

*If* $f, g \in L^2(\mathbb{R}_+; \mathsf{k} \oplus \bar{\mathsf{k}})$ *are right-continuous step functions and* $T \in \mathbb{R}_+$ *then*

$$\lim_{n\to\infty} \lim_{h\to 0^+} \sup_{t\in[0,T]} \left\| E^{\varepsilon(f)} \left( X_t^{\widetilde{R}\widetilde{\pi}(G_n(h)^n)\widetilde{R}^*,h} - X_t^{Q^*\Psi(H)Q} \right) E_{\varepsilon(g)} \right\| = 0,$$

*where* $X^{Q^*\Psi(H)Q} = (X_t^{Q^*\Psi(H)Q})_{t\geqslant 0}$ *is the Markov-regular QS cocycle with generator* $Q^*\Psi(H)Q$*,*

$$\begin{aligned}
\Psi(H) :=& \Delta^\perp \widetilde{R}\widetilde{\pi}(H)\widetilde{R}^*\Delta^\perp + \Delta\widetilde{R}(\widetilde{\pi}\circ\delta^\perp)(H)\widetilde{R}^*\Delta^\perp \\
&+ \Delta^\perp\widetilde{R}(\widetilde{\pi}\circ\delta^\perp)(H)\widetilde{R}^*\Delta, \\
\textit{and} \quad H :=& F + \frac{1}{2}\delta\big(\delta^\perp(F)\delta^\perp(F)\big).
\end{aligned}$$

# Chapter 5

# Quasifree random walks

In [5, Proposition 8, p. 278] Attal and Joye showed that the quantum stochastic cocycle (or equivalently the strong solution of the constant coefficient QSDE) obtained in [5, Theorem 7, p. 272] induces a non-Fock quasifree representation of the CCR algebra. Motivated by their work we generalise their result by giving the necessary and sufficient conditions when QS cocycles (obtained as a continuous-time limit from repeated interactions) induces quasifree representations of the CCR algebra. In contrast to [5], we obtain the representations which induce a gauge-invariant quasifree state, as well as, a squeezed quasifree state.

## 5.1 Transpose lemma

Let $\mathsf{k}$ be a Hilbert space. Denote by $U_{\mathsf{k}}$ the isomorphism $|u\rangle\langle v| \mapsto u \otimes \overline{v}$ between $\mathrm{HS}(\mathsf{k})$ and $\mathsf{k} \otimes \overline{\mathsf{k}}$. The Hilbert space $\mathrm{HS}(\overline{\mathsf{k}})$ is also isomorphic to $\mathsf{k} \otimes \overline{\mathsf{k}}$, the isomorphism $U_{\overline{\mathsf{k}}} \colon \mathrm{HS}(\overline{\mathsf{k}}) \to \mathsf{k} \otimes \overline{\mathsf{k}}$ is such that $U_{\overline{\mathsf{k}}}(|\overline{u}\rangle\langle\overline{v}|) = v \otimes \overline{u}$ for all $u, v \in \mathsf{k}$.

**Lemma 5.1.1.** *For any $M \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$ and $N \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \overline{\mathsf{k}})$ we have the following equalities:*

1. $(M^* \otimes I_{\overline{\mathsf{k}}})E_x = \left(I_{\mathfrak{h}} \otimes U_{\overline{\mathsf{k}}}^{-1}(x)\right) M^{\mathsf{c}}$
2. $E^x(M \otimes I_{\overline{\mathsf{k}}}) = M^{\mathsf{T}} \left(I_{\mathfrak{h}} \otimes \left(U_{\overline{\mathsf{k}}}^{-1}(x)\right)^*\right)$
3. $(N^* \otimes I_{\mathsf{k}})(I_{\mathfrak{h}} \otimes \Pi)E_x = \left(I_{\mathfrak{h}} \otimes U_{\mathsf{k}}^{-1}(x)\right) N^{\mathsf{c}}$

*4.* $E^x(I_{\mathfrak{h}} \otimes \Pi^{-1})(N \otimes I_{\mathsf{k}}) = N^{\mathsf{T}}\left(I_{\mathfrak{h}} \otimes \left(U_{\mathsf{k}}^{-1}(x)\right)^*\right)$

*for all* $x \in \mathsf{k} \otimes \overline{\mathsf{k}}$*, where* $\Pi$ *is the tensor flip on* $\mathsf{k} \otimes \overline{\mathsf{k}}$*.*

*Proof.* Let $c, d \in \mathsf{k}$ and $u \in \mathfrak{h}$. By Lemma 2.1.8 we have that $M^* E_c = E^{\overline{c}} M^{\mathsf{c}}$, which gives *(1)* when $x$ is a simple tensor. The general case follows by taking limits. We obtain the identity (*2*) by taking the adjoint of *(1)*. The proofs of *(3)* and *(4)* are analogous. □

**Remark 5.1.2.** Since $M \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$ then $ME_u \in \mathrm{HS}(\mathbb{C}; \mathfrak{h} \otimes \mathsf{k})$ for all $u \in \mathfrak{h}$. Therefore, $M \in \mathrm{Mat}(\mathsf{M}, \mathsf{H})_{\mathbb{C},\mathsf{k}}$ according to the notation used in (2.4) and by Definition 2.1.5 the partial transpose $M^{\mathsf{T}}$ exists and so is the associated conjugate $M^{\mathsf{c}} = (M^{\mathsf{T}})^*$. Hence, we don't have to make an extra assumptions associating our operators with $\mathrm{Mat}(\mathsf{M}, \mathsf{H})_{\mathbb{C},\mathsf{k}}$.

