Elementary Evolutions in Banach Algebra

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Abstract. An elementary class of evolutions in unital Banach algebras is obtained by integrating product functions over Guichardet’s symmetric measure space on the half-line. These evolutions, along with a useful subclass, are characterised and a Lie–Trotter product formula is proved. The class is rich enough to form the basis for a recent approach to quantum stochastic evolutions.

Introduction

In this note we identify and analyse a simple class of evolutions in unital Banach algebras along with a useful subclass. They have infinitesimal generators, in terms of which they are characterised, and we establish a Lie–Trotter product formula for such evolutions. Our approach is via Guichardet’s symmetric measure space ([Gui]) of the Lebesgue space \( \mathbb{R}^+ \). Apart from the merits of simplicity, one motivation is the fact that the theory forms the basis for a recent approach to quantum stochastic evolutions ([DLT], [DL]) in which quantum stochastic Trotter product formulae are proved (cf. [LSi]), characterisations of stochastic cocycles are established (cf. [LSk]) and convergence theorems for scaled quantum random walks are proved (cf. [Bel]).

After a brief section of preliminaries where notations are fixed, the basic theory occupies Section 2, and the product formula is proved in Section 3.

1. Preliminaries

For a step function \( f \) with domain \( \mathbb{R}^+ = [0, \infty] \) we write Disc \( f \) for the (possibly empty) complement of the set of points \( t \) where \( f \) is constant in some neighbourhood of \( t \); for a vector-valued function \( f \) on \( \mathbb{R}^+ \) and subinterval \( J \) of \( \mathbb{R}^+ \), \( f_J \) denotes the function on \( \mathbb{R}^+ \) which agrees with \( f \) on \( J \) and vanishes outside \( J \). For a Banach space \( X \), \( B(X) \) denotes the unital Banach algebra of bounded operators on \( X \). The symbol \( \sim \) is used (for both elements of, and subsets of, an algebra) to denote ‘commutes with’ ([RSz]), \( \# \) denotes cardinality, and \( \subset \subset \) stands for subset of finite cardinality. For sets \( A \) and \( B \), we write \( F(A; B) \) rather than \( B^A \), for the set of functions from \( A \) to \( B \), and for \( f \in F(A; B) \), we denote its range, \( f(A) \), by \( \text{Ran} f \).

Finally, we use the following notation for simplices: for \( n \in \mathbb{N} \) and \( t \geq r \geq 0 \), set

\[
\Delta^{(n)}_{[r,t]} := \{ a \in [r,t]^{-n} : a_1 < \cdots < a_n \} \quad \text{and} \quad \Delta^{[n]} := \{ a \in (\mathbb{R}^+)^n : a_1 \leq \cdots \leq a_n \}.
\]

The uniqueness result below will serve us well. In Section 2 we give a very convenient representation of the equation’s well-known solution.

**Theorem 1.1.** Let \( x_0 \in \mathcal{X} \) and \( a \in L^1_{\text{loc}}(\mathbb{R}^+; \mathcal{A}) \) for a right Banach \( \mathcal{A} \)-module \( \mathcal{X} \).

(a) The following integral equation has at most one solution \( f \in C(\mathbb{R}^+; \mathcal{X}) \):

\[
f(t) = x_0 + \int_0^t ds f(s) a(s) \quad (t \in \mathbb{R}^+). \tag{1.1}
\]
(b) Let $f \in C(\mathbb{R}^+; \mathcal{A})$. Then $f$ satisfies (1.1) if
\[
f(0) = x_0 \text{ and } f'(s) = f(s)a(s) \quad (s \in \mathbb{R}^+ \setminus \mathcal{N}),
\]
for a Lebesgue-null Borel subset $\mathcal{N}$ of $\mathbb{R}^+$ satisfying Haus $f(\mathcal{N}) = 0$, where Haus denotes one-dimensional outer Hausdorff measure.

(a) is straightforward and classical; for a proof of (b), see [Vol]. The condition Haus $f(\mathcal{N}) = 0$ is automatic if either $\mathcal{N}$ is countable or $\mathcal{N}$ is Lebesgue-null and $f$ is locally Lipschitz; for us, $a$ will be a step function, so that $\mathcal{N}$ is finite.

2. Evolutions in Banach algebra

In this section we consider norm-continuous evolutions in a unital Banach algebra and analyse two sub-classes. To this end we introduce Guichardet’s symmetric measure space of the Lebesgue spaces of subintervals of $\mathbb{R}^+$.

For the rest of the paper $\mathcal{A}$ is a fixed unital Banach algebra; its group of invertible elements is denoted $\mathcal{A}^\times$.

**Definition.** An evolution $E$ in $\mathcal{A}$ is a family $(E_{r,t})_{0 \leq r \leq t}$ in $\mathcal{A}$, or function from $\Delta^2$ to $\mathcal{A}$, such that
\[
E_{r,r} = 1_{\mathcal{A}} \text{ and } E_{r,s} E_{s,t} = E_{r,t} \quad (0 \leq r \leq s \leq t);
\]
The class of evolutions in $\mathcal{A}$ is denoted $\text{Evol}(\mathcal{A})$.

**Example.** Let $\alpha = (\alpha_t)_{t \geq 0}$ be an $E_0$-semigroup on a von Neumann algebra $\mathcal{M}$, that is, a one-parameter semigroup of endomorphisms of $\mathcal{M}$ (which is pointwise ultraweakly continuous), and let $V = (V_t)_{t \geq 0}$ be a family of contractions in $\mathcal{M}$ forming an $\alpha$-cocycle, thus $V_0 = 1$ and $V_{s+t} = V_s \alpha_s(V_t)$ ($s,t \geq 0$) ([Arv]). Then the family $(\alpha_t(V_{t-r}))_{0 \leq r \leq t}$ forms an evolution in $\mathcal{M}$.

