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From numerical concepts to concepts of number

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Abstract: Many experiments with infants suggest that they possess quantitative abilities, and many experimentalists believe that these abilities set the stage for later mathematics: natural numbers and arithmetic. However, the connection between these early and later skills is far from obvious. We evaluate two possible routes to mathematics and argue that neither is sufficient: (1) We first sketch what we think is the most likely model for infant abilities in this domain, and we examine proposals for extrapolating the natural number concept from these beginnings. Proposals for arriving at natural number by (empirical) induction presuppose the mathematical concepts they seek to explain. Moreover, standard experimental tests for children's understanding of number terms do not necessarily tap these concepts. (2) True concepts of number do appear, however, when children are able to understand generalizations over all numbers; for example, the principle of additive commutativity ($a + b = b + a$). Theories of how children learn such principles usually rely on a process of mapping from physical object groupings. But both experimental results and theoretical considerations imply that direct mapping is insufficient for acquiring these principles. We suggest instead that children may arrive at natural numbers and arithmetic in a more top-down way, by constructing mathematical schemas.

Keywords: acquisition of natural numbers; mathematical concepts; representations of mathematics; theories of mathematical cognition

1. Introduction

Natural numbers are the familiar positive whole numbers: 1, 2, 3, . . . (or, on some treatments, the non-negative whole numbers: 0, 1, 2, 3, . . .); and they clearly play an essential part in many mathematical activities, for example, counting and arithmetic. In addition to their practical role, natural numbers also have a central place in mathematical theory. Texts on set theory use the natural numbers to construct more complicated number systems: the integers, rationals, reals, and complex numbers (e.g., Enderton 1977; Hamilton 1982), and even the surreal numbers (Knuth 1974). For example, we can represent the integers (positive, negative, and zero) as the difference between pairs of natural numbers (e.g., $-7 = 2 - 9$). Similarly, we can represent the rationals as the ratio of two integers ($-7/9$) and, thus, as the ratio of the differences between two natural numbers (e.g., $[2 - 9]/[10 - 1]$). Children may not necessarily learn the integers, rationals, or (especially) reals in terms of natural numbers, but the availability of these constructions is an important unifying idea in mathematical theory, testifying to natural numbers' central foundational role.

Given the central position of the natural numbers in practice and theory, how do children learn them? We argue here that although research on number learning is an extremely active and exciting area within cognitive developmental psychology, there is a gap between the numerical concepts studied in children and the natural-number concepts they use in later life. There is a lack of conceptual fit between the properties of the natural numbers and the properties of what psychologists have identified as precursor representations of quantity. These representations may be useful to nonhuman animals, infants, children, and even adults for certain purposes, such as estimating amounts or keeping track of objects, but they are not extendible by ordinary inductive learning to concepts of natural numbers. Moreover, the tasks psychologists have used to determine whether children have natural-number concepts do not necessarily tap these concepts. We argue that psychologists should look elsewhere for a basis for number concepts, and we suggest a possible starting point.

After some preliminaries in the first section, we start building a case for this point of view in section 2 by sketching what we think is the most complete current model of infants' early quantitative abilities. We then try to show in the third section what the obstacles are for using this model to capture the concept of natural number. Deficiencies in this respect have become increasingly evident in recent work, and the proposals for bridging between these early representations and more mature ones have grown correspondingly more complex. We argue that the difficulty of constructing such a bridge is a principled one and that the "precursor" representations are not precursors. In the fourth section, we consider another popular way of thinking about how children acquire number concepts, by mapping them from groupings of physical objects. We argue that this mapping view is also up against difficult problems. The fifth section speculates about the route to a more adequate theory.

Although we think there is no way to get from current proposals about early quantitative representations to mature number concepts, this claim should not be confused with more sweeping or dismissive ones. In particular, we will not be claiming that early quantitative representations are unimportant or irrelevant to adult performance. There is evidence, for example, that magnitude representations, which many psychologists believe underlie infants' quantitative abilities, also play a role in adults' mathematics. Our concern here is solely with whether psychological research is on the right track in its search for the cognitive origins of natural number, as we think there's a good chance that it is not.

1.1. Words and numbers

In exploring this terrain, we stick to a few terminological restrictions, since the uses of key terms such as "number" and "counting" are far from uniform in everyday language. First, we refer to the number of elements in a set as the set's cardinality, which can be finite or infinite. (Other authors use the terms "numerosity" or "set size" for what we take to be the same concept.) Second, we follow the trend in psychology of using natural numbers for the positive integers (1, 2, 3, ...). In most formal treatments, the natural numbers start with 0, rather than 1, but for psychological purposes it is useful to think of 1 as the first natural number, since it is unclear whether children initially view 0 as part of this sequence (see sect. 5.3.1 for further discussion). In any case, we eliminate from consideration as the natural numbers any sequence that fails to have: (a) a unique first element (e.g., 1); (b) a unique immediate successor for each element in the sequence (e.g., 905 is the one and only immediate successor of 904), (c) a unique immediate predecessor for each element except the first (e.g., 904 is the one and only immediate predecessor of 905), and (d) the property of (second-order) mathematical induction. The latter property essentially prohibits any element from being a natural number unless it is the initial number or the successor (... of the successor) of the initial number. We discuss these requirements in section 5. It might be reasonable to place further restrictions on the natural numbers, but systems that fail to observe the four (a) to (d) requirements just mentioned are simply too remote from standard usage in mathematics to be on topic.

Finally, counting. The term "counting" has an intransitive use ("Calvin counted to ten") and a transitive one ("Martha counted the cats"). In this article, we reserve the term for the intransitive meaning, and we distinguish two forms of counting in this sense. One form, which we call *simple counting*, consists of just reciting the number sequence to some fixed numeral, for example, "ten" or "one hundred." The second form, which we call *advanced counting*, is the ability to get from any numeral "n" to its successor "$n + 1$" in some system of numerals for the natural numbers. Thus, an advanced counter who is given the English term "nine hundred four" could supply the successor "nine hundred five," and an advanced counter with Arabic numerals who is given "904" could supply "905." Advanced counting, but not simple counting, implies knowledge of the full system of numerals for the natural numbers. (For studies of numerical notation, such as the Arabic or Roman numerals, see Chrisomalis [2004] and Zhang & Norman [1995]; for studies of number terms in natural language, see Hurford [1975] and the contributions to Gvozdanović [1992].)

For clarity, we use the term "enumerating" for the transitive meaning of counting – determining the cardinality of a collection – and it is enumerating which is the focus of much developmental research on the origins of mathematics, notably Gelman and Gallistel's (1978) landmark book. Enumerating typically involves pairing verbal numerals with objects to reach a determinable total, but research has also considered various forms of nonverbal enumeration. In some theories, for example, some internal continuous quantity (e.g., activation strength) is adjusted, either serially or in parallel, to achieve a measure of a set's cardinality. We use the term "mental magnitude" (or magnitude for short) in this article to denote such a continuous mental representation, and we contrast this with countable representations, such as the numerals in standard systems (e.g., Arabic numerals or natural-language terms for natural numbers). Mental magnitudes could, of course, represent many different properties, such as duration, length, or volume, but unless we indicate

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otherwise, the mental magnitudes at issue will be representations of cardinality.

2. Possible precursors of natural numbers

Nearly all cognitive research on the development of number concepts rests on the idea that such concepts depend on enumerating objects. To be sure, there is plenty of debate about how infants assess the cardinality of object collections and about how these early abilities give place to more sophisticated ones. However, there seems to be no serious disagreement that enumerating is the conceptual basis for number concepts.

There may be theoretical reasons, however, to question a necessary link between enumerating and number. Recent structuralist theories in the philosophy of mathematics take numbers to be, not cardinalities, but positions in an overall structure – a structure that obeys the axioms of the system in question (Parsons 2008; Resnik 1997; Shapiro 1997). The number five, for example, is what occupies the fifth position in a system that obeys the characteristics (a) to (d) that we listed in section 1.1. Psychologists seem to have been influenced instead by the conception of natural numbers as sets of all equinumerous sets of objects (see Frege 1884/1974; Russell 1920). According to one version of this conception, for example, five is the set of all five-membered sets of objects. As a theoretical account of natural numbers, however, this one runs into difficulty because it presupposes an infinite number of objects (since there are an infinite number of natural numbers). To circumvent the problem that there might not be enough objects to go around, Frege had to posit the idea that numbers were themselves objects (see Dummett 1991, pp. 131–33), and Russell had to posit an Axiom of Infinity, guaranteeing that there are infinite sets of a certain sort. In either case, numbers are not simply sets of sets of physical objects, such as tables or trees. It virtually goes without saying that not all philosophers of mathematics agree on the merits of the structuralist approach over earlier ones.[1] Nevertheless, the structuralist view suggests that the concept of natural number is not necessarily defined by cardinality or enumeration, and we develop this suggestion in section 5.

But, although we think that the structuralist approach is the most plausible current theory about the meaning of number terms, we do not presuppose it in examining psychological accounts. Instead, we argue that even if natural numbers are cardinalities (and cardinalities sets of sets of physical objects), most current accounts fail to provide a satisfactory explanation of how children move from their initial quantitative abilities to a mature concept of natural number. For these purposes, then, let us temporarily assume (in sects. 2–4) that "one," "two," "three," and so on denote the appropriate cardinalities, and consider proposals for learning the natural number concept in these terms. Of course, one psychological theory that would be theoretically adequate has NATURAL NUMBER as an innate concept. We briefly consider this possibility in section 2.1, but most psychologists believe that NATURAL NUMBER is constructed from other innate starting points. Section 2.2 sketches the view of infants' quantitative abilities that seems to us in best accord with current theory and research in this area. Sections 3 and 4 then consider the prospects that children could use these abilities to construct the natural number concept.

2.1. Innate natural number concepts

Infants might start off with pre-specified number concepts that represent cardinalities, one concept representing all sets with one element, a second representing all sets with two elements, and so on. A simple mental system of this sort might be diagrammatic, with a symbol such as "■" standing for all one-item sets, "■■" standing for all two-item sets, and continuing with a new item added to the previous one to form the next number symbol.[2] Of course, such a system could never represent each of the infinitely many cardinalities by storing separate concepts for each. Nevertheless, a simple generative rule might be available for deriving new symbols from old ones (by adding a "■" to form the successor) that would allow the infant to represent in a potential way all cardinalities (see the grammar in sect. 3.2 for an explicit formulation). A system of this kind is consistent with Chomsky's (1988, p. 169) suggestion that "we might think of the human number faculty as essentially an 'abstraction' from human language, preserving the mechanism of discrete infinity and eliminating the other special features of language." It is easy to see how an infant could use such a system for enumerating things and for simple arithmetic operations (e.g., concatenating two such symbols to obtain the symbol for their sum); and it is possible that older children's reflections on the system could lead them to an understanding of other mathematical domains (e.g., the positive and negative integers or the rational numbers). A less artificial example comes from a recent proposal by Leslie et al. (2007) that there is an innately given internal symbol for the integer value 1 and an innate successor function that generates the remaining positive integers (subject to some further psychological constraints).

However, although there is evidence that infants have an early appreciation of cardinality (as we will see in the next subsection), several investigators have argued against innate number concepts based on "discrete" (i.e., countable) representations. For example, Wynn (1992a) concludes on the basis of a longitudinal study of 2- and 3-year-olds that there is a phase in which children interpret "two," "three," and higher terms in their own counting sequence to stand for some cardinality or other without knowing which specific cardinality is correct. They may know, for instance, that "three" represents the cardinality of a set containing either two elements or a set containing three elements, and so on; however, they may not be able to carry out the command to point to the picture with two dogs when confronted by a pair of pictures, one with two dogs and the other with three. These children can, of course, perceptually discriminate the pictures; their difficulty lies in understanding the meaning of "two" in this context.

Wynn's (1992a) evidence is that children in this dilemma can already perform simple counting (e.g., can recite the number terms "one" through "nine"), and they already understand that "one" can refer to sets containing just one object. They also know that "one" contrasts in meaning with "two" and other elements in their list of count terms. The argument is that if children already had a countable internal representation of the natural numbers, there should not be a delay between the time they understand "one" and the time they understand "two" (and between the time they understand "two" and "three") in such tasks. But since there is, in fact, such a lag, younger children's understanding of cardinality must occur by means of a system that differs from that of the natural numbers. Wynn opts for a representation in which mental magnitudes (degrees of a continuous or analog medium) represent cardinality.

One point worth noting is that Wynn's argument was not directed against innate numbers in general, but, rather, against a more specific proposal attributed to Gelman and Gallistel (1978). This proposal included not only a countable representation but also a set of principles for using the representation to enumerate sets of objects. If children have (a) an innate representation for natural numbers, (b) an algorithm for applying them to enumerate sets, (c) knowledge of the initial portion of the integer sequence in their native language (e.g., "one," "two,"... ,"nine"), and (d) knowledge that the first term of the natural-language sequence ("one") maps onto the first term of their innate representation, then it is difficult to see why they do not immediately know which cardinality "two" ("three,"..., "nine") denotes. The evidence tells against (a) to (d), considered jointly, but leaves it open whether children's delay between understanding "one dog" and "two dogs" is due to incomplete knowledge of the principles for enumerating sets (Le Corre et al. 2006) or due to processing difficulties in applying the principles to larger sets (Cordes & Gelman 2005), rather than the result of a lack of a countable representation of natural numbers.

We think the possibility of an innate system for the natural numbers should not be dismissed too quickly. Such a theory, however, is clearly out of favor among psychologists (though see Leslie et al. [2007] for a reappraisal). According to many current views, children build the natural-number concept from preliminary representations with very different properties, and it is accounting for the transition between these preliminary representations and the mature ones that creates the theoretical gap with which we are concerned.

2.2. Magnitudes and object individuation

Many current theories in cognitive development see children's understanding of number as proceeding from concepts that do not conform to the structure of the natural numbers. On the one hand, there is the claim that numerical ability in infants rests on internal magnitudes – perhaps some type of continuous strength or activation – that nonhuman vertebrates also use for similar purposes (Dehaene 1997; Gallistel & Gelman 1992; Gallistel et al. 2006; Wynn 1992b). On the other hand, infants' math-like skills may also draw on discrete representations for integer values less than four (Carey 2001; Spelke 2000). Either approach requires some account of how children arrive at natural numbers from these beginnings.

There seems little doubt that infants are sensitive to quantitative information in their surroundings. For example, 10- to 12-month-old infants demonstrate their awareness of quantities in an addition-subtraction task: If the infants see an experimenter hide two toys in a box and then remove one, they will search longer in the box (presumably to find the remaining hidden toy) than if the experimenter hides only one toy in the box and then removes it (Van de Walle et al. 2000). Similarly, in habituation experiments, infants see a sequence of displays, with each display containing a fixed number of dots (e.g., 8 dots) in varying configurations. After the infants habituate, they see new arrays containing either the same number of dots (8) or a new number (e.g., 16). Under these conditions (and with overall surface area controlled), infants as young as 6 months look longer at the novel number of items, as long as the ratio of dots in the two kinds of display exceeds some critical value (e.g., Xu 2003; Xu & Spelke 2000; Xu et al. 2005).

Controversy surrounds the reason for the infants' success. Wynn (1992b) argued that infants keep track of the number of objects in the addition-subtraction task by means of internal continuous magnitudes, using the magnitudes to predict what they will find. A magnitude representation of this sort has also the advantage of accounting for the results from animal studies of cardinality detection (see Gallistel et al. 2006, for a review) and for experiments on number comparison by adults (e.g., Banks et al. 1976; Buckley & Gillman 1974; Moyer & Landauer 1967; Parkman 1971). In the latter studies, participants see a pair of single-digit numerals (e.g., 8 and 2) on each trial and must choose under reaction-time conditions which numeral represents the larger number. Mean response times in these experiments are faster, the larger the absolute difference between the digits; for example, participants take less time to compare 8 and 2 than 4 and 2. This symbolic distance effect is what we should expect if participants make their judgment by comparing two internal magnitudes, one for each digit. If the magnitudes include some amount of noise, then the larger the absolute difference between the digits, the more clear-cut the comparison and the faster the response times. The mental-magnitude idea also accords with people's ability to provide rough estimates of cardinality in situations in which an exact count is difficult or impossible (e.g., Conrad et al. 1998). People may produce these magnitude representations in an iterative way by successively incrementing the magnitude for each item to be enumerated (an accumulator mechanism), but they could also produce a magnitude representation in parallel as a global impression of a total (for details of this issue, see Barth et al. 2003; Cordes et al. 2001; Whalen et al. 1999; Wood & Spelke 2005). We use the term *single-mechanism theories* for all such models in which magnitudes are infants' sole means of keeping track of quantity.

Carey (2001; 2004), Spelke (2000; 2003), and their colleagues, however, have argued that infants' ability to predict the total number of objects in small sets (less than 4) depends, not on internal magnitudes, but on attentional or short-term memory mechanisms that represent individual objects as distinct entities (see, also, Scholl & Leslie 1999). One such representation is maintained for each object within the four-object capacity limit. Infants seem unable to anticipate the correct number of objects in addition-subtraction tasks for cardinalities of four or more (Feigenson & Carey 2003; Feigenson et al. 2002a), even though they can discriminate much larger arrays of items (e.g., 8 vs. 16 dots) in habituation tasks (Xu 2003; Xu & Spelke 2000; Xu et al. 2005). Carey and Spelke therefore argue that infants' failure in the former tasks is due to the infants' tendency to engage object representations (rather than magnitudes) for small numbers of objects. In the original formulation, these were pre-conceptual object-tracking devices – called object files or visual indexes – that record objects' spatial position and perhaps other properties (Kahneman et al. 1992; Pylyshyn 2001). In more recent formulations (Le Corre & Carey 2007), these are working memory representations of sets of individual objects. Success with larger arrays depends instead on a magnitude mechanism that correctly distinguishes sets only if the sets' ratio is large enough (e.g., greater than 3 : 2 for older infants; Lipton & Spelke 2003; Xu & Arriaga 2007). We call this account *the dual mechanism view* (see Feigenson et al. 2004).

Should we conclude, then, that infants' knowledge of number is built on magnitude information alone, on magnitude information in combination with discrete object-based representations, or on some other basis? One issue concerns small numbers of objects. Recent addition-subtraction and habituation experiments with two or three visually presented objects have also controlled for surface area, contour length (i.e., sum of object perimeters), and other continuous variables. Some of these studies, however, have found that infants respond to the continuous variables rather than to cardinality (Clearfield & Mix 1999; Feigenson et al. 2002b; Xu et al. 2005). According to the dual-mechanism explanation, small numbers of objects selectively engage infants' discrete object-representing process, and this process operates correctly in this range. So why don't the infants attend to cardinality? Feigenson et al. (2002b; 2004) suggest that infants do employ discrete object representations in this situation but attend to the continuous properties of the tracked objects when these objects are not distinctive. When the objects do have distinctive properties (Feigenson 2005) or when the infants have to reach for particular toys (Feigenson & Carey 2003), the individuality of the items becomes important, and the infants respond to cardinality. This suggests a three-way distinction among infants' quantitative abilities: (a) With small sets of distinctive objects, infants use discrete representations to discriminate the objects and to maintain a trace of each. (b) With small sets of nondistinctive items, however, infants feed some continuous property from the representation (e.g., surface area) into a mental magnitude and remember the total magnitude. (c) With large sets of objects, infants form a magnitude for the total number. According to (b), infants should fail in discriminating small numbers of nondistinctive objects (e.g., 1 vs. 2 dots) under conditions that control for continuous variables, as they are relying on an irrelevant magnitude, such as total area. But by (c), they should succeed with larger numbers (e.g., 4 vs. 8 dots) under controlled conditions, as they are using a relevant magnitude (total number of items). In fact, there is evidence that this prediction is correct (Xu 2003).

A second issue has to do with large numbers of objects. People's ability to respond to the cardinality of large sets, as well as small sets, depends on individuating the items in the set, barring the kinds of confounds just discussed. This follows from the very concept of cardinality (as Schwartz [1995] has argued). Even a magnitude representation for the total number of objects in a collection must be sensitive to individual objects; else it is not measuring cardinality but some other variable. Individuating objects, in the sense we use here, means determining, for the elements of an array, which elements belong to the same object. Thus, individuation is the basis for deciding when we are dealing with a single object and when we are dealing with more.

Investigators in this area have concluded that object files or working-memory representations cannot be the only means infants have to individuate objects. If they were, then the limitations of these mechanisms would appear in experiments with sets of larger cardinalities containing nondistinctive objects (Barth et al. 2003; 2006; Wood & Spelke 2005; Xu 2003; Xu & Arriaga 2007; Xu & Spelke 2000; Xu et al. 2005). If these object representations simply output information about some continuous variable such as surface area (as they do for small numbers of nondistinctive items), then infants should also fail to distinguish the number of items in large sets in studies that implement appropriate controls (see Mix et al. [2002a] and Xu et al. [2005] for debate about the controls' appropriateness). Moreover, current experiments with both infants (Wood & Spelke 2005) and adults (Barth et al. 2003) find that increasing the cardinality of large arrays does not necessarily increase the time required for discriminating the arrays, provided that the sets to be compared maintain the same ratio (e.g., 2 : 1).

In order to handle the problem of dealing with large sets, we apparently need a mechanism for individuating items, but one that is not subject to the capacity limits of working memory or object files. One possibility is that some perceptual mechanism is able to individuate relatively large numbers of items in parallel, with the output of this analysis fed to a magnitude indicator. Dehaene and Changeux (1993) propose a parallel analysis of this sort, and parallel individuation is also consistent with estimates that adults can attentionally discriminate at least 60 nondistinctive items in the visual field (Intriligator & Cavanagh 2001).[3]

The model in Figure 1 provides a summary of infants' quantitative abilities based on this account. According to this model, infants first segregate items in the visual field by means of a parallel attentive mechanism, similar to that discussed by Intriligator and Cavanagh (2001) or Dehaene and Changeux (1993). Infants will quickly forget the results of this analysis once the physical display is no longer in view. But while the display is visible, infants assign a more permanent object representation if the total number of items is less than four. The Figure 1 model therefore predicts the results for small numbers of objects in the same way as the simpler theory considered earlier (though the use of both parallel segregation and object representations suggests that individuating objects may be a more complex process than might first appear). If the number of items is four or more, however, infants cannot employ object representations but may, instead, use the output from the initial parallel analysis to produce a single measure of approximate cardinality. Thus, the lower track accords with Barth et al.'s (2003) and Wood and Spelke's (2005) findings of constant time to discriminate large displays when the ratios between them are equal. The model assumes ad hoc that object representations take precedence over the global cardinality measure for small arrays. However,

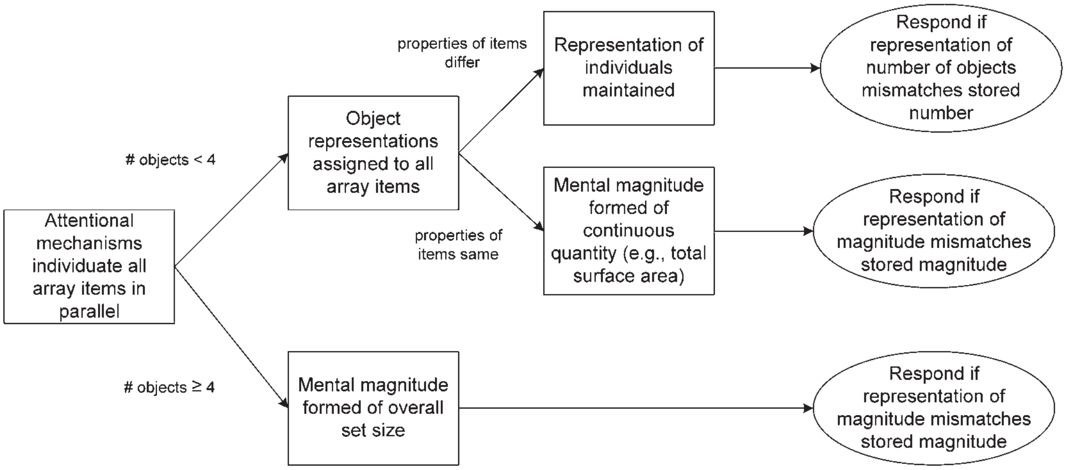

Figure 1. A model for infants' quantitative abilities. Response rules in ovals indicate conditions under which infants look longer in addition-subtraction or habituation tasks. They are not meant to exhaust possible uses of these representations.

perhaps an explanation for this co-opting behavior could be framed in terms of the functional importance of keeping track of individual objects compared to treating them as a lump sum.[4]

We do not mean to suggest that the tracks in Figure 1 are the only quantitative processes that people (especially adults) can apply to an array of items. While the display is visible, adults can obviously enumerate the elements verbally. Similarly, children and adults may also use a nonverbal enumeration mechanism similar to that described by Cordes et al. (2001), Gallistel et al. (2006), and Whalen et al. (1999) in some conditions. Which strategies people employ may depend on properties of the display, task demands, and other factors, and we have not tried to capture this interaction in the figure. (We should add that the model in Fig. 1 is our attempt to understand current empirical results within the dual-mechanism framework, and it may not be an accurate depiction of the views of specific dual-mechanism theorists. Our aim is to clarify the implications of such theories rather than to provide an exact account of a particular version of the model.)

We have tried in this section to understand what mechanisms underlie infants' performance in tasks that aim to assess their numerical concepts. Although there are many uncertainties about the Figure 1 model, it appears to account for much of the available data. Our goal, however, is not to defend the model but to examine its implications for later learning. The model is useful because it presents a provisional survey of the components that, according to developmental research, children bring to learning more sophisticated mathematical notions. The issue for us is this: Many psychologists believe that people's mathematical thinking originates from the components in Figure 1. But, if the picture in Figure 1 is even approximately correct, it presents some extremely difficult problems for how children acquire the concept of natural number. These problems are next on our agenda.

3. The route to concepts of number

Let us suppose the Figure 1 model or some close relative correctly captures infants' sensitivity to cardinality. Should we then say that they have the concept of natural number? Dual-mechanism theorists tend to answer "no" (Carey [2004] and Carey & Sarnecka [2006] are explicit on this point). Neither magnitudes nor short-term representations of individual objects have the properties of the natural numbers; hence, according to these theories, children's quantitative concepts have to undergo conceptual change in order to qualify as true number representations. The task for these theorists is then to specify the nature of this change. Some single-mechanism accounts claim that although magnitudes do not represent natural numbers, they do represent continuous quantity, perhaps even real numbers (Gallistel et al. 2006). The route to natural numbers in this case involves transforming a continuous representation into a countable one. In this section, we extend dual theorists' skepticism about the relation between the natural number concept, on the one hand, and object files, magnitudes, and similar representations, on the other. Not only do the latter fail to qualify as representations of numbers in their own right, but also there is no straightforward way to get from them to natural numbers.

In examining proposals about the acquisition of natural number (and related arithmetic principles in sect. 4), we repeatedly make use of a simple methodological rule that it might be worth describing in advance. In explaining how a person acquires some idea Q, cognitive scientists often claim that people make an inductive inference to Q from some body of information P, which these people already possess. If people already know P and if the inference from P to Q is plausible to them, then the inference is a potential explanation of how they acquire Q. However, rival inferences can undercut such an explanation. Suppose there is also a body of information P' (possibly equal to P) and an inference from P' to a contrary idea *non-*Q. Then if P' is as believable and as salient as P and if the inference from P' to *non-*Q is as plausible as the one from P to Q, then the inference from P to Q fails to explain Q adequately. We call this rule the *no-competing-inference test* for psychological explanations. To take a nonmathematical example, suppose we want to explain people's belief that their deity is omnipotent. We might hypothesize that this idea comes from previous knowledge of a powerful parent, plus a conscious or unconscious inference from the parent to the deity. But, although this may be the right account, we should also consider possible competing inferences. In everyday experience, we encounter only individuals (even parents) with limited power. So why don't people draw the inference from a person with limited power to a deity who is non-omnipotent? There could, of course, be considerations that favor the first inference over the second (e.g., Freud [1927/1961] believed that people's fear and need for protection motivates the inference to an omnipotent deity). However, unless we can supply such a reason – a reason why the selected inference is more convincing than potential competing ones – the initial explanation is incomplete.

It is understandable why theories sometimes violate the no-competing-inference test. Because we ordinarily know the final knowledge state Q that we want to explain, it is natural to look for antecedents P that would lead people to Q. Because we are not trying to explain *non*-Q, we do not seek out antecedents for these rivals. It is also clear that cognitive scientists who work on mathematical thinking are no more prone than others to violate the no-competing-inference principle. Still, we find this principle helpful in evaluating the strengths and weaknesses of existing theories in this domain.

3.1. Numerical concepts versus concepts of numbers

As dual-mechanism theorists have pointed out, analog magnitudes are too coarse to provide the precision associated with specific natural numbers (Carey 2004; Carey & Sarnecka 2006; Spelke 2000; 2003). The magnitude representation of 157 would barely differ from that of 158 (if a magnitude device could represent them at all), so they would not have the specificity of a unique natural number and its successor. Short-term object representations do have the discreteness of natural numbers, but they are not unitary representations. Without further apparatus, having one, two, or three such active representations does not amount to a representation of oneness, twoness, or threeness. If a child is tracking three objects, he or she has one object representation per object but nothing that represents the (cardinality of the) set of three. Unless such representations build in the concept of a unified set of individuated elements, there is nothing to represent number. According to the dual-mechanism story, then, it is only after children learn to count and to combine the precursor representations that they have true concepts of natural numbers.