**Example 5.1.3.** Let $\tilde{\xi}$ be a unit vector defined in (4.3). Then by the preceding lemma, for any $M \in \mathcal{B}(\mathfrak{h}, \mathfrak{h} \otimes \mathsf{k})$ we obtain

$$(M^* \otimes I_{\overline{\mathsf{k}}})E_{\tilde{\xi}} = \frac{1}{\sqrt{1-\gamma_0}}\left(I_{\mathfrak{h}} \otimes \sum_{i=1}^{N} \sqrt{\gamma_i}\, |\overline{e_i}\rangle \langle \overline{e_i}|\right) M^{\mathsf{c}}.$$

where $N \in \mathbb{N} \cup \{\infty\}$.

## 5.2 Quasifree setup

According to Example 4.4.4 the unitary QS cocycle $X^F$ which we obtained through convergence of random walks has the generator

$$F = \begin{bmatrix} F_0^0 & F_+^0 \\ F_0^+ & 0 \end{bmatrix} \in \begin{bmatrix} \mathcal{B}(\mathfrak{h}) & \mathcal{B}(\mathfrak{h} \otimes (\mathsf{k} \oplus \overline{\mathsf{k}}); \mathfrak{h}) \\ \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes (\mathsf{k} \oplus \overline{\mathsf{k}})) & \mathcal{B}(\mathfrak{h} \otimes (\mathsf{k} \oplus \overline{\mathsf{k}})) \end{bmatrix}, \tag{5.1}$$

where

$$\begin{aligned}
F_0^0 &= -\mathrm{i}H_{\mathrm{sys}} - \mathrm{i}\langle \xi, H_{\mathrm{par}}\xi\rangle I_{\mathfrak{h}} \\
&\quad -\frac{1}{2}\Big(\gamma_0 C^*C + \sqrt{\gamma_0(1-\gamma_0)}(C^*G + G^*C) + (1-\gamma_0)G^*G\Big), \\
F_+^0 &= -\mathrm{i}\Big(\sqrt{\gamma_0}C^* + \sqrt{1-\gamma_0}G^*\Big), \\
F_0^+ &= -\mathrm{i}\Big(\sqrt{\gamma_0}C + \sqrt{1-\gamma_0}G\Big).
\end{aligned} \tag{5.2}$$

In Example 4.4.4 we have also emphasized the special case of $X^F$, obtained from the natural setup of repeated quantum interactions. The corresponding generator is given by $F$ as in (5.1), where

$$\begin{aligned}
F_0^0 &= -\mathrm{i}H_{\mathrm{sys}} - \mathrm{i}\omega(H_{\mathrm{par}})I_{\mathfrak{h}} \\
&\quad -\frac{1}{2}\Big(\gamma_0 V^*V + (1-\gamma_0)E_{\tilde{\xi}}(VV^* \otimes I_{\overline{\mathsf{k}}})E_{\tilde{\xi}}\Big), \\
F_+^0 &= -\mathrm{i}\Big(\sqrt{\gamma_0}\begin{bmatrix} V^* & 0 \end{bmatrix} + \sqrt{1-\gamma_0}\begin{bmatrix} 0 & E^{\tilde{\xi}}(V \otimes I_{\overline{\mathsf{k}}}) \end{bmatrix}\Big), \\
F_0^+ &= -\mathrm{i}\left(\sqrt{\gamma_0}\begin{bmatrix} V \\ 0 \end{bmatrix} + \sqrt{1-\gamma_0}\begin{bmatrix} 0 \\ (V^* \otimes I_{\overline{\mathsf{k}}})E_{\tilde{\xi}} \end{bmatrix}\right).
\end{aligned} \tag{5.3}$$

for some $V \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$ and self-adjoint $H_{\mathrm{sys}} \in \mathcal{B}(\mathfrak{h})$, $H_{\mathrm{par}} \in \mathcal{B}(\mathsf{k})$.

The results in this section will answer the question when the generators $F$ defined in (5.1) with matrix coefficients as in (5.2) or (5.3) can be written in the form (2.30), and therefore when the unitary QS cocycle $X^F$ is a $\Sigma$-quasifree cocycle (Definition 2.3.15).

**Remark 5.2.1.** Recall that $j\colon \mathsf{k} \to \overline{\mathsf{k}}$ stands for the conjugation, that is, it is given by $c \mapsto \overline{c}$.