A family $(E_{r,t})_{0 \leq r \leq t}$ in $\mathcal{A}$ is called an opposite evolution if instead
\[
E_{r,r} = 1_{\mathcal{A}} \text{ and } E_{r,s} E_{s,t} = E_{r,t} \quad (0 \leq r \leq s \leq t).
\]
An evolution is invertible if it is $\mathcal{A}^\times$-valued, and continuous, respectively Lipschitz, if the following maps are continuous, respectively Lipschitz continuous,
\[
[r, \infty) \to \mathcal{A}, \quad s \mapsto E_{r,s} \quad \text{and} \quad [0,t] \to \mathcal{A}, \quad s \mapsto E_{s,t} \quad (r,t \in \mathbb{R}_+).
\]
We denote these classes by $\text{Evol}(\mathcal{A}^\times)$, $\text{Evol}_c(\mathcal{A})$ and $\text{Evol}_{lc}(\mathcal{A})$ respectively.

**Remarks.** For $E \in \text{Evol}(\mathcal{A}^\times)$, $((E_{r,t})^{-1})_{0 \leq r \leq t}$ defines an opposite evolution; also $E$ extends to an evolution $(E_{r,t})_{r \leq t}$ (where $r$ and $t$ now range over $\mathbb{R}$) by the prescription
\[
E_{r,t} := \phi_r^{-1} \phi_t \quad \text{where} \quad \phi_s := \begin{cases} E_{0,s} & \text{if } s \geq 0 \\ (E_{0,-s})^{-1} & \text{if } s \leq 0. \end{cases}
\]

**Proposition 2.1.** The map \( \{ \phi \in F(\mathbb{R}^+; \mathcal{A}^\times) : \phi(0) = 1_{\mathcal{A}} \} \to \text{Evol}(\mathcal{A}^\times) \) given by \( \phi \mapsto (\phi_r^{-1} \phi_t)_{0 \leq r \leq t} \) is bijective, and restricts to a bijection
\[
\{ \phi \in C(\mathbb{R}^+; \mathcal{A}^\times) : \phi(0) = 1_{\mathcal{A}} \} \to \text{Evol}_c(\mathcal{A}).
\]

**Proof.** All that needs to be proved is that if $E \in \text{Evol}_c(\mathcal{A})$, then $E_{0,t} \in \mathcal{A}^\times$ for all $t \in \mathbb{R}_+$. Thus let $E \in \text{Evol}_c(\mathcal{A})$ and suppose for a contradiction that $E_{0,s} \notin \mathcal{A}^\times$ for some $s \in \mathbb{R}_+$. Set $t := \inf\{ s \in \mathbb{R}_+ : E_{0,s} \notin \mathcal{A}^\times \}$. In view of the facts that the set $\mathcal{A} \setminus \mathcal{A}^\times$ is closed, the map $s \mapsto E_{0,s}$ is right continuous at 0, and $E_{0,0} = 1_{\mathcal{A}} \in \mathcal{A}^\times$, it follows that $E_{0,t} \notin \mathcal{A}^\times$ and $t > 0$. Since $E_{t,t} = 1_{\mathcal{A}} \in \mathcal{A}^\times$, the openness of $\mathcal{A}^\times$ and left continuity of the map $s \mapsto E_{s,t}$ at $t$ imply that, for small enough $h > 0$, the evolution identity $E_{0,t} = E_{0,t-h} E_{t-h,t}$ expresses a noninvertible element as a product of invertibles, and we have our contradiction. \( \square \)
Remarks. Thus continuous evolutions are invertible, and invertible evolutions are actually one-parameter objects.

Evolutions generalise one-parameter semigroups, in the sense that every (norm-continuous) one-parameter semigroup $(p_t)_{t \geq 0}$ in $\mathcal{A}$ defines a (continuous) evolution $(p_{t-r})_{0 \leq r < t}$. However—in stark contrast to the well-known simple structure of continuous semigroups: $(e^{sa})_{s \geq 0}$ ($a \in \mathcal{A}$) (see e.g. [Rud])—continuous evolutions are continuous one-parameter semigroups $(\mathcal{C}E)$ for $X$ in $\mathcal{B}(X)$ which is exponentially bounded, i.e. where there is $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|E_{r,s}\| \leq Me^{\omega(t-r)}$ $(r \leq t)$, the prescription $(T_t^E)(s) := E_{s-t,s}f(s-t)$ defines a $C_0$-semigroup on the Banach space $\mathcal{C}E$ satisfying $T_t^EM_\varphi = MT_t^E_\varphi T_t^E$ ($\varphi \in \mathcal{C}e_0(\mathbb{R}; X)$, $t \in \mathbb{R}_+$) where $T$ is the right-shift semigroup on $\mathcal{C}e_0(\mathbb{R})$ and $M$ denotes (scalar) multiplication operator; every such semigroup arises in this way (see [EnN]). An interesting question then is—how might norm continuity of an evolution $E$ be recognised in its semigroup $T^E$?

Using Guichardet’s symmetric measure space, we shall embed the class of evolutions given by semigroups in a much wider class. For $(r,t) \in \Delta^2$, set

$$\Gamma_{r,t} := \{ \sigma \in [r,t]: \#\sigma < \infty \} \text{ and } \Gamma^{(n)}_{r,t} := \{ \sigma \subset [r,t]: \#\sigma = n \} \quad (n \in \mathbb{Z}_+),$$

with measurable structure and measure induced from that of Lebesgue measure on each simplex $\Delta^{(n)}_{r,t}$, via the bijection

$$\Delta^{(n)}_{r,t} \to \Gamma^{(n)}_{r,t}, \quad s \mapsto \{s_1, \ldots, s_n\} \quad (n \in \mathbb{N}),$$

and letting $\varnothing \in \Gamma^{(0)}_{r,t}$ be an atom of measure one ([Gui]). Thus $\Gamma^{(n)}_{r,t}$ and $\Gamma_{r,t}$ have measure $(t-r)^n/n!$ and $\exp(t-r)$ respectively. We use the abbreviations $\Gamma$, $\Gamma^{(n)}$, $\Gamma^{\geq 1}$ and $\int ds \sigma$ for $\Gamma_{[0,\infty]}$, $\Gamma^{(n)}_{[0,\infty]}$, $\bigcup_{n \geq 1} \Gamma^{(n)}$ and integration with respect to the symmetric measure on $\Gamma$. Each function $\varphi : \mathbb{R}_+ \to \mathbb{C}$ determines a function

$$\pi_\varphi : \Gamma \to \mathbb{C}, \quad \sigma \mapsto \prod_{s \in \sigma} \varphi(s).$$