Some single-mechanism theorists credit infants (and nonhuman animals) with more mathematical sophistication. For example, Gallistel et al. (2006, p. 247) assert that "when we refer to 'mental magnitudes' we are referring to a real number system in the brain." Although we tend to think of real numbers as more advanced concepts than natural numbers, this may reverse the true developmental progression. The reals may be the innate system, with natural numbers emerging later as the result of counting or through other means.

However, some of the criticisms that dual-mechanism theorists level against magnitudes as representations of natural numbers also apply to magnitudes as representations of the reals. Because the mental magnitudes become increasingly noisy and imprecise as the size of the number increases, larger numbers are less discriminable than smaller ones. For example, if we consider 157 and 158 as real numbers (i.e., as 157.000… and 158.000…), they will be much less discriminable than two smaller but equally spaced numbers, such as 3 and 4 (3.000… and 4.000…). In Gallistel et al.'s view, this imprecision is the result of the way a mental magnitude is retrieved rather than a property of the magnitude itself. This is of no comfort, however, to the idea that infants can represent real numbers. If cognitive access to this representation is always noisy or approximate, it is unclear how the system could attribute the correct real-number properties to the representation without some independent concept of the reals. People cannot skirt the retrieval step because, as Gallistel et al. consistently emphasize, the representation of a number cannot be inert but has to play a role in arithmetic reasoning. An analogy may be helpful on this point. Suppose you have access to some continuously varying quantity, such as the level of water in a tub, and suppose, too, that the viewing conditions are such that the higher the level of water, the greater the perceived level randomly deviates from the true level. Could you use such a device to represent the real numbers – in order to perform arithmetic? Although you could combine two quantities of water to get a larger quantity, the representation of the sum would be even fuzzier than that of the original quantities (Barth et al. 2006) and has few of the properties of real-number arithmetic. For example, real-number addition is a function that takes two reals as inputs and yields a unique real as output. But addition with noisy magnitudes is not a function at all; for any two real input values, it can yield any value within a distribution as a possible output.[5] Of course, if you already knew some statistics, you might be able to use this tool to compensate for the deviations, but this depends on a pre-existing grasp of real-number properties.

These considerations suggest that if prelinguistic infants start from the components in Figure 1, then there is no reason to think that they have concepts of either the natural or the real numbers. Many theorists believe, however, that once children have learned language or, at least, language-based counting, they are in a position to attain true concepts of natural numbers and that they have acquired these concepts when they are able to perform tasks such as enumerating the items in an array or carrying out simple commands (e.g., "Give me six balloons"). In what follows, we suggest that neither of these ideas stands up to scrutiny. Language is unable to transform magnitudes or object representations to true number concepts, and tests involving small numbers of objects do not necessarily tap concepts of natural number.

3.2. The role of language and verbal counting

We have mentioned Chomsky's (1988) hypothesis that mathematics piggybacks on language, making use of the ability of syntax to generate countably infinite sequences. In more recent work, Chomsky and colleagues (see Hauser et al. 2002) take what seems a different view of the relation between language and mathematics – one in which both systems spring from an underlying ability to perform recursive computations; we consider this idea in section 5. However, many theorists continue to see language as necessary in shaping a true understanding of the natural numbers. Considering this issue draws us back into an arena of active controversy.

3.2.1. Language as sufficient for number concepts. There are several reasons why the language-to-math hypothesis is attractive. First, natural languages possess properties that are also crucial in mathematics and that are difficult to obtain from experience with everyday objects and actions. The grammatical resources of language can easily generate the type of countably infinite sequence that can represent the natural numbers. For example, the nearly trivial grammar in (1) produces the "square language" we introduced earlier:

(1)
$$S \rightarrow \blacksquare + F$$
$$F \rightarrow \blacksquare + F$$
$$F \rightarrow \varnothing$$

We originally used this language in section 2.1 to represent cardinality, but it could also serve more generally as a system of numerals. As an example, these phrase-structure rules generate the tree structure in Figure 2 as the representation of three.

The role of the *F* symbol in this grammar illustrates the way recursion is useful in generating the natural numbers. The symbol *F* can be embedded as many times as necessary in order to produce the correct number of squares. What makes Figure 2 represent three is in part that it occupies the third position in the sequence of such strings that the grammar of (1) generates. (Of course, we are not proposing the square language as a cognitively plausible representation but only as a simple illustration of the generative capacity that such representations would require; we consider other ways to formulate the natural numbers in sect. 5.) This tie to language would clearly be helpful in accounting for math properties that depend essentially on the infinite size of the natural numbers (see sect. 4). Along similar lines, Pollmann (2003) and Wiese (2003) have pointed out that the natural numbers, like certain parts of language, are an inherently relational system in which the meaning of any numeral depends on its position in the system as a whole. Language furnishes a type of relationally determined meaning in which a sentence, for example, depends on the grammatical relations among its constituents (e.g., "The financier dazzled the actress" differs in meaning from "The actress dazzled the financier"). Thus, language can set the stage for understanding mathematics.

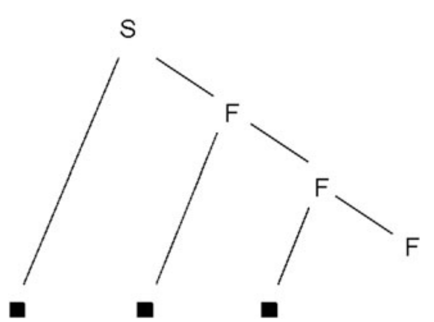

Figure 2. A representation of the number three, according to the grammar rules in (1).

It is possible to imagine a strong version of the language-to-mathematics hypothesis in which possessing a natural language is not only necessary but also sufficient for the development of concepts of natural numbers. According to this type of theory, language is the sole source of number concepts. Psychologists who see a role for language in acquiring number concepts have more often taken the "catalyst" view that we describe shortly (in the next subsection), but the stronger position may be implicit in the idea that "the human number faculty [is] essentially an 'abstraction' from human language"(Chomsky 1988, p. 169). Evidence against this possibility comes from recent studies of native Brazilian peoples who appear to lack concepts for exact numbers greater than four (Gordon 2004; Pica et al. 2004). These people have no number terms that distinguish between, for example, six and seven; instead, they use words such as "many" for larger numbers of items. In tasks that require knowledge of approximate quantity, members of these cultures perform in a way that is comparable to Americans or Europeans. For example, Pica et al. (2004) report that the Mundurukú are able to point to the larger of two sets of 20 to 80 dots with accuracy that is nearly the same as French controls. However, in tasks that require exact enumeration, accuracy is relatively low. If participants see a number of objects placed in a container and then see a subset of the objects withdrawn, they have difficulty predicting how many (Pica et al. 2004) – or whether any (Gordon 2004) – objects remain. These experiments suggest that the Mundurukú and Pirahã peoples use a system for dealing with cardinality roughly similar to that of Figure 1. They treat large cardinalities (and, perhaps, small cardinalities as well) as approximate quantities. As in the case of findings with Western infants, it is possible to question whether difficulties in assessing the cardinality of a set imply lack of a concept of natural numbers (see sects. 2 and 3.3). But in this case, it is difficult to argue for knowledge of natural numbers in the absence of evidence for more than four discrete representations for numerical properties. Gelman and Butterworth (2005) suggest that such counterevidence might be obtainable. However, taking the Brazilian results at face value, we need to explain why natural language shows up in such cultures but natural numbers do not, if there is an innate linguistic basis for a countable number system.[6,7]

3.2.2. Language as a catalyst for number concepts. A weaker, and more plausible, hypothesis is that children need, not language in general, but some type of language-based enumeration technique in order to form number concepts. Initially, children verbally enumerate items by means of simple counting (see sect. 1.1 for our distinction between simple and advanced counting). They match a small fixed list of numerals to the elements of a collection.

Adults, of course, can enumerate by advanced counting, and there is no doubt that advanced counting could be helpful in conveying the concept of natural numbers. Once children have mastered advanced counting, they have a model of the natural numbers that is much closer than anything in the world of (finite) physical experience. This is because the elements of advanced counting (the numerals of the counting system) are in a one-to-one correspondence with the natural numbers – a correspondence that preserves the successor relation (i.e., the successor relation on the numerals corresponds to that on the natural numbers). We are not claiming that children attain the concept of natural number by learning advanced counting: We think it more likely that children learn an underlying set of principles that facilitates both advanced counting and the concept of natural number (see sect. 5). However, advanced counting, not simple counting, provides the numerals that are the obvious counterpart of the natural numbers.

Most psychologists believe, however, that children acquire the natural number concepts long before they master advanced counting either in natural language or in explicit mathematical notation. Thus, if language-based counting plays a role in forming these concepts, it must be simple counting and associated enumerating that are responsible. How are they able to produce this effect? Some recent dual theories of a counting-to-number link suggest that enumerating items with natural-language count terms provides a conceptual bridge between magnitudes and object representations, giving rise to a new sort of mental representation (Spelke 2000; 2003; Spelke & Tsivkin 2001). Magnitudes bring to this marriage the concept of a set, object representations bring the concept of an individual, and the result is the concept of cardinality as a measure of a set of distinct individuals:

> To learn the full meaning of *two*, however, children must combine their representations of individuals and sets: they must learn that *two* applies just in case the array contains a set composed of an individual, of another, numerically distinct individual, and of no further individuals.... The lexical item *two* is learned slowly, on this view, because it must be mapped simultaneously to representations from two distinct core domains. (Spelke 2003, p. 301)

But it is difficult to understand how conjoining these systems could transform number representations in the desired way (see Gelman & Butterworth [2005] and Laurence & Margolis [2005] for related comments). Suppose the meaning of a number word like "two" connects to both a fuzzy magnitude and two object files. According to this theory, magnitude information must transform the representation of two separate objects into an adult-like representation of a single set of two. But why doesn't the fuzziness of the magnitudes lead the children to believe that "two" means approximately two individuals (or a few)? Why do magnitudes lead to sets rather than to some other form of composite, such as a part-whole grouping? Why is language necessary if even infants can treat individual items as parts of chunks (Feigenson & Halberda 2004; Wynn et al. 2002)? Unless we can somehow answer these questions, the explanation trips over what we called the no-competing-inference rule, since there are many competing conclusions about the meaning of number words that children could draw from the same data. We might do better to discard magnitudes and to think of the resulting representations as drawing on some more direct form of set-like grouping. Along these lines, Carey (2004; Carey & Sarnecka 2006; Le Corre & Carey 2007) has proposed that children use the resources of natural-language quantifiers to combine object representations into sets, so that children come to represent one as {a}, two as {a, b}, and so on – representations which we will refer to as "internal sets."

> It is language that spurs the creation of an internal symbol whose meaning is *that which is common to all situations where a pair of individuals are being tracked at the same time*. Associating linguistic markers with unique states of the parallel individuation system is only possible for up to three objects, because the parallel individuation system can only keep track of up to three individuals at once. (Carey & Sarnecka 2006, p. 490, emphasis in original)

Some single-mechanism theories describe infants as already having true natural number concepts for smaller numbers; so the role of language is more plausibly confined to extending these concepts to the rest of the integers (Bloom 2000; see also Hurford 1987 for a related account). According to Bloom (2000, p. 215), for example, "Long before language learning, ... [babies] have the main prerequisite for learning the smaller number words: they have the concepts of oneness, twoness, and threeness. Their problem is simply figuring out the names that go with these concepts."

The crucial question for both single-mechanism and dual-mechanism theories is then whether simple counting and enumerating allow children to extend their knowledge of number beyond these first three to a full concept of natural number. Suppose, in other words, that at a critical stage, children have worked out facts like those in (2):

(2) "one" represents one

"two" represents two

"three" represents three

According to the assumptions we have temporarily adopted, words occupy the left-hand side of these relations, and cardinalities occupy the right-hand sides (e.g., one is the size of singleton sets, two is the size of two-member sets, etc.). The concepts that mediate the relations in (2) depend on the theory in question. For single-mechanism theories, internal magnitudes underlie these associations; for example, "two" denotes two because children learn that "two" represents what the corresponding internal magnitude does. For dual-mechanism theories, the associations depend on preliminary combinations of object representations and magnitudes (Spelke 2003) or object representations and set-like groupings (Carey 2004; Carey & Sarnecka 2006; Le Corre & Carey 2007).

In all cases, though, the outcome of these linkages is that children acquire the denotations in (2). Then, by correlating the sequence of words in the count series with the regular increase in cardinality, the children arrive at something like the generalization in Principle (3):

(3) For any count word "n," the next count word "$s(n)$" in the count sequence refers to the cardinality $(n + 1)$ obtained by adding one element to collections whose cardinality is denoted by "n."

Carey (2004) and Hurford (1987) have detailed formulations along these lines. Similar suggestions appear in Bloom (1994) and Schaeffer et al. (1974). According to Carey and Sarnecka (2006, p. 490), "This idea (one word forward [in the count list] equals one more individual) captures the successor principle." Notice, though, that Principle (3) depends on the concept of the next count word, which we have referred to as "$s(n)$," for any count term "n" (if "n" is "five," "$s(n)$" is "six"; if "n" is "ninety," "$s(n)$" is "ninety-one"; etc.). For these purposes, simple counting won't do as a guide to "$s(n)$," as simple counting uses a finite list of elements. For example, if a child's count list stops at "nine," then Principle (3) can extend the numeral-cardinality connection through nine. In order to capture all the natural numbers, however, Principle (3) requires advanced counting: an appreciation of the full numeral system. But at this point the trouble with the counting hypothesis comes clearly into view, for at the point at which children are supposed to infer Principle (3) – at a little over 4 years of age – they have not yet mastered advanced counting. There is nothing that determines for such a number learner which function or sequence specifies the natural number words (i.e., the function that appears as "$s(n)$" in Principle [3]).

In learning ordinary correlations or functions, children induce a relation between two pre-existing concepts, for example, degree of hunger and time since lunch. By contrast, what lies behind the proposal that children induce Principle (3) from (2) is the bootstrapping hypothesis that they are simultaneously learning advanced counting along with (and because of) the correlation with cardinality. But it is unclear how this is possible in the case of natural numbers (see Rips et al. 2006; 2008). Suppose, for example, that the count system that the child is learning is not one for the natural numbers but, instead, for arithmetic modulo 10, so that adding 1 to 0 produces 1, . . . , and adding 1 to 8 produces 9, but adding 1 to 9 produces 0, and so on in a cyclical pattern. In this case, Principle (3) is still a valid generalization of (2) if we interpret "$s(n)$" as the next numeral in the modular cycle, but then what has been learned is not the natural numbers.

The generalization in (3) can seduce you if you think of the child as interpreting it (after a year of struggle) as "Aha, I finally get it! The next number in the count sequence denotes the size of sets that have one more thing." But "next number in the count sequence" isn't an innocent expression since the issue is, in part, how children figure out from (2) that the next number is given by the successor function for the numerals corresponding to the natural numbers and not to a different sequence (e.g., the numbers mod 10 or mod 38 or mod 983). You might be tempted to reply that this problem is no different from any other case of (empirical) induction, where there is normally an infinite choice of extrapolations. There have to be some constraints on induction to make learning humanly possible. But although this is true, it is unclear what general constraints could steer Principle (3) toward the natural numbers, especially because the function successor-mod-10 and many others seem less complicated than the successor function for natural numbers, with its infinite domain. (See Rips et al. [2006] for a discussion of the relation between the bootstrapping problem and more general problems of induction and meaning.)[8]

We have been concentrating on the relation between numerals and cardinalities, as the issues are clearest in this context, but the same difficulties appear if we look at acquisition of number meanings from the perspective of the mental representations that support them. The theories we are examining suppose that the mental representations are latched to external cardinalities, so that larger internal magnitudes or larger internal sets always correlate with larger external set sizes. On this assumption, if we are learning the standard system for natural numbers, "nine" will come to be associated with a single magnitude or internal set, whereas if we are learning the mod_{10} system, "nine" will be associated with a collection of internal magnitudes or sets. Principle (3) does not tell us which of these connections is correct.

Some theorists may understand Principle (3) as a way of transcending, rather than extending, the initial representations. On this understanding, mental magnitudes or internal sets are no longer needed once children arrive at this principle. To the extent that properties of the initial representations carry over to later ones, however, they bring additional difficulties to number concepts. If we start with a magnitude representation for (2) and extend it by (3), we get increasingly noisy representations as we go to higher numbers. There is nothing about the representation that gives us the ingredients we need to formulate the correct hypothesis of a countable sequence (as Leslie et al. [2007] point out). If we start with a set-like representation for (2) and try to extend it by (3), we run into the problem that we can't possibly represent in this way more than a small initial segment of the natural numbers. Our ability to represent individual sets (e.g., {a}, {a, b}) must come to an end because of memory limits, but the natural numbers keep on going. To take up the slack, the concepts have to go generative, as in (1). But since the right generative principle is not supposed to be available beforehand, it is unclear what guarantees a structure that will continue infinitely. To represent the natural numbers, though, we need a representation for a sequence that is both countable and infinite.

This is not to say that the generalization in (3) is false or that it is unhelpful to number learners. The generalization is true, but it does not serve to fix the meaning of the numerals for children, because at this point they do not know what function "$s(n)$" is. For this reason, Principle (3) cannot tell them what the natural numbers are; Principle (3) is indeterminate for them. For practical purposes of enumerating objects, of course, it is important for children to realize that there is some systematic relation or other that holds between the numeral sequence and the cardinalities, and (3) could mark this recognition. However, realizing that there is such a link does not fully specify it. Theories of number acquisition rely on Principle (3) both because they take the meaning of a numeral to be a cardinality and because they suppose (3) specifies this meaning for the natural numbers. But (3) is incapable of performing this function, since it presupposes knowledge of the very structure that it is supposed to create. This suggests that enumerating might be less crucial to the development of natural number than might first appear. Enumerating – pairing numerals to cardinalities – cannot create the natural numbers, as many forms of enumerating that are consistent with Principle (3) lead to nonstandard systems (see sect. 5.3.4).

3.3 Do tests of "how many objects?" require concepts of natural number?

Suppose, though, that the child finally succeeds in the standard tests of number comprehension, performing correctly when asked to "Point to the picture with six dogs" or to "Give me six balloons." Should we now say that he or she has the concept of natural number? The answer seems to be "no" when we are dealing with the small collections that these experiments employ. A textbook exercise in first-order logic asks students to paraphrase sentences such as "There are (exactly) two dogs" or "There are (exactly) three hats."[9] An answer to the first of these exercises appears in Proposition (4):

(4) $(\exists x)\,(\exists y)\,(x \text{ is a dog})\ \&\ (y \text{ is a dog})\ \&\ (x \neq y)\ \&\ (\forall z)\,(z \text{ is a dog} \supset [(z = x) \vee (z = y)])$.

Sentences like this one do not contain references to numbers or any other mathematical objects but get along with concrete objects, such as dogs. The quantifiers and variables in (4) make clear its commitments about the existence of objects – (4) is committed to dogs but not to numbers (see Hodes 1984; Parsons 2008; Quine 1973) – but the representation for two dogs as an internal set (e.g., {a, b}) or as a magnitude presumably carry the same commitments.

We should be careful to acknowledge that children's quantitative abilities extend beyond concrete physical objects like dogs. Even infants are sensitive to the number of tones in a sequence (e.g., Lipton & Spelke 2003) and to the number of jumps of a puppet (Wynn 1996). They also keep track of sums of entities appearing in different modalities – for example, visual objects plus tones, at least if they have previously witnessed the tones paired with the objects (Kobayashi et al. 2004). In this sense, the infants' numerical skills are more abstract than what is required to enumerate visually presented items. However, this type of abstractness does not affect the present argument, since the infants can accomplish all these tasks by representing objects, tones, or jumps, rather than number.

Our goal in this article is to find out how people attain the concept of natural number. To summarize our interim conclusions about this, let us consider hypothetical children who have made the inference to Principle (3) and can correctly understand requests, such as "Give me *n* balloons," for "*n*" up to "nine" (or the last of the children's current set of count terms).

Do such children have the concept NATURAL NUMBER?

No, since many definitional properties of the natural numbers are unknown to them (e.g., that the numbers don't loop around).

Could the children have partial knowledge of NATURAL NUMBER?

Yes, in the sense that they could know some properties of this concept. There is no reason to think that knowledge of natural numbers is all or none. Although children must have a certain body of information to be said to have the natural number concept (see sect. 1.1), they may assemble the components of this information over an extended period of time.

Do such children have the concept of ONE (or TWO or… or NINE)?

Not that we can discern from the results of tests such as "Give me *n*." Although children may have such concepts, the range of tasks that we have reviewed does not reveal their presence. To put this in a slightly different way, the developmental studies may have revealed numerical concepts but not concepts of numbers. It may be only when children make mathematical judgments *about numbers* (rather than about objects) that we can study the nature of these concepts. For example, whereas it is easy to express the idea that there are two dogs by means of proposition (4) without using concepts of numbers, it is more difficult to express the ideas that one is the first number, that one is less than two, that for any number there is a larger one, and so on.[10]

Our distinction between numerical concepts and concepts of number partially resembles others that have appeared in the literature on number development. Gelman (1972; Gelman & Gallistel 1978, Ch. 10), for example, separates children's ability to determine the number of elements in a collection from their ability to reason about the resulting cardinality. For instance, deciding that there are three books in one pile and five in another requires enumerating the books, but deciding that the two piles have different numbers of books is a matter of numerical reasoning, in Gelman's terminology. The distinction we are driving at here, however, differs in that even numerical reasoning (in Gelman's sense) does not necessarily involve concepts of number. It would be possible to determine that two piles have different numbers of books by employing concrete representations of books rather than representations of number. Compare this judgment with the idea that five is greater than three, which does seem to require concepts of numbers. Closer to our own distinction is Gelman and Gallistel's account of numerical versus algebraic reasoning: "Numerical reasoning deals with representations of specific numerosities. Algebraic reasoning deals with relations between unspecified numerosities" (Gelman & Gallistel 1978, p. 230). However, even algebraic reasoning on this account is about the cardinalities of physical objects rather than about numbers themselves. Gelman and Gallistel (1978, p. 236) do note, however, "the conceivable existence of another stage of development… In this stage arithmetic is no longer limited to dealing with representations of numerosity. It now deals with that ethereal abstraction called number."

To forestall a possible misunderstanding, we are not asking whether children have conscious access to the principles governing the natural number system or other mathematical domains, and we are not asking when (or if) children are able to behave like "little mathematicians" in explicitly wielding such principles in reasoning or computation. Of course, a child's explicit formulation of such principles would be excellent evidence that he or she had concepts of natural numbers, and it would place an upper bound on when he or she had acquired these concepts. But there is no reason why the child couldn't display evidence of such concepts indirectly – for example, evidence of a correct understanding of the sentence, "Three is less than four." Gelman and Greeno (1989) have clarified this point concerning mathematical principles, and the analogous case with respect to knowledge of linguistic rules is too well known to need replaying here. What we are interested in probing is whether children have any concept whatsoever of numbers, implicit or explicit, and our review of research on infants and preschool children has turned up no evidence that allows us to decide this issue. This is due to limitations in the nature of the experimental tasks. To find such evidence, then, we need to look at how children make mathematical judgments that have a more complex structure, as we do in the following section.

One objection to this line of reasoning can be summarized thus:

> It is impossible that early quantitative abilities are disconnected from true concepts of number, since evidence for these precursors appears even in adults' mathematics. For example, adults' judgments of which of two digits is larger yield distance effects on reaction times (see the studies cited in sect. 2.2). Assuming that some magnitude-like representation is responsible for this effect, magnitude must be part of adults' natural-number concept. For this reason, some proposals about number representation in adults have included these magnitudes, along with other ingredients (e.g., Anderson 1998; McCloskey & Lindemann 1992).

Adults might well find magnitude representations useful, for example, in carrying out tasks that call for estimation of quantity or amounts; but we do not find it convincing that because number terms are associated with magnitudes, magnitudes are responsible for number concepts. There may be a sense of "concept" in psychology in which anything can be part of a concept, as long as a corresponding expression reminds us of it. But what proponents of magnitudes-as-precursors-of-natural-numbers have to claim is not just that magnitude is associated with natural number (in the way, e.g., that BREAD is associated with JAM), but also that it plays a causal role in children's acquisition of this concept – that NATURAL NUMBER is built on a foundation of magnitude – and we see no reason for believing this is true. For example, natural number includes the notion that each such number has a unique successor, but there is nothing about magnitudes that enforces this idea (since magnitudes don't have successors), and there is no easy way for magnitudes to be conjoined with this idea to produce the adult concept of natural number.

Here is a related objection:

> Some of the components of Figure 1 seem likely to be part of adults' ability to enumerate objects using advanced counting. For example, they must use object individuation to discriminate the to-be-enumerated items, and they may need object representations or magnitudes as well. Granted: these resources are not sufficient for adults' (or even children's) object enumeration, since this requires further knowledge, such as Gelman and Gallistel's cardinal principle (the last element of the count sequence represents the cardinality of a collection). Nevertheless, some of the Figure 1 processes are surely part of the story of adult enumeration and, hence, must be part of adults' concept of natural number.

This objection is initially tempting because of the assumptions that we have temporarily adopted: that numbers are cardinalities and that cardinalities are sets of sets of physical objects. The components of Figure 1 that determine object representations no doubt carry over to adult performance in enumerating objects (i.e., determining the cardinality of groups of objects). But this only makes the difficulties we have just seen more acute. The lack of a plausible story about how children graduate from the representations and processes of Figure 1 to an adult concept of natural number suggests that the assumptions themselves are incorrect. As in the case of the previous objection, this one works only if you assume that adults' enumerating figures into the concept NATURAL NUMBER. What we suggest in section 5 is that the natural number concept, and even concepts of particular numbers such as TEN, may not necessarily depend on enumeration, either definitionally or empirically. Before exploring this idea, however, we first examine a different route from objects to number.

4. Knowledge of mathematical principles

The ability to perform simple counting and enumerating probably will not suffice as evidence of concepts of numbers for the reasons we have just seen. Even early arithmetic may be too restricted a skill to demand number concepts: A child's first taste of arithmetic may involve object tracking, mental manipulation of images of objects, counting strategies, or mental look up of sums that do not require the numerals to refer to numbers. This may seem to raise the issue of whether even adults have or use the concept of natural number outside very special contexts, such as mathematics classes. Certainly, older children and adults continue to use number words in phrases such as "three stooges" for which no concepts of number may be in play. However, older children and adults also appear to have a range of knowledge about numbers, which they can use in nontrivial arithmetic, numerical problem solving, and other tasks, and a look at this knowledge may give us some ideas about how the natural number concept first appears.

One place to search for evidence of concepts of numbers is knowledge of general statements that hold for infinitely many numbers. Understanding generalizations of the form "for any number x, $F(x)$" forces people to deal with concepts that carry a commitment to numbers rather than to physical objects, as these generalizations are overtly about numbers. Statements of this sort include those that define the numbers (e.g., every natural number has just one immediate successor) and those that state arithmetic principles that adults can express with algebraic variables (e.g., additive commutativity: $a + b = b + a$; additive inverse principle: $a + b - b = a$). Statements of the first sort have an especially important role here, as they bear on the issue of when people can be said to have the concept NATURAL NUMBER, and we return to them in section 5. General arithmetic principles, though, are also of interest because the infinite scope of such principles makes it difficult to paraphrase them purely in terms of statement about physical objects (at least not without additional mathematical apparatus). Children's knowledge of these principles can provide evidence that they have a concept of number, whether or not this exactly coincides with the natural numbers. In this section, we consider as an example the additive commutativity principle because there is a substantial body of research devoted to how children acquire it. (We also consider briefly the additive inverse principle in Note 11.) Bear in mind, however, that many other principles could serve the same purpose.

We are not requiring that children be able to compute the answers to specific arithmetic problems in order to demonstrate understanding of math principles: It is enough that they recognize the necessity of the rule itself. Although children would, of course, have to possess the notion of addition in order for them to recognize that $a + b = b + a$, there is no need for them to be able to compute correctly that, say, $946 + 285 = 1{,}231$ and $285 + 946 = 1{,}231$. What's crucial is that they understand that, *for all natural numbers*, reversing the order of the addends does not change their sum.

4.1. Acquisition of the commutativity principle

Commutativity appears to be one of the few general relations to attract researchers' attention, probably because of its close ties to children's early addition strategies. It may also be one of the first general properties of addition that children acquire (Canobi et al. 2002; see Baroody et al. [2003] for an extensive review of children's concept of commutativity.)