**Theorem 5.2.2.** *Let*

$$Z = \begin{bmatrix} \sqrt{I+T^2} & 0 \\ 0 & jTj^* \end{bmatrix}$$

*be as Examples 2.2.4 for some positive operator $T \in \mathcal{B}(\mathsf{k})$, and let $F = \begin{bmatrix} F_0^0 & F_+^0 \\ F_0^+ & 0 \end{bmatrix}$ be as in (5.2), where $H_{\mathsf{sys}} \in \mathcal{B}(\mathfrak{h})$ and $H_{\mathsf{par}} \in \mathcal{B}(\mathsf{K})$ are self-adjoint, $C := \mathrm{i}\begin{bmatrix} M \\ N \end{bmatrix}$*

*and* $G := \mathrm{i}\begin{bmatrix} R \\ S \end{bmatrix}$ *for some* $M,\ R \in \mathcal{B}(\mathfrak{h};\mathfrak{h}\otimes\mathsf{k})$ *and* $N,\ S \in \mathcal{B}(\mathfrak{h};\mathfrak{h}\otimes\overline{\mathsf{k}})$.
*If*

- $H = H_{\mathsf{sys}} + \omega(H_{\mathsf{par}})I_{\mathfrak{h}}$,
- $\sqrt{\gamma_0}M + \sqrt{1-\gamma_0}R = \left(I_{\mathfrak{h}} \otimes \sqrt{I+T^2}\right)L$,
- $\sqrt{\gamma_0}N + \sqrt{1-\gamma_0}S = -\left(I_{\mathfrak{h}} \otimes jTj^*\right)L^{\mathsf{c}}$ *for some* $L \in \mathcal{B}(\mathfrak{h},\mathfrak{h}\otimes\mathsf{k})$,

*then the unitary Markov-regular QS cocycle with generator* $F$ *is quasifree with covariance* $I_{L^2(\mathbb{R}_+)} \otimes Z$.

*Proof.* Let $L \in \mathcal{B}(\mathfrak{h},\mathfrak{h}\otimes\mathsf{k})$. We obtain

$$\begin{aligned}
F_0^+ &= -\mathrm{i}\left(\sqrt{\gamma_0}C + \sqrt{1-\gamma_0}G\right) \\
&= \begin{bmatrix} \sqrt{\gamma_0}M + \sqrt{1-\gamma_0}R \\ \sqrt{\gamma_0}N + \sqrt{1-\gamma_0}S \end{bmatrix} \\
&= \left(I_{\mathfrak{h}} \otimes \begin{bmatrix} \sqrt{I+T^2} & 0 \\ 0 & jTj^* \end{bmatrix}\right)\begin{bmatrix} L \\ -L^{\mathsf{c}} \end{bmatrix} \\
&= (I_{\mathfrak{h}} \otimes Z)\begin{bmatrix} L \\ -L^{\mathsf{c}} \end{bmatrix}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
F_+^0 &= \begin{bmatrix} \sqrt{\gamma_0}M^* + \sqrt{1-\gamma_0}R^* & \sqrt{\gamma_0}N^* + \sqrt{1-\gamma_0}S^* \end{bmatrix} \\
&= \begin{bmatrix} L^* & -L^{\mathsf{T}} \end{bmatrix}(I_{\mathfrak{h}} \otimes Z).
\end{aligned}$$

Now, observe that

$$
\begin{aligned}
&-\frac{1}{2}\Big(\gamma_0 C^*C + \sqrt{\gamma_0(1-\gamma_0)}(C^*G + G^*C) + (1-\gamma_0)G^*G\Big) \\
&= -\frac{1}{2}(\sqrt{\gamma_0}C^* + \sqrt{1-\gamma_0}G^*)(\sqrt{\gamma_0}C^* + \sqrt{1-\gamma_0}G^*) \\
&= -\frac{1}{2}\begin{bmatrix} \sqrt{\gamma_0}M^* + \sqrt{1-\gamma_0}R^* & \sqrt{\gamma_0}N^* + \sqrt{1-\gamma_0}S^* \end{bmatrix}\begin{bmatrix} \sqrt{\gamma_0}M + \sqrt{1-\gamma_0}R \\ \sqrt{\gamma_0}N + \sqrt{1-\gamma_0}S \end{bmatrix} \\
&= -\frac{1}{2}\begin{bmatrix} L^* & -L^{\mathsf{T}} \end{bmatrix}(I_{\mathfrak{h}} \otimes Z^2)\begin{bmatrix} L \\ -L^{\mathsf{c}} \end{bmatrix}.
\end{aligned}
$$

Hence, we obtain that

$$
F_0^0 = -\mathrm{i}H - \frac{1}{2}\begin{bmatrix} L^* & -L^{\mathsf{T}} \end{bmatrix}(I_{\mathfrak{h}} \otimes Z^2)\begin{bmatrix} L \\ -L^{\mathsf{c}} \end{bmatrix}.
$$

Therefore by virtue of Corollary 2.3.18, the unitary QS cocycle with generator (5.1) is $\Sigma_T$-quasifree. □

All the remaining results in this chapter will exploit Corollary 2.3.18. Henceforth, $N \in \mathbb{N} \cup \{\infty\}$.

**Corollary 5.2.3.** *Let*

$$
Z = \begin{bmatrix} \sqrt{\gamma_0}\sum_{i=1}^{N} \alpha_i \,|e_i\rangle\langle e_i| & 0 \\ 0 & \sum_{i=1}^{N} \sqrt{\gamma_i}\alpha_i \,|\overline{e_i}\rangle\langle\overline{e_i}| \end{bmatrix}
$$

*for some* $(\alpha_n) \in \ell^\infty$ *and let* $F = \begin{bmatrix} F_0^0 & F_+^0 \\ F_0^+ & 0 \end{bmatrix}$ *be as in (5.3), where* $H_{\mathsf{sys}} \in \mathcal{B}(\mathfrak{h})$ *and* $H_{\mathsf{par}} \in \mathcal{B}(\mathsf{K})$ *are self-adjoint, and* $V \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$.