Thus $\pi_0 = \delta_\varnothing$ and the mapping $\varphi \mapsto \pi_\varphi$ respects measure equivalence classes. For $\varphi \in L^1(\mathbb{R}_+)$, $\pi_\varphi \in L^1(\Gamma)$, $\int \pi_\varphi = \exp\int \varphi$ and $\|\pi_\varphi\|_1 = \exp\|\varphi\|_1$. In particular, for nonnegative functions $\varphi, \psi \in L^1(\mathbb{R}_+),$

$$\|\pi_{\varphi+\psi}\|_1 = \|\pi_\varphi\|_1\|\pi_\psi\|_1 \text{ and } \|\pi_{\varphi+\psi} - \pi_\varphi\|_1 = \|\pi_\psi\|_1(\|\pi_\varphi\|_1 - 1). \quad (2.1)$$

Remark. For $\varphi \in L^2(\mathbb{R}_+)$, let $\varepsilon_\varphi = (1, \varphi, (2!)^{-1/2}\varphi^2, \ldots)$ denote the exponential vector in the symmetric Fock space $\Gamma(L^2(\mathbb{R}_+))$. Then the prescription

$$\varepsilon_\varphi \mapsto \pi_{\varphi} \quad (\varphi \in L^2(\mathbb{R}_+)),$$

extends to a unitary map $\Gamma(L^2(\mathbb{R}_+)) \to L^2(\Gamma)$. For a Hilbert space $k$, this tensorises to give an isometry from $\Gamma(L^2(\mathbb{R}_+; \varphi)) = \Gamma(L^2(\mathbb{R}_+) \otimes k)$ to $L^2(\Gamma; \Phi_k)$, where $\Phi_k$ denotes the full (unsymmetrised) Fock space over $k$; its image is

$$\{ f \in L^2(\Gamma; \Phi_k) : \forall \sigma \in \Gamma \ f(\sigma) \in k^{\#\sigma} \}.$$

For more on Guichardet space analysis, see [LiP], [Mey] and references therein; a cornerstone is the integral-sum formula which we state next—for a proof see [LiP].

Lemma 2.2. Let $n \in \mathbb{N}$ and $H \in L^1(\Gamma^n; X)$ for a Banach space $X$. Then

$$\int d\sigma_1 \cdots \int d\sigma_n H(\sigma_1, \ldots, \sigma_n) = \int d\sigma \sum H(\sigma_1, \ldots, \sigma_n)$$

where the sum is over all $n^{\#\sigma}$ partitions of $\sigma$ into $n$ subsets $\sigma_1, \ldots, \sigma_n$. 
In particular, for $H \in L^1(\Gamma \times \Gamma; X)$

$$\int d\alpha \int d\beta \ H(\alpha, \beta) = \int d\sigma \sum_{\alpha \subset \sigma} H(\alpha, \sigma \setminus \alpha).$$

Note that the integral-formula for functions $H$ of the form $(\alpha_1, \ldots, \alpha_n) \mapsto \pi_{\varphi_1}(\alpha_1) \cdots \pi_{\varphi_n}(\alpha_n) x$, where $x \in X$ and $\varphi_1, \ldots, \varphi_n \in L^1(\mathbb{R}^+)$, reduces to the simple identity $(\prod_{i=1}^n \exp \int \varphi_i) x = \{ \exp \int \varphi \} x$, where $\varphi = \sum_{i=1}^n \varphi_i$.

The composition of $A$-valued functions on $\Gamma$ defined by

$$f \circ g : \sigma \mapsto \sum_{\alpha \subset \sigma} f(\alpha) g(\sigma \setminus \alpha) \quad (2.2)$$

enjoys the following properties: if $f \subset \Gamma_I$ and supp $g \subset \Gamma_J$ for disjoint sets $I$ and $J$, then

$$(f \circ g)(\sigma) = f(\sigma \cap I) g(\sigma \cap J) \text{ for } \sigma \in \Gamma,$$

whereas, by the integral-formula, if $f, g \in L^1(\Gamma; A)$ then

$$f \circ g \in L^1(\Gamma; A), \quad \int f \circ g = \int f \int g \quad \text{and} \quad \|f \circ g\|_1 \leq \|f\|_1 \|g\|_1. \quad (2.4)$$

**Definition.** Let $a \in L^1_{\text{loc}}(\mathbb{R}^+; A)$. Its associated product functions $\pi_a$ and $\sigma_\pi$ in $L^1_{\text{loc}}(\Gamma; A)$ are defined by $\pi_a(\varnothing) = \sigma_\pi(\varnothing) = 1_A$ and for $\sigma = \{ s_1 < \cdots < s_n \}$, $\pi_a(\sigma) = a(s_1) \cdots a(s_n)$ whereas $\sigma_\pi(\sigma) = a(s_n) \cdots a(s_1)$; in short,

$$\pi_a(\sigma) := \prod_{s \in \sigma} a(s) \quad \text{and} \quad \sigma_\pi(\sigma) := \prod_{s \in \sigma} a(s).$$