Evidence on commutativity suggests that most 5-year-olds know that the left-to-right order of two groups of objects is irrelevant to their total. Three ducks on the right and two ducks on the left have the same sum as two on the right and three on the left. Children recognize the truth of this relation even when they can't count the number of items in one or both groups – for example, because the experimenter has concealed them (Canobi et al. 2002; Cowan & Renton 1996; Ioakimidou 1998, as cited in Cowan 2003; Sophian et al. 1995). Of course, children do not read any mathematical notation in these studies; they simply make same/different judgments about the total number of objects. Hence, any potential difficulties in coping with explicit mathematical variables do not come into play in the way they might for beginning algebra students (MacGregor & Stacey 1997; Matz 1982). However, the lack of explicit mathematical connections raises the issue of whether children's judgments about the spatial or temporal order of the combination reflect the same notion of commutativity as their later understanding that $a + b = b + a$. Children who succeed in these grouping tasks have apparently understood the idea that for two disjoint collections of concrete objects, A and B, certain spatial or temporal rearrangements do not change the cardinality of their union. But the commutativity of addition is the statement that for any two numbers, a and b, the number produced by adding a to b is the same as that produced by adding b to a.

This difference between generalizing over objects and over numbers does not imply that knowledge of the spatial or temporal commutativity of objects is irrelevant in learning the commutativity of addition. In working out the relation between them, however, it is good to keep in mind that not all binary mathematical operations are commutative. For example, subtraction, division, and matrix multiplication are not; even addition of ordinal numbers is not commutative (Hamilton 1982, p. 216). Similarly, not all physical grouping operations are commutative in the sense of preserving cardinalities. The total number of objects in a pile may depend on whether fragile objects are put on before or after heavy ones. This suggests that any transfer of commutativity from physical to mathematical operations must be selective rather than automatic. There seems to be little possibility that children could first discover that physical grouping of objects is commutative with respect to totals and then immediately generalize commutativity to addition of numbers. Children would have to hedge the initial "discovery" in ways that might be difficult to anticipate before they had some knowledge of addition itself, and they would have to transfer the properties to some mathematical operations but not to others.

This difference between commutativity in the physical and mathematical domains helps account for some of the empirical findings. Many children are able to pass a commutativity test involving sums of hidden objects, as in the experiments cited earlier, before they are able to solve simple addition problems (Ioakimidou 1998, as cited in Cowan 2003). Once they have learned addition, however, they do not automatically recognize the commutativity of specific totals (e.g., that $2 + 5 = 5 + 2$). This is true even when the addition strategies they use presuppose commutativity. For example, some children solve addition problems by finding the larger of the two addends and then counting upwards by the smaller addend; that is, these children solve both 2+5 and 5+2 by starting with 5 and counting up two more units to 7. However, children who use this strategy of counting on from the larger addend do not always see that addition is commutative when directly faced with this problem. For example, although they may use counting from the larger addend to solve both 2+5 and $5 + 2$, they may not be able to affirm that $2 + 5 = 5+2$ without performing the two addition operations separately and then comparing them (Baroody & Gannon 1984). In fact, some children seem to discover the commutativity of addition only after noticing that these paired sums turn out to be the same over a range of problems (Baroody et al. 2003).

We argued in section 3 that there is no evidence from studies of infants that they possess concepts of numbers. Even tasks with older children that require them to determine the cardinality associated with specific number words do not necessarily reveal their presence. Of course, it is still quite possible that preschool children have such concepts. The available experimental techniques may simply not be the right ones to detect them. The studies on commutativity are of interest in this respect because tasks involving this principle do seem to require concepts of numbers in order for children to appreciate the principle's generality. The results of these studies suggest, however, that children do not automatically recognize the validity of the principle when they first confront it.[11]

We are about to explore the issue of where such principles come from. But the findings about additive commutativity already suggest that people's understanding of mathematical properties cannot be completely explained by their nonmathematical experience. This partial independence is in line with the relative certainty we attach to mathematical versus nonmathematical versions of these properties. We conceive of the commutativity of addition for natural numbers as true in all possible worlds but not the commutativity of physical grouping operations.

4.2. Mapping of mathematical principles from physical experience

Most psychological theories of math principles (e.g., commutativity) portray them as based on knowledge of physical objects or actions. In this respect, these theories follow Mill's assertion that:

> the fundamental truths of [the science of Number] all rest on the evidence of sense; they are proved by showing to our eyes and fingers that any given number of objects – ten balls, for example – may by separation and re-arrangement exhibit to

> the senses all the different sets of numbers the sum of which is equal to ten. (Mill 1874, p. 190)

Of course, nearly all contemporary theories in this area credit children with some innate knowledge of numerical concepts (e.g., via magnitudes), as we have seen in section 2. Unlike Mill's proposal, these theories do not try to reduce all mathematical knowledge to perceptual knowledge. Nevertheless, all theories of how children acquire arithmetic principles, such as the commutativity or the additive inverse principle, view these principles as based, at least in part, on physical object grouping. In the case of the commutativity of addition, these theories typically see spatial-temporal commutativity for sums of objects as a precursor, though they may also acknowledge the role of other psychological components, such as experience with computation (e.g., Gelman & Gallistel 1978, p. 191; Lakoff & Núñez 2000, p. 58; Piaget 1970, pp. 16–17; Resnick 1992, pp. 407–408). Theories of this sort must then explain the transition from knowledge of the object domain to the mathematical domain. An account of the empirical-to-mathematical transition is pressing in view of the evidence that this transition is not automatic. How does this transformation take place?

According to Lakoff and Núñez (2000), general properties of arithmetic depend on mappings from everyday experience. These mappings begin with simple correlations between a child's perceptual-motor activities and a set of innate, but limited, arithmetic operations (roughly the ones covered by the Fig. 1 model). The child experiences the grouping of physical objects simultaneously with the addition or subtraction of small numbers. This correlation is supposed to produce neural connections between cortical sensory-motor areas and areas specialized for arithmetic, and these connections then support mapping of properties from object grouping to arithmetic. Lakoff and Núñez call such a mapping a "conceptual metaphor" – in this case, the "Arithmetic is Object Collection" metaphor. This metaphor transfers inferences from the domain of object collections to that of arithmetic, including some inferences that do not hold for the innate part of arithmetic. For example, closure of addition – the principle that adding any two natural numbers produces a natural number – does not hold in innate arithmetic, according to Lakoff and Núñez, because innate arithmetic is limited to numbers less than four. The metaphor Arithmetic is Object Collection, however, allows closure to be transferred from the object to the number domain, expanding the nature of arithmetic:

> [T]he metaphor [Arithmetic is Object Collection] will also extend innate arithmetic, adding properties that the innate arithmetic of numbers 1 through 4 does not have, because of its limited range – namely, *closure* (e.g., under addition) and what follows from closure...The metaphor will map these properties from the domain of object collections to the expanded domain of number. The result is the elementary arithmetic of addition and subtraction for natural numbers, *which goes beyond innate arithmetic*. (Lakoff & Núñez 2000, p. 60, emphasis in original.)

A key issue for the theory, though, is that everyday experience with physical objects, which provides the source domain for the metaphors, does not always exhibit the properties that these metaphors are supposed to supply. Closure under addition, for example, does not always hold for physical objects, as there are obvious restrictions on our ability to collect objects together. The mappings in question are unconscious ones: They do not require deliberative reasoning about object collections or mathematics; and they are not posited specifically for arithmetic. Still, given everyday limits on the disposition of objects, why don't people acquire the opposite "nonclosure" property – that collections of objects *cannot* always be collected together – and project it to numbers? Acquiring the closure property cannot rest on a child's experience that it is always possible "in principle" to add another object, since it is exactly this principle that the theory must explain. The theory seems to run up against the no-competing-inference test that we outlined at the beginning of section 3.

The Lakoff-Núñez theory also contains a metaphor that produces the concept of infinity from experience with physical processes: "The Basic Metaphor of Infinity." This metaphor projects the notion of an infinite entity (e.g., an infinite set) from experience with repeated physical processes, such as jumping. The repeated process is conceived in the metaphor as unending and yet as having, not only intermediate states, but also a final resultant state. Mapping this conception to a mathematical operation yields the idea of an infinitely repeated process (e.g., adding items to a set) and an infinite resulting entity (e.g., an infinite set). Lakoff and Núñez do not invoke the Basic Metaphor of Infinity in their initial explanation of closure under addition (Lakoff & Núñez 2000, pp. 56–60), perhaps because closure does not necessarily require an infinite set (e.g., modular arithmetic is closed under addition, even though only a finite set of elements is involved). But they do use this metaphor later in dealing with what they call "generative closure," which would include additive closure as a special case (pp. 176–78). Closure of addition over the natural numbers does involve an infinite set; so perhaps the Basic Metaphor of Infinity is needed in this context. However, this additional apparatus encounters the same difficulty from the no-competing-inference test as does their earlier explanation. Although there may be a potential metaphorical mapping from iterated physical processes to infinite sets of numbers, it is at least as easy to imagine other mappings from iterated processes to finite sets. Why would people follow the first type of inference rather than the second? The Lakoff-Núñez theory is part of a more encompassing framework of cognitive semantics and embodied cognition (see Lakoff & Núñez 2000 for references to this literature), and it includes many more conceptual metaphors. As far as we can see, however, there is nothing about this background that would allow us to resolve this question. This does not mean that such mappings are worthless. Math teachers can exploit them to motivate complex ideas by emphasizing certain metaphors over their rivals ("Don't think about limits that way, think about them this way..."). But without some method for making rival inferences less plausible than the chosen one, the mappings do not explain acquisition.

The principles that make trouble for mapping theories are precisely the ones that are of central interest for our purposes: They are generalizations over all numbers within some math domain. To see that these principles are true, people cannot simply enumerate instances but must grasp, at least implicitly, general properties of the number system. Because the domain of ordinary physical objects and actions contains no counterpart to these principles, people cannot automatically transfer them from that domain. It is possible, of course, that cognitive theories could get around these difficulties by envisioning a different kind of relation between the physical and mathematical realms. In particular, ideas about mathematical objects may be the result of idealizing or theorizing about concrete experience – a view that goes along with certain strains in the philosophy of mathematics (e.g., Putnam 1971; Quine 1960). But it is not easy to get a clear picture of how the psychological theory-building process works. It is unclear, for example, how a mental math theory compensates for the messiness of object grouping to obtain the crisp properties of addition, such as commutativity, additive closure, and so on, that allow mathematical reasoning to proceed. Some versions of the theory idea in psychology depend on postulating a metacognitive process that allows people to reflect on lower-level mental representations and to create a new higher-level representation that generalizes their properties (e.g., Beth & Piaget 1966; Resnick 1992). However, once the abstracting begins, how does this system know which features to preserve, which to regularize or idealize, and which to discard?

In this section, we have been exploring possible ways for children to arrive at math generalizations. This is because these generalizations provide evidence that children have concepts of number. If our present considerations are correct, however, children cannot reach such generalizations by induction over physical objects, and we should therefore consider more direct ways of reaching them. It is also worth noting that many of these same concerns apply to theories in which abstracting over physical objects yields, not mathematical principles like commutativity, but the numbers themselves. Suppose that children initially notice that two similar sets of objects – for example, two sets of three toy cars – can be matched one-to-one. At a later stage, they may extend this matching to successively less similar objects – three toy cars matched to three toy drivers – and eventually to one-to-one matching for any two sets of three items. This could yield the general concept of sets that can be matched one-to-one to a target set of three objects – a possible representation for three itself. In this way, learning the number three could be seen as a concept forming process similar to, but more abstract than, the formation of other natural language concepts (see Mix et al. 2002b). But, as we have already mentioned in section 2, the theoretical view that a number is a set of equinumerous sets of physical objects is on shaky grounds, and even if it is possible to learn the concepts of small natural numbers (e.g., THREE) in this way, there is no possibility that the abstraction process is sufficient psychologically for learning all natural numbers one by one.

5. Math schemas

We suggested that early quantitative skills may reveal more about object concepts than about math concepts, and we also suggested that children cannot bootstrap their way from these beginnings to true math concepts by means of empirical induction. For this reason, we looked at beliefs that are more directly about numbers and other mathematical entities. Principles such as the commutativity of addition are cases in point, as are others that generalize over all numbers. Most psychological theories of math suppose that people acquire such generalizations from their experience with physical objects (with the aid of innate numerical concepts, such as magnitudes), but an inspection of these theories revealed gaps in their explanations. These theoretical problems go hand in hand with psychological evidence that questions the possibility of abstracting math from everyday experience. What's left as an account of concepts of number?

5.1. An alternative view of number knowledge

We believe a better explanation of how people understand math takes a top-down approach. Instead of attempting to project the natural numbers from knowledge of physical objects or from partial knowledge of the numerals and cardinality, children form a schema for the numbers that specifies their structure as a countably infinite sequence. Once the schema is in place, they can use it to reorganize and to extend their fragmentary knowledge. The schema furnishes them with a representation for the natural numbers, because the elements of the structure play just such a role.

This view contrasts with the bottom-up approaches that we have canvassed in sections 3 and 4. These approaches suggest that children achieve knowledge of the natural number concept by extrapolating from their early skills in enumerating objects (or manipulating them in other ways). Some form of inductive inference transforms these skills into a full-fledged grasp of the natural numbers. Our review turned up no plausible proposals about the crucial inference, and our suspicion is that this gap is a principled one. Children's simple counting and enumerating does not provide rich enough constraints to formulate the right hypothesis about the natural numbers (Rips et al. 2006; 2008). Investigators could, of course, agree that a pure bottom-up approach cannot be the whole story and that early numerical concepts have to be supplemented with further constraints in order for children to converge on the right hypothesis. But, although this hybrid idea might be correct, the constraints that are needed (which we discuss in the following section) are themselves sufficient to determine the correct structure. Why not suppose, then, that children build a schema for the natural numbers on the basis of these constraints and then instantiate the schema to their preliminary number knowledge?

What is distinctive to the approach we are exploring is that the natural number schema is understood directly as generalizations about numbers rather than in terms of operations on physical objects, such as enumerating or grouping. According to our view, it is no use trying to reduce number talk to object talk, or number thought to object thought. Of course, early numerical concepts could help motivate children to search for math schemas as a way of dealing with their experience. On the present view, however, although these concepts may play a motivational role, they do not provide direct input to schema construction; they do not play a role in framing hypotheses about the concept NATURAL NUMBER.

A kind of caricaturized version of our hypothesis is that children learn axioms for math domains, having come equipped with enough logical concepts to be able to express these axioms and with enough deductive machinery to draw out some of their consequences. It is impossible to give a full theory at this point, as not enough evidence is available about the key principles, but in what follows, we consider in a tentative and speculative way what some of the components of such a theory might be, attempting to fill in enough gaps to make it seem less like a caricature. In section 4, we looked at principles that refer to numbers in general, exploring proposals about where these concepts come from. In this section, we narrow our focus to principles that define the concept NATURAL NUMBER.

5.2. Starting points

A first approximation is to think of a knowledge schema for a mathematical domain as knowledge of the definitions or axioms for that field, plus inference rules for applying them. But, of course, in the case of knowledge of the natural numbers, we obviously don't introduce children to the topic by giving them axioms, definitions, and inference rules. They therefore do not start out with a schema for natural numbers in the sense in which an undergraduate who has just learned the axioms of set theory has a schema for set theory. Instead, children gradually acquire the information they need to understand the meanings of numbers. What are the starting points for learning this information if, as we believe, they are not the quantitative abilities discussed in sections 2 to 4? We will assume that children have an innate grasp of concepts that allow them to express the notions of uniqueness (there is one and only one *P* such that...) and mapping (for every *P* there is a *Q* such that...). These resources would allow children to formulate the idea of a function (for every *P*, there is one and only one *Q* such that...) and a function that is one-to-one. It is important for our purposes that these representations contain variables for individuals and predicates, since it is in this sense that the representations are schematic.

We also assume that children have innate processing abilities for combining and applying these representations. The crucial built-in operation for math is recursion. A particular token operation may need to carry out other tokens of the same operation in the course of its execution. The system must maintain procedures that keep track of potential levels of embedding, so that execution of the highest-level operation can resume when the second-level finishes after the third level finishes... after the lowest-level finishes. The same operations can also be used to perform simple iterative tasks. The importance of recursion for understanding natural numbers comes from its close relation to the successor function, as we noticed in connection with the grammar for the square language for natural numbers in (1) (see sect. 3.2.1). Our reading of the proposal by Hauser et al. (2002) is that natural language, mathematics, and navigation all draw on a more basic recursive capacity, and if so, this proposal seems consistent with the present suggestions. Of course, recursion alone is not sufficient for producing the natural numbers, but it may well be a necessary part of people's ability to use these structures.

Like many theories that include an innate component, this one has to deal with the fact that children tend to develop mathematics relatively late and in a relatively variable way, compared to skills like comprehending their native language. In addition to the built-in aspects, however, children must still assemble the schematic or structural information that is specific to a domain of mathematics (see the following subsection). We typically expect children to acquire abilities such as these in a measured way that depends in part on their exposure to the key information. Moreover, by taking the ultimate source of countable infinity to be a math schema rather than language, we gain some flexibility in accounting for the psychological facts. For one thing, we needn't worry why mathematics is not distributed in the same universal way as natural language. We take no position on the exact relationship between language learning and mathematics learning; but from the point of view of Hauser et al. (2002), in which language and mathematics both draw on the same recursive resources, the issue is not why mathematics is slow and effortful but why language is fast and easy.

5.3. Math principles

What information must children include in their math schema in order to possess the concept of natural number? As we mentioned earlier, it is hard to escape the conclusion that they need to understand that there is a unique initial number (0 or 1); that each number has a unique successor; that each number (but the first) has a unique predecessor; and that nothing else (nothing other than the initial number and its successors) can be a natural number. These are the ideas that the Dedekind-Peano axioms for the natural numbers codify (Dedekind 1888/1963), and our top-down approach suggests that these principles (or logically equivalent ones) are acquired as such – that is, as generalizations – rather than being induced from facts about physical objects. However, to repeat our earlier warning, there is no reason to think that children have to be consciously aware of these ideas, to have them in a formalized language, to cite them explicitly in reasoning, or to come upon them all at once. People also supplement these basic ideas with many elaborations rather than deriving all their number knowledge from basic principles. Without something like a tacit grasp of these central ideas, however, it is simply unclear what it would mean to claim that children had a concept of natural number. For this reason, it is striking how little research has been devoted to these principles. Here we summarize the state of knowledge of such principles, partly to identify where gaps exist in research.

5.3.1. The first number. Children may appreciate quite early in their mathematical career that the unique starting number is one. By the time they are 3 years old, they can recite short counting sequences beginning with "one," and they are able to understand phrases such as "one dog" (Fuson 1988; Wynn 1992a). As we have emphasized, however, these abilities do not necessarily indicate that children think of "one" as a number. The functions "one" performs in sentences such as "Give me one balloon" are similar to those of determiners such as "a" ("Give me a balloon"), which are not numbers (e.g., Carey 2004; Carey & Sarnecka 2006). Evidence seems to be lacking about when children use number terms in expressions such as "One is the first number" or "One is less than two," that are prima facie about numbers rather than about (physical) objects. Even when children are able to affirm that one is a number, it is unclear at what point they have distinguished numbers from the numerals they see in picture books, puzzles, and games. In ordinary talk, number terms are ambiguous in this respect ("The number one is to the left of the number two" refers to numerals, but "The number one is less than the number two" refers to numbers).

Although most psychological theories consider "one" to be the first number term because of its position in the standard sequence of count terms and because of its role in enumerating, it is not completely clear that this should rule out zero as a possible initial number for children. On the one hand, there is evidence that zero presents some conceptual difficulties (Wellman & Miller 1986). On the other, children seem to have an early understanding of quantifiers such as "none" or "no" (as in "There are no cookies") that express a cardinality of zero items (Hanlon 1988). On the assumptions that numbers are cardinalities and that numbers derive from natural language quantifiers, it is mysterious why zero should be so difficult.

5.3.2. The successor function is one-to-one. It is the one-to-one nature of the successor function that makes the natural numbers unending. Children must learn that each natural number has just one successor (so the successor relation is a function) and that each natural number except one has just a single predecessor (so the successor function is one-to-one). Because of these constraints, the sequence of natural numbers cannot stop or double back. Evidence concerning children's appreciation of these facts suggests that they appear rather late (Hartnett 1991). Although children in kindergarten are often able to affirm that you can keep on counting or adding 1 to numbers, it takes them a while – perhaps as long as another year or two – to work out the fact that this implies that there cannot be a largest number. Counting skill is not a good predictor of the ability to understand the successor function, although knowledge of numbers larger than 100 does seem predictive. It may be, as Hartnett suggests, that children who can grapple with larger numbers have learned enough about the generative rules of the numeral sequence (i.e., advanced counting) to understand their implications about the infinity of the numbers. As we would expect, there is a relation between knowledge of advanced counting and knowledge of constraints on the successor function, but the exact form of this interaction cannot be determined from present evidence.

5.3.3. Math induction. In its usual formal presentations, this closure principle takes the form: "For all properties P: if $P(0)$ and if $P(k)$ implies $P(k+1)$ for an arbitrary natural number k, then for all natural numbers n, $P(n)$." In view of the importance of mathematical induction for an understanding of natural numbers, it is odd that psychologists have given this principle so little attention. We know of only one recent study that purports to investigate children's understanding of mathematical induction (Smith 2002), but unfortunately, it actually examines a quite different logical principle – universal generalization – as we have argued elsewhere (Rips & Asmuth 2007). It may seem strange even to suppose that children just learning the natural numbers could cope with a principle as complex as math induction, which they typically encounter only much later as a proof rule in high school. But math induction is equivalent to the following idea (the Least Number Principle), given other facts about the natural numbers (Kaye 1991): For all properties P: If $P(n)$, then there is a smallest number m such that $P(m)$. The Least Number Principle does not seem out of reach of children.

5.3.4. Other principles? Mention of the Least Number Principle should make it clear that we are not claiming that the Dedekind-Peano axioms are the only ones that are sufficient for producing the natural numbers or that they are the most cognitively plausible for the job. However, we do not know of systematic attempts to find substitutes in the psychological literature. One might suggest that Gelman and Gallistel's (1978) counting principles (the one-one, stable-order, and cardinal principles) define a successor relation and that research in this area has concentrated on these principles for just this reason. The counting principles, of course, are crucial in understanding children's ability to enumerate objects and are a worthy subject of investigation in their own right, but, as a definition of the successor relation for natural numbers, their status is similar to Principle (3) and is subject to the same argument that appears in section 3.2.2. The principles map the terms in a count list onto the numerosities they denote, so the next term in the count list comes to be connected with a cardinality that has one more element than the last. This induces a function on the cardinalities. Moreover, Gelman and Gallistel's one-one principle (one and only one numeral is used for each element in an array) would prohibit sequences that violate the successor function by looping around. For example, the one-one principle would prevent counting sequences such as "one, two, three, one, two, three, . . ." instead of "one, two, three, four, five, six, . . ." (though Gelman & Gallistel [1978, p. 132] do report children's occasional use of such sequences). Does this yield the structure of the natural numbers? Not necessarily, as there is no guarantee that the sequence will continue indefinitely. Gelman and Gallistel's (1978) original treatment may have assumed an innate sequence of mental count terms ("numerons") that do have the structure of the natural numbers and will therefore produce the correct successor function. But, in that case, it is the structure of the numerons (along with the counting principles) that are responsible, not the count principles alone.[12] This goes along with our hunch that advanced counting, but not enumerating, is closely linked with knowledge of the natural numbers.

5.4. Competition among schemas

The natural numbers are, of course, not the only structures that children are learning at this age. They must also cope with linear but finite sequences (e.g., the letters of the alphabet), circular structures (e.g., the days of the week or the hours of the day), partial orders (e.g., object taxonomies or part-whole relations), and many others. In top-down learning of the structure of the natural numbers, children must decide which of these schemas is the right one. Their preliminary numerical concepts cannot decide this, as simple counting and enumerating small, finite sets are compatible with several distinct structures. Finite linear lists and circular structures are both compatible with their experience, provided the number of elements in these structures is greater than the number they have so far encountered. For this reason, we suspect that external clues are probably necessary to determine the right alternative. We can expect children to be undecided about whether there is a last number or whether numbers circle back and to experiment with different schemas, as they sometimes do (Harnett 1991). What decides them in favor of a countably infinite sequence may be hints from parents or teachers (e.g., that there is no end to the numbers) or more implicit clues about the numerals or arithmetic. Children are able to absorb this information because they already have access to schemas that are potentially relevant.

Once children know the right schema, they are in a position to make inferences about the natural numbers that would have seemed unwarranted earlier. These include the kinds of generalizations that we encountered in section 4: closure under addition, the property that any two numbers can be ordered under $\leq$, and many others. Likewise, they can infer new facts about the numerals, such as the existence of numerals beyond those in their current count list. There should be a burst of such inferences following children's discovery of the natural number schema, but current data about such properties are too thin to trace this time course.

6. Concluding comments

Thanks to analytic work by Dedekind (1888/1963), Frege (1884/1974), and others, we have a firm idea about the constituents of the natural number concept. Psychological research on number, however, has not always taken advantage of these leads. We hope to refocus effort in this area by outlining a framework that can accommodate research on such issues. The math schema idea obviously does not amount to a full-fledged theory of people's knowledge of natural numbers, much less a theory for all mathematics, but we hope it points to the kind of information that we need to fill in.

If our picture is approximately correct, though, it may have some fairly radical consequences for current cognitive theory.[13] How does the natural number concept depend on object files, internal magnitudes, experience with concrete objects, and mental models or internal sets of such objects? A potential answer that we believe is consistent with the evidence is that there is no dependency whatsoever. The early representations may simply not be conceptually responsible for, or part of the meaning of, the concept of natural number.

You might view as a paradoxical consequence of this position that it cuts off some everyday numerical activities – both in adults and children – from the concept of number. Activities such as estimating the number of objects in a collection or even exactly enumerating these objects may proceed without drawing on natural number concepts. Number concepts may come into play only at a more abstract level – for example, in arithmetic – where the focus is on the numbers themselves rather than on physical objects. However, people can bring to bear different analyses in numerical contexts. Moreover, we need not view such a consequence as belittling investigations of either sort of activity or the research that targets them. Estimating and enumerating objects are well worth studying, even if they do not directly support number concepts. Number concepts are worth studying because of their role in mathematical reasoning, even if mathematical reasoning is not the whole of numerical cognition. Separating these forms of thinking is meant to clarify their origins and interrelations. In particular, understanding the natural number concept may allow us to avoid trying to derive it from unwieldy raw material from which no such derivation is possible.

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NOTES

1. See Dummett (1991, pp. 52–53) for a defense of the idea that natural numbers denote cardinalities. There are also nominalist proposals (e.g., Field 1980) that avoid positing abstract structures.

2. Hilbert (1922/1996; 1926/1983) introduced strings of symbols such as these as a basis for all mathematics. The strings were supposed to be "extralogical concrete objects which are intuited as directly experienced prior to all thinking" (1926/1983, p. 192). Because these strings are concrete and easily surveyable, Hilbert believed they provided a better foundation for numbers than more abstract items, such as sets. See, also, Resnik (1997, Ch. 11) for a "quasi-historical" account of the development of Greek mathematics using strings of this kind, and Parsons (2008) for an examination of Hilbert's notion.

3. Wood and Spelke (2005) point out, however, that such a device has difficulty explaining on its own why infants are unable to discriminate small numbers of nondistinctive items in addition-and-subtraction and habituation tasks. A parallel individuating process should presumably work as well or better in dealing with 1 versus 2 objects than in dealing with 8 versus 16. Barth et al. (2003) and Wood and Spelke (2005) suggest instead that the number of objects (e.g., dots) in a large array is computed from the array's global properties, such as its area and density. Although some studies control both density (e.g., dots per square inch) and area (e.g., total number of square inches in the array), observers might compute the product of these quantities, which is a measure of the total number of dots.

But the issue here re-arises in the way in which observers calculate density. Observers could unconsciously count the number of dots and divide by the area to determine density and then use density in further computations, but this would beg the question of how they determine their initial count. However, if observers determine density by a truly global property – a sense of visual crowding in the display – then there is no reason to think that the proposed calculation could yield anything like a veridical measure of cardinality. The same sense of perceptual crowding that arises from a set of 40 dots in an area of 150 cm^2 could also come from one irregularly shaped strand, weaving back and forth within the same area. Although the magnitude system may deliver approximate measures of cardinality, it cannot deliver arbitrary measures. The problem here is not that the output of multiplying density and area is still a measure of density (area) rather than cardinality. The problem is that the output would be an utterly unreliable measure of cardinality. If the density that is input to the computation is based on visual crowding and is the same for a one-item strand as for a 40-dot array, then output of the computation will be unable to distinguish one item from 40.

A third alternative for computing density is sampling. If the items to be enumerated were evenly spaced (e.g., dots on a line at equal intervals), then density could be calculated from a single interval between an item and its neighbor. (Church & Broadbent [1990] suggest such an algorithm for evenly spaced tones.) In experiments with large numbers of visual elements, however, the displays randomly distribute the elements. Using a fixed number of elements to calculate density would, in general, lead to widely varying estimates of the same cardinality. (See also Bemis et al. [submitted] for empirical evidence against sampling.)

Barth et al. (2003) and Wood and Spelke (2005) may be right that people use global properties of large displays to determine approximate cardinality. But further research would be necessary to determine whether such an algorithm is both a reasonable guide to cardinality and consistent with the discrimination data.