*Then the unitary Markov-regular QS cocycle with generator* $F$ *is quasifree with convariance* $I_{L^2(\mathbb{R}_+)} \otimes Z$ *if and only if*

$$
V = -\mathrm{i}\Big(I_{\mathfrak{h}} \otimes \sum_{i=1}^{N} \alpha_i \,|e_i\rangle\langle e_i|\Big)L
$$

*for some* $L \in \mathcal{B}(\mathfrak{h}, \mathfrak{h} \otimes \mathsf{k})$ *and* $\alpha_n = \frac{1}{\sqrt{\gamma_0 - \gamma_n}}$ *for each* $n \geqslant 1$.

*Proof.* According to the notation used in Theorem 5.2.2, $V = \mathrm{i}M$, $N = R = 0$ and by applying Lemma 5.1.1

$$S = -\frac{1}{\sqrt{1-\gamma_0}} \left( I_{\mathfrak{h}} \otimes \sum_{i=1}^{N} \sqrt{\gamma_i} \left| \overline{e_i} \right\rangle \left\langle \overline{e_i} \right| \right) M^{\mathrm{c}}.$$

Set $T := \sum_{i=1}^{N} \beta_i \left| e_i \right\rangle \left\langle e_i \right|$ for some sequence $(\beta_n)$ of positive numbers. Observe that

$$\begin{aligned} \sqrt{\gamma_0} M &= \left( I_{\mathfrak{h}} \otimes \sqrt{I + T^2} \right) L \\ \sqrt{1-\gamma_0} S &= -\left( I_{\mathfrak{h}} \otimes jTj^* \right) L^{\mathrm{c}} \end{aligned}$$

if and only if $M = (I_{\mathfrak{h}} \otimes \sum_{i \geqslant 1} \alpha_i \left| e_i \right\rangle \left\langle e_i \right|) L$ for some $L \in \mathcal{B}(\mathfrak{h}, \mathfrak{h} \otimes \mathsf{k})$ and a sequence $(\alpha_n)$ of positive numbers such that

$$\begin{aligned} \sqrt{\gamma_0} \alpha_i &= \sqrt{1 + \beta_i^2} \\ \sqrt{\gamma_i} \alpha_i &= \beta_i . \end{aligned}$$

Hence,

$$\begin{aligned} \sqrt{\gamma_0} M &= \left( I_{\mathfrak{h}} \otimes \sqrt{I + T^2} \right) L \\ \sqrt{1-\gamma_0} S &= -\left( I_{\mathfrak{h}} \otimes jTj^* \right) L^{\mathrm{c}} \end{aligned}$$

if and only if

$$V = -\mathrm{i}M = -\mathrm{i}\Big( I_{\mathfrak{h}} \otimes \sum_{i=1}^{N} \alpha_i \left| e_i \right\rangle \left\langle e_i \right| \Big) L,$$

where

- $\alpha_n = \frac{1}{\sqrt{\gamma_0 - \gamma_n}}$ and
- $T = \sum_{i=1}^{N} \frac{\sqrt{\gamma_i}}{\sqrt{\gamma_0 - \gamma_i}} \left| e_i \right\rangle \left\langle e_i \right|$.

□

Note that $T = \sum_{i=1}^{N} \frac{\sqrt{\gamma_i}}{\sqrt{\gamma_0 - \gamma_i}} \left|e_i\right\rangle \left\langle e_i\right|$ is a Hilbert–Schmidt operator.

**Example 5.2.4.** Let $\mathsf{k} = \mathbb{C}$ and the covariance be induced by a number $\sigma$ such that $\sigma^2 > 1$ and

$$\sigma^2 = \lambda^2 + \mu^2 \quad \text{and} \quad \lambda^2 - \mu^2 = 1,$$

where $\lambda$ and $\mu$ are positive real numbers. According to Theorem 2.2.3 we obtain a representation of $\mathrm{CCR}(\mathcal{F}^{\mathbb{C}})$

$$w_f \mapsto W(f) := W_0(\lambda f) \oplus W_0(-\mu \overline{f})$$

and a gaug-invariant quasifree state $\varphi$ given by

$$\varphi(w_f) = \langle \Omega, W(f)\Omega \rangle = e^{-\frac{1}{2}\sigma^2 \|f\|^2}.$$

Now, by applying the preceding corollary, to calculate $\sigma$ we will have to solve the following system of equations

$$\begin{cases} \gamma_0 \alpha^2 + (1-\gamma_0)\alpha^2 = \sigma^2 \\ \gamma_0 \alpha^2 - (1-\gamma_0)\alpha^2 = 1 \end{cases},$$

for some $\alpha \in \mathbb{C}$.