For $a \in L^1_{\text{loc}}(\mathbb{R}^+; A)$ define $E^a$ and $\partial E$ in $C(\Delta^2; A)$ as follows.

$$E^a_{r,t} := \int_{[r, t]} \pi_a = \int_{[\pi_r, \pi_t]} \pi_a \quad \text{and} \quad \sigma_\pi := \int_{[\pi_r, \pi_t]} \sigma_\pi.$$

**Remark.** If $a = \varphi(\cdot) 1_A$, for a function $\varphi \in L^1_{\text{loc}}(\mathbb{R}^+)$, then

$$E^a_{r,t} = \int_{[\pi_r, \pi_t]} \pi_\varphi 1_A = e^{\int_{\pi_r}^{\pi_t} \varphi} 1_A.$$

**Lemma 2.3.** Let $c, d, b \in L^1(\mathbb{R}^+; A)$ and $a, b \in L^1_{\text{loc}}(\mathbb{R}^+; A)$.

(a) $\|\pi_c\|_1 \leq \exp ||c||_1$ and $\|\pi_{c+d} - \pi_c\|_1 \leq ||\pi_c||_1 \left( ||\pi_c||_1 - 1 \right)$.

(b) $\pi_c \circ \pi_d = \pi_{c+d}$ if Ran $d = \text{Ran } c$, whereas

$\pi_c \circ \sigma_\pi = \pi_{c+\sigma_\pi}$ provided that $d(s_1) \sim (c+d)(s_2)$ when $s_2 > s_1 > 0$.

(c) $E^a$ is the unique continuous solution of the integral equations (2.5) below (in turn, for each fixed $r$, and each fixed $t$); $\partial E$ is likewise for (2.6).

$$E^a_{r,t} = 1_A + \int_r^t ds \ E^a_{r,s} \ a(s) = 1_A + \int_r^t ds \ a(s) \ E^a_{s,t}, \quad (2.5)$$

$$E_{r,t} = 1_A + \int_r^t ds \ b(s) \ E_{r,s} = 1_A + \int_r^t ds \ E_{s,t} b(s). \quad (2.6)$$

(d) For $(r, t), (u, v) \in \Delta^2$, setting $I := [r, t]$ and $J := [u, v]$, the following hold:

(i) $\|E^a_{r,t} - E^b_{u,v}\| \leq \exp ||a_f||_1 (\exp \|b-a\|_1 + \exp ||a_{fJ}||_1 - 1)$.

(ii) $E^a_{r,s} E^b_{s,t} = E^a_{r,t}$ where $c := a_{[r,s]} + b_{[s,t]}$.

(iii) $E^a_{r,t} E^b_{s,t} = E^{a+b}_{r,t}$ if $b(s_1) \sim (a+b)(s_2)$ for $r < s_1 < s_2 < t$.

(iv) $E^a_{r,t} E^b_{r,t} = E^{a+b}_{r,t}$ where $e_r(s) := (-b)E_{r,s} a(s) E^a_{s,t} + b(s)$.

(v) $E^{L_{w,a}}_{r,t} = L_{r+w,t+w}$ for $w \in [r, \infty]$, where $L_{w,a}$ is given by

$$\langle L_{w,a} \rangle (s) = \begin{cases} \ a(s + w) & \text{if } s + w \geq 0 \\ \ 0 & \text{otherwise}. \end{cases}$$
Proof. Note the following binomial-type identities, for functions \(a_1, a_2 : \mathbb{R}_+ \to \mathcal{A}\):

\[
\pi_{a_1 + a_2}(\sigma) = \sum_{i \in \{1, 2\}^n} a_i(s_1) \cdots a_n(s_n) \quad \text{for} \quad \sigma = \{s_1 < \cdots < s_n\} \tag{2.7}
\]

\[
= \sum_{\alpha \in \sigma} \pi_{a_1}(\alpha) \pi_{a_2}(\sigma \setminus \alpha) \quad \text{if} \quad \text{Ran} \ a_1 \sim \text{Ran} \ a_2. \tag{2.8}
\]

(a) The first estimate follows from submultiplicativity of the norm. For the second, note that (2.7) implies that

\[
\| \pi_{c+h}(\sigma) - \pi_c(\sigma) \| \leq \| \pi_{C+H}(\sigma) - \pi_C(\sigma) \|
\]

for \(\sigma \in \Gamma\), where \(C := \|c(\cdot)\|\) and \(H := \|h(\cdot)\|\). Thus, by (2.1),

\[
\| \pi_{c+h} - \pi_c \|_1 \leq \| \pi_{C+H} \|_1 (\| \pi_H \|_1 - 1) = \| \pi_c \|_1 (\| \pi_h \|_1 - 1).
\]

(b) The first identity follows from (2.8). The second follows easily from the fact that, under the given commutation assumption,

\[
\pi_{c+d}(s \cup \tau) = (c(s) \pi_{c+d}(\tau) + \pi_{c+d}(\tau)d(s),
\]

when \(s < \tau\) (meaning \(s < t\) for all \(t \in \tau\)).

(c) All four of the required identities follow from the integral-sum formula. For example, for the first one, define \(\mathbb{1}(\alpha, \beta)\) to be 1 if \(\#\beta = 1\) and \(a < b\) for all \(a \in \alpha\) and \(b \in \beta\), and to be 0 otherwise, then

\[
\int_r^t ds E_{r,s}^a(s) = \int_r^t ds \int_{[r,s]} d\alpha \pi_a(\alpha \cup \{s\})
\]

\[
= \int d\alpha \int d\beta \pi_{a \cup \beta}(\alpha \cup \beta) \mathbb{1}(\alpha, \beta)
\]

\[
= \int d\alpha \sum_{\sigma \supset \alpha} \pi_{a \cup \sigma}(\alpha) \mathbb{1}(\alpha, \sigma \setminus \alpha) = \int_{[r,t]} d\sigma \pi_{a \cup \sigma}(\sigma) = E_{r,t}^a - 1_A.
\]

Uniqueness for the first and last follows from Theorem 1.1; uniqueness for the other two follows from the left module sister version of Theorem 1.1.