4. Modifications to the Figure 1 model would also be needed to capture effects of chunking or grouping on enumeration. First, data from Halberda et al. (2006) suggest that adults can simultaneously group up to three subsets of dots based on color and can enumerate each subset separately using the magnitude system. Thus, the attentional mechanisms in the first part of Figure 1 must be able to partition on the basis of color (and presumably other low-level visual properties), in addition to individuating items within the groups. This could be handled in Dehaene and Changeux's (1993) system by coding for (a limited number of) colors in addition to item location and size. Second, the upper tracks of the system can assign representations to groups or chunks, as well as to objects. Wynn et al. (2002) found that 5-month-olds are sensitive to the number of groups (2 vs. 4) of dots in a habituation task, where each group was a small set of dots moving in a swarm. Likewise, Feigenson and Halberda (2004) found that infants can succeed in an addition-subtraction task with four objects if these objects initially appeared in two spatially separated groups of two. This suggests that Figure 1 should frame the division between the upper and lower tracks in terms of number of chunks, rather than in terms of number of objects.

5. A magnitude system of this sort would yield only a single output at a time, so one could think of magnitude addition as a three-place function of two addends and a time. This would preserve magnitude addition as a function. But in such a system, the sum of two numbers would differ from one instant to the next (i.e., $+(5, 7, t) \neq +(5, 7, t+\Delta)$), whereas real number addition is constant over time. Any way you look at it, arithmetic with mental magnitudes lacks some of the familiar properties of arithmetic with reals. In particular, real-number addition does not have time as a parameter. (Similar considerations affect the suggestion that the output of magnitude addition is a unique distribution of values.)

6. Perhaps we could draw the same moral from the fact that, although natural-language semantics seems to depend heavily on Boolean operations, such as union and intersection (e.g., Chierchia & McConnell-Ginet 1990), it seems to depend less heavily on specifically arithmetic operations, such as addition and multiplication (except, of course, for sentences that are explicitly about numbers).

7. Current neuropsychological evidence is also potentially relevant to the relation between language and number, but these results are partially conflicting and difficult to interpret. On the one hand, a close connection between calculation and language goes along with functional magnetic resonance imaging and evoked potential data showing activation of the left inferior frontal region (implicated in verbal association tasks) during exact calculation. Tasks involving numerical approximation recruit instead the bilateral intraparietal lobes (Dehaene et al. 1999). On the other hand, there appear to be empirical dissociations between linguistic and calculation abilities and between linguistic abilities and the appreciation of mathematical principles (Donlan et al. 2007). For example, Rossor et al. (1995) and Varley et al. (2005) describe global aphasic patients who are nevertheless able to perform correctly on written addition, subtraction, and multiplication problems, including those that depend on structural grouping (e.g., $50 - [\{4+7\} \times 4] = ?$). There are also clinical cases of relatively normal language development with little numerical ability (Grinstead et al., submitted). It may be too early to draw any strong conclusions from the results of such studies.

8. One complaint about this argument (Margolis & Laurence 2008) is that it incorrectly assumes that children treat the small number terms (e.g., "one") in (2) as ambiguous or as not truly denoting the corresponding number. This is because children would have to revise the denotations later when they find that "one" can also denote eleven, twenty-one, and so on, in case they find themselves learning the mod_{10} system. However, it is easy to formulate the argument without such an assumption. Imagine that what the child is learning is not the natural numbers, but a system in which the numerals loop back after the numerals that the child has already learned (e.g., suppose the child is able to recite the count sequence to "nine"; then "one" denotes one, "two" two, [... and so on till] "nine" nine, but "ten" denotes ten, twenty, thirty, etc., "eleven" denotes eleven, twenty-one, thirty-one, etc., ... and "nineteen" denotes nineteen, twenty-nine, thirty-nine, ...). Of course, you could also hold that children are unable to learn a cyclical system of this sort or that there is a general prohibition, such as mutual exclusivity, against using the same term to apply to two different individuals. However, this claim flies in the face of children's success in learning the days of the week, the months of the year, the tones in a major or minor scale, and other such circular lists (see Rips et al. 2008). Finally, you could assert that, prior to making the inference in (3), children not only know that the first three numerals have fixed designations, but they also know that no looping in the numeral sequence is possible (the sequence continues without end) and that nothing other than the "$s(n)$" sequence can be a numeral. Such knowledge, however, implies that the child's numerals already satisfy the axioms for the natural numbers (see sect. 5.3). The inference in (3) maps these numerals onto cardinalities, but the child already has a representation of the natural numbers before performing the inference. This latter idea, however, is something that most psychologists deny, as we have mentioned in section 2.1.

9. There is current debate in linguistic semantics and pragmatics as to whether noun phrases like "two dogs" denote exactly two dogs, at least two dogs, or an indeterminate meaning that is decided by context. The lower-bounded (at least two) sense can be obtained from (4) by omitting the last conjunct. We take no stand on the correct interpretation here, though such issues may become important if the child's understanding of number depends on knowledge of the natural-language count terms. See Carston (1998) and Musolino (2004) for accounts of this debate.

10. For recent debate about what the "How many?" task reveals about children's numerical competence, see Cordes and Gelman (2005) and Le Corre et al. (2006); however, our present point is independent of these more empirical issues. See also Carey and Sarnecka (2006) for cautions about inferring a type of number concept from experimental evidence.

11. The results on commutativity also seem to hold of other mathematical principles, though the data are much more incomplete than in the case of commutativity. A second example might be the additive inverse principle (in the form $a + b - b = a$), first studied by Starkey and Gelman (1982). Until they are about 4 years old, children are not able to appreciate the inverse relation between addition and subtraction in an addition-and-subtraction task (Vilette 2002, Experiment 1). Although practice observing the counteracting effects of adding and subtracting the same number of concrete objects helps 3-year-olds perform more accurately, the benefit is no greater than that of observing the separate effects of addition and of subtraction (Vilette 2002, Experiment 2). This suggests that successful children initially deal with the inverse relation $a + b - b$ by literally adding and then subtracting b objects. By contrast, 4- to 5-year-olds recognize the answers to such problems more easily than comparable ones of the form $a + b - c$ that do not allow them to use the inverse relation as a shortcut (Bryant et al. 1999; Rasmussen et al. 2003). In the latter studies, the terms of the problems are given in numeric form (with or without objects present), and children provide numeric answers (e.g., "How many invisible men do we have if we start with 14, add 7 more, and then take away 7?"). The gap between the performance of younger and older children makes it reasonable to conjecture that awareness of the inverse principle depends on some prior (but not necessarily school-taught) arithmetic.

12. We thank Susan Carey for pointing out the relevance of the How to Count principles in this context.

13. Another consequence of our position concerns the relation between logical reasoning and mathematics. We have argued that math concepts may depend on an underlying cognitive framework that includes recursion. Typical production systems for handling problem spaces (e.g., Anderson 1983; Klahr & Wallace 1976; Newell 1990) also include limited logical abilities for dealing with conditionals and conjunctions and for instantiating or binding variables. The usual form of a production rule is: "If Condition_1 and Condition_2 and… and Condition_k, then take action A," in which mental action A occurs (e.g., a symbol is stored in working memory) if Conditions_{1-k} are met by the current contents of memory. Although there are competing cognitive architectures (e.g., connectionist ones), production systems and other classical systems have an advantage of providing basic resources like these that mathematical reasoning can build on. Much the same can be said about the resources that people need for explicit deductive reasoning in tasks that depend on logical connectives and quantifiers. One of us has suggested (Rips 1994; 1995) that certain aspects of explicit deductive reasoning (e.g., rules like modus ponens) are especially natural because they are inherited directly from this background architecture. Work in the foundations of mathematics appears to show that attempts to reduce mathematics to logic are problematic at best (see Giaquinto 2002, for a review), and there is no reason to suppose that mathematical reasoning can be reduced to logical reasoning in any simpler way. But our current perspective suggests, nonetheless, that there may be important indirect connections between them, because of their very tight dependence on a common pool of cognitive resources. We know few systematic attempts in psychology to trace the relations between logical and mathematical thought (see Houdé & Tzourio-Mazoyer 2003, for a start), but there is every reason to try to do so.

Open Peer Commentary

Finger counting: The missing tool?

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Abstract: Rips et al. claim that the principles underlying the structure of natural numbers cannot be inferred from interactions with the physical world. However, in their target article they failed to consider an important source of interaction: finger counting. Here, we show that finger counting satisfies all the conditions required for allowing the concept of numbers to emerge from sensorimotor experience through a bottom-up process.

Can the principles underlying the structure of natural numbers be inferred from interactions with the physical world? According to Rips et al., the transition from sensorimotor experience to mathematical concepts is largely undetermined and cannot account for the acquisition of natural numbers. We argue that fingers may provide the missing tool to apprehend numbers in the physical world, and we put forward striking similarities between finger-counting strategies and the principles underlying the structure of natural numbers.

Children spontaneously use a stable sequence of finger movements while counting, presumably due to the joint influence of motor constraints and cultural habits. The stable-order principle emerging from the use of consistent finger-counting strategies involves the understanding that natural numbers include a unique first element (e.g., thumb or little finger, depending on cultural habits), a unique immediate successor for each element in the sequence (e.g., from thumb to little finger, there is only one successor for each finger), and a unique immediate predecessor for each element except the first (e.g., from index to little finger, there is only one preceding finger). Finger-based representations therefore preserve cardinality and maintain a one-to-one correspondence between the physical world and symbolic systems.

Moreover, once the sequence of numbers from 1 to 10 is instantiated in finger counting, fingers can be used to keep track of the successors of any other number, providing an empirical ground for mathematical induction. Finger counting goes even farther, as it allows the children to infer the base-10 mathematical system. Historically, the base 10 has been the most widely used, as evidenced by the traces left by Amorites, Incas, and Tibetans (Ifrah 1981). There is no reason to explain this choice relative to others, such as the more convenient base 12, which has more divisors, except that the base-10 system has a physical counterpart in finger use.

The influence of finger-counting strategies on number representations is supported by several empirical facts. A first evidence comes from the origin of number words in many languages (e.g., in English, *five* comes from a common root of *finger* and *fist*, *digit* means at the same time *number* and *finger*; Menninger 1969). This embodied vocabulary suggests that counting originates from the use of fingers rather than using arbitrary quantitative words. Developmental studies showed that the score obtained in finger discrimination tasks is the best predictor of arithmetical performance in children (Fayol et al. 1998; Noël 2005). Moreover, the errors made by children in mental calculation often reveal a 5-unit difference relative to the correct response (e.g., $18 - 7 = 6$), which is reminiscent of the representation of intermediary results on a full-opened hand (Dohmas et al. 2008). Furthermore, training children with poor mathematical achievement to discriminate their fingers was found to improve both their finger and number knowledge (Gracia-Bafalluy & Noël 2008).

The question arises as to whether finger-counting strategies still influence numerical cognition in adults. To address this issue, we conducted an experiment in which participants had to identify Arabic digits by pressing the keyboard with a different finger for each number. Results showed that responses were faster and more accurate when the finger assigned to each number matched the finger-counting strategy of the participants (Di Luca et al. 2006). In another experiment, we showed that number naming was faster when participants were presented with canonical finger configurations, that is, configurations conforming to their personal finger-counting habits, rather than non-canonical ones. Moreover, we observed that comparing Arabic digits was faster when the digit display was preceded by the unconscious presentation of either canonical or non-canonical finger configurations. However, only the priming effect induced by canonical configurations generalized to new, never consciously seen, numerosities, which provides clear evidence for an automatic access to semantic numerical knowledge from finger-counting configurations (Di Luca & Pesenti 2008). Finally, we showed that canonical configurations are processed as symbolic systems and activate a place-coding semantic representation of numbers, whereas non-canonical configurations activate a summation-coding semantic representation (Di Luca 2008; Di Luca & Pesenti, under revision; Roggeman et al. 2007).

The hypothesis that knowledge of natural numbers is grounded in finger representations is also supported by neuroimaging data showing that, in numerical tasks, activation was observed in the parieto-precentral circuits classically associated with finger movements (Pesenti et al. 2000). Electrophysiological results converge to the same conclusion (Andres et al. 2007; in press). For example, Sato et al. (2007) found that the amplitude of muscle twitches induced by transcranial magnetic stimulation (TMS) in the right hand increased when participants performed a verbal parity judgment on numbers 1–4 relative to numbers 6–9. Because all participants reported starting counting with their right hand, this result could reflect a specific contribution of hand motor circuits to number processing.

Although these findings favor a unique relationship between numbers and fingers, Rips et al. may still argue that finger counting is mediated by innate and abstract representations of natural numbers. However, the assumption that natural numbers are preconfigured is contradicted by the finding that some Amazonian populations do not develop the complete number sequence (Pica et al. 2004). Interestingly, this limited numerical knowledge was found to co-occur with a rudimentary finger-counting strategy, which supports the idea that finger counting may critically contribute to understanding natural numbers. Accordingly, neuropsychological studies have shown a joint deficit in calculation and finger discrimination following a vascular damage (Gerstmann 1940) or a virtual TMS lesion of the parietal cortex (Rusconi et al. 2005).

In conclusion, we showed that finger-counting strategies are sufficient to allow the development of numerical representations that integrate the basic properties of natural numbers as well as higher-order properties such as the base-10 concept. We therefore argue that it is not necessary to presuppose an abstract schema specifying the structure of natural numbers to explain the development of the concept of numbers. The finding that finger-counting strategies still influence number processing in adults rather suggests that knowledge of natural numbers may build up on finger-based representations through a bottom-up process.

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In defense of intuitive mathematical theories as the basis for natural number

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Abstract: Though there are holes in the theory of how children move through stages of numerical competence, the current approach offers the most promising avenue for characterizing changes in competence as children confront new mathematical concepts. Like the science of mathematics, children's discovery of number is rooted in intuitions about sets, and not purely in analytic truths.

Rips et al. present a thought-provoking assessment of the current debate on the origin of numerical concepts in language development. The article's main challenge is to the hypothesis that number word meanings are bootstrapped from systems of nonlinguistic number representation. On many fronts the authors' arguments are compelling, and the points they raise offer important challenges to current models. I disagree, however, that problems with current developmental theories are the result of an inherently flawed approach, or that they are misled in their general trajectory. Instead, I argue that there is no better way to understand human knowledge of natural number than to witness its development in young children.

The problem is to determine the explananda to the theory of number knowledge. Rips et al.'s general thesis is that knowledge of natural number is defined in terms of an inferential system, and that therefore developmentalists should focus their efforts on evaluating how children come to manipulate numbers as syntactic objects, independent of their particular denotations. Stipulating that this particular knowledge should act as a metric of competence, however, is entirely arbitrary, and unprecedented in developmental psychology. In the study of human intuitive theories of biology, physics, and psychology, an implicit distinction is made between common sense understanding and scientific understanding. To untangle the two, developmentalists interested in human knowledge of biology, for example, have investigated children's initial intuitive theories, and how these theories react and change as children are exposed to new vocabulary and concepts (e.g., Carey 1985; Piaget 1929/2007).

Developmentalists have also looked back in scientific history to the earliest biological theories, and have examined parallels between conceptual change in ontogeny and scientific history. This approach allows us to differentiate theories that come spontaneously to each child from those that are discovered scientifically and transmitted from generation to generation, whether explicitly in school or implicitly in how we talk and reason about biology in the presence of children. It is doubtful that we would benefit from stipulating what should count as "knowledge of biology," since there is no such static object. However, we can make progress by asking how children initially reason about biological phenomena, how biological reasoning has changed over human history, and how children's theories evolve as they are exposed to culturally transmitted scientific knowledge.

The same arguments can be made for mathematics. To stipulate what should count as "knowledge of number" would risk blurring the line between common sense understanding and scientific understanding (see Husserl's discussion of psychologism in the study of logic; Husserl 1970). We can, however, ask about children's intuitive knowledge, about the scientific history of mathematics, and about how children's mathematical theories evolve as they are exposed to new vocabulary (e.g., the count list) or concepts (e.g., addition, subtraction, etc.). This approach does not differ from that of linguistics, which seeks to characterize core properties of language by separating properties that are exhibited by all human languages from those that vary from language to language. Using this approach, the best way to understand human knowledge (of number, biology, or language) is to distinguish the components that come naturally to each child from those that have evolved idiosyncratically over human history, and to study their interaction in child development. Developmental theories are perhaps short on details regarding how this happens, but ultimately are, I believe, on the right track.

What is this track, in the case of number? As Rips et al. note, there has been an explosion of progress in understanding pre-linguistic systems of number. Also, we have made steady progress in our characterization of how children initially interpret number words. These studies have perhaps not convinced us of how core systems are implicated in number word learning, but they should convince us that children's first hypotheses about number make contact with representations of cardinalities, and that this connection only grows with age. It therefore remains possible that mental representations of number are always rooted in cardinalities, and that higher-order principles (such as commutativity) amount to beliefs about number, rather than being constitutive of number knowledge. Although cardinalities may indeed be irrelevant to defining natural number scientifically (i.e., in mathematics), we cannot assume a priori that concepts in the psychology of number take the same form as concepts in mathematics. To determine whether they do, we must examine human knowledge empirically, as it unfolds in development.

Progress in mathematics is owed mainly to the unbounded inferential power of its formal syntactic representations (many of which are beyond our intuitive grasp). Still, the science would arguably not exist if its basic truths did not satisfy human intuition, and were not relatable to our experience. Kant, hardly an empiricist, argued that our mathematical concepts are responsible to intuitions about objects and sets in the world: "The concept of twelve is by no means obtained by merely thinking the union of seven and five. . . We must go beyond these concepts, and have recourse to an intuition which corresponds to one of the two – our five fingers, for example. . ." (Kant 1781/1934, p. B15). This is not to say that our mathematical intuitions (e.g., pertaining to cardinalities) are *derived* from experience (*à la* Mill). Instead, the logic that governs experience may constrain the scientific theories that we formulate to explain it. To the extent that this is true, it should hardly be surprising if human knowledge of number remains bound to intuitions about sets and cardinalities throughout development. Connecting "object talk" to formal symbols may be the intuitive basis not only for mathematics, but also for mental representations of number, as humans move from reasoning about physical objects as infants to reasoning about mathematical objects as adults.

Do mental magnitudes form part of the foundation for natural number concepts? Don't count them out yet

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Abstract: The current consensus among most researchers is that natural number is not built solely upon a foundation of mental magnitudes. On their way to the conclusion that magnitudes do not form any part of that foundation, Rips et al. pass rather quickly by theories suggesting that mental magnitudes might play some role. These theories deserve a closer look.

Rips et al. have written a valuable critique of current theory and research, but in concluding that mental magnitudes form no part of the foundation of natural number concepts, the authors consider only briefly theories positing a partial role for mental magnitudes. These theories deal with children's early counting: Because mastery of verbal counting implies knowledge of critical aspects of natural number, studies of counting acquisition should be useful to researchers interested in hunting down the origins of natural number.

Gelman, Gallistel, and colleagues have argued that "very early in the prolonged process of learning verbal counting children recognize that the structure and function of verbal counting are the same as the structure and function of their pre-verbal counting system. Both processes honor the one-one, stable order, and cardinal principles, and both processes deliver symbols that are subject to arithmetic processing" (Gallistel 2007, p. 439). They hold that this recognition of common structure and function allows children to link count words to their corresponding mental magnitudes, but that the count words are numerically meaningful prior to the mapping (Gallistel & Gelman 1992).

Other researchers have criticized this account on various grounds. First, a serial enumeration process is often assumed to be critical to the idea that young children (like nonhuman animals) have access to a nonverbal counting process (e.g., Le Corre & Carey 2008). There is evidence that representations of the numerosity of spatial arrays are not constructed serially, in adults and in infants (Barth et al. 2003; Wood & Spelke 2005), but as Cordes and Gelman (2005) point out, a serial counting process is not required at all. A parallel enumeration mechanism (e.g., Dehaene & Changeux 1993) that adheres to the formal structure of a counting process would do the trick. Of course, a parallel process also makes the relation between verbal and nonverbal counting less apparent – but for Gelman and her colleagues, a parallel process does count.

A potentially more serious problem may arise if analog magnitude representations of numerosity are generated by a process that could not reasonably be called counting. Barth et al. (2003) and Wood and Spelke (2005) suggest that the best explanation of experimental results to date may rest in a mechanism that computes numerosity through global non-numerical properties, rather than by enumerating individuals, but their data do not speak directly to this point. This is an important empirical question which remains open; despite the arguments of Rips et al., there is no *a priori* reason to reject density-based models. The visual system's mechanism for extracting density need bear no resemblance to the common definition of density as number per unit area, or to the authors' description of "visual crowding."

Would direct evidence against the existence of any nonverbal counting mechanism (serial or parallel) defeat the analog magnitude structure-mapping hypothesis (e.g., Gelman & Lucariello 2004)? Clearly the version of the hypothesis that depends specifically upon nonverbal counting would not survive such evidence. Important components of the hypothesis could nevertheless remain: we need not choose between a nonverbal counting system and an unstructured conglomeration of disconnected magnitudes. An analog magnitude system with representations generated by computations over global display properties would possess an implicit ordinal structure, with larger magnitudes systematically corresponding to larger sets in the world. Structure mapping would be a more difficult problem with such a system, but it would still be possible – and children's prolonged learning of these mappings does imply that they are solving a difficult problem (Wynn 1992b).

The nature of the process that generates mental magnitudes may not necessarily decide the fate of the analog mapping hypothesis, broadly conceived. What evidence would argue forcefully against it? What is certainly critical is that children must recognize the structures of the mental magnitude and verbal counting systems and the relation between these structures. Researchers have criticized the hypothesis in part because children do not appear to exhibit an early understanding of the rule that positions later in the verbal count list should correspond to larger magnitudes (Condry & Spelke 2008; LeCorre & Carey 2007; 2008; an understanding of this "later/greater" rule is also important for related proposals, e.g., Spelke 2003; Wynn 1992b). These findings would seem to spell trouble for structure-mapping accounts (LeCorre & Carey 2008).

However, it is possible that children's failure to demonstrate any later/greater knowledge in the tasks used to date reflects something other than incompetence. These studies have shown that beginning counters do not have an early understanding of the fact that numbers around 10 should map to larger sets than numbers around 5. Does this finding necessarily mean that such children have no understanding of the parallel structures of the counting words and the magnitude system? Possibly not, for a number of reasons – one of which is the relatively compressed numerical range tested. Consider that children might initially map "the next number in the verbal count list" to "the noisy mental magnitude associated with the next easily discriminable numerosity." If so, the corresponding change in magnitude that initially maps to the next number in the count list may be too great for the 5–10 range to reveal children's later/greater knowledge. Children might possess knowledge of the later/greater rule even while failing utterly to demonstrate that knowledge in the 5–10 range: their extreme underestimation, in combination with noise in their estimates, could conceal all evidence of a later/greater understanding. Somewhat paradoxically, larger sets might be needed to reveal this knowledge. Therefore, additional converging evidence from similar tasks using larger sets would be desirable in order to rule out the analog mapping hypothesis (but see Gallistel 2007, for discussion as to why such evidence would not be sufficient). Of course, the account lacks the critical notion of a unique next number, but it may be premature to reject the possibility of any role for the analog magnitude system, especially if the idea of a mental unit magnitude is included in such a model (e.g., Leslie et al. 2007).

Math schemata and the origins of number representations

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Abstract: The contrast Rips et al. draw between "bottom-up" and "top-down" approaches to understanding the origin of the capacity for representing natural number is a false dichotomy. Its plausibility depends upon the sketchiness of the authors' own proposal. At least some of the proposals they characterize as bottom-up are worked-out versions of the very top-down position they advocate. Finally, they deny that the structures that these putative bottom-up proposals consider to be sources of natural number are even precursors of concepts of natural number. This denial depends upon an idiosyncratic, and mistaken, idea of what a precursor is.

Rips et al. criticize a "bottom-up" approach to the origin of the capacity for representing natural number. According to the bottom-up view that they believe characterizes most current work on the development of numerical cognition (including mine), representations of natural number are supposedly derived, by empirical induction, from representations of sets, objects, and the quantitative resources of Figure 1 of the target article (parallel individuation of small sets, analog magnitude representations of number). While agreeing that the representational systems sketched in Figure 1 exist, and underlie a variety of behaviors of infants and young children, Rips et al. deny that these (or even the explicit representational scheme they call "simple counting") are precursors to representations of natural number. Rips et al. propose an alternative "top-down" account, in which math schemas that are the equivalent of Peano's axioms are somehow directly acquired without involving the representations of Figure 1 or of simple counting.

Rips et al. wildly misconstrue my proposal. Although I hold that the schemata of Figure 1 and of simple counting are precursors to representations of natural number, these do not exhaust the innate machinery brought to bear in this achievement; my proposal does not bottom out in these structures. My position is more of a worked-out version of Rips et al.'s top-down approach than a bottom-up approach (and thus I agree with much they have to say in their target article).

My proposal depends upon a particular form of bootstrapping process (Quinian bootstrapping) that has been well studied in the literature on the history and philosophy of science (Carey, in press; Nersessian 1992; Quine 1960). Carey (2004; in press) illustrates the role of Quinian bootstrapping in the acquisition of simple counting, which results in a representational schema that goes beyond the resources of Figure 1. Simple counting is the first schema that represents even a finite subset of the natural numbers, and the bootstrapping episode that creates this schema is only one of several that eventually result in the capacity for representing natural number. A second bootstrapping episode described in Carey (in press) underlies the integration of simple counting with the analog magnitude representations of Figure 1. These do not complete the story; however, they illustrate how it works.

All episodes of Quinian bootstrapping require top-down creation of explicit placeholder structures, the symbols of which get their meaning entirely from conceptual roles within those structures. Besides the resources needed for the construction of the placeholder structures, Quinian bootstrapping involves modeling processes through which these structures are infused with mathematical meaning. For example, nothing in Figure 1 captures the child's capacity to create ordered lists of symbols. The meaningless list, "one, two, three, four..." is a placeholder structure, the meaning of which is exhausted by its conceptual role as an ordered list. Other computational resources are drawn upon in the process of creating meaning for this placeholder structure – various logical capacities, recursion, a variety of processes that model the representations of Figure 1 in terms of the placeholder structure, as well as induction.

As Rips et al. point out, their own proposal for the innate building blocks for number representation includes knowledge that is "tacit." Their proposal suffers, however, from the lack of any hint of what they might mean by this. How are the innate math schemas they presuppose represented? What are the symbols like (format, content), and what computations do they enter into? What is tacit knowledge? The schemata instantiated in Figure 1 provide answers to those questions. The actual symbols in parallel individuation represent individuals, but the system as a whole tacitly embodies arithmetical knowledge in the processes that pick out and manipulate sets, compute a one-to-one correspondence, and compute numerical equality and inequality. The actual in analog magnitude representations represent approximate cardinal values of sets of individuals, but, again, the system as a whole tacitly embodies arithmetical knowledge in the processes that compute arithmetic functions over these values, including sums and ratios. Ditto for simple counting; much of the knowledge that ensures that simple counting constitutes a representation of a finite subset of the natural numbers is tacit, captured in the counting principles characterized in Gelman and Gallistel's (1978) classic work.

Rips et al. claim that neither simple counting, nor the representational systems depicted in Figure 1, are precursors to natural number, arguing that the concept of natural number cannot be defined in terms of structures of Figure 1, nor be derived from them by empirical induction. However, the mastery of simple counting is a necessary prerequisite for the mastery of complex counting, which Rips et al. agree is likely to be a necessary part of acquiring the math schema of natural number. The mastery of simple counting draws on the resources of Figure 1 (plus others), and, in this sense, these structures are all part of the precursors to natural number. The authors point out that on the mathematical ontology they favor, the content of a mathematical symbol is given entirely by computational role (its place in the system), and on this view, simple counting and the structures in Figure 1 play no role in the mathematical concept of natural number (which is exhausted by the concept of a unique first number, the concepts of successor and predecessor, and mathematical induction). However, aspects of these latter component concepts are implicit in the computations carried out over the schemata captured in Figure 1 and in simple counting, and provide constraints in the modeling processes through which the placeholder structures created by top-down processes come to have meaning.

My proposal, like theirs, assumes that conceptual role is the main source of content for mathematical concepts. My proposal concerns how new primitive symbols are coined and how they come to have the appropriate conceptual role. Contrary to Rips et al., I believe that the content of each symbol for a positive integer is determined both by conceptual role and by the capacity to represent cardinalities of sets of actual individuals. This hypothesis makes sense of one of the most striking facts about mathematical development: Mathematical development, both historically and in ontogenesis, often occurs in the course of modeling the world. It is no accident that Newton invented the calculus and Newtonian mechanics, or that Maxwell invented the mathematics needed to field theories in the course of modeling Faraday's electromagnetic phenomena. In the end, the big mistake that Rips et al. make is methodological: they miss the fact that modeling activities can give placeholder structures meaning, even if in the end the structures involved in these modeling processes, such as the schemata of Figure 1, are part of an acquisition ladder that is not essential to the conceptual role constructed. This is what developmental precursors really are – representations that play a role in the bootstrapping process.

Before Rips et al. have offered an viable alternative to the picture they criticize, they owe us some idea of what the math schemas they advocate are like, at Marr's algorithmic level of description, and how they are acquired.

What is still needed? On nativist proposals for acquiring concepts of natural numbers

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Abstract: Rips et al.'s analyses have boosted the plausibility of proposals that the human mind embodies some critical properties of natural numbers. I suggest that such proposals can be further evaluated by infant studies, neuropsychological data, and evolution-based considerations, and additionally, that Rips et al.'s model may need to be modified in order to more completely reflect infants' quantitative abilities.