Hence, we obtain

$$\sigma = \frac{1}{\sqrt{2\gamma_0 - 1}}, \qquad \lambda = \frac{\sqrt{\gamma_0}}{\sqrt{2\gamma_0 - 1}}, \qquad \mu = \frac{\sqrt{1-\gamma_0}}{\sqrt{2\gamma_0 - 1}}.$$

**Theorem 5.2.5.** *Let*

$$Z = \begin{bmatrix} \sqrt{I + T^2}U \cosh(D) & -\sqrt{I + T^2}UK \sinh(D) j^* \\ -jTUK \sinh(D) & jTU \cosh(D) j^* \end{bmatrix}$$

*be as in Example 2.2.5 for some positive operators $D$, $T \in \mathcal{B}(\mathsf{k})$, unitary $U \in \mathcal{B}(\mathsf{k})$*

*and an involution $K$ on $\mathsf{k}$, and let $F = \begin{bmatrix} F_0^0 & F_+^0 \\ F_0^+ & 0 \end{bmatrix}$ be as in (5.2), where $H_{\mathsf{sys}} \in \mathcal{B}(\mathfrak{h})$ and $H_{\mathsf{par}} \in \mathcal{B}(\mathsf{K})$ are self-adjoint, $C := \mathrm{i}\begin{bmatrix} M \\ N \end{bmatrix}$ and $G := \mathrm{i}\begin{bmatrix} R \\ S \end{bmatrix}$ for some $M$, $R \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$ and $N$, $S \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \overline{\mathsf{k}})$.*
*If*

- $H = H_{\mathsf{sys}} + \omega(H_{\mathsf{par}}) I_{\mathfrak{h}}$.
- $$\begin{aligned} \left(\sqrt{\gamma_0} M + \sqrt{1-\gamma_0} R\right) &= \left(I_{\mathfrak{h}} \otimes \sqrt{I + T^2} U \cosh(D)\right) L \\ &+ \left(I_{\mathfrak{h}} \otimes \sqrt{I + T^2} U K \sinh(D) j^*\right) L^{\mathsf{c}} \end{aligned},$$
- $$\begin{aligned} \left(\sqrt{\gamma_0} N + \sqrt{1-\gamma_0} S\right) &= -\left(I_{\mathfrak{h}} \otimes jTUK \sinh(D)\right) L \\ &- \left(I_{\mathfrak{h}} \otimes jTU \cosh(D) j^*\right) L^{\mathsf{c}} \end{aligned}$$

*for some $L \in \mathcal{B}(\mathfrak{h}, \mathfrak{h} \otimes \mathsf{k})$, then the unitary Markov-regular QS cocycle with generator $F$ is quasifree with convariance $I_{L^2(\mathbb{R}_+)} \otimes Z$.*

*Proof.* Let $M$, $N$, $R$ and $S$ be as claimed in the theorem and take an arbitrary $L \in \mathcal{B}(\mathfrak{h}, \mathfrak{h} \otimes \mathsf{k})$. Then we arrive at

$$\begin{aligned} F_0^+ &= -\mathrm{i}\left(\sqrt{\gamma_0} C + \sqrt{1-\gamma_0} G\right) \\ &= \begin{bmatrix} \gamma_0 M + \sqrt{1-\gamma_0^2} R \\ \gamma_0 N + \sqrt{1-\gamma_0^2} S \end{bmatrix} \\ &= \begin{bmatrix} \left(I_{\mathfrak{h}} \otimes \sqrt{I+T^2} U \cosh(D)\right) L + \left(I_{\mathfrak{h}} \otimes \sqrt{I+T^2} UK \sinh(D) j^*\right) L^{\mathsf{c}} \\ -\left(I_{\mathfrak{h}} \otimes jTUK \sinh(D)\right) L - \left(I_{\mathfrak{h}} \otimes jTU \cosh(D) j^*\right) L^{\mathsf{c}} \end{bmatrix} \\ &= \left(I_{\mathfrak{h}} \otimes \begin{bmatrix} \sqrt{I+T^2} U \cosh(D) & -\sqrt{I+T^2} UK \sinh(D) j^* \\ -jTUK \sinh(D) & jTU \cosh(D) j^* \end{bmatrix}\right) \begin{bmatrix} L \\ -L^{\mathsf{c}} \end{bmatrix}. \end{aligned}$$

By simple algebraic operations we can show that the other entries of the matrix in (5.1) have the desired form. □

**Corollary 5.2.6.** *Let $K$ be an involution on $\mathsf{k}$, let*

$$Z = \begin{bmatrix} \sum_{i=1}^N \sqrt{\gamma_0} \alpha_i \left|e_i\right\rangle \left\langle e_i\right| & -\sum_{i=1}^N \sqrt{\gamma_0} \beta_i K \left|e_i\right\rangle \left\langle e_i\right| j^* \\ -j \sum_{i=1}^N \sqrt{\gamma_i} \beta_i K \left|e_i\right\rangle \left\langle e_i\right| & j \sum_{i=1}^N \sqrt{\gamma_i} \alpha_i \left|e_i\right\rangle \left\langle e_i\right| j^* \end{bmatrix}$$

*for some* $(\alpha_n) \in \ell^\infty$, $(\beta_n) \in \ell^\infty$ *and let* $F = \begin{bmatrix} F_0^0 & F_+^0 \\ F_0^+ & 0 \end{bmatrix}$ *be as in (5.3), where* $H_{\mathsf{sys}} \in \mathcal{B}(\mathfrak{h})$ *and* $H_{\mathsf{par}} \in \mathcal{B}(\mathsf{K})$ *are self-adjoint, and* $V \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$.