(d) (i) follows from Part (a). (ii) follows from (2.4), (2.2) and the identity \(\pi_n(\sigma \cap [r, s]) \pi_n(\sigma \cap [s, t]) = \pi_n(\sigma)\); with (i) it implies that \(E^a \in \text{Evol}(\mathcal{A}) \subset \text{Evol}(\mathcal{A}^x)\).

(iii) follows from Part (b) and identity (2.4). In particular, since \(E^b\) is invertible, this implies that

\[
(E_{r,b}^b)^{-1} = (-b)E_{r,s}(r, s) \in \Delta^2. \tag{2.9}
\]

To prove (iv), set \(E\) equal to the pointwise product \(E^a E^b\). Integrating by parts using Part (c), the assumed commutation relations, and (2.9), we have

\[
E_{r,t} = 1_A + \int_r^t ds (E_{r,s}^a E_{r,s}^b + E_{r,s}^b E_{r,s}^a b(s)) = 1_A + \int_r^t ds E_{r,s}e_{r,s}(s).
\]

Therefore (iv) follows from uniqueness in Part (c). With a simple change of variable, (v) follows from the identity

\[
(L_{w})_{r,t}(s) = a_{r+w,t+w}(s + w) \quad (s \in \mathbb{R}_+). \tag*{□}
\]

The summarising proposition below now follows easily.

**Proposition 2.4.** Let \(a, b \in L^1_{\text{loc}}(\mathbb{R}_+; \mathcal{A})\), \(c \in L^\infty(\mathbb{R}_+; \mathcal{A})\) and \((r, t) \in \Delta^2\). Then

(a) \(E^a \in \text{Evol}_c(\mathcal{A})\) and \(E^c \in \text{Evol}_c(\mathcal{A})\).

(b) \(E^a \circ E_{r,t} = E_{r,t}^a b\) if \(b(s) \sim (a + b)(s)\) for \(r < s_1 < s_2 < t\), in particular,

\[
(e^{lt} \circ E_{r,t})^{-1} = (e^{lt} \circ E_{r,t})^{-1} \quad \text{and} \quad e^{lt} \circ E_{r,t} = E_{r,t}^{a+\varphi(1)A} \quad \text{for} \quad \varphi \in L^1_{\text{loc}}(\mathbb{R}_+)\.
\]

(c) \(E_{r,t}^a = E_{0,t-r}^{L_{0,1}^a} \text{ and } E_{0,s+u}^a = E_{0,u}^a E_{0,s}^{L_{0,u}}\), for \(s, u \in \mathbb{R}_+\).
**Definition.** An evolution of the form \( E^a \) where \( a \in L^1_{\text{loc}}(\mathbb{R}^+; \mathcal{A}) \) will be called **elementary, with generator** \( a \); we denote this class of evolutions by \( \text{Evol}_e(\mathcal{A}) \).

The following example is of considerable historical importance (see e.g. [EnN]).

**Example.** Let \( a : \mathbb{R}^+ \to B(X) \) be strongly continuous, for a Banach space \( X \). Then, by the Banach-Steinhaus Theorem, \( a \) is locally bounded, and by (2.6),
\[
aE_{r,t} = I_X + \int_r^t ds \, a(s) aE_{r,s} \quad (0 \leq r \leq t).
\]
In particular, for all \( x \in X \), the nonautonomous abstract Cauchy problem
\[
u'(t) = a(t)u(t) \quad (t \geq 0), \quad u(0) = x,
\]
has unique “classical” solution \( aE_{0,t} \mathbb{1} \mathbb{1}(\mathbb{R}^+; X) \).

Noting that \( \text{Evol}_e(\mathcal{A}) \subset \text{Evol}_c(\mathcal{A}) \), we characterise the class of elementary evolutions next.

**Theorem 2.5.** Let \( E \in \text{Evol}_c(\mathcal{A}) \) and set \( \phi_t := E_{0,t} \ (t \in \mathbb{R}^+) \). Then the following are equivalent:

1. There is a function \( c \in L^1_{\text{loc}}(\mathbb{R}^+; \mathcal{A}) \) such that
\[
\phi_t - \phi_r = \int_r^t ds \, c(s) \quad (0 \leq r \leq t).
\]
2. \( E \in \text{Evol}_e(\mathcal{A}) \).

In this case \( c(s) = E_{0,s} a(s) \ (s \in \mathbb{R}^+) \), where \( a \) is the generator of \( E \).

**Proof.** Multiplying (2.5) on the left by \( E_{0,r} \) we see that (ii) implies (i).

Suppose that (i) holds. By Proposition 2.1, \( \text{Ran} \phi \subset \mathcal{A}^\prime \), and so we may define a function \( a \in L^1_{\text{loc}}(\mathbb{R}^+; \mathcal{A}) \) by \( a(s) := (\phi_s)^{-1}c(s) \). Since \( E \) and \( E^a \) are both continuous evolutions, it suffices to show that \( \phi_t = E^a_{0,t} \) for all \( t \in \mathbb{R}^+ \). Now
\[
\phi_t = 1_\mathcal{A} + \int_0^t ds \, a(s) \quad (t \in \mathbb{R}^+)
\]
so, by Part (c) of Lemma 2.3 (uniqueness), it follows that \( \phi_t = E^a_{0,t} \) for all \( t \in \mathbb{R}^+ \), as required.