Rips et al.'s extensive analyses adopting a "top-down approach" not only question the bootstrapping hypotheses, but also strengthen the possibilities that some defining properties of natural numbers are innate to human cognitive structure. In the authors' proposal, innate resources – including the ability of recursion – enable the construction of a math schema that embodies the one-to-one successor function, a critical property of natural numbers (sect. 5.2). In an alternative proposal (Leslie et al. 2007), the innate basis of natural-number concepts is a mental magnitude system ("the speedometer"), which incorporates a discrete digital representation that is aligned and calibrated to, but distinct in nature from, the continuous analog representation. Current data, however, are too limited to evaluate these proposals. Here I suggest some directions for investigation.

Infant studies still play a critical role. Although mathematics is typically learned late, Rips et al. have argued that to children, math principles are not necessarily consciously available, expressible in language, or acquired all at once (sect. 5.3, para. 1), leaving it open whether some knowledge of such principles is present in infancy. Accordingly, predictions can be developed and tested. However, there are great practical difficulties in testing infants, because the tests must be directly about number concepts, as Rips et al. have emphasized.

Neuropsychological data are also relevant. According to Rips et al., mental magnitudes are too imprecise to be associated with specific natural numbers (sect. 3.1), suggesting that construction of the natural-number schema is relatively independent from the magnitude-representation system. In contrast, the model proposed by Leslie et al. (2007), which incorporates a digital representation as part of the magnitude-representation system, suggests a strong tie between operation of the system and acquisition of natural-number concepts. As such, neural imaging data and specific clinical cases may differentiate the two proposals.

Evolutionary perspectives may offer a plausibility check. Several domains of knowledge are considered to be both essential to survival and ones for which innate mental modules have evolved. For knowledge of natural numbers, Rips et al. have proposed that recursion is a critical underlying capacity, which may have undergone a domain-specific-to-domain-general shift in human evolutionary history (Hauser et al. 2002), whereas Leslie et al. (2007) have proposed that a digital-integer representation is a built-in aspect of the evolved magnitude-representation system. These proposals for innate structures that embody properties of natural numbers can be examined with respect to which one could plausibly argue that such structures would have adaptive values.

Lastly, it should be pointed out that some findings seem to be inconsistent with the processes depicted in Rips et al.'s model (Fig. 1 of the target article). These findings deserve consideration in building a more complete model and for examining its implications. The model in Figure 1 is primarily based on infants' processing of visual arrays, such that the attentional mechanisms for parallel individuation function is the starting point, and the outcome, depending on the number of objects (>4 or not), follows separate routes of processing. However, numerous studies have shown that infants can quantify sequences of items such as sounds and actions. For example, 6-month-olds discriminate between sequences at a 1 : 2 ratio when the numbers of items were ≥ 4 (e.g., 4 vs. 8, 8 vs. 16), and the discrimination precision increases to a 2 : 3 ratio (e.g., 4 vs. 6, 8 vs. 12) by age 9 months. In sharp contrast, when the number of items was ≤ 4, infants did not discriminate at the ratio that they did when the number was ≥ 4: 6-month-olds failed discrimination at the 1 : 2 ratio (e.g., 2 vs. 4), and 9-month-olds succeeded at that but failed at the 2 : 3 ratio (2 vs. 3) (e.g., Lipton & Spelke 2003; 2004; Wood & Spelke 2005). Note that earlier studies have shown that 6-month-olds can discriminate between 2- and 3-action sequences (Sharon & Wynn 1998; Wynn 1996), although some have argued that non-numerical factors may account for these results (e.g., Wood & Spelke 2005). In sum, none of these results can be accounted for by parallel individuation processes of attention; such processes cannot be the starting point for the quantitative processing of sequences of items.

In addition, the failure to discriminate between small-number sequences does not provide direct evidence that magnitude representation is not formed of small numerosities. This is because how many items a sequence consists of is not determined until the sequence ends and becomes defined – magnitude representation of a sequence, if formed, must be a result of constant updating from the input of the very first item on. Thus, other factors may be responsible for this finding.

The same possibility applies to infants' processing of visual arrays. The model in Figure 1 implies that for sets containing fewer than 4 objects, magnitude representations of set size are not formed, and thus object representations (object-files) must be responsible for infants' discrimination, as suggested by the findings that 10- and 12-month-old infants succeeded in discriminating among 1 to 3 objects, but failed with comparisons involving 4 objects (e.g., 1 vs. 4) (Feigenson & Carey 2005; Feigenson et al. 2002a). However, a recent study has shown that 5-month-olds can discriminate between arrays of 2 versus 8 and 3 versus 12 objects (Cordes & Brannon 2007). This suggests that infants do form magnitude representations of small-number sets, but require a greater ratio of 1 : 4 (as compared with that of 1 : 2 for large-number sets) for successful discrimination. It also casts some doubt on the claim that numerical discrimination between small-number arrays is based only on representations of individuals that are subject to a 3-item limit.

The model also implies that for arrays containing 4 or fewer objects, discrete object representations are maintained only if they have distinctive properties, allowing number-based responses; otherwise, infants' responses are driven by magnitude representation of continuous quantities of the array (Feigenson 2005). However, such a deficit in numerical discrimination of homogenous arrays has not been found in most studies with infants or preschool children, and many studies actually reported a relative advantage for homogeneous sets (for discussion, see Gelman & Gallistel 1978 and Cantlon et al. 2007). For instance, Starkey (1992) found that pre-counting 24- and 30-month-olds performed better on a nonverbal addition-and-subtraction task with homogeneous sets than with heterogeneous sets. Thus, Rips et al.'s model is in need of revision to address these findings.

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From magnitude to natural numbers: A developmental neurocognitive perspective

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Abstract: In their target article, Rips et al. have presented the view that there is no necessary dependency between natural numbers and internal magnitude. However, they do not give enough weight to neuroimaging and neuropsychological studies. We provide evidence demonstrating that the acquisition of natural numbers depends on magnitude representation and that natural numbers develop from a general magnitude mechanism in the parietal lobes.

Rips et al. have provided a considered review, and a theoretical framework for the way in which humans acquire (or do not acquire) natural numbers. However, although they refer to cognitive and developmental studies, they ignore an important component, which can constrain developmental and cognitive theories: the human brain. Given a decade of advances in understanding the neural basis of number processing, any accounts of numerical development and abilities that do not address the neural instantiation of number processing are necessarily incomplete. Here, to show the limitations of Rips et al.'s view, we consider the parietal lobes, a key area in number processing, with regard to their role in numerical magnitude and their contribution to general magnitude processing. On the basis of evidence from neuroimaging, neuropsychological, and behavioral studies, we challenge Rips et al.'s position that there is no dependency at all between natural number and internal magnitude (target article, sect. 6).

It is indisputable that adults in Western culture have natural number concepts based on symbolic representation of numbers. A great deal of evidence indicates that the parietal lobes, and especially the intraparietal sulcus (IPS), play a key role in numerical cognition (Brannon 2006; Cohen Kadosh et al. 2008; Dehaene et al. 2003; Nieder 2005). In imaging studies, activity in IPS is modulated bilaterally as a function of the distance effect, the numerical priming effect, and the size congruity effect (see Cohen Kadosh et al. [2008] for a review). All these effects stem from numerical representation. The IPS is also active during the processing of other magnitudes, such as physical size, time, space, and luminance (for reviews, see Cohen Kadosh et al. 2008; Walsh 2003). This led Walsh (2003) to suggest that the parietal lobe, especially at the right hemisphere, embodies the basis of a common metric for space, time, and numbers, and many neuroimaging studies have shown that symbolic numbers, non-symbolic numbers, and other magnitudes activate the IPS (see Cohen Kadosh et al. [2008] for a meta-analysis). Notably, the similarity between numbers and other magnitudes is not confined only to the neuronal level but is also observed during behavioural tasks, as indicated by similar effects and response functions (Cohen Kadosh et al. 2008).

These similarities between numbers and other magnitudes, both at the behavioural and neuronal levels, are not mere coincidence; they are indicative of the basic knowledge of magnitude upon which more specific and conceptual representations are built. As Rips et al. (sect. 2.2) have noted, during infancy, numbers do not represent a unique feature. Other dimensions, separately or jointly, can serve as cues in order to detect changes in quantity/magnitude. In our view, this is because of a shared magnitude mechanism that precedes numerical representation. In this scenario, different magnitudes are jointly represented from infancy (Feigenson 2007). Later, the child develops neuronal circuits dedicated to numerical information, and acquires the understanding of natural numbers. This specialized representation, which emerges as we understand natural number concepts, is possible due to interactions between non-tuned neuronal substrates originally dedicated to general magnitude representation, with areas in the left hemisphere involved in language, and ventral occipito-temporal areas involved in symbolic processing (Cohen Kadosh et al. 2007a). This conceptualization of human functional brain development is in line with other fields that have examined neuronal specialization, such as face perception (Cohen Kadosh & Johnson 2007), and is termed the Interactive Specialization Approach (Johnson 2001). The general framework of this theory suggests that some cortical regions that are initially functionally poorly defined are partially activated in a wide range of different contexts and tasks as cortical development proceeds.

The possibility of acquiring the concept of natural numbers via interaction between general magnitude and language is dismissed by Rips et al. Although they suggest why either magnitude *or* language might not play a role in the formulation of natural numbers, they do not explain why general magnitude *and* language cannot play an integrative role in shaping the understanding of natural numbers.

As with previous cases of the interactive specialization approach (Thomas & Johnson 2008), the interactive specialization approach for numerical cognition is supported by investigations of developmental disorders. People with developmental dyscalculia – the inability to process numerical information adequately – have problems not only with automatic processing of numerical quantity (Cohen Kadosh et al. 2007b; Rubinsten & Henik 2005; 2006), but also with other magnitudes (Cohen Kadosh et al. 2007b). Non-dyscalculic adults show similar impairments as developmental dyscalculics when their right IPS (rIPS) is stimulated by transcranial magnetic stimulation (Cohen Kadosh et al. 2007b), thus demonstrating a causal relationship between rIPS abnormalities and developmental dyscalculia. The rIPS plays an active role during infancy and childhood: It is involved in numerical processing as early as 3 months postnatal (Izard et al. 2008), and it is still dominant in numerical tasks at 4 years (Cantlon et al. 2006). After children acquire natural number concepts, via interaction with language, which is mainly left lateralized, the left IPS is also activated (Ansari & Dhital 2006).

The lack of early brain specialization is not the only developmental factor to affect the representation of numerical information. For example, synaesthesia, another developmental atypicality, leads one to similar conclusions. Adults with digit-colour synaesthesia, who experience colour whenever they see numbers (e.g., 2 in red), have a lack of neuronal specialization in the magnitude system (Cohen Kadosh et al., in press). Their symbolic representation of digits is affected by luminance – similar to the tendency shown by children up to two years of age (Cohen Kadosh et al. 2007c).

Rips et al.'s idea, while interesting, does not reflect the current state of neuroscientific knowledge either about numbers or development in general (Johnson 2001), and nor does it accurately represent the link between the cognitive abilities of infants and the development of number concepts. Abnormality in the magnitude processing system or in the language system can lead to developmental dyscalculia – a view that is supported by some of the evidence we present here, as well as by the comorbidity between dyscalculia and dyslexia (Geary 1993).

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Differences between the philosophy of mathematics and the psychology of number development

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Abstract: The philosophy of mathematics may not be helpful to the psychology of number development because they differ in their purposes.

These are interesting times for psychologists who research number development. Ingenious studies are revealing capabilities in infants and nonhuman species that are challenging to describe and explain. Whether and how these studies relate to later development is an important issue.

Rips et al. are not the first to think that the philosophy of mathematics will prove helpful in guiding the psychology of number development. It is true that explaining the nature of mathematical knowledge and how it can be acquired has concerned some philosophers since the beginnings of Western philosophy. Mathematical concepts and relations have been variously portrayed in perspectives such as Platonism, nominalism, intuitionism, constructivism, logicism, set theory, and structuralism. My understanding is that no position has successfully dealt with the issues philosophers feel a philosophy of mathematics should manage. Some problems with using philosophy as a guide are that there are several possibilities, and they point in different directions; and it is unclear whether any have reached the destination of interest. The existence of such continuing disputes suggests that there can be no simple connection between mathematical proficiency and philosophical conceptions of natural number.

Another problem is that philosophical discourse is particularly ill-suited for operationalisation into empirical tests. Any conversation with children about the natural numbers may be fraught with ambiguity: they may be talking about numbers as abstract entities defined only in relation to each other, or they may be tacitly referring to quantities of objects. One can be more confident when discussing the multiplication of negative numbers that reference to the real world has been abandoned, but there are ingenious real-world analogies for this, which would need ruling out.

Whereas philosophers are concerned with explicitly conscious and communicable constructions of number systems, the authors are proposing to use their ideas to characterise the unconscious intuitions of children. This risks treating a conception of natural number as equivalent to a set of components. A set of intuitions is no more an implicit concept of number than a set of components is an implicit computer.

The most serious disadvantages of trying to base the psychology of number development on the philosophy of mathematics derive from their fundamentally different purposes. The philosophy of mathematics is not interested with the development of the individual or even in explaining the history of mathematics.

In contrast, any psychology of number development is bound to be deficient unless it recognizes that children do not develop their ideas about number in a vacuum: They grow up in environments replete with uses of number and numerals. The natural number sequence, also known as the counting numbers, is a cultural product that has developed over a long period and embodies considerable knowledge of number. Some time after acquiring it, children learn the knowledge implicit in it, for example, the relative magnitude of numbers (Siegler & Robinson 1982) and the number-after rule (Baroody 1995).

It is not just exposure to cultural tools such as the natural number sequence that is responsible for children's development. If it were, then they would be in the same predicament as archaeologists who are still unsure about the mathematical significance of the Ishango bone. Instead of reconstructing number on their own, children benefit from conversations with number system users and activities which may be deliberately or incidentally instructive (Harris & Koenig 2006). Not enough is known about how others support children in appreciating the properties of cultural tools, but the success of interventions such as Rightstart (Griffin et al. 1994) demonstrates how effective such support can be. When children progress from conceiving of number as an adjective to thinking of number as a noun (Resnick 1992), it is likely to be through their participation in classroom communities that create mathematics such as those described by Ball and Bass (2003).

Neo-Fregeanism naturalized: The role of one-to-one correspondence in numerical cognition

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Abstract: Rips et al. argue that the construction of math schemas roughly similar to the Dedekind/Peano axioms may be necessary for arriving at arithmetical skills. However, they neglect the neo-Fregean alternative axiomatization of arithmetic, based on Hume's principle. Frege arithmetic is arguably a more plausible start for a top-down approach in the psychological study of mathematical cognition than Peano arithmetic.

In the early days of developmental psychology, psychologists were in close contact with logicians and mathematicians working on the foundations of mathematics. Karl and Charlotte Bühler participated in the discussions of the Vienna Circle, which included among its members, Rudolf Carnap, Kurt Gödel, and Karl Menger. Piaget's work was heavily influenced by Bourbaki, the French structuralist school, and Piaget had many discussions with Jean Dieudonné, one of its leading members (see Piaget 1968; cf. Aczel 2007). In recent decades, developmental psychology and foundational work in mathematics have grown apart, sometimes resulting in outright antagonism (Dehaene 1997). Rips et al. restore the connection between two research fields. In their top-down approach, the cognitive math principles are based on the Dedekind-Peano axioms. They argue that these axioms or mathematically equivalent minor variations (e.g., the Least Number Principle) should be studied more seriously by psychologists. My major worry is that they have missed an important new development in foundational studies in mathematics.

Rips et al. briefly consider Frege's conception of numbers as sets of all equinumerous sets of objects, and rightly conclude that this view is untenable (Frege 1884/1974; 1893/1967). Frege's system was flawed, because of the inconsistency of the notorious Law V. However, Crispin Wright (1983) pointed out that Frege's arithmetic can be derived from Hume's principle (HP). Frege (1893/1967) contains all the essential inferences for a valid deduction of the laws of arithmetic, and Frege's inconsistent Law V can thus be sidestepped. The second-order Peano postulates can be derived in a consistent second-order system with only one extra-logical predicate N ("is the number of") and one non-logical axiom HP, stating that the number of Fs is equal to the number of Gs, in case there is a one-to-one correspondence between the Fs and the Gs. This result is known as Frege's theorem, and it implies that HP can serve as the basis of arithmetic (Heck 1993; Zalta 2008).

From a mathematical point of view, second-order Peano arithmetic (PA) and Frege arithmetic (FA) (second-order logic + HP) are almost equivalent (for their relative strength, see, e.g., Heck 2000). From an epistemological or cognitive point of view, PA and FA are very different. The basic concept in PA is the successor relation. PA is thus strongly related to the ordinal conception of number or to the cognitive abilities of enumeration and counting. FA, on the other hand, is based on equinumerosity, or one-to-one correspondence. The basic cognitive ability underlying HP is the ability to judge whether the objects of two sets can be put into a one-to-one relation, or, the ability to relate every object of a set to a single object of another set. For example, a set of knives is equinumerous to a set of forks if one can form pairs of forks and knives, without remaining forks or knives. Equinumerosity judgments are thus possible without enumerating the two collections. In the philosophy of mathematics, this discovery has triggered a lively epistemological debate on the question whether HP is an *a priori* or conceptual truth (Boolos 1997; Demopoulos 1998; Heck 2000). Neo-Fregeans claim that HP is central in mathematical knowledge.

The problem with the neo-Fregean program is that its claims about mathematical knowledge are not based on psychological evidence. Nevertheless, HP and Frege's theorem may be quite important in the empirical study of mathematical cognition. Since the 1970s, psychologists have regarded counting as the basis of numerical skills. Gelman and colleagues have argued that three principles underlie the ability to count: the one-to-one correspondence principle, the stable-order principle, and the cardinality principle (Gelman & Gallistel 1978; Gelman & Greeno 1989; Gelman & Meck 1983; Gelman et al. 1986). These principles constitute an enumeration procedure and suffice to explain various numerical skills. However, the fact that these studies are largely based on the numerical skills of infants that have been taught to count at a very early age may have biased this research. It is perfectly possible that the ability to make one-to-one correspondence judgments (HP) is more basic and relevant than has generally been assumed. Two (arbitrarily chosen) examples should suffice to illustrate that HP is often overlooked as an explanation for certain numerical abilities.

Gordon (2004) carried out several matching tasks during his stay with the Pirahã. Gordon would put a certain number of objects (1–10) below a line, and the participant had to put an equal amount of objects on the other side of the line. In general, there is a considerable decrease in performance for larger numbers, with one striking exception, namely, the line match. This would fit well with the participants having mastered a one-to-one correspondence procedure. As Gordon tampered with the matching condition, performance decreased. Gordon's results clearly indicate that non-numerates can employ a one-to-one correspondence procedure, without using enumeration (despite his claim to the contrary; Gordon 2004, p. 497). The uneven line match is especially noteworthy. If rectangles below the line were put in a line with unequal distances, performance was very good until four, and then dropped below 50% for five and six, and later went up again to almost correct performance for higher numbers. An explanation might be that for small numbers, the Pirahã use the more or less precise subitizing, with the effect that performance decreases rapidly above three. For larger sets, a one-to-one correspondence procedure takes over, leading to almost perfect performance. This is a hypothesis that can rather easily be tested, because one would assume that participants can be taught to use this strategy also for smaller numbers, with a resulting overall excellent performance.

Second, Jordan and Brannon (2006) have demonstrated that 7-month-old infants can already recognize cross-modal one-to-one correspondences for low numerosities. Although Brannon and Jordan explain this result as evidence for a number representational system, it can more easily be interpreted as the mastery of HP.

In conclusion, the claim made by Rips et al. that math schemas, roughly based on the Dedekind-Peano axioms, may be important in the study of (mature) mathematical cognition deserves further empirical scrutiny. However, there is a non-trivial, almost equivalent axiomatic approach that is arguably very different from a cognitive point of view. In a top-down approach towards mathematical cognition focusing on knowledge of mathematical principles, it seems more promising to start with a psychological study of one-to-one correspondence (FA or Heck's roughly equivalent system; Heck 2000), than with Peano's mathematical induction or commutativity.

Bridging the gap between intuitive and formal number concepts: An epidemiological perspective

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Abstract: The failure of current bootstrapping accounts to explain the emergence of the concept of natural numbers does not entail that no link exists between intuitive and formal number concepts. The epidemiology of representations allows us to explain similarities between intuitive and

formal number concepts without requiring that the latter are directly constructed from the former.

Rips et al. have rightly pointed out a lack of fit between the properties of the natural numbers as defined by axiomatic systems in number theory and the unlearned representations of magnitude in infants. However, to conclude from this that there is "no dependency whatsoever" (sect. 6, para. 2) between them seems premature. Typically, the domains of intuitive knowledge that developmental psychologists have uncovered (such as intuitive psychology or number) are underdetermined. For example, although children are born with cognitive biases that lead them to attend to the actions and goals of others, it remains as yet unclear how they reliably develop a fully-fledged belief-desire psychology that differentiates an agent's mental states from the actual state of the world.

As an alternative to Rips et al.'s and to traditional bootstrapping approaches, it may be fruitful to examine number from the perspective of the epidemiology of representations. Its basic idea is that there is a strong causal link between the mental representations of the individual members of a culture and the public representations they share – in this case natural number concepts and their symbolic denotations (number words, numerical notation systems). To acquire a novel concept, learners partly draw on pre-existing knowledge. Thus, each time a cultural representation is transmitted, it has to pass the bottleneck of the pool of mental representations within the minds of individual learners. Representations with a poor fit to the pool of knowledge are less likely to be understood, and hence transmitted, than those with a good fit. Cognitive biases that are universal in humans likely play an important role in this process. As Nichols (2002) has demonstrated for etiquette norms, our universal feel of disgust for bodily excretions makes rules that limit contact with them (e.g., prohibitions to spit in public) more attractive than norms that do not stir our evolved emotional responses (e.g., placement of the napkin to the left or right of the plate). Importantly, Nichols (2002) does not claim that etiquette norms are directly based on or constructed from universal human predispositions. Rather, their good fit with our evolved drive to avoid disgusting situations has promoted their cultural success.

In the case of number, unlearned quantificational skills might similarly constrain and guide the cultural transmission of numerical concepts. If number concepts were based on axiom-like schemas, as Rips et al. suggest, we would expect some cultures to develop nonstandard numbers – which satisfy Peano's axioms in all respects but which we would yet not call numbers; however, apart from Western mathematics, there is no evidence that nonstandard models of arithmetic were ever developed. Unlearned intuitions of number may promote the cultural fitness of some numerical representations in favor of others. Evidence from educational psychology (Vlassis 2004) suggests that adolescents have difficulties grasping the concept of negative integers: they make more mistakes when solving equations that involve negative terms, and especially those that yield negative solutions. Although adults can compare the magnitudes of pairs of natural numbers quasi-automatically, their performance drops markedly when one or both digits are negative (Fischer 2003). These difficulties are hard to explain from a purely formal point of view, as the negative integers' properties are in many respects similar to the natural numbers', such as closure (i.e., $a + b$ is a natural number/integer for any natural number/integer a and b), commutativity, and associativity. From an ecological point of view, however, conceptualizing negative integers is less relevant for organisms than conceptualizing positive quantities. If our evolved intuitions of number continue to play a role in learning processes, it becomes easier to understand why negative integers were historically less widespread than positive integers. Indeed, negative numbers were actively resisted despite their usefulness in calculations in cultures as disparate as 16th-century Europe, Han-dynasty China, and the medieval Islamic world.

What, then, is the relationship between our innate magnitude representations and natural numbers? One possibility which seems consistent with anthropological data is that although natural numbers are supported by unlearned inductive inferences (De Cruz 2006), there is considerable cultural variation in the degree to which public representations of number actually support them. For example, humans are equipped with the ability to discriminate between continuous and discrete quantities (Castelli et al. 2006). In some cultures (e.g., Western culture), children are confronted with a variety of symbolical representations for number, such as number words, Arabic digits, or even finger counting. These public representations provide external instantiations of the discreteness of natural numbers, leading children to understand that large numbers that are close together are yet distinct. Indeed, Western 5-year-olds who typically only count to 20 infer that numbers above their counting range apply to specific, unique cardinal values: if a set has 61 members, it cannot contain 65 elements (Lipton & Spelke 2006). In contrast, in some Amazonian or Australian aboriginal cultures this distinction is not made, leading people growing up in these communities to rely on approximate numerical skills only. Consequently, they cannot discriminate between quantities if the ratio between them is small. While our intuitive quantification skills are not sufficient for natural number concept formation, they do support inductive inferences that promote an understanding of natural numbers in cultures that use symbolic representations that denote exact cardinalities.

Not all basic number representations are analog: Place coding as a precursor of the natural number system

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Abstract: Rips et al.'s arguments for rejecting basic number representations as a precursor of the natural number system are exclusively based on analog number coding. We argue that these arguments do not apply to place coding, a type of basic number representation that is not considered by Rips et al.

We commend Rips et al.'s initiative to put to the fore how a conceptual understanding of the natural numbers is achieved by our cognitive system. This is a necessary step towards the integration of scientific progress in the now largely separate domains of basic number representations and more complex forms of numerical cognition. In this respect, it is a crucial question whether basic number representations constitute the basis for the development of a complete knowledge of the natural number system.

We do not agree, however, with the authors' conclusion that magnitude representations cannot be the precursor of understanding the properties of the natural number system. Our main point is that the authors are selective in regarding the magnitude representations they envisage. The authors' argument is exclusively built on analog magnitude representations. Although the biological reality of analog coding has been demonstrated (Roitman et al. 2007), there are reasons to believe that its functional importance in numerical cognition is limited. In a behavioral priming experiment, we have shown that in a naming task, dot displays evoke a priming pattern that is consistent with an analog magnitude code, but that Arabic digit primes do not (Roggeman et al. 2007). Moreover, analog magnitude coding falls short in explaining some behavioral effects observed in number processing tasks (Verguts et al. 2005). In a computational modeling study, we have proposed that analog magnitude coding is primarily an intermediate processing step that is necessary to transform visual input into another type of magnitude representation: namely, place coding (Verguts & Fias 2004).

Place coding is another type of basic magnitude representation that is neglected by Rips et al., although it contains characteristics that make it well-suited as a precursor for full knowledge of the concept of natural numbers. In place coding, a number is represented not in an analog way, but rather, by demarcating a position on a numerical continuum. This is accomplished by neurons that are tuned to a particular numerosity (Nieder et al. 2002). For instance, a neuron tuned to numerosity 3 responds maximally to three objects, less strongly to two or four objects, and even more weakly to one or five objects. Neural place coding has been demonstrated in monkeys (Nieder et al. 2002), children (Cantlon et al. 2006), and adults (Piazza et al. 2004), and applies to the coding of symbolic and non-symbolic numbers (Piazza et al. 2007).

A first important property is that, because place-coding neurons code number independently from continuous variables that are correlated with numerosity (Nieder et al. 2002), continuous magnitude is kept isolated from place coding, thereby allowing the identification of a unique first element. Second, number tuning is not perfect, such that when a given number is presented, not only this number but also neighboring numbers become activated; it naturally follows that predecessor and successor elements are determined.

More required properties are obtained when symbolic numbers are introduced. First, as pointed out by Rips et al., for a magnitude representation system to serve as a precursor for the concept of number, it is necessary that the magnitude representation system can code number in a precise way. Although the place coding system is only approximate for nonsymbolic stimuli (Nieder et al. 2002; Piazza et al. 2004), Verguts and Fias (2004) have pointed out that the use of language, or symbols more generally, leads to a strong enhancement of the representational accuracy of place coding. We simulated the learning of the numerical meaning of symbols by simply associating number symbols with nonsymbolic number inputs (collections of dots) that were already trained to be processed by place-coding neurons. After training, the behavior of the place-coding neurons was investigated when presented with symbolic input alone. It was found that the place-coding neurons encoded symbolic numbers much more precisely than nonsymbolic numbers, although, importantly, precision was not perfect such that the predecessor/successor properties were present as well (e.g., a neuron coding for symbolic number 3 was also slightly active for its predecessor 2 and successor 4). Recent neurophysiological studies in humans (Piazza et al. 2007) and also in macaque monkeys (Diester et al. 2007) confirm the predictions of the model. In a subsequent modeling study, we showed that the resulting system explains adult human behavior in a variety of number processing tasks (Verguts et al. 2005). Hence, a key contribution of language to a mature conceptual understanding of natural numbers is that it makes the basic number representations more precise.

Second, Rips et al. note that some form of mathematical induction must be assumed, which essentially entails that properties learned in the context of some numbers can be generalized to others. If larger numbers are represented as combinations based on the building blocks 1 to 9, such generalization is in principle possible: Verguts and Fias (2006) showed how a neural network can discover and apply such generalizations in the context of number naming. Empirically, it has been found that even two-digit numbers appear to be represented by such building block combinations, rather than holistically (Nuerk et al. 2001; Ratinckx et al. 2005, Verguts & De Moor 2005).