*Then the unitary Markov-regular QS cocycle with generator* $F$ *is quasifree with convariance* $I_{L^2(\mathbb{R}_+)} \otimes Z$ *if and only if*

$$V = -\mathrm{i}\left(I_{\mathfrak{h}} \otimes \sum_{i=1}^{N} \alpha_i \left|e_i\right\rangle \left\langle e_i\right|\right) L - \mathrm{i}\left(I_{\mathfrak{h}} \otimes \sum_{i=1}^{N} \beta_i K \left|e_i\right\rangle \left\langle e_i\right| j^*\right) L^{\mathsf{c}}$$

*for some* $L \in \mathcal{B}(\mathfrak{h}, \mathfrak{h} \otimes \mathsf{k})$, *where*

- $\frac{\beta_i}{\alpha_i} \geqslant 0$ ,
- $|\alpha_n| \geqslant \frac{1}{\sqrt{\gamma_0 - \gamma_n}}$, $|\beta_n| \geqslant 0$,
- $|\alpha_n|^2 - |\beta_n|^2 = \frac{1}{\gamma_0 - \gamma_n}$ *for each* $n$.

*Proof.* According to the notation used in Theorem 5.2.5, the above $V = \mathrm{i}M$, $N = R = 0$ and by applying Lemma 5.1.1

$$S = -\frac{1}{\sqrt{1-\gamma_0}}\left(I_{\mathfrak{h}} \otimes \sum_{i=1}^{N} \sqrt{\gamma_i}\left|\overline{e_i}\right\rangle \left\langle \overline{e_i}\right|\right) M^{\mathsf{c}}.$$

Set

- $T := \sum_{i \geqslant 1} c_i \left|e_i\right\rangle \left\langle e_i\right|$, where each $c_i \geqslant 0$,
- $D = \sum_{i \geqslant 1} |d_i| \left|e_i\right\rangle \left\langle e_i\right|$, where $\{d_n\} \in \ell^\infty$,
- $U = \sum_{i \geqslant 1} w_i \left|e_i\right\rangle \left\langle e_i\right|$, where each $w_i \in \mathbb{C}$ is such that $|w_i| = 1$.

Observe that

$$\sqrt{\gamma_0} M = \left(I_{\mathfrak{h}} \otimes \sqrt{I + T^2} U \cosh(D)\right) L + \left(I_{\mathfrak{h}} \otimes \sqrt{I + T^2} U K \sinh(D) j^*\right) L^{\mathsf{c}}$$
$$\sqrt{1-\gamma_0} S = -\left(I_{\mathfrak{h}} \otimes jTUK \sinh(D)\right) L - \left(I_{\mathfrak{h}} \otimes jTU \cosh(D) j^*\right) L^{\mathsf{c}}$$

if and only if $M = \left(I_{\mathfrak{h}} \otimes \sum_{i=1}^{N} \alpha_i \left|e_i\right\rangle \left\langle e_i\right|\right) L + \left(I_{\mathfrak{h}} \otimes \sum_{i=1}^{N} \beta_i K \left|e_i\right\rangle \left\langle e_i\right| j^*\right) L^{\mathsf{c}}$ for some $L \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$ and sequences $(\alpha_n)$, $(\beta_n) \in \ell^\infty$ such that

$$\begin{array}{rcl} \sqrt{\gamma_0}\alpha_i &=& \sqrt{1+c_i^2}w_i\cosh(|d_i|) \\ \sqrt{\gamma_0}\beta_i &=& \sqrt{1+c_i^2}w_i\sinh(|d_i|) \\ \sqrt{\gamma_i}\beta_i &=& c_i w_i\sinh(|d_i|) \\ \sqrt{\gamma_i}\alpha_i &=& c_i w_i\cosh(|d_i|) \end{array}.$$

Hence,

$$\sqrt{\gamma_0}M = \left(I_{\mathfrak{h}} \otimes \sqrt{I+T^2}U\cosh(D)\right)L + \left(I_{\mathfrak{h}} \otimes \sqrt{I+T^2}UK\sinh(D)j^*\right)L^{\mathsf{c}}$$
$$\sqrt{1-\gamma_0}S = -\left(I_{\mathfrak{h}} \otimes jTUK\sinh(D)\right)L - \left(I_{\mathfrak{h}} \otimes jTU\cosh(D)j^*\right)L^{\mathsf{c}}$$

if and only if

$$V = -\mathrm{i}M = -\mathrm{i}\left(I_{\mathfrak{h}} \otimes \sum_{i=1}^{N}\alpha_i\,|e_i\rangle\,\langle e_i|\right)L - \mathrm{i}\left(I_{\mathfrak{h}} \otimes \sum_{i=1}^{N}\beta_i K\,|e_i\rangle\,\langle e_i|\,j^*\right)L^{\mathsf{c}},$$