**Remarks.** Evolutions of the above type are a.e.-weakly differentiable in the following sense. By Lebesgue’s Differentiation Theorem, for all \( \omega \in \mathcal{A}^\prime \) there is a null set \( \mathcal{N}_\omega \) in \( \mathbb{R}^+ \) such that for all \( t \in \mathbb{R}^+ \setminus \mathcal{N}_\omega \),
\[
\omega(h^{-1}(\phi_{t+h} - \phi_t) - c(t)) \to 0 \text{ as } h \to 0.
\]
Conversely, it follows from Theorem 1.1 that (ii) holds if there is a Lebesgue-null Borel subset \( \mathcal{N} \) of \( \mathbb{R}^+ \) such that \( \phi \) is differentiable on \( \mathbb{R}^+ \setminus \mathcal{N} \), \( \phi' \in L^1_{\text{loc}}(\mathbb{R}^+; \mathcal{A}) \) and Haus \( \phi(\mathcal{N}) = 0 \).

The next result applies to finite-dimensional Banach algebras. A convenient reference for the Radon–Nikodým property is [DiU]; for differentiability of Lipschitz functions, see [LPT].

**Corollary 2.6.** Let \( E \in \text{Evol}_e(\mathcal{A}) \) where \( \mathcal{A} \) has the Radon–Nikodým property, and set \( \phi_t := E_{0,t} \ (t \in \mathbb{R}^+) \). Then the following are equivalent:

1. \( E \in \text{Evol}_e(\mathcal{A}) \); respectively, \( E \in \text{Evol}_c(\mathcal{A}) \) with locally bounded generator.
2. There is an absolutely continuous \( \mathcal{A} \)-valued measure \( m \) on \( \mathbb{R}^+ \) of locally bounded variation such that \( m([r,t]) = \phi_t - \phi_r \ (0 \leq r \leq t) \); respectively, \( \phi \) is locally Lipschitz, so \( E \in \text{Evol}_{1,e}(\mathcal{A}) \).
We next identify a subclass of elementary evolutions which is useful in applications. To this end, and for use in the next section, we adopt the following notation.

**Notation.** Let \( D = \{T_1 < \cdots < T_N\} \subseteq [0,\infty[ \) and set \( T_0 := 0 \) and \( T_{N+1} := \infty \). For \( u \in \mathbb{R}^+ \), define \( m = m(u) \in \mathbb{Z}^+ \), \( n = n(u) \in \mathbb{N} \) and \( \{u^D_k : m \leq u \leq n\} \) by \( \{u^D_{m+1} \leq \cdots \leq u^D_n\} \) for any such \( \{u^D_k \} \) and \( D \subseteq [0,\infty[ \). The piecewise-semigroup evolutions are therefore those evolutions which enjoy the semigroup decomposition property (2.10). Note that the set \( \{u^D_k \} \subseteq \mathbb{R}^+ \) for any such \( \{u^D_k \} \) and \( D \subseteq [0,\infty[ \).

**Definition.** We call \( E \) a piecewise-semigroup evolution if there are associated time point and semigroup sets
\[
D = \{T_1 < \cdots < T_N\} \subseteq [0,\infty[ \quad \text{and set} \quad T_0 := 0 \quad \text{and} \quad T_{N+1} := \infty,
\]
where \( T_0 := 0 \) and each \( P(T) \) is a semigroup in \( \mathcal{A} \), for which the following holds:
\[
E_{r,t} = \begin{cases} \prod_{s=0}^{j} P(r,s)r \left( P(t_{s+1}) \cdots P(t_{r-s}) \right) P(t_{s+1}) & \text{if } r^D = t^D_0 \\ \prod_{s=0}^{j} P(r,s)r \left( P(t_{s+1}) \cdots P(t_{r-s}) \right) P(t_{s+1}) & \text{otherwise.} \end{cases} \tag{2.10}
\]
Note that, for any such \( D \) and \( \{P(T)\} \), (2.10) defines an evolution. Let \( \text{Evol}_{\text{pws}}(\mathcal{A}) \) denote the resulting collection; thus \( \text{Evol}_{\text{pws}}(\mathcal{A}) \cap \text{Evol}_{\text{c}}(\mathcal{A}) \subseteq \text{Evol}_{\text{c}}(\mathcal{A}) \).

The piecewise-semigroup evolutions are therefore those evolutions which enjoy the semigroup decomposition property (2.10). Note that the set \( D \) can be empty, and it is only the minimal such set \( \mathcal{D} \) that is determined by the evolution \( E \). We have the following elementary characterisation.

**Proposition 2.7.** Let \( E \in \text{Evol}_{\text{c}}(\mathcal{A}) \). Then the following are equivalent:

(i) \( E \in \text{Evol}_{\text{pws}}(\mathcal{A}) \).

(ii) \( E \in \text{Evol}_{\text{c}}(\mathcal{A}) \), with piecewise constant generator.

In this case, the associated minimal time point and semigroup sets of \( E \) are respectively, \( \text{Disc} a \) and \( \{e_s(t) : s \geq 0 ; t \in [0] \cup \text{Disc} a\} \), where \( a \) is the (right-continuous version of) the generator of \( E \).

**Proof.** Suppose that (ii) holds and let \( a \) be the generator of \( E \). Let \( D = \text{Disc} a = \{T_1 < \cdots < T_N\} \), set \( T_0 := 0 \) and \( T_{N+1} := \infty \), and let \( (r,t) \in \Delta^2 \). By the evolution property,
\[
E_{r,t} = \begin{cases} E_{r,t}^D \left( E_{r,t}^D \cdots E_{r,t}^D \right) E_{r,t}^D & \text{if } r^D = t^D_0 \\ \text{otherwise.} \end{cases} \tag{2.11}
\]
Now, for \( k = 0, \cdots, N \), \( a \) is constant on \( [T_k, T_{k+1}] \) so, for \( [u,v] \subseteq [T_k, T_{k+1}] \),
\[
E_{u,v} = \int_{[u,v]} d\sigma \pi_{a}(\sigma) = \int_{[u,v]} d\sigma \pi_{a}(\sigma)^{\#} = e^{(v-u)a_0(T_k)} = P(t_{T_k}^{-u-a}),
\]
where \( P(T) \) denotes the semigroup generated by \( a(T) \). Thus (2.11) becomes (2.10), showing \( E \) to be a piecewise-semigroup evolution with associated time and semigroup sets as claimed.