Finally, we believe that language also facilitates linking the place-coding system to other types of information that may further constrain it towards the principles of the natural number system. For instance, numbers are frequently displayed from left to right (as on rulers, keyboards, etc.). There is now good evidence that the coding system becomes systematically associated with this spatial information through development. For instance, small numbers are preferentially responded to with the left hand, and large numbers with the right hand (for review, see Fias & Fischer 2005). Such number-space associations make the deduction that natural numbers represent alternative systems such as modular arithmetic, unlikely.

Although much work remains to be done to capture the empirical and theoretical details of the developmental trajectory from the place-coding system to a complete conceptual understanding of natural numbers, we argue that the place-coding system needs to be considered as a serious candidate to serve as a precursor for the natural numbers.

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A spatial perspective on numerical concepts

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Abstract: The reliable covariation between numerosity and spatial extent is considered as a strong constraint for inferring the successor principle in numerical cognition. We suggest that children can derive a general number concept from the (experientially) infinite succession of spatial positions during object manipulation.

We agree with Rips et al. that we can best understand the nature of number concepts by studying judgments about numbers (sect. 3.3). Studies of the distance and size effects have long established that our number concepts are not of an abstract logical nature, as is envisioned by the authors; instead, they remain analogue psychological representations of the quantities they reflect. More recently, it has been shown that numbers are habitually spatially coded, and this could help to explain how number concepts are generalized. We briefly review the evidence for spatial coding of numbers before explaining its relevance to the issue at hand.

The notion of a link between our mental representations of number and space dates back to Galton (1880) and has been highlighted by the recent surge of interest in the Spatial-Numerical Association of Response Codes (SNARC) effect. The SNARC effect entails faster left-side responses to small numbers and faster right-side responses to larger numbers, for example, when classifying them by parity or magnitude. Since its discovery by Dehaene et al. (1993), this pervasive association between numbers and space has been shown to affect cognitive processes from attention allocation to movement execution, in a wide range of tasks (Fischer & Mills, submitted). Importantly, the SNARC effect is automatic in nature, as it occurs even when number magnitude is task-irrelevant. Examples of such automaticity include spatial biases when searching for phonemes in a number's name (Fias et al. 1996) or when bisecting long strings made of either large or small digits (Fischer 2001). However, a number's spatial association is probably not part of its conceptual representation because its spatial mapping is rather context-dependent and changes flexibly (for a recent review, see Wood et al., in press).

The lesson from SNARC is that, while the direction of the spatial mapping of numbers onto space is flexible, the mapping itself appears to be obligatory. This fundamental constraint of our number comprehension reflects our experience that every increment in numerosity is also an increment in spatial extent, and vice versa for decrements. Such spatial coding of numerosity is more consistent than other inferential transfers from physical to mathematical operations considered by Rips et al. (sect. 4). This reliable correlation between numerosity and spatial extent in addition and subtraction might provide the missing constraint to rule out competing inferences: It suggests that children can derive the successor principle for numbers from the unlimited succession of spatial positions during object manipulation.

Much more than merely a conceptual metaphor, spatial cognition is engrained in our number comprehension, and this is also reflected in the way we first learn to understand numbers by using our fingers, an experience that still shapes the SNARC effect in adulthood (Fischer 2008). Spatial–numerical associations seem to emerge at around 3 years of age, well before formal schooling and reading acquisition; in Western cultures, they consist of counting from left to right, adding from left to right, and subtracting from right to left. Importantly, this spatial preference predicts performance in integer-matching and object-matching tasks with numerosities exceeding 4 (Opfer & Thompson 2006).

Rips et al. suggest that "number concepts may come into play only at a more abstract level – for example, in arithmetic – where the focus is on the numbers themselves rather than on physical objects" (sect. 6, para. 3). It is thus important that the spatial biases observed with single numbers do indeed predict performance at the level of arithmetic. For example, adults point to a number's location farther to the right on a visually presented number line when it is the result of an addition, and farther to the left when it is the result of a subtraction (Pinhas & Fischer, in press). We predict that this spatial bias is also present in children once they have acquired a general number concept.

In conclusion, we think that a closer look at situated and embodied cognition in number acquisition can dispense of innate concepts to account for the development of number comprehension.

Music training, engagement with sequence, and the development of the natural number concept in young learners

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Abstract: Studies by Gardiner and colleagues connecting musical pitch and arithmetic learning support Rips et al.'s proposal that natural number concepts are constructed on a base of innate abilities. Our evidence suggests that innate ability concerning sequence ("Basic Sequencing Capability" or BSC) is fundamental. Mathematical engagement relating number to BSC does not develop automatically, but, rather, should be encouraged through teaching.

Evidence addressing development of natural number conceptualization comes from improvements in math found related to musical pitch skill learning in first and second graders (Gardiner 2000; 2008a; Gardiner et al. 1996). The specificity of connection could not be explained fully by improved general learning. But since both these areas of skill profit at this stage from mental engagement (Gardiner 2008a) involving sequential representation, even if in different ways, this similarity may well account for the pitch–math connection (Gardiner 2000; 2003; 2008a). This and further evidence to be discussed supports Rips et al.'s proposal in the target article, that innate processing capability has an essential role in the development of natural number concept; but it implies that innate capability concerning sequence is still more foundational than the capability at math-related abstraction that Rips et al. propose.

To briefly review the learning evidence: First-grade students receiving arts skill "test arts" training (1 hr 40 min/week) accelerated significantly at math, but not similarly at reading, passing controls who received arts teaching that emphasized appreciation rather than skill (50 min/week) (Gardiner 2000; Gardiner et al. 1996); this occurred whether the skill training students were at the bottom, middle, or top of the class leaving kindergarten. At the end of second grade, those with two years of test arts did best, those with one year next best, and those with no test arts most poorly at math, but again not at reading. The test arts component that developed arts skill most substantially was the Kodaly method music training (Chosky 1981).

Studies with Kodaly alone again found math improvements related to the extent of the Kodaly training (Gardiner 2000). Students receiving Kodaly and Kodaly-related training since at least the first grade have now been found to perform significantly better on math at the end of the third grade, compared to all other students in their district as a group (79% vs. 48% at grade level, $p < .001$), and 17 – 34% better, regardless of whether or not they had diagnosed learning disability, or familial poverty (Gardiner 2008b).

First- and second-grade math progress correlated significantly with Kodaly-trained capability involving musical pitch but not rhythm (Gardiner 2000). This may seem surprising, as musical rhythm is often taught as involving counting. But Kodaly method avoids rhythm counting, instead emphasizing discrimination between rhythm pattern types.

I have proposed (Gardiner 1998; 2000; 2008a) that the positive effects of pitch and math learning on one another reflect an interaction that moves mental engagements advantageously toward incorporating sequence, away from less favorable options. In math, engagement with the sequential nature of the natural numbers provides a foundation for developing engagement, with addition and subtraction as operations both involving movement, but in opposite directions along the natural number sequence. In pitch engagement, the sequential representation of ordering of discrete pitches from lower to higher according to vibration frequency within the musical scale, underlies the developing sense of upward and downward movement in perceived melody and helps the young learners remember and sing accurately many different melodies built from the same scale.

The beginning math student has other options for engaging with arithmetic; for example, memorizing production rules like "$1 + 2$ gives 3." The student learning melodies may develop and memorize the sequence of vocal gestures producing each melody separately. But such strategies are ultimately limiting. Further development in both mathematics and music builds advantageously upon the sequential conceptualizations of natural numbers and musical scales, respectively.

As Rips et al. discuss, conceptualization of the natural numbers must capture the basic Dedekind-Piano axioms (Dedekind 1888/1963) of first element, successor, predecessor, and inductive continuation. Note to what extent the notion of sequence is at the heart of these particular axioms. Besides the possibility of infinite continuation, the elements of a musical scale also satisfy these axioms – even though the choice of musical pitches for a scale has no relationship to set cardinality, but rather, has to do with frequency interrelationships of interest to music. Indeed, most scales rise in pitch not only through "whole," but also through "half" steps.

Our data thus suggest that innate "Basic Sequencing Capability (BSC)," adapted in so many different ways in skill learning (Gardiner 2007), is also foundational to the development of the natural number concept. Like advances in counting and the arithmetic operations, development of further abstractions concerning the natural numbers can then profit from mental representation capturing the sequential nature of natural numbers.

That representation concerning sequential properties of natural number is especially foundational, is further implied by the variety of engagement applications involving the ordering of natural numbers that must be developed. Natural numbers are directly related to set cardinality when discrete objects are counted; however, ordinal applications (first, second, third, etc.) concern position in a counting sequence but not set cardinality, and applications concerning measurement require a further bridge, again to sequence (Gardiner 2007).

If innate capability concerning sequence is critical to natural number development, as I suggest, then the extensive development and practice by young children at counting (Gelman & Gallistel 1978; cf. Gelman & Butterworth 2005), which already connects sequential operation to progression through natural numbers, may prepare for further development. The next step could then be to internalize the mechanisms involved in counting out loud, substituting for sequential auditory verbalization purely mental acts concerning sequential position and movement along sequence (Gardiner 2007). Our evidence then could imply that experience with musical-scale engagement that also involves sequence, even if differently, can accelerate the development of a sequence based mental internalization of natural number engagement (Gardiner 2008a).

Rips et al. have comprehensively reviewed research showing that young learners have means other than counting for differentiating magnitudes and for establishing cardinality of small sets of objects. Such capabilities are important in their own ways, still exist beyond childhood, and can be disrupted along with capabilities based on natural numbers by injury or by experimental methods causing localized temporary disruption of brain processing capability (Cappelleti et al 2007). This can imply brain architecture that places processing involving natural number near other forms of quantitative processing, perhaps to facilitate interaction, but not necessarily that the quantification built on natural numbers is developed from these other engagement forms.

Further potentially valuable opportunities for mental interaction with the arts as learners advance in math capability is discussed elsewhere (Gardiner 2007; 2008c).

Counting and arithmetic principles first

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Abstract: The meaning and function of counting are subservient to the arithmetic principles of ordering, addition, and subtraction for positive cardinal values. Beginning language learners can take advantage of their nonverbal knowledge of counting and arithmetic principles to acquire sufficient knowledge of their initial verbal instantiations and move onto a relevant learning path to assimilate input for more advanced, abstract understandings.

"Can you do addition?" The White Queen asked. "What's one and one and one?"
"I don't know," said Alice, "I lost count."
"She can't do addition," the Red Queen interrupted.
— Lewis Carroll

Rips et al. are correct: "Children's simple counting and enumerating does not provide rich enough constraints to formulate the right hypothesis about the natural numbers" (sect. 5.1, para. 2). However, their presentation of my position (e.g., Gallistel & Gelman 2005; Gelman 1993; 2006; Gelman & Gallistel 1978; Leslie et al. 2008) leaves out a critical feature of it – that I steadfastly have maintained that the meaning and function of counting are subservient to the arithmetic principles of ordering, and addition and subtraction.

The proposal that nonverbal counting principles facilitate learning of verbal counting does not imply that there is an immediate mapping from the nonverbal conceptual system to the verbal one. The presence of any relevant mental structure facilitates attention to, and learning about, data that share the same structure. The exact same counting principles, and their performance constraints, are shared by both the nonverbal and verbal systems. Therefore, learners have a way to collect structure-relevant data, including the fact that number words are recognized as being in their own class (Wynn 1992b). Similarly, beginning language learners can make sense of the fact that although the partitive follows a cardinal word, as well as quantity terms, this is but a statistical fact. The partitive occurs with other part–whole constructions such as *side of*, *mother of, color of*, and so on.

If losing one's count were the same as not being able to do addition, the Red Queen would be right. But, we know that she is not. To develop my reply to Rips et al., I start with evidence that young children can relate their use of natural numbers to arithmetic. Second, I take up the ubiquity of erroneous counts and ask: Do these reflect limits on competence alone, or are their other sources of systematic variability? Finally, I return to the question of from where the understanding of verbal counting principles and arithmetic comes.

Bullock and Gelman (1977) showed that even two-and-a-half year-old children could understand that the property of being numerically more or numerically less, defined which of two displays was the "winner." In the transfer condition, when children first encountered the novel set sizes of 3 and 4, they responded on the basis of ordering. Further, more than 60% of the children counted and used the related cardinal values (Gelman 1993). This is evidence that very young children, who are still very poor counters, already can map common numerical relations to each other. Zur and Gelman (2004) reported that 3- and 4-year-olds counted to check predictions about the effect of adding and subtracting items. No child ever tried to make their count be consistent with their predictions. The 4-year-olds did problems that contained values as large as 15; the maximum N for the younger group was 5. Both age groups were sensitive to the inverse relation (e.g., $5-1$ followed by $5+1$, or $12+3$ followed by $15-3$), even though these problems were not presented one after the other. Zur (2004) has also shown that when 4- and 5-year-olds encounter a second pair of commuted problems – that is, one followed by another, different problem that shares the property of commutativity – they are faster and more accurate than they are with pairs of problems that do not share this property. Further, this transfers to a target analogy problem.

In short, I agree with Rips et al. that the principles of arithmetic cannot be induced from the rote learning of counting or from the simple knowledge of the referents of the first few count words. It is rather the reverse: Children understand what counting is about because they can make at least implicit use of the principles of arithmetic reasoning. I think the authors seriously misestimate the age at which children have some nontrivial understanding of cardinality and the successor principle.

Sure, children take a long time to fully grasp the rules for framing the next count word in English – a non-trivial number of eighth graders have not fully mastered these skills (Harnett & Gelman 1998). When anyone replies that the next number after one quadrillion is two quadrillion, it is unlikely that they think that adding one to an unimaginably large number doubles its value. Because they do not know how to generate the word for the next number, and because, like adults, they can fail to distinguish between the word and the concept itself, they can appear to not believe that continued addition always generates a still-larger number. The large majority of Hartnett's children who were scored as "Ambiguous" regarding their understanding of the successor principle, were either concerned about the fact that they did not know the word which would result, or they would run into physical limitations. They gave answers like, "Well, the number would be make-pretend," or "if you make up numbers, then it's alright," or "I don't really know, because I've never really counted to where it stops."

Rips et al. highlight the extent to which the road to counting mastery is strewn with numerous mistakes. I have shown that errors are influenced by variations in set size, density, rate of counting, time of presentation, and "touchability" of items – all variations that influence one's ability to keep entities in a display separate from each other, and/or to keep track of the counted from the to-be-counted items. So, too, does the requirement that the one-one and stable ordering outputs stay lock-stepped together. Most results bearing on these variables are reviewed in Gelman and Gallistel (1978) and Cordes and Gelman (2005).

No matter what, children do have to tackle the task of memorizing a sequence of terms wherein there is nothing about a given entry in the list that systematically predicts what the next one after it must be, and the next, and so on, for a very long time. Humans, unlike computers, are not good at memorizing long lists of arbitrary sounds that must be repeated in the exact same order, trial after trial. Young children also have to become fluent counters, decode task variables, and so on (Gelman & Greeno 1989). But at least they will be on a relevant learning path and be able to disambiguate other relevant data – for example, the fact that count words are in a separate class (Wynn 1992b) – and could use the distributional fact that number words can be followed by the partitive "of," despite its low frequency of usage by adults (Bloom 2000; Bloom & Wynn 1998) and its legitimate usage after many quantifiers or phrases, such as "a side of," "the color of," and so on. Therefore, when principles of arithmetic organize the domain, we can say that the counting principles serve to facilitate learning the verbal version of these arithmetic principles. Subsequently, more and more formal understandings of cardinality will develop, but this does not preclude an early level of understanding.

Look Ma, no fingers! Are children numerical solipsists?

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Abstract: I ask whether it is necessary that principles of number be mentally represented and point to the role of language in determining cultural variation. Some cultures possess extensive counting systems that are finite. I suggest that learning number principles is similar to learning conservation and, as such, might be derived from learning about the empirical properties of objects and other individuals in combinations.

The target article provides an important discussion of the potential pitfalls in taking experimentally obtained behaviors and making general inferences about numerical cognition in infants and children. All too often, researchers assume that the practices of counting and differentiation evidenced in experiments directly reflect an underlying cognitive system that is a prototype of mature adult numerical cognition. One cannot assume that the internal systemic properties of number are bootstrapped from experience with real objects. Could it be the case that concepts of number are acquired by pure rationalism in the unconscious mind as children attempt to make sense of the number system and its properties?

The question we must ask is whether the abstract formal properties of number theory are a necessary part of the representation of number, as opposed to being part of a meta-analysis of the number system that is taught in schools. It is one thing for children to behave as if they had a principle of associativity or commutativity; it is another to explicitly represent such a principle in a formal logical mental theory-structure. If the number principles are learned in a natural way or are innate, then where are the boundaries of complexity? Must we require induction of the commutative principle, but not Goedel's theorem? Perhaps the boundaries lie just at the level of figuring out what changes and does not change upon minor rearrangements of configuration.

To reject the idea that number principles are arrived at through mathematical induction, we need an account of how number properties can be acquired without the formal machinery. Notice that we are not talking about bootstrapping as a stepping-stone to a formal representation, but about having the right truth-value structure in numerical reasoning without needing the formal inductive mechanisms described in the target article.

Let us consider whether the formal machinery gets us to the right place in accounting for human numerical cognition. We are told that it follows from the successor function that there is no final number, since the set of all numbers must be infinite. But this is just not true for all cultures. We are now familiar with cultures that have very few numbers, such as the Pirahã (Everett 2005; Gordon 2004) and Munduruku (Pica et al. 2004), who clearly lack knowledge of numerical principles. Some cultures count by body parts, such as the Yupno of Papua New Guinea, who finish counting at the name of the right testicle (Menninger 1898/1969). More intriguing are the Polynesian cultures whose original language had a counting system that went to a million, but stopped dead there (Beller & Bender 2008). The correct answer to the question: "Is there a largest number?" in these cultures is "Yes." Surely such cultures had something like a successor function, but if such a function guarantees discrete infinity of the number system, then how did these Polynesians end up with a finite counting system? We must take into account the linguistic basis of counting and how that directly affects the nature of mathematical principles that apply, modulo the culture. Therefore, if a counting system is generative, but not fully recursive, you end up with a number set that is very large but finite. In the same way, if a natural language has a generative grammar that lacks recursion, as might be the case for Pirahã (Everett 2005) and possibly Hixkarayana (Derbyshire 1979; Geoff Pullum, personal communication), then the number of sentences generated by the language is also very large, but finite.

How, then, might the child figure out that associativity and commutativity hold, absent any formal instruction or complex machinery of mathematical induction? Perhaps if we step back a bit, we can see that these principles are quite similar to the principle of conservation of number (Piaget 1952). The principle of conservation tells us that quantities do not change as a result of irrelevant transformations. The commutative and associative principles for addition tell us that adding two quantities, or numbers, does not change their sum under irrelevant transformation (switching the addends or regrouping). In Carey's (1987) reanalysis of conservation phenomena, she points out that conservation does not follow from logical necessity, as Piaget had proposed, and so learning conservation could not therefore follow from logical analysis of the domain. Instead, conservation is an empirical fact about the world and turns crucially on the notion of "relevance" of the transformation. A favorite teaching trick of mine after a class on conservation, is to ask students whether the area bounded by a piece of string is changed upon altering the shape of the string slightly. Most students will say "no" and, of course, will be shown to be wrong as the string is flattened into a shape with bounded area of zero. The point is made more forcefully if you then invite a 3-year-old into the room and they get the correct answer!

If conservation is learned empirically rather than logically, then figuring out that substances or individuals do not change quantity upon being poured into a different container or spatially rearranged, would involve learning about the nature of matter and objects in the physical world. It isn't so much experience with objects in counting situations that bootstraps these number principles, nor that these principles are arrived at autonomously within the unconscious theorem-generating mental number space, but that the indirect process of trying to figure out the nature of objects as objects and individuals within sets might be the catalyst for arriving at these principles in the relatively informal manner that we also acquire knowledge of other domains of folk-science. The inference that these principles apply universally to all quantities or numbers, might be understood at the folk-math level in the same way that we have faith that the sun will rise tomorrow, by everyday Hume-style, deductively invalid, inductive generalization. The formal use of successor functions and mathematical induction/deduction might only come into play when children are taught explicitly in school.

Set representations required for the acquisition of the "natural number" concept

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Abstract: Rips et al. consider whether representations of individual objects or analog magnitudes are building blocks for the concept NATURAL NUMBER. We argue for a third core capacity – the ability to bind representations of individuals into sets. However, even with this addition to the list of starting materials, we agree that a significant acquisition story is needed to capture natural number.

Rips et al. paint a picture of math knowledge as divorced from the object tracking and simple enumeration abilities of young children. Something is right here, and something is wrong. What is right is that arithmetic is a domain of relations across numbers, numbers (at least on many stories) are abstract objects, and the truths of mathematics are not dependent on anything in the world, like objects or approximation abilities. But what is wrong with this picture is that neither humans over the course of history nor children over the course of development initially understand mathematics as a domain concerning abstract entities divorced from the world. The ability to describe relationships across real-world collections of objects was presumably the historical impetus for discovering arithmetic, it is how children begin engaging in mathematics, and for most humans, it remains the major forum for applying arithmetic knowledge. *A fortiori*, a bridging mechanism between mathematical truths about numbers (i.e., abstract objects) and collections (i.e., things in the world) is required if such knowledge is to be made relevant to real-world calculation.

We suggest that the concept set and ordered relations across sets can serve as a bridging mechanism (another possibility is plural variables). Rips and colleagues do not list set representations among the early capacities potentially relevant for math knowledge. We suggest that the concept set is required and that this notion cannot come from object tracking, the approximate number system, or language.

Conceiving of a set requires representing the hierarchical relationship between individual items and the larger structure into which they are bound (note that this is distinct from perceptual grouping or recoding, both processes that destroy the individuals comprising the group). Whereas individuals exist in the world, sets are abstractions that exist in the mind; it takes a mind to maintain multiple hierarchically ordered construals.

The object tracking system (and working memory) will not provide the concept set. This system will not return a representation of one-ness, two-ness, or three-ness when it is engaged in tracking one, two, or three objects. This is because the objects are completely separate individuals in the world; maintaining this separateness was one of the main motivators for positing a tracking system in the first place.

Nor will the approximate number system (ANS), which produces analog magnitude representations, provide the concept set. The representation of 1 in the ANS is a continuous distribution of activation (e.g., Gallistel and Gelman [2000] describe this system as instantiating the real numbers, not the integers). Also, the representation for 8 in the ANS (the 8-distribution) is in no way made up of 8 individual distributions of 1. Further, on some models, the entities (in the world) that enter into this representation are not even individuated (in the mind) prior to creating this estimate (rather, they are Gaussian activations passed through a continuous normalizing filter; Dehaene & Changeux 1993). With no discrete representation of 1, no compositionality, and, on some models, no individuation prior to enumeration, the ANS cannot provide the concept set.

Nor can language do the job. "SET" is a notion with content. The syntax provides algorithms for manipulating content. Although the language faculty might provide relevant computational possibilities, including embedding, agreement, and recursion, these are all content-free syntactic operations. One must maintain a distinction between the computation that binds any two individuals into a set and the concept of a set of two individuals. The former might reside in the language faculty, but not the latter, and it is the latter that is needed to bridge math to the world.

What then is needed to connect set with number? If we begin with object tracking, which will provide us with representations of discrete and separate individuals, one might take these separate individuals (e.g., two separate object files) and bind them into a set of individuals. Rather than tracking, for example, Joe and Jenny, one could then index the set of Joe and Jenny. Our recent work suggests that 14-month-old infants make this distinction between tracking individuals and tracking sets of individuals (Feigenson & Halberda 2004; in press). Importantly, this work provides evidence that infants maintain representations of the individuals comprising the set, thereby distinguishing set representations from demonstrations of perceptual grouping and tracking (e.g., Wynn et al. 2002), in which there is no evidence that infants retain representations of a group's components.

Next, the learner must abstract away from any particular instance of tracking, for example, a set of two particular individuals, to represent the more general case of tracking a set of any two individuals. Presently, we know of no evidence that infants construct this more general representation, but we see no computational limitation that makes this abstraction impossible. Last, the learner must order set representations with, for example, 1, 2, and 3 individuals according to the successor function. With this ordering, set representations become bona fide representations of one-ness, two-ness, and three-ness (modulo Rips et al.'s comments concerning mod_{10}). This ordering appears to occur in the course of learning to enumerate collections.

The set-binding computation we discuss may require that items be represented in parallel prior to binding. Limits on parallel individuation or on working memory would therefore limit these set representations to smaller numbers of items. Our guess is that precise large numbers will be known only through a meta-language, or schemas along the lines Rips et al. suggest. Large numbers may derive their semantic content via the functional role they play in relations with other numbers, but ultimately they make contact with the smaller cardinalities, for which set representations are also available.

We suggest that the concept SET will be a requirement for any theorist who tries to build an acquisition story from early enumeration abilities to NATURAL NUMBER. Furthermore, any story that divorces natural number from these early abilities will require the concept SET (or some other bridge), applicable to real-world collections, in order to connect purely mathematical knowledge to the real world.

Recursive reminding and children's concepts of number

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Abstract: According to the recursive reminding hypothesis, repetition interacts with episodic memory to produce memory representations that encode – and recursively embed – experiences of reminding. These representations provide the rememberer with a basis for differentiating among the first time something happens, the second time it happens, and so on. I argue that such representations could mediate children's understanding of natural number.

If Rips et al.'s target article is right, children's understanding of natural number cannot be derived from numerical abilities that they displayed as infants. Two crucial elements that are missing in the article are a basis for the concept of successor and a schema for the starting value of one. I suggest here that ideas from the field of memory may help to fill in these gaps. Specifically, I propose that everyone with a functioning episodic memory possesses a pre-symbolic analogue of counting, which could mediate the acquisition of number concepts.

To start in the memory laboratory, consider an experiment in which the items to be enumerated are not simultaneously displayed in front of the subject, but are dispersed over time. A standard experiment might use 50 different items (e.g., words or pictures), presented in a list totaling 150 study trials. Different stimuli are presented different numbers of times, haphazardly interleaved with presentations of the other items. After the list has ended, subjects are shown each item and asked how many times it appeared. People are surprisingly good at making such frequency judgments. Telling subjects to attend to presentation frequency ahead of time has little or no effect on performance, and 4-year-olds judge frequency about as well as adults (Hasher & Chromiak 1977). "Memory strength" does not appear to be the basis of this ability, because one cannot trick subjects into saying that a once-occurring item appeared two times merely by presenting it recently (Wells 1974) or for a long study duration (Hintzman 2004). It is not plausible that subjects are counting, in the strict sense of the term, because this would require maintaining running totals for many different items, and the usual study instructions give subjects no reason to count.

How can we understand such findings? Hasher and Zacks (1979) proposed that humans are specially prepared by evolution to acquire frequency information. A different hypothesis (Hintzman 2004) is that the ability is not innate, as such, but derives directly from the operation of more basic memory processes, which yield recursive representations of repeated experience.

I assume that everyone with a functioning episodic memory has three basic abilities: (1) to maintain a complex mental state, (2) to encode a representation of that state into memory, and (3) to retrieve the representation reflexively, when given a matching retrieval cue. I also assume that the mental state that results from retrieval has two parts: one that refers to the present moment (i.e., the encounter with the cue) and one that refers to the past (the reminding of an earlier experience). This distinction between present and past is a fundamental aspect of the rememberer's subjective experience. Schematically, if A represents the subject's mental state upon first encountering a particular stimulus, then the mental state upon encountering the stimulus for the second time is $A + R(A)$.

A crucial consequence of the second encounter with the stimulus is that the corresponding experience, $A + R(A)$, is itself encoded in memory. Now if the same stimulus is encountered yet again, the person is reminded of having been reminded on the previous trial, so the resulting mental state is $A + R[A + R(A)]$, which is itself encoded in memory, and so on. In this notation, the R outside brackets is the current reminding, and the R inside brackets is the reminding that is remembered from the previous trial. It is important to understand that according to the hypothesis, the memory representation itself is recursive, not just the process that generates the representation. Each new mental state – and therefore each new encoding – is a compound that may include (or refer to) previous mental states. It appears that assumptions (1) to (3) stated above will automatically generate recursive representations. A theorist who adopts those assumptions but does not want recursive representations would have to include a restriction that specifically rules them out.

I assume that on a frequency judgment test, the subject retrieves the test item's latest memory representation and maps its depth of embedding or recursion onto the number scale (Hintzman 2004). There may be no limit to the number of levels of reminding that a memory can represent. Frequency judgments become less and less accurate in an absolute sense as frequency goes higher, but there is no evidence that they ever level off. Certainly, memory mechanisms are subject to several kinds of "noise," so if there is a memory limit, it may be more akin to a limit of resolution imposed on the entire representation than to a strict limit on the number of levels of recursion that can be encoded in a memory trace.

Of course, a child seldom encounters a series of events as uniform as a list of items that includes exact repetitions. The child does, however, have many novel or first-time experiences, and many experiences that are repetitive or routine. I assume that most experiences of the latter type include remindings of similar experiences from the past. So, if the hypothesis of recursive reminding is applicable to list learning experiments, it may also be applicable to a child's everyday life. The only precondition is a functioning episodic memory. With iterative application of encoding and retrieval, recursive representations will essentially build themselves.