where

- $|\alpha_i|^2 - |\beta_i|^2 = \frac{1}{\gamma_0-\gamma_i}$ for each $i$,
- $\frac{\beta_i}{\alpha_i} \geqslant 0$,
- $T = \sum_{i=1}^{N}\frac{\sqrt{\gamma_i}}{\sqrt{\gamma_0-\gamma_i}}\,|e_i\rangle\,\langle e_i|$,
- $U = \sum_{i=1}^{N}\frac{\alpha_i}{|\alpha_i|}\,|e_i\rangle\,\langle e_i| = \sum_{i=1}^{N}\frac{\beta_i}{|\beta_i|}\,|e_i\rangle\,\langle e_i|$ ,
- $\cosh(D) = \sum_{i=1}^{N}|\alpha_i|\sqrt{\gamma_0-\gamma_i}\,|e_i\rangle\,\langle e_i|$,
- $\sinh(D) = \sum_{i=1}^{N}|\beta_i|\sqrt{\gamma_0-\gamma_i}\,|e_i\rangle\,\langle e_i|$.

□

# Appendix A

Let $\mathcal{A}$ be a unital Banach algebra with the unit $\mathbf{1}$.

**Lemma A.0.7.** *Let $a \in \mathcal{A}$ and let $b\colon (0,\infty) \to \mathcal{A}$ be such that*

$$\lim_{h\to 0^+} \|b(h) - a\| = 0$$

*then*

$$\lim_{h\to 0^+} \sup_{t\in[0,T]} \|e^{ta} - e^{tb(h)}\| = 0,$$

*for all $T \geqslant 0$.*

*Proof.* Note first that for any $k \in \mathbb{N}$ we have

$$\begin{aligned} &a^k - b(h)^k \\ =&(a-b(h))a^{k-1} + b(h)(a-b(h))a^{k-2} + \ldots + b(h)^{k-1}(a-b(h)). \end{aligned}$$

By taking the norm of the above expression and applying the triangle inequality we obtain

$$\begin{aligned} &\|a^k - b(h)^k\| \\ \leqslant&\|a-b(h)\|(\|a\|^{k-1} + \|b(h)\|\|a\|^{k-2} + \ldots + \|b(h)\|^{k-2}\|a\| + \|b(h)\|^{k-1}). \end{aligned}$$

Fix $T > 0$, let $\delta > 0$ be such that $\|b(h)\| \leqslant \|a\| + 1$ whenever $0 < h < \delta$ and take

$t \in [0, T]$. If $0 < h < \delta$ then we arrive at

$$
\begin{aligned}
\|e^{tb(h)} - e^{ta}\| &\leqslant \sum_{k \geqslant 1} \frac{t^k \|a^k - b(h)^k\|}{k!} \\
&\leqslant t\|a - b(h)\| \sum_{k \geqslant 1} \frac{t^{k-1}(\|a\| + 1)^{k-1}}{k!} \\
&\leqslant t\|a - b(h)\| \sum_{k \geqslant 0} \frac{t^k(\|a\| + 1)^k}{k!} \\
&\leqslant t\|a - b(h)\| e^{t(\|a\|+1)} \\
&\leqslant T\|a - b(h)\| e^{T(\|a\|+1)} \\
&\overset{h \to 0^+}{\longrightarrow} 0.
\end{aligned}
$$

□

**Lemma A.0.8.** *Let $x_0 \in \mathbb{R}_+$ and let $x \colon (0, \infty) \to \mathbb{R}_+$ be such that*

$$\lim_{h \to 0^+} x(h) = x_0.$$

*Then*

$$\lim_{h \to 0^+} \sup_{t \in [0,T]} \left| e^{tx(h)} - (1 + hx(h))^{\lfloor \frac{t}{h} \rfloor} \right| = 0$$

*for all $T \geqslant 0$, where $\lfloor \frac{t}{h} \rfloor := \max\{m \in \mathbb{Z} : m \leqslant \frac{t}{h}\}$.*

*Proof.* The proof is inspired by [85, Example 2.3, p. 32].

First note that since $\binom{\lfloor \frac{t}{h} \rfloor}{k} h^k \leqslant \frac{\lfloor \frac{t}{h} \rfloor^k}{k!} h^k \leqslant \frac{t^k}{k!}$ for all $k \in \{0, \ldots, \lfloor \frac{t}{h} \rfloor\}$ and $\frac{t}{h} - 1 \leqslant \lfloor \frac{t}{h} \rfloor$ then we have

$$0 \leqslant e^{tx(h)} - (1 + hx(h))^{\lfloor \frac{t}{h} \rfloor} \leqslant e^{tx(h)} - (1 + hx(h))^{\frac{t}{h} - 1}.$$

for all $t \geqslant 0$.