Suppose conversely that (i) holds, and let \( D = \{T_1 < \cdots < T_N\} \) be the associated minimal time point and semigroup sets of \( E \). Since \( E \in \text{Evol}_{\text{c}}(\mathcal{A}) \), each of these semigroups is norm continuous. Let \( a \) be the piecewise constant function \( \sum_{k=0}^{N} a(T_k) \) where, for \( k = 0, \cdots, N \), \( a(k) \) is the generator of \( P\left(T_k\right) \). Then \( E^a \) also satisfies (2.10), and so \( E = E^a \).
Thus the evolutions with piecewise constant generators are the continuous evolutions which enjoy a semigroup decomposition. We characterise a slightly wider class of evolutions next. By **piecewise continuity** for a Banach-space valued function \( x \) defined on \( \mathbb{R}_+ \), we mean that there is a finite subset \( D \) of \([0, \infty[\) such that \( x \) is continuous on \( \mathbb{R}_+ \setminus D \) and the limits \( a(0+) \), \( a(-) \) and \( a(s+) \) exist, for \( s \in D \). For definiteness, we take the unique right-continuous (i.e. càdlàg) version of each piecewise continuous function.

**Proposition 2.8.** Let \( E \in \text{Evol}_c(\mathcal{A}) \). Then the following are equivalent:

(i) \( s \mapsto E_{0,s} \) has piecewise continuous derivative on \( \mathbb{R}_+ \).

(ii) \( E \in \text{Evol}_c(\mathcal{A}) \) with piecewise continuous generator.

**Proof.** By Proposition 2.1, \( E \) is invertible. Assume that (i) holds and define \( a : \mathbb{R}_+ \to \mathcal{A} \) to be the piecewise continuous function \( s \mapsto (E_{0,s})^{-1} \frac{d}{ds} E_{0,s} \). Then \( a \in L^1_\text{loc}(\mathbb{R}_+; \mathcal{A}) \) and (ii) holds since \( s \mapsto E_{0,s} \) and \( s \mapsto E_{0,s}^\pi \) both satisfy the conditions of Theorem 1.1, Part (b), with \( N := \text{Disc} a \).

The converse is clear. \( \square \)

### 3. Lie–Trotter Product Formula

In this section we prove a Trotter product formula and an Euler-type formula, for elementary evolutions. The following notation is convenient for handling Trotter products of evolutions.

**Notation.** Let \( D \subset \subset [0, \infty[ \), in other words \( D \in \Gamma_{[0, \infty[} \), and let \( G \in F(\Delta^2; \mathcal{A}) \). Then, in the notation associated with the diagram in Section 2, define \( G \)’s \( D \)-fold product function by

\[
G^D : \Delta^2 \to \mathcal{A}, \quad G^D_{r,t} = \begin{cases} G_{r,t}^D & \text{if } r^D < t^D \\ 1_A & \text{otherwise.} \end{cases}
\]

In particular, if \( G \) is an evolution then \( G^D_{r,t} \) equals \( G_{r,t}^D \) if \([r, t] \cap D \) is nonempty, and equals \( 1_A \) otherwise.

**Definition.** We say that a sequence \( (D(n))_{n \geq 1} \) in \( \Gamma_{[0, \infty[} \setminus \{ \emptyset \} \) converges to \( \mathbb{R}_+ \) if

\[
\min D(n) \to 0, \quad \max D(n) \to \infty \quad \text{and} \quad \text{mesh } D(n) \to 0.
\]

Similarly, a family \( (D[h])_{h > 0} \) in \( \Gamma_{[0, \infty[} \) converges to \( \mathbb{R}_+ \) if, as \( h \to 0 \),

\[
\min D[h] \to 0, \quad \max D[h] \to \infty \quad \text{and} \quad \text{mesh } D[h] \to 0.
\]

Here mesh \( D \) is defined to be \( \max \{|s - t| : s, t \in D, s \neq t\} \) (or \( \infty \) if \( #D = 1 \)).

**Theorem 3.1.** Let \( a_1, a_2 \in L^1_\text{loc}(\mathbb{R}_+; \mathcal{A}) \), let \( (D(n))_{n \geq 1} \) be a sequence in \( \Gamma_{[0, \infty[} \setminus \{ \emptyset \} \) converging to \( \mathbb{R}_+ \), and let \( T \in \mathbb{R}_+ \). Then

\[
\sup_{[r,t] \subset [0,T]} \| E_{r,t}^{a_1 + a_2} - 1.2E^D(n) \| \to 0, \quad \text{where } 1.2E_{u,v} := E_{u,v}^{a_1} E_{u,v}^{a_2}.
\]

**Proof.** Set \( a = a_1 + a_2 \) and \( A = A_1 + A_2 \), where \( A_i := \|a_i(\cdot)\| \in L^1_\text{loc}(\mathbb{R}_+) \) \( (i = 1, 2) \), and set \( \pi := \pi_{a_1} \circ \pi_{a_2} \), for the composition defined in (2.2). Thus \( \pi \in L^1_\text{loc}(\Gamma; \mathcal{A}) \) with

\[
\pi(\emptyset) = 1_A \quad \text{and} \quad \pi(\{s\}) = a(s) \quad \text{for } s \in \mathbb{R}_+,
\]

so the functions \( \pi \) and \( \pi_a \) agree on \( \Gamma^{<1} \). Also, by (2.4),

\[
1.2E_{u,v} = \int_{\Gamma_{[u,v]}} \pi(\sigma) \quad \text{for } ((u, v) \in \Delta^2).
\]
By further application of the integral-sum formula—more specifically (2.4), and (2.3),

\[ \frac{1}{2}E_{r,t}^{D(n)} = \int_{\Gamma_{r,t}} d\sigma \pi^{(n)}(\sigma), \]

where

\[ \pi^{(n)}(\sigma) := \pi(\sigma \cap [r_1^{D(n)}, r_2^{D(n)}]) \cdots \pi(\sigma \cap [r_{t-1}^{D(n)}, r_0^{D(n)}]). \]