How could this help the child acquire the concepts of one and successor? Let us suppose that all remindings possess a common element, R, which is tacitly understood to reference the past. The mental states $A + R(A)$ and $B + R(B)$, are similar because they include this common element. By way of contrast, the mental states A and B are similar in that the element of reminding is absent. Children know what it means to remember, and what it means to not remember, before the age of three (Fivush &

Hudson 1990). On the basis of this generalization, a child could form two schemas: X for a novel or initial experience, and X + R(X) for a repeated experience. Once formed, these schemas could be mapped onto one (or first) and greater than one. In a like manner, if less easily, schemas for X + R(X) and X + R[X + R(X)] could be differentiated and mapped onto two (or second) and greater than two, and so on. Limitations on representations and the processes that act on them make precise mappings less and less accurate as one moves up the scale, but even where errors are unavoidable, rough mappings can be learned.

If everyone with a normal memory knows what it means to remember, then they should know that the process is the same, regardless of the represented level of recursion. Consider three memory states: (a) A, (b) A + R(A), and (c) A + R[A + R(A)]. The rememberer understands that the process that leads from (b) to (c) is the same as the process that led from (a) to (b) on the earlier occasion. An appreciation of this identity could be generalized into a schema for the concept of successor. Thus, the common element, R, in remindings may help the child acquire one and successor, both concepts that Rips et al. identify as fundamental to number.

It may be especially easy to associate the "one-two-three" rhyme with number concepts through recursive reminding, because counting and reminding are both discrete and sequential (stepwise) processes. To count objects in a spatial display, one must focus attention on the objects in a particular, often arbitrary, order. This demand is absent when time imposes the order.

An obvious consequence of this hypothesis is that a child will not come to understand natural number, as described in the target article, until the child has developed a normal episodic memory. However, even rats may have the capacity for recursive reminding (see Capaldi 1994). The human understanding of number presumably requires coordination of this capacity with the use of linguistic symbols.

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On some concepts associated with finite cardinal numbers

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Abstract: I catalog several concepts associated with finite cardinals, and then invoke them to interpret and evaluate several passages in Rips et al.'s target article. Like the literature it discusses, the article seems overly quick to ascribe the possession of certain concepts to children (and of set-theoretic concepts to non-mathematicians).

Natural numbers serve both as finite cardinal numbers (cardinals) and finite ordinal numbers, and belong to (or embed into) other number systems. However, following Rips et al., I will consider the natural numbers only as finite cardinals.

Each natural number is associated with several concepts. There are the concepts expressed in English by the numeral determiners (e.g., "at least three," "at most three," and "exactly three"); call these finite-cardinality quantifier (hereafter FCQ) concepts, as their semantic values are cardinality quantifiers. There are concepts expressed by set-theoretic predicates (e.g., "three-membered"); their semantic values are properties of sets. The syntax and role in reasoning of the nouns such as "3" and "three" at least suggest that they are singular terms. If so, they express concepts such as that of the number three "itself"; I'll call these numerical-individual concepts (hereafter NI). At many places in this article, I was unsure of which of these concepts the authors were considering. Finally, metalinguistic concepts (e.g., of Arabic numerals like "3") deserve mention.

Three further comments are pertinent in relation to this:

1. In various contexts, we use phrases of the form "three *N*s" to mean "exactly three *N*s," or "at least three *N*s," or even "at most three *N*s." Such default uses might create the impression that there is a "root" concept of three-ness from which we form the three FCQ concepts by some sort of supplementation. Not so! Just consider the definitions of these quantifiers in terms of the apparatus of standard first-order logic (presented in many places; e.g., Barwise & Etchemendy 1999, pp. 366–68).
2. A quotation from Bloom (in the target article's sect. 3.2.2, para. 5, referring to Bloom 2000, p. 215) seems to ascribe FCQ concepts to pre-linguistic infants. This would be astonishing: having FCQ concepts requires having concepts of existence, universality, identity, and some connectives, as well as being disposed to find certain inferences involving these notions compelling (see Hodes 2004; Peacocke 1992, Ch. 1).
3. Having the concept of being three-membered requires possession of the concepts of being a set and of set-membership. The latter concepts form a "local holism" (see Peacocke 1992) that entered mathematics only in the late nineteenth century. These concepts are vastly more sophisticated than FCQ concepts. (They constitute a device to bring plurality within the purview of "singularist" logic [see McKay 2006], allowing us to replace plurals – for example, "The Three Tenors" – with a singular expression – for example, "the set whose members are exactly Domingo, Pavarotti, and Carreras.") Perhaps this emergence of set-theoretic concepts built upon some sort of "implicit conception" (see Peacocke 1996; 2005) of set-hood with older roots[1]; but such a conception would also be sophisticated. In places (e.g., sect. 2.2, para. 4), the authors seem to suggest that reputable psychologists believe children to possess these concepts (or their precursor conceptions)! I recommend that psychologists concern themselves with the concept of being three-membered only when investigating the psychology of professional mathematicians.

Perhaps young children are in a psychological state that makes the model presented in Figure 1 applicable to them. But Rips et al. seem to presuppose that this state involves concept-possession.[2] This presupposition is implausible. Having concepts requires the ability to engage in conceptual thought. Very young children might be able to discriminate between situations in which there are three toys in a box and those in which there are two toys in a box; but such ability is not by itself evidence of possession of FCQ concepts. It might not even be evidence of possession of the concept expressed by the binary determiner "more/than" (as in "More sheep are in the meadow than cows are in the barn").

In section 3.1, Rips et al. attribute to "many theorists" these theses: (i) "[O]nce children have learned language, or, at least, language-based counting, they are in a position to attain true concepts of natural numbers"; (ii) "[W]hen they are able to perform tasks such as enumerating the items in an array or carrying out simple commands (e.g., 'Give me six balloons')," they have acquired these concepts (sect. 3.1, para. 4). By being "in a position to attain" a concept, the authors seem to mean being exactly one cognitive step away from possessing that concept. By "true concepts of natural numbers," they might mean the FCQ concepts, but more probably they mean the NI concepts, associated with at least some natural numbers (maybe extending as far as the person in question can easily count).

Possession of the NI concept associated with a number probably requires prior possession of the corresponding FCQ concepts. Surely the latter requires the ability to enumerate at least certain objects. Enumeration is already at least one cognitive step beyond language-based counting itself, and beyond that lies the ability to deliver the "upshot" of enumeration.

(When my daughter was acquiring FCQ concepts, or at least the ability to express them, I'd give her three toys and ask, "How many?" She'd enumerate them: "One, two, three," and then in response to my question, say "Four"!) Does this "upshot" ability indicate possession of the "exactly" FCQ concepts? The authors, in rejecting the aforementioned thesis (ii), might be answering "No," and "No" seems plausible to me. Even if they would answer "Yes," I think that they would agree that having lots of FCQ concepts leaves their possessors some distance from having the corresponding NI concepts. I doubt that many American middle-school algebra students possess NI concepts, even for natural numbers less than 10. Any experiments demonstrating possession of NI concepts would have to distinguish such possession from possession of corresponding metalinguistic concepts; for example, of Arabic numerals.

Whether theses (i) and (ii) refer to the FCQ or the NI concepts, Rips et al. show good sense in rejecting them. They correctly point out (in sect. 3.2.2) that a child can "arrive at something like" the generalization to which they refer as Principle (3) only after mastering advanced counting. They then address this thesis: (iii) Children simultaneously learn to do advanced counting and to correlate the numerals to which they can count with the FCQ concepts associated with those numerals. They say that it is unclear how thesis (iii) could be true (in sect. 3.2.2, para. 10), and go on to argue against it. Their argument seems beside the point. As I understood thesis (iii), the advanced counting in question is in a numeral system for the natural numbers, not one for arithmetic mod_{10}. If so, how things go for a child who learns to count cyclically ("Zero, one, . . ., nine, zero, one, . . .") is irrelevant; it certainly is not a way of counting on which Principle (3) would be true.

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NOTES

1. I didn't understand the authors' notion of a knowledge schema for the natural numbers; but it occurred to me that it might have some connection with Peacocke's notion of an implicit conception.

2. One should beware of conflating a concept with a mental representation that underlies possession of that concept. Such conflation might have encouraged some psychologists to classify other mental representations, whose work is pre-conceptual, as concepts, and thus to overascribe concept possession.

The role of the brain in the metaphorical mathematical cognition

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Abstract: Rips et al. appear to discuss, and then dismiss with counterexamples, the brain-based theory of mathematical cognition given in Lakoff and Núñez (2000). Instead, they present another theory of their own that they correctly dismiss. Our theory is based on neural learning. Rips et al. misrepresent our theory as being directly about real-world experience and mappings directly from that experience.

In their target article, Rips et al. appear to discuss, and then dismiss with counterexamples, the brain-based theory of mathematical cognition provided in Lakoff and Núñez (2000). In fact, they do not discuss our theory at all, but instead present another theory of their own – which they correctly dismiss.

Our account is based on neural computation and neural learning, as discussed in Feldman (2006). The neural theory of conceptual metaphor and metaphor learning is discussed at length in Lakoff and Johnson (1999). On this account, the neural learning of primary metaphors is based on repeated correlated experiences that activate two different brain regions. The repetition of experience leads, via spreading activation and repeated synaptic strengthening along existing pathways, to the formation of neural circuitry linking the areas in an appropriate way so that the circuitry formed physically constitutes the conceptual metaphorical mappings that we observe in hundreds of cases in languages throughout the world. The mathematical cases are just special cases predicted from the general account.

We argue that, for arithmetic, there are four such primary metaphors that are neurally bound via best-fit principles of neural computation. The result is a metaphor system that yields the so-called abstract properties of arithmetic. Many of the properties of arithmetic arise from the metaphorical overlap of inferences, but some come from different metaphors in the system. We specifically show that not all the inferences can arise from metaphorical inferences based on any one area of experience.

Núñez and I argue further that infinite entities, such as infinite decimals, or the set of all integers, or the set of all sums, arise from a quite general metaphor that characterizes infinite entities in general in all branches of mathematics. This Basic Metaphor of Infinity is simple and arises naturally from the neural theory of metaphor – outside of mathematics per se. It is based on Srini Naranayan's neural computational theory of aspect in natural languages (Narayanan 1997). In that theory, processes in the brain are computed via circuitry for what he calls "X-schemas," which is short for "executing schemas." Each process has an initial state, an iterated action (with no particular bound on the iteration), and an optional final state. Those without final states are called "imperfective" in linguistics – such as walking. Those with final states – like walking 100 steps – are called "perfective." Because of the overlap in the initial state and iteration, the activation of perfectives also activates the circuitry for imperfectives – over and over. The result is a neural metaphorical mapping in which unbounded iteration (unbounded infinite processes) is understood as having a metaphorical final state – an infinite entity, with special cases like the set of all integers and the set of all sums.

The central argument of Rips et al. is that the general properties of arithmetic cannot arise directly from real-world experiences in themselves. We agree. It is the real-world experiences as registered in human brains that results in learned circuitry, which constitutes conceptual metaphors that yield the properties of arithmetic.

Rips et al. do not argue against our theory. They act instead as if our theory were a version of the literal experience account that they correctly reject. Here is what they say (sect. 4.2, para. 3):

> According to Lakoff and Núñez (2000), general properties of arithmetic depend upon mappings from everyday experience.

They then continue (sect. 4.2, para. 4):

> A key issue for the theory, though, is that everyday experience with physical objects, which provides the source domain for the metaphors, does not always exhibit the properties that these metaphors are supposed to supply. Closure under addition, for example, does not always hold for physical objects, as there are obvious restrictions on our ability to collect objects together.

This is true of direct experience in the world, but not of the neural circuitry learned on the basis of repeated successful "small" cases of object collection, taking steps in a given direction, and so on. Our theory holds despite such physical limitations on large collections in the world. Rips et al., however, consistently misconstrue our theory as a version of the real-world experience induction theory that they are arguing against.

I believe that Núñez and I got it right, not just for arithmetic, but for all the forms of higher mathematics we discuss. The general theory we give applies outside of mathematics per se, applies in languages and conceptual systems throughout the world, and also happens to work for mathematics as a special case. What we put forth is an explanatory theory that starts from the apparently inborn capacities for subitizing and baby arithmetic and adds neural learning theory, which gives rise to an account of the learning of primary metaphors in general, and primary conceptual metaphors for arithmetic in particular.

If Núñez and I are right, then Rips et al. are dead wrong. What is at issue is much broader than the Rips et al. theory. The question is, what is the nature of mathematics in general, the advanced branches as well as the simple ones?

Why cardinalities are the "natural" natural numbers

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Abstract: According to Rips et al., numerical cognition develops out of two independent sets of cognitive primitives – one that supports enumeration, and one that supports arithmetic and the concepts of natural numbers. I argue against this proposal because it incorrectly predicts that natural number concepts could develop without prior knowledge of enumeration.

In their article, Rips et al. argue that inductive accounts of the development of the natural numbers suffer from an irreparable defect: they cannot explain how children arrive at the intuition that there are infinitely many numbers. Moreover, the authors argue that these accounts confuse cardinalities with natural numbers. Indeed, according to Rips et al., the natural numbers cannot be acquired from representations of cardinalities, because natural numbers are not individuated by their reference to cardinalities but, rather, by the system of rules that govern their behavior (e.g., the Dedekind-Peano axioms). Rips et al.'s solution to these problems is two-fold: (1) They postulate that human minds comprise two autonomous systems of numerical reasoning – one dedicated to the representation of cardinalities, the other dedicated to the representation of the natural numbers; and (2) they argue that the development of the system that represents the natural numbers is governed by a schema of innate constraints.

In my view, Rips et al.'s theory must answer multiple challenges. First, it does not seem to satisfy the "no competing inference" criterion rightly imposed by the authors themselves on theories of learning. As far as I can tell, the initial state they propose (in sect. 5.2) is potentially consistent with many more systems than just the natural numbers.

Second, because it postulates that the cognitive system for the natural numbers is completely independent from the system dedicated to representations of cardinalities, Rips et al.'s account predicts that one could grasp the natural numbers without having first developed a system for representing cardinalities. This seems to be wrong both historically and developmentally. Each of the histories of number notations I have read reports that, in culture after culture, the first numerical symbols were "external sets" that were connected to sets in the world via one-to-one correspondence (Dantzig 1967; Hurford 1987; Ifrah 1985; Kaplan 1999). None of these report cultures in which the meaning of the first symbols was determined by inferential rules.

As Rips et al. point out, little work has been done on children's grasp of infinity, and even less has been done on their grasp of rules that apply over the entire infinite domain. However, available research on these issues (including the research reviewed in their article) suggests that children learn symbols for cardinalities long before their use of symbols shows evidence of being governed – explicitly or implicitly – by rules such as the Dedekind-Peano axioms or the rules of arithmetic. By now, multiple studies have shown that, by age 4, many children have learned how to use counting to represent cardinalities (e.g. Le Corre et al. 2006; Wynn 1990; 1992b). In contrast, one of the rare studies on children's understanding of the successor principle showed that it is not until about 3 years later that the majority of children understand that there is no largest number (Hartnett & Gelman 1998). Similarly, the study of children's understanding of the commutativity of addition cited in Rips et al.'s article suggests that many kindergarteners do not yet understand that addition is commutative. Finally, Hughes (1986) reports that children understand statements like "four dogs plus three dogs is seven dogs" before they understand statements about the numbers themselves (e.g., "four plus three is seven"). In sum, it seems that the normal sequence of acquisition of number symbols is the same in childhood as in history; that is, in both cases, the meaning of number symbols is determined by their relation to cardinalities of sets long before the symbols are of the kinds of formal systems Rips et al. have in mind.

Furthermore, I have recently shown that analog magnitudes play a causal role in the acquisition of knowledge of a basic inferential relation between large number words (beyond "four"), namely, numerical order (e.g., knowledge that "ten fish" is more than "six fish"). That is, I have found that children must map these number words onto analog magnitudes (Le Corre, under review) to be able to order them. These mappings are formed many months after children have learned how to use counting to determine the numerical size of a set. Therefore, children spend a period knowing how counting works but not knowing how to order large number words relative to each other. For example, some children know that correctly counting a collection of fish as "one, two, three, four, five, six" results in collection called "six fish" and that correctly counting a collection as "one, two, three, four, five, six, seven, eight, nine, ten" results in a collection called "ten fish," but they do not know that the expression "ten fish" denotes more fish than the expression "six fish." It is only when they can verbally estimate the numerical size of large sets without counting – a sign that they have mapped large number words to large analog magnitudes – that they can order expressions containing large number words relative to each other. Given that (1) analog magnitudes are representations of cardinalities, and that (2) knowledge of the order of linguistic expressions could be the antecedent of knowledge of the order of numerical symbols (e.g., $6 < 10$), this suggests that knowledge of formal relations between numbers could be constructed out of representations of cardinalities.

In sum, it seems that Rips et al.'s solution to the challenge they have posed is not the right one, because their natural numbers are not the natural ones – they are not the ones that came to mind first to inventors of numerical notations, nor are they the ones that children acquire first. Of course, I have not answered Rips et al.'s challenge; we still do not know how children could induce an infinite system from a finite learning set. Yet, I suggest that the consistent appearance of symbols for cardinalities prior to more formal systems is not an accident, but rather, reveals how the former is a scaffold for the construction of the latter. I think simple learning considerations argue for the same point. Indeed, if quantificational uses of numbers were not relevant to the acquisition of the natural numbers, how would children ever learn that symbols like the number words or Arabic digits are symbols for the natural numbers?

Early numerical representations and the natural numbers: Is there really a complete disconnect?

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Abstract: The proposal of Rips et al. is motivated by discontinuity and input claims. The discontinuity claim is that no continuity exists between early (nonverbal) numerical representations and natural number. The input claim is that particular experiences (e.g., cardinality-related talk and object-based activities) do not aid in natural number construction. We discuss reasons to doubt both claims in their strongest forms.

Rips et al. argue that the concept of natural number, which includes formal properties such as the successor function and commutativity, is not grounded in non-symbolic (nonverbal) numerical representations involving object files and internal magnitudes. Rather, the natural numbers are constructed "top-down" on the basis of innate constraints on processing (e.g., recursion) that lead to "math schemas," which encompass various formal properties. Although we agree with Rips et al. (and others) that nonverbal numerical representations alone will not allow for the construction of the concept of natural number, we disagree with two claims central to their proposal: (1) *the discontinuity claim* that there is no continuity between early numerical representations and natural number, and (2) *the input claim* that particular experiences (e.g., cardinality-related talk and object-based activities) do not support natural number construction.

The discontinuity claim. Although Rips et al. acknowledge that adults use magnitude representations on tasks such as numerical estimation and comparison, they argue that this does not provide evidence for continuity between internal magnitudes and natural number, as these tasks could engage the magnitude system alone and not abstract mathematical knowledge. Furthermore, they emphasize that magnitude representations do not serve as precursors to natural number because they do not instantiate key principles such as the successor function. However, evidence that abstract mathematical reasoning might be influenced by magnitude-related information would lend support to greater continuity. In fact, Landy and Goldstone (2007) have shown that adults' success in solving algebraic problems is influenced by the distances between symbols; people are better (and faster) at solving problems for which the spacing is consistent with the order of operations (see our Fig. 1). Algebraic problem-solving involves a learned system of ordered operations, and there is nothing about magnitude that reinforces these operations. And, yet, this type of abstract mathematical reasoning is clearly grounded in spatial-perceptual cues.

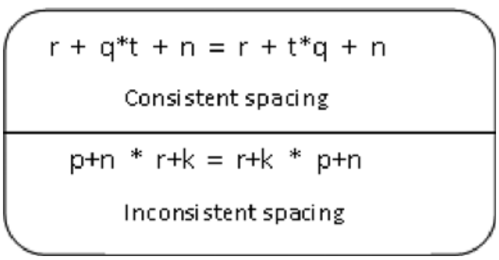

Figure 1. (Lourenco & Levine). Algebraic math problems with consistent and inconsistent spatial cues, as used in Landy and Goldstone (2007).

Other research highlights the predictive value of early numerical competence for particular natural number principles. It has been shown that children who have higher levels of mathematical knowledge at the start of preschool (when this knowledge is largely focused on objects) are those who show higher mathematical knowledge throughout the elementary school years (when the focus is on the numbers themselves) (Duncan et al. 2007; cf. Denton & West 2002). It has also been shown that children's ability to solve nonverbal addition and subtraction problems (for which physical objects are used) develops earlier than their ability to solve parallel symbolic calculations. Importantly, however, in support of continuity between early nonverbal numerical representations and natural number concepts, performance on nonverbal, object-based calculation tasks are highly correlated with performance on number fact problems ($r = .65$, $p < .001$) and word problems ($r = .63$, $p < .001$; Levine et al. 1992).

The input claim. Based largely on the view that there is discontinuity between early numerical representations and the concept of natural number, Rips et al. argue that input related to early representations does not support construction of the natural numbers. They point out that other cultures such as the Mundurukú and Pirahã have "natural language" (although, see Everett 2005) and yet do not have natural number. Although we would not suggest that linguistic experience per se allows for natural number construction, particular linguistic experiences, such as those involving the coordination of a count list with successively larger sets of objects, may have a significant impact (Carey 2004; Le Corre & Carey 2007).

Parent and teacher talk about number is focused mostly on counting and the cardinality of object sets (Klibanoff et al. 2006; Levine et al., under review; Suriyakham 2007). If such talk were irrelevant to the construction of natural number, the prediction would be that children who experience little of this input would still perform as well in math as children who experience a lot of it. Such a prediction is counterintuitive, and, more importantly, not supported by existing data (Ehrlich 2007; Klibanoff et al. 2006). Furthermore, the top-down construction of math schemas proposed by Rips et al. is consistent with the prediction that more abstract talk about natural number should lead to a better understanding of the formal properties of natural number than what they call "object talk." Would it really be more beneficial to talk to children about 3 being one more than 2 than about 3 dogs being one more dog than 2 dogs? This seems unlikely, especially since more abstract talk about number principles occurs when concrete objects serve to instantiate these principles (Mix 2008; Thompson 1994).

Cognition is not context-free, and even adults rely on supportive structures in the environment to enhance thinking and problem-solving. Moreover, in solving addition problems (e.g., $5 + 2 = 7$), preschool children employ a variety of strategies, including using their fingers to represent each addend (counting all of them), and more sophisticated strategies such as counting on from the larger addend (Siegler & Jenkins 1989). When addition problems are represented using fingers and other objects, the underlying principles can be made more concrete, allowing children to reflect on them by reversing the addition process and repeating it (Mix, in press). Moreover, object representations or "manipulatives" may lead to automatic retrieval of basic number facts, which, in turn, may lead to increased reliance on abstract numerical properties, rather than concrete objects, to solve these problems.

Summary. Rips et al. lay out a cogent, logical argument for discontinuity between early nonverbal numerical representations and the later concept of natural number, with all its formal properties. Although their argument is appealing, existing empirical findings are, in our opinion, not consistent with complete discontinuity. The associations in adults and children described above suggest magnitude- and object-based grounding of natural number knowledge. Furthermore, the studies of early numerical experiences suggest that object-based numerical input is predictive of (and perhaps causally related to) later mathematical achievements.

Specific and general underpinnings to number; parallel development

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Abstract: In this commentary, we outline an epistemological continuum between earlier and later number concepts, showing how empirical findings support the view that specific and general underpinnings to number develop in parallel in children; and we raise the question, based on cross-syndrome comparisons in infancy, as to whether exact or approximate number abilities underlie these later skills.

Post-Piagetian "top-down" approaches to number development require the definition of an epistemological continuum that relates, within the same framework, simpler to more complex mathematical schemas and concepts. The word *continuum* is key, as it brings together both "bottom-up" and top-down strands of research and enables the empirical testing of domain-specific and domain-general hypotheses about number development in the same groups of children. Rips et al. have not yet outlined a framework to relate their "simple" and "advanced" counting schemes, which limits their approach, but such an outline is possible using our epistemological continua framework.

Finding a conceptual continuum between Gelman and Gallistel's (1978) domain-specific number tasks such as counting or cardinality, on the one hand, and the Piagetian (1952) domain-general logical requirements, on the other, is not an easy task considering that their definitions of cardinality, for example, are different and conceptually asymmetric (Bryant 1994; Karmiloff-Smith 1992). Alternatively, however, it is possible to find conceptual compatibility between counting and the classic Piagetian (1952) task of class inclusion, provided that the latter is re-interpreted within a numerical context such as children's understanding of the structure of the numeration system (SNS, or the natural numbers). This is viable considering that the idea that any group of 10 units may be regrouped as one of the next unit (i.e., 10-ones become 1-ten, 10-tens become 1-hundred, and so on), is quite similar to the Piagetian schema of a hierarchical classification system of inclusive relations (Resnick 1983), in which the class containing only one element is included in the class containing two elements, which in turn is included in the class containing three, and so on (Piaget & Szeminska 1952).

The continuum between counting and knowledge of the SNS can also be analysed in terms of their logical invariants' similarities and differences. Whereas counting implies the use of units of the same denomination (or size, i.e., ones), mastery of the SNS implies the ability to count and combine units of the same and different denominations; that is, ones, tens, hundreds, and so on (Martins-Mourao & Cowan 1998; Nunes & Bryant 1996). Second, whereas counting only allows children to relate numbers sequentially, either as larger or smaller in terms of their immediate position in the number-line, knowledge of the SNS empowers them to interpret numerals as a composition of other numbers or as a composition of units of different denominations, long before any knowledge of written numbers (Martins-Mourao 2000; Nunes & Bryant 1996). Third, the idea that any numeral can be composed by the addition of any smaller number that came before it in the number system (e.g., "125" $= 100 + 10 + 10 + 1 + 1 + 1 + 1 + 1$) defines, from the child's point of view, a break with simpler concepts of the past, and a re-conceptualisation of number itself (Hiebert & Behr 1988). It is this notion, which is also equivalent to the part-whole schema specifying that any quantity can be divided into parts as long as the combined parts neither exceed nor fall short of the whole (Resnick 1983), that enables children to generate any number in the system without having to memorise the number-line in its entirety (i.e., Rips et al.'s advanced counting).

Hence, the definition of an epistemological continuum between domain-specific skills such as counting and domain-general knowledge such as the part-whole schema, is possible provided that it is framed within children's natural progression from counting small sets of objects (simple counting) to being able to generate any number in the system (advanced counting).

Our research suggests that both domain-specific and domain-general knowledge may develop in parallel almost simultaneously – instead of sequentially – between the ages of 3 and 5, although the latter may only be explicit to children much later (Karmiloff-Smith 1992). Evidence to support this argument must show that children have developed at least a simple version of the part-whole schema at a quite young age but not yet have learned all of the situations to which they may apply it successfully (Resnick 1983).

Instead of looking at knowledge of Gelman and Gallistel's (1978) counting principles, Martins-Mourao and Cowan (1998) examined children's developmental stages in counting ability and found important conceptual changes defined by the progression from the unbreakable chain level to the breakable chain level, between the ages of 3 and 5 (Fuson 1988). When they asked 4- to 6-year-olds, "What numbers come after 10?," those at the unbreakable chain level were unable to interrupt the number-line and had to count up from one (i.e., "1, . . ., 10, **11, 12, 13**!"), whereas those at the breakable chain level said "10, 11, 12, 13, . . ." thereby showing the ability to manipulate the number-line and judge 10 as a unit of different size, composed of 10 ones. Martins-Mourao and Cowan then asked the same group to provide an answer to an additive composition task that required them to pay for items in a shop. In a typical item, the child was given three 10p coins and six 1p coins to pay for an item costing 14p. Success in this task required decomposing the total amount to be paid into one unit of 10 and several ones, and then composing the quantity from units of different denominations (i.e., ones, tens, and hundreds). Supporting our argument, results showed that no child at any point passed the additive composition task without also being able to continue counting up from an arbitrary number in the list, which suggests that continuation of counting allows the child to establish simpler part-whole relations much earlier than previously thought.

How can researchers trace the developmental pathways underpinning number development? Our cross-syndrome comparison of Down syndrome (DS) and Williams syndrome (WS) yielded interesting results about such pathways. Older children and adults with DS outstrip those with WS in all number tasks (Paterson et al. 2006). Although children with WS learn to count fluently, they have serious problems understanding cardinality, for instance (Ansari et al. 2003). Yet, in infancy WS children are as successful as healthy controls in discriminating changes in exact small-number displays, whereas DS infants fail (Paterson et al. 1999). However, the same WS infants fail to discriminate approximate large-number displays differing in ratio ($1:2/2:3$) (Van Herwegen et al., in press). Given all the subsequent number problems experienced by older children with WS, this suggests that approximate number abilities may be a more important underpinning to subsequent conceptual number development than early discreet number abilities.

In sum, to understand the development of number abilities, we need to consider full developmental trajectories of both domain-specific and domain-general abilities that develop in parallel.

The origins of number: Getting developmental

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Abstract: Rips et al. raise important questions about the relation between infant quantification and achievement of natural number concepts. However, they may be oversimplifying the interactions that characterize actual development in real time. Though they propose a worthwhile agenda for future research, its explanatory power will be limited if it does not address developmental issues with greater sensitivity.