Now assume that $T > 1$ and let $t \in [0, T]$. Let $1 \leqslant \alpha < \beta < \infty$ define a function $f \colon [\alpha, \beta] \to \mathbb{R}_+$ by setting $f(z) = z^t$. It is clearly continuous and differentiable on $[\alpha, \beta]$. Hence, by the mean value theorem there exists $\gamma \in (\alpha, \beta)$

such that

$$\frac{\beta^t - \alpha^t}{\beta - \alpha} = f'(\gamma).$$

We obtain

$$\left|\beta^t - \alpha^t\right| = f'(\gamma)(\beta-\alpha) = t\gamma^{t-1}(\beta-\alpha) \leqslant T\gamma^{T-1}(\beta-\alpha) \leqslant T\beta^{T-1}(\beta-\alpha). \quad \text{(A.1)}$$

Let $\delta_1 > 0$ be such that $e^{x(h)} \leqslant e^{x_0} + 1$ whenever $0 < h < \delta_1$ and let $\delta_2 > 0$ be such that $(1 + hx(h))^{\frac{1}{h}} \leqslant e^{x_0} + 1$ whenever $0 < h < \delta_2$.
If $0 < h < \delta$, where $\delta = \min\{\delta_1, \delta_2\}$ then we arrive at

$$e^{x(h)} \leqslant e^{x_0} + 1 \quad \text{and} \quad (1 + hx(h))^{\frac{1}{h}} \leqslant e^{x_0} + 1.$$

Therefore,

$$\begin{aligned}
&\left|e^{tx(h)} - (1 + hx(h))^{\lfloor \frac{t}{h} \rfloor}\right| \\
\leqslant &\left|e^{tx(h)} - (1 + hx(h))^{\frac{t}{h}-1}\right| \\
= &\left|e^{tx(h)} - (1 + hx(h))^{\frac{t}{h}} \left(1 + hx(h))^{-1}\right)\right| \\
\leqslant &(1 + hx(h))^{-1} \left|(e^{x(h)})^t - \left((1 + hx(h))^{\frac{1}{h}}\right)^t\right| + (1 + hx(h))^{-1} hx(h)e^{Tx(h)}.
\end{aligned}$$

Hence, by applying (A.1) we obtain

$$\begin{aligned}
&(1 + hx(h))^{-1} \left|(e^{x(h)})^t - \left((1 + hx(h))^{\frac{1}{h}}\right)^t\right| + (1 + hx(h))^{-1} hx(h)e^{Tx(h)} \\
\leqslant &T(e^{x_0} + 1)^T (1 + hx(h))^{-1} \left|e^{x(h)} - (1 + hx(h))^{\frac{1}{h}}\right| + (1 + hx(h))^{-1} hx(h)e^{Tx(h)} \\
\to &0 \quad \text{as } h \to 0^+.
\end{aligned}$$

□

**Theorem A.0.9** (Euler's formula)**.** *Let $a \in \mathcal{A}$ and let $b\colon (0, \infty) \to \mathcal{A}$ be such that*

$$\lim_{h\to 0^+} \|b(h) - a\| = 0.$$

*Then*

$$\lim_{h\to 0^+} \sup_{t\in[0,T]} \left\| (1 + hb(h))^{\lfloor \frac{t}{h} \rfloor} - e^{ta} \right\| = 0 \tag{A.2}$$

*for all* $T \geqslant 0$.

*Proof.* For each $n \in \mathbb{N}$ let $c(k,n) := \begin{cases} \binom{n}{k} & \text{if } k \in \{0,\dots,n\} \\ 0 & \text{otherwise} \end{cases}$.

We have

$$\left\| e^{ta} - (1 + hb(h))^{\lfloor \frac{t}{h} \rfloor} \right\| \leqslant \left\| e^{ta} - e^{tb(h)} \right\| + \left\| e^{tb(h)} - (1 + hb(h))^{\lfloor \frac{t}{h} \rfloor} \right\|,$$

where the first term of the right-hand side tends to 0 as $h \to 0^+$, uniformly in $t$ by Lemma A.0.7.

Now, since $\binom{n}{k} \leqslant \frac{n^k}{k!}$ for $1 \leqslant k \leqslant n$ then for each $t \in \mathbb{R}_+$ the expression

$$\frac{t^k}{k!} - c\left(k, \left\lfloor \frac{t}{h} \right\rfloor\right) h^k$$

is always non-negative. Therefore, we obtain

$$\begin{aligned} \left\| e^{tb(h)} - (1 + hb(h))^{\lfloor \frac{t}{h} \rfloor} \right\| &\leqslant \sum_{k\geqslant 0} \left( \frac{t^k}{k!} - c\left(k, \left\lfloor \frac{t}{h} \right\rfloor\right) h^k \right) \|b(h)\|^k \\ &= e^{t\|b(h)\|} - (1 + h\|b(h)\|)^{\lfloor \frac{t}{h} \rfloor} \\ &\overset{h\to 0^+}{\to} 0. \end{aligned}$$

By applying Lemma A.0.10 we obtain the uniform convergence of (A.2) for $t$ on each compact subinterval of $\mathbb{R}_+$. □

Let $\mathcal{A} = \mathcal{B}(X)$, where $X$ is a Banach space.

**Corollary A.0.10** (Euler's formula for semigroups)**.** *Let* $(T_t)_{t\geqslant 0}$ *be a norm continuous semigroup of operators on* $X$ *with generator* $A \in \mathcal{B}(X)$.

*If* $B\colon (0,\infty) \to \mathcal{B}(X)$ *is such that*

$$\lim_{h\to 0^+} \|B(h) - A\| = 0$$

*then*

$$\lim_{h\to 0^+} \sup_{t\in[0,T]} \left\|(I + hB(h))^{\lfloor \frac{t}{h} \rfloor} - T_t\right\| = 0 \tag{A.3}$$

*for all* $T \geqslant 0$.

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