Now,

\[ \|\pi^{(n)}(\sigma)\| \leq \pi_A(\sigma) \quad (n \in \mathbb{N}, \sigma \in \Gamma). \]

Thus \( \pi^{(n)} \in L^1_{\text{loc}}(\Gamma; A) \), \( \pi^{(n)}(\emptyset) = 1_\mathcal{A} = \pi_a(\emptyset) \) and, for \( \sigma \in \Gamma_{[0,\infty]} \setminus \{\emptyset\} \), the equality

\[ \pi^{(n)}(\sigma) = \pi_a(\sigma) \]

holds—as soon as \( n \in \mathbb{N} \) is sufficiently large that

\[ \min D(n) < \min \sigma, \quad \max D(n) > \max \sigma \quad \text{and} \quad \text{mesh } D(n) < \text{mesh } \sigma. \]

The result therefore follows from the Dominated Convergence Theorem:

\[ \sup_{[r,t] \subset [0,T]} \left\| E_{r,t}^{n} - \frac{1}{2}E_{r,t}^{D(n)} \right\| \leq \int_{[0,T]} d\sigma \left\| \pi_a(\sigma) - \pi^{(n)}(\sigma) \right\| \to 0. \]

In order to handle Euler-type products we define, for \( a \in L^1_{\text{loc}}(\mathbb{R}_+; \mathcal{A}) \), the truncated evolution:

\[ \tilde{E}^a : \Delta^2 \to \mathcal{A}, \quad \tilde{E}_{r,t}^a := \int_{[r,t]} \tilde{\pi}_a \quad \text{where} \quad \tilde{\pi}_a := 1_{r \leq s \leq t} \pi_a. \quad (3.1) \]

Thus \( \tilde{E}_{r,t}^a = 1_{\mathcal{A}} + \int_{r}^{t} ds a(s) \).

**Theorem 3.2.** Let \( a_1, a_2, (D(n))_{n \geq 1} \) and \( T \) be as in Theorem 3.1. Then

\[ \sup_{[r,t] \subset [0,T]} \left\| E_{r,t}^{n} - (N)E_{r,t}^{D(n)} \right\| \to 0, \quad \text{where} \quad (N)E_{u,v}^{n} := E_{u,v}^{n_1} \cdots E_{u,v}^{n_{t-[u,v]}}, \quad ((u,v) \in \Delta^2), \]

and similarly for the truncations.

**Proof.** A proof is obtained as follows. In the proof of Theorem 3.1 replace \( \pi_{a_1}, \pi_{a_2}, \pi, 1_\mathcal{A}E \) and \( \pi^{(n)} \) by \( \tilde{\pi}_{a_1}, \tilde{\pi}_{a_2}, \tilde{\pi}, 1_\mathcal{A}\tilde{E} \) and \( \tilde{\pi}^{(n)} \) respectively, where \( \tilde{\pi} \) is defined as \( \pi \) is but with \( \tilde{\pi}_{a_1} \) and \( \tilde{\pi}_{a_2} \) in place of \( \pi_{a_1} \) and \( \pi_{a_2} \), and \( \tilde{\pi}^{(n)} \) is defined as \( \pi^{(n)} \) is, but with \( \tilde{\pi} \) in place of \( \pi \). In short, drawing on the definitions (3.1), retrace the argument with all \( \pi \)'s and \( E \)'s endowed with tildes.

**Remarks.** The above two proofs need little adjustment to deliver the following generalisation. For \( a = a_1 + \cdots + a_N \) where \( a_1, \cdots, a_N \in L^1_{\text{loc}}(\mathbb{R}_+; \mathcal{A}) \), and \( T \in \mathbb{R}_+ \),

\[ \sup_{[r,t] \subset [0,T]} \left\| E_{r,t}^{n} - (N)E_{r,t}^{D(n)} \right\| \to 0, \quad \text{where} \quad (N)E_{u,v}^{n} := E_{u,v}^{n_1} \cdots E_{u,v}^{n_{t-[u,v]}} \quad ((u,v) \in \Delta^2), \]

and similarly for the truncations.

The above proofs also yield corresponding results for a continuous-parameter family \( (D[h])_{h \geq 0} \). In particular, taking \( a_1 \) and \( a_2 \) constant, respectively \( a_2 = 0 \) and \( a_1 = a_1 \) a constant, then gives the following limits

\[ e^{ha_1 e^{ha_2}}(t^n h^{D(h)} - t^{n D(h)})/h \to e^{(t-r)(a_1+a_2)} \]

and \( (1_\mathcal{A} + ha)(t^n h^{D(h)} - t^{n D(h)})/h \to e^{(t-r)a} \)

as \( h \to 0 \); the classical Lie–Trotter product formula ([ReS], Theorem VIII.29) and Euler formula emerge upon taking \( r = 0 \) and \( D[h] = \{ nh : 1 \leq n \leq N \} \) where \( N = \lfloor 1/h \rfloor \):

\[ e^{ha_1 e^{ha_2}}(t/h) \to e^{(t)(a_1+a_2)} \]

and \( (1_\mathcal{A} + ha)(t/h) \to e^{ta} \).

The close connection between the Trotter product and Euler formulæ was richly investigated, at the deeper level of \( C_0 \)-semigroups, by Chernoff (see [Che]).
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