Rips et al. bring a fresh perspective to the study of numerical development, and their points are well taken. In particular, they make a thorough and compelling case against the possibility of a direct transition from infant quantification to natural number concepts, proposing instead that these concepts are induced from verbal counting. This is a worthy proposal, indeed, and one that merits empirical study. However, in and of itself, it is unlikely to provide a complete account of the way number concepts are constructed. In short, achieving a genuinely developmental account is going to require greater sensitivity to developmental issues.

One of these issues is timing, and, more specifically, how inputs, contexts, and responses interact in developmental time. It can be tempting to oversimplify these relations to make the researcher's task more tractable. For example, we might ask, as many – including the present authors – have, whether language comes before or after concepts. The problem is that elements of both are in the mix, right from the beginning, and there is no clear guideline for determining when a child "has" either one (see Mix et al. 2005). It is true that modern theorists have given short shrift to the contribution of verbal counting. However, shifting the explanatory weight from nonverbal processes to words is just as unbalanced. Taking such a view obscures the subtle layering of experiences and insights that occur as development unfolds over real time (see also Mix 2002).

Second, developmentalists have found that cognitive change rarely reduces to a single cause. Instead, it emerges from within a complex, ever-shifting, multifactorial system (e.g., Smith & Thelen 2003). This means there is no single basis for number concepts. Advanced counting is no more *the* basis of number concepts than is experience matching object sets, object tracking, or estimates of continuous amount. The empirical fact is that children have access to all of this information and more. Moreover, by adulthood, all these inputs have been coordinated into an interrelated conceptual structure, suggesting that none of them winds up cognitively vestigial. Thus, although there may be a key role for advanced counting, it is not the only relevant input. The challenge is to explain how these various streams of information coalesce into stable behaviors. On a related note, it is not clear why researchers need to adopt one definition of number over another. Though it is true that recent research has emphasized number as category, developmentalists have traditionally defined number as the combination of class and seriation. Piaget, for example, studied both aspects extensively and focused on how children integrate the two rather than assigning precedence to one or the other.

The present argument is similarly overdrawn with respect to the competence-performance distinction. Its crux seems to be whether quantification in infancy constitutes "true number concepts," and if it does not, what does. The only way to evaluate this question is to assume we have some way of defining true number concepts; that there is a clear division between the advent of veritable number concepts and their illegitimate precursors. However, development is not like turning on a light: Children do not lack concepts one day, and then experience them full-blown the next. Instead, they acquire new insights in fits and starts, enjoying moments of competence in one context and then losing it in others, until finally their performance is stable across a range of situations (see Mix 2002; Sophian 1997; Thelen & Smith 1993). They may grasp various pieces of the concept at different times, such that partial understandings interact for years before children reliably exhibit what might be considered mature understanding (Mix 2002; Mix et al. 2005). How, then, can we say whether infants have true number concepts? And what does it profit us to do so? We can admit a role for what infants and young children seem to understand, along with a role for advanced counting, and lose nothing.

An analogy from motor development might be helpful. Children crawl before they can walk. Crawling is not walking, and it is unclear how a child could suddenly start walking with only a background in crawling. However, the experience of crawling is not developmentally irrelevant. It tells babies about navigation through space, the properties of various surfaces, the feeling of balance, and so forth – information they will need when they start to walk. Furthermore, there is a temporary bridge that helps many babies transition from crawling to walking, called cruising – moving while upright but using furniture for support. Again, cruising is not "true" walking, but it binds the previous experience with navigation (crawling) with the new skill to be learned (walking), via a self-generated scaffold. What if number development is the same? Maybe young children have a way to track objects and estimate continuous amounts. This is not the same as natural number, and, as Rips et al. contend, they may need advanced counting to get to there. But these precursors give children important information about one-one correspondence, ordinality, and equivalence – ideas they will need to make sense of conventional counting. What might bridge the two (i.e., nonverbal and verbal enumeration) is labeling the cardinality of small sets; abundant research tells us children can do so (see Mix et al. 2005). And researchers long ago proposed that the juxtaposition of labeling with counting was the mechanism for coordinating class and seriation, or cardinality and counting, in the verbal realm (see Klahr & Wallace 1976; Schaeffer et al. 1974). This critical insight, that a set of three is counted 1-2-3, seems necessary for making sense of advanced counting, but it can only be achieved via these precursor experiences.

In summary, Rips et al. raise important questions that require answers. However, these answers are likely to be more complex than the present analysis suggests.

Making numbers out of magnitudes

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Abstract: We argue that number principles may be learnable instead of innate, by suggesting that children acquire probabilistically true number concepts rather than algorithms. We also suggest that non-propositional representational formats (e.g., mental models) may implicitly provide information that supports the induction of numerical principles. Given probabilistically true number concepts, the problem of the acquisition of mathematical principles is eliminated.

Rips et al. state that the principles of higher-order mathematics cannot be acquired without sufficient innate knowledge (e.g., number principles). We suggest that this conclusion only arises given two assumptions about the acquisition of mathematics: (1) "natural numbers" are algorithms and (2) principles of mathematics are processed as propositional representations.

The authors suggest that early quantity representations "are not extendible by ordinary inductive learning to concepts of natural numbers" (target article, sect. 1, para. 2). Yet it is unclear exactly how natural numbers are cognitively instantiated. One possibility is that natural number concepts (NNC) are context-free

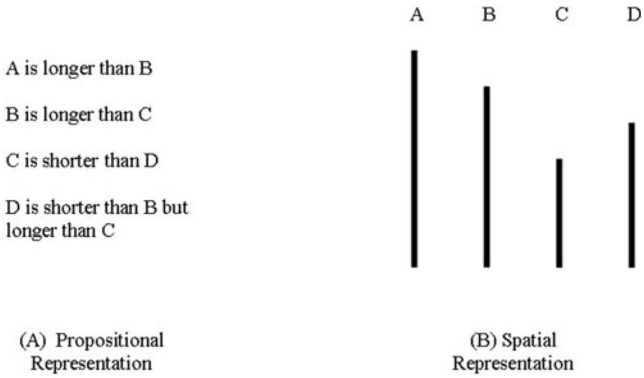

Figure 1. (Morris & Masnick) A comparison of propositional and model-based representations of the same transitive relation.

and algorithmic (i.e., applied appropriately, this concept will output a specific solution). Such concepts would not be learnable by induction and would lead to mathematics errors only when misapplied (demonstrating performance failure rather than lack of competence). This suggestion resembles Rips's (1994) suggestion in logical reasoning (i.e., Natural Logic), that logical inferences are drawn using innate algorithmic logical rules. Natural logic rules return a conclusion when matched to structure extracted from natural language (e.g., "If John mows the lawn, he will get 10 dollars" matches *P*, *Q*).

Returning to number, the authors state that understanding the Dedekind-Peano axioms is necessary for natural number and that children are unable to acquire them through experience (see sect. 5.3). This assumption, like natural logic, is an application of the poverty of stimulus argument, and rejects the possibility of learning a set of deterministic rules or concepts by observing specific examples. Stated differently, algorithmic concepts are only possible given a set of examples that exhaustively map the space of possible concepts; thus, induction can only yield probabilistically true concepts (Gold 1967). A second possibility is that NNC themselves are not algorithms. Thus, error may arise from conceptual inadequacy (i.e., lack of competence) or misapplication (i.e., performance). If true, then such concepts are learnable through induction, which undermines the argument for innate number knowledge.

Even if we grant that NNC are unlearnable, we contend that algorithmic concepts are unnecessary for mathematical processing. An analogy can be made between this characterization of mathematical concepts and nativist notions of grammatical rules. When the necessity of algorithmic rules is removed from the acquisition of language, it also eliminates the problem of the poverty of the stimulus (cf. Tomasello 2003). If the end point of language acquisition is not grammar algorithms, but an approximation of the language of other native speakers, then probabilistically true rules can be induced from input that should approximate natural language. Similarly, the assumption of algorithmic NNC seems unrealistic for human performance. Instead, concepts induced from positive evidence (i.e., probabilistically useful concepts) would provide roughly equal explanatory power as natural number algorithms in all but the most abstract cases.

A second, related assumption made by Rips et al. is that there is an unresolved problem in mapping magnitudes to the propositional representations necessary to support natural number inferences. This problem may only occur if it is necessary to use propositional representations for numerical inferences – an assumption which we argue is unnecessary. Humans and nonhuman animals use non-propositional representations of quantity to perform sophisticated computations. For example, rats and pigeons can track the number of events regardless of perceptual modality (Church & Meck 1984; Emmerton et al. 1997), and bumblebees and certain birds (e.g., juncos) track means and variance in foraging areas (Shafir et al. 1999; Waddington & Gottlieb 1990), presumably without using propositions.

Two viable candidates for potential non-propositional representations of natural number are distributed representations found in neural networks and mental models. Several of the authors' principles have been demonstrated to emerge from distributed representations. For example, the authors cite a neural network model that initially contained no mathematical principles and no specific number detectors, and that was able to develop the ability to compare numbers (Dehaene & Changeux 1993). This model demonstrated that experience with objects (not numbers) resulted in specific numerosity detectors. A second model, ESpaN (Grossberg & Repin 2003), accounts for single- and multiple-number effects by creating associations between spatial number representations (a simulated "Where" stream) with verbal categories for number (a simulated "What" stream). In both models, successor functions emerge from implicit representation in the spatial models mapped onto number functions.

Model-based representations also create different conditions for drawing inferences. Representing transitive relations demonstrate that different formats are related to different levels of processing complexity. Transitive relations represented as propositions require many successive inferential steps to draw a putative conclusion – perhaps more than a child could simultaneously hold in working memory. For example, one must represent the relationship between A and B, and the relationship between B and C, and then one must add an additional representation of the relationship between A and C (see Fig. 1A). If one posits a spatial representation of the premises, then this problem no longer holds (see Fig. 1B). Solving a transitive relation becomes immediately apparent simply by scanning the model of the premises, allowing a solution from direct perception (Morris & Schunn 2005). Model-based representations make transitive relations obvious because the relations between objects are conveyed in the models themselves, reducing the need for secondary inferential processing (Johnson-Laird 1983). Model-based representations of magnitudes provide implicit information about relations such as numerical succession, which would be helpful for inducing principles about ordinal – and perhaps even cardinal – relations (see Mix & Sandhofer 2007). Although the result of such inductions would be probabilistically true concepts, rather than natural number principles, further experience (e.g., language, formal schooling) would provide additional positive and negative evidence for their veracity.

The quantity system that represents magnitudes such as number and duration is well suited for representing "natural" instances in which quantity distinctions arise. It does not require propositions and may produce given effects directly (as in models) or in the emergence of structure through experience (as in neural networks). In this way, experience with magnitudes would likely produce highly consistent information (e.g., 4 is always larger than 3) that would allow the induction of probabilistic, but not algorithmic, concepts. If the requirement of algorithms is eliminated, instead allowing for probabilistic ally true concepts, the problem of the acquisition of mathematical principles is also eliminated.

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Don't throw the baby out with the math water: Why discounting the developmental foundations of early numeracy is premature and unnecessary

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Abstract: We see no grounds for insisting that, because the concept natural number is abstract, its foundations must be innate. It is possible to specify domain general learning processes that feed into more abstract concepts of numerical infinity. By neglecting the messiness of children's slow acquisition of arithmetical concepts, Rips et al. present an idealized, unnecessarily insular, view of number development.

As adults, we treat the concept of cardinal number as an endogenous support for children's engagement with mathematical reasoning. This is fine, as long we can then exogenously disengage children from treating number words solely as the answer to the question "How many?" on the grounds that reasoning about integers cannot be reduced to operations on set sizes (Haylock & Cockburn 2003). We read Rips et al. as endorsing a notion of disengagement of some part of the child's representational activity from physical numerousness as a prerequisite to representing natural number within an abstract generative system that reaches up to infinity. However, Rips et al.'s speculative proposal amounts to a non-engagement mechanism being endogenously provided. The appeal for an innate specification of natural number seems surprisingly premature given Rips et al.'s insistence that an account of number concepts must be developmental, but where the developmental inputs are yet to be specified.

We cannot see that Rips et al.'s radically innate proposal generates testable hypotheses, though we hope that it will, in the interests of adversarial testing. Still, Rips et al. assume that the two ways of looking through the lens – through a philosophical telescope towards infinity or through a developmental microscope at the earliest cognitive building blocks – have been resolved from the perspective of mathematics. We, however, would prefer to hold off, pending an evaluation of an orderly set of modelings that vary in the richness of initial-state representations (as propounded by Margolis & Lawrence 2008) and of subsequently reiterated representational operations on them (as propounded by Karmiloff-Smith 1992). If magnitude projection entails the extension of a number concept onto external items (e.g., the fingers on my hand) to yield the cardinal "five," then the proposed disengagement looks like yielding an analysis of the abstract structure that got me to "five."

On the other hand, there is every reason to welcome Rips et al.'s focus on reasoning with – and about – numerical concepts. Disengaging mental representations of number from the real-world objects that provide numerical input is a start, but one must also be in a position to operate over such representations. Consider the following test concerning the successor function. If a child watches a puppet miscounting five items, "1, 2, 3, 4, 6," and if he or she knows that (a) the puppet thinks there are 6, and that (b) there are really 5, the child is starting to disengage one of the numerical representations of this event from the veridical how-many representation. A child who answers correctly multiple variations of these questions (i.e., where the miscounts and the discrepancy between factual and counterfactual representations of cardinality vary), conceivably appreciates the precise way in which the errors violate the unique successor function (Freeman et al. 2000).

One naturally wants to look for symptoms of children spontaneously disengaging themselves from the numerical representation of plurality of objects, towards a concept of natural number as specified by Rips et al., and there is certainly a paucity of evidence within psychology. As Rips et al. astutely point out, counting ability is not a good predictor of this. We can add that neither is it likely to be (Muldoon et al. 2003). But insight does appear to be associated with an emerging ability to reflect on how the accuracy of the counting procedure both identifies ordinal relations and determines the validity of the cardinal representation of the set (Muldoon et al. 2003). It is from this perspective that we start to get a picture of how discourse provides domain general endogenous input for the mental activity necessary to "disengage."

Age-appropriate tests can also be amended to link in with set theory's insights into the relational nature of sets. A weakness in Rips et al.'s position is that they appear to be concerned primarily with the structure of a single infinite sequence. But number has a relational aspect to sequence that is equally important, and an appreciation of this rests on the criterion that is rightly identified as second-order mathematical induction. Preschoolers' engagement with the logical necessity of number is evident when they share objects out using a "one-for-you, one-for-me" procedure to produce sets in 1-1 correspondence and show some understanding that by counting one set, the number of objects in the other set can be inferred. Crucially, children as young as 4 years are sensitive to (1) the fact that numerical comparisons between two or more counted sets rest on adherence to the rules governing within-set enumeration (i.e., a unique first term and a unique successor); and (2) how the logic underpinning the ordinal relations between sequential count terms is tied to the concept of object-to-object correspondence (Muldoon et al. 2005; 2007). As with their appreciation of the successor function, this is not associated with procedural mastery of counting or sharing – although these are probably necessary skills – but with the ability to reason appropriately about numerically relevant versus irrelevant inputs.

In the final analysis, the presence of any predictors of number development begs the question of whether prompting children to reason about numerical data feeds into, or draws from, a schema for recursion and its extension to problems of numerical relationships. This is the key question that needs answering. There is evidence that such reasoning does not emerge spontaneously (Muldoon et al. 2007), and the processes of learning – particularly in social settings – appear to us equally valid candidates for schema input as the innate representations that Rips et al. propose. Just because psychology has not revealed the origins of the natural number concept in its work to date, does not mean yet that we have to abandon a bottom-up approach in favor of a top-down one. We do not have to turn the telescope round and look through the smaller end towards infinity; we need to train the developmental microscope on the right target and carefully adjust the focus.

The innate schema of natural numbers does not explain historical, cultural, and developmental differences

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Abstract: Rips et al.'s proposition cannot account for the facts that (1) a historical look at the word number systems suggests that the concept of natural numbers has been progressively elaborated; (2) people from cultures without an elaborate counting system do not master the concept of natural numbers; (3) children take time to master natural numbers; and (4) the competing advantage of the postulated math schema in the natural selection process is not obvious.

As Rips et al. have rightly underlined, humans are not only able to approximate numerosities as many other animal species do, but they also have the specific possibility to develop the concept of natural numbers. For Rips et al., natural numbers correspond to sequences that have a unique first element, a unique immediate successor for each element in the sequence, and a unique immediate predecessor for all but the first element. For them, this type of information would be included in an innate math schema. Yet, this theoretical proposition cannot account for different facts.

First, it cannot account for the fact that the construction of word number systems in all cultures is a very slow process (Hurford 1987) that shows evidences of a progressive elaboration with common stages across cultures. For instance, in most languages, small numerosities are expressed by numerals that do not express, in their morphological structure, a successor function (e.g., two is not expressed as "one unit after one") and do not correspond to the name of fingers. This suggests that the first numerals have probably not been immediately used as items of an ordered series characterized by a successor principle, but as independent lexical units corresponding to the immediate perception of small numerosities (the subitizing range). The schema proposed by Rips et al. does not predict such a distinction between the first three number words and the other ones. Moreover, when a language possesses a complete system of numerals organized in series, a regular structure is observed: a lexical segment of variable length and then an additive structure that ends at some point at which a multiplicative structure appears. Interestingly, when a multiplicative structure appears, it modifies the way of representing the additive structure just at that point in the series. For example, in many languages, the first way to indicate numerosities beyond the base 10 is the creation of particular words reflecting, in their structure, an additive relation (for instance, "third-teen" in thirteen). But this way of progressing in the structure will be modified just where the multiplication structure appears: after that point the additive structure is created by adding the unit to the decade word (e.g., twenty-two). These irregularities indicate that, in the past, individuals have introduced an additive structure without being aware that many years later, it would be necessary to introduce a multiplicative structure. If we had an innate schema of natural numbers, why would it take so long and be so chaotic to build a well-structured numeral system? Hence, this description of the different numeral systems indicates that the conquest of natural numbers with their complete formal characteristics has been a long and effortful story and not simply the expression of an innate schema.

Second, Rips et al.'s proposition is also challenged by the transversal study of different cultures. Indeed, individuals who are living in societies in which the development of a counting system is still at the initial stages show similar abilities as people in Western and other modern societies in tasks requiring to approximate numerosities, but no signs of a true mastery of the concept of natural numbers (see the studies of the Mundurukù by Pica et al. [2004] and those of Pirahà by Gordon [2004]). For instance, when asked how many dots are presented, the Mundurukù do not use the few numerals of their lexicon in a counting sequence. Faced with five dots, they produce the word "five" only in 28% of the trials. This same numeral is also used when 6, 7, 8, or 9 dots are presented. Similarly, in simple subtraction, they might select 2 dots as the answer for 6–3 dots. In the same way, the Pirahà can be precise when asked to select the same number of batteries as presented in a model if it involves less than 3 batteries, but beyond that point, their answer is approximate. Accordingly, how can we ascertain whether these people know that each number has only one immediate successor and one immediate predecessor if the label connected to a specific numerosity is variable or if the construction of a collection equivalent to another one is only approximate? In another study of a group from Papua New Guinea, Wassmann and Dasen (1994) compared the addition skills of the different members of the Yupno communities. The Yupno count neither days, nor people. To them, counting is only meaningful for the exchange of the bride price. The old men know the traditional body-part counting system and can use it for solving simple additions. The children go to math classes, where they learn the Tok Pisin number system (close to English) and use it to calculate. Yet, the young men who have not been to school know neither the Tok Pisin, nor the old body-counting system, and cannot solve any addition. Thus, here again, natural numbers seem only accessible to the people of the community who, given their specific environment, have been exposed to, and can master, a counting system.

Third, if we look at the development of math abilities in children who are raised in cultures in which numbers are meaningful and an elaborate counting system exists, it appears that it takes them time to grasp the meaning of natural numbers. If we all had an innate schema of natural numbers, why does it take so long, even after the learning of the counting string, to grasp the meaning of natural numbers? For Rips et al., this delay is explained by the fact that, in addition to this innate schema, children would need to be exposed to the "key information." However, what *is* this key information? Why and how would the exposure to this key information be crucial for the development of that math schema? If the failure to be exposed to this key information explains the very limited number comprehension of Mundurukùs or Pirahàs, what is that key information, if not a number system developed and shared by people in that culture?

In summary, the innate schema of natural numbers postulated by Rips et al. does not account for a large body of historical, cultural, and developmental observations. This model, based on the Dedekind and Peano axioms of natural numbers, provides only a description of the mature understanding of natural numbers; it is not helpful in explaining the phylogenetic and ontogenetic construction of natural numbers. Moreover, from the natural selection viewpoint, the competing advantage of such an innate model is not obvious.

Proto-numerosities and concepts of number: Biologically plausible and culturally mediated top-down mathematical schemas

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Abstract: Early quantitative skills cannot be directly extended to provide the richness, precision, and sophistication of the concept of natural number. These skills must interact with top-down mathematical schemas, which can be explained by bodily grounded everyday mechanisms for abstraction and imagination (e.g., conceptual metaphor, blending) that are both biologically plausible and culturally shaped (established beyond the child's mind).

There is a widespread belief that by studying the basis of the "counting" numbers we learn about mathematics itself (Butterworth 1999; Dehaene 2002). Many experimentalists in child psychology and number neuroscience think that the concept of natural number is bootstrapped from early quantitative skills such as estimating magnitudes and enumerating. These basic skills, however, cannot be directly extended to provide the richness, precision, and sophistication of the concept of natural number, let alone that of more complex mathematical concepts. In their target article, Rips et al. lucidly explain this point and conclude that children construct the concept of natural number and arithmetic relying on top-down processes and by constructing "mathematical schemas." I agree. However, between facts and conclusion – as in good *panini* – the interesting stuff is in the middle: Where do these schemas come from?

A major problem in explaining the "acquisition" of the concept of number is that scholars often introduce crucial elements of the *explanans* in the *explanandum* (e.g., taking number systems as pregiven). Gallistel et al. (2006), for instance, speak of "mental magnitudes" referring to a "real number system in the brain" (p. 247); the very real numbers are taken for granted. The system of (*infinitely* precise) real numbers is an extremely sophisticated concept, shaped over centuries with technical notions such as *completed order field* and the *least upper bound axiom*. How could such a system be simply "in the brain"? For the purposes of a biological brain dealing with magnitudes in the real world, the dense *ordered field* of rational numbers – with infinitely many rationals between any two rationals – would suffice. But, again, rational numbers cannot be taken for granted, either. The point is that in explaining the acquisition of the concept of number, not even the natural numbers and their properties can be taken for granted. Rips et al. are aware of this, and go on to propose a top-down approach based on mathematical schemas. Their characterization of schema, however, is quite abstract and generic, leaving unanswered questions such as: (1) What known cognitive mechanisms make such schematic abstractions possible? (2) What is the biological plausibility of these schemas? (3) Can top-down constraints established beyond the child's mind shape the consolidation of such schemas? If yes, how?

Embodied cognition and cognitive semantics have been concerned with questions of precisely this kind. Regarding mathematical concepts, George Lakoff and I have suggested that mathematical abstractions and idealizations such as Rips et al.'s mathematical schemas can be investigated through everyday cognitive mechanisms for human imagination such as conceptual metaphor and blending (Fauconnier & Turner 2002), among others (Lakoff & Núñez 2000). Not only do these mechanisms sustain human imagination (Q. 1) (Lakoff 1993; Núñez 2006), but they are also biologically plausible (Q. 2) (Gallese & Lakoff 2005; Núñez et al. 2007). Moreover, they can be culturally shaped (Núñez & Sweetser 2006) with their inferential organization established beyond the individual proper (Q. 3). Rips et al. are right that the concept of natural number cannot be, via empirical induction alone, a mere extension of everyday object manipulation. However, their dismissal of conceptual mapping theory is based on an incomplete and superficial understanding of it. We suggest that the laws of arithmetic for natural numbers (e.g., additive commutativity) come not just from one empirical mapping involving object manipulation (i.e., Arithmetic is Object Collection, as in *gathering two and three balls to get five*), but rather, from the isomorphic structure of four different source domains of primary experiences, one of which does not deal with objects but with motion along a path (as in *taking two steps and then three to land five steps away*; Lakoff & Núñez 2000, pp. 71–80). Such isomorphism provides structural correspondences across the source domains of four different grounding metaphors, yielding equivalent numerical results. This understanding is not about a mere phrasing of operations on physical objects, it is about the abstracted numbers. Moreover, from a neural perspective, the conflation of these isomorphic primary experiences presumably involves coactivations of brain areas that sustain those experiences, resulting in relevant neural links. Finally, this approach explains the precise and ubiquitous nature (even in technical domains) of numerical metaphorical expressions such as "greater" than, "smaller" than, "between" π and $-\pi$, and so on. Rips et al.'s mathematical schema does not.

Regarding conceptual mappings, Rips et al. rightly ask why selected inferences are more convincing than potential competing ones (i.e., the "No-competing-inference test for psychological explanations"; see sect. 3). Arguing that our theory fails this test, they miss that this concern is indeed central in conceptual mapping theory; and in mathematics, it is this very issue that shows how the field is largely driven by top-down processes. Consider an essential property of natural, rational, and real numbers: order. Then, like Tartaglia in sixteenth-century Italy, you bump into $\sqrt{-1}$, which, being a nonzero entity, is neither greater nor smaller than 0. $\sqrt{-1}$ simply fails the usual order test. Two competing inferences stand out: $\sqrt{-1}$ is not a number, because it violates order, or it is a number, but "number" must be redefined by giving up order. Both inferences are perfectly viable, but it is the latter one that over centuries the mathematical community has sanctioned as the more desirable one. Clashing inferences leading to competing bodies of knowledge permeates the history of mathematics, from post-Pythagorean mathematics with irrational numbers (Lakoff & Núñez 2000, p. 71) to Georg Cantor (but not Galileo) creating transfinite numbers (Núñez, in press). Often the mathematical community ends up adopting one inferential avenue and burying the other one, but this is not always the case. Competing-inference cases can also coexist. There are internally consistent but mutually inconsistent set theories, including one, for instance, that rules out self-membership (axiom of foundation) and another – hypersets, which allows self-membership (anti-foundation axiom; Lakoff & Núñez 2000; Núñez 2008).

Looking at how children "acquire" the concept of integers (natural numbers including zero and their negatives) with addition and multiplication may be informative here. The usual "negative times negative yields positive" rule cannot be derived either from early number skills or from direct mappings of physical experience. It is a matter of top-down mechanisms that impose precise, previously sanctioned inferential structure (which usually is taught dogmatically). Rarely explained in textbooks, this rule is a convention sanctioned by (adult) mathematicians – not children – who saw the fruitfulness of extending the distributive property of multiplication over addition to negative numbers. For the developing child who had no say in the sanctioning, the rule seems arbitrary. Similar, although more subtle, top-down dynamics must be accounted for in explaining the child's "acquisition" of the concept of natural number.

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Natural number concepts: No derivation without formalization

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Abstract: The conceptual building blocks suggested by developmental psychologists may yet play a role in how the human learner arrives at an understanding of natural number. The proposal of Rips et al. faces a challenge, yet to be met, faced by all developmental proposals: to describe the logical space in which learners ever acquire new concepts.

Rips et al. rely on a notion of "deriving" concepts, as when they say that, "understanding the natural number concept may allow us to avoid trying to derive it from unwieldy raw material from which no such derivation is possible" (target article, sect. 6, para. 3). But, *prima facie*, derivation requires both (1) starting points, and (2) something like a logic that permits introduction of defined notions. The possibilities for licensing particular derivations may be limited by either (1) or (2). It is a familiar point that a chain of reasoning not licensed by one system (e.g., a propositional calculus) may be licensed by another (e.g., a predicate calculus). Analogous points are relevant when reflecting on the capacity of human learners to "derive" the concept of natural number from the starting points suggested by developmental psychologists (object tracking, approximate number representations, etc.). Rips et al. suggest, in effect, that these "premises" are inadequate. What concerns us is the need of a plausible description of the relevant "background logic" that is somehow implemented by human psychology.

Our point is not merely that Rips et al. do not provide such a description, but that (so far as we know) nobody knows how to characterize the logical space of cognitive change – be it human concept acquisition or animal learning. In one sense, delineating this space is a major focus of ongoing work throughout cognitive science (e.g., debates concerning eliminative connectionism). However, not even optimists can think we have a remotely adequate description of the mental logic governing potential derivations of new concepts from available cognitive resources. Of course, in the absence of an understood mental logic, it is especially hard to know whether the resources currently posited by developmental psychologists are adequate starting points for human learners.

At the beginning of their "Concluding comments" (sect. 6), Rips et al say that "Thanks to analytic work by Dedekind (1888/1963), Frege (1884/1974), and others, we have a firm idea about the constituents of the natural number concept." The target article ends with the suggestion referred to earlier, that "understanding the natural number concept may allow us to avoid trying to derive it from unwieldy raw material from which no such derivation is possible." These statements may be too bold (current work in philosophical logic suggests disagreement), and a remark about the analytic work may be informative.

Given a consistent fragment of Frege's second-order logic, the Dedekind axioms for arithmetic follow from (i) the general principle that (the extensions of) two concepts are equinumerous if and only if they correspond one-to-one, and (ii) suitable definitions, in notation made available by the logic, of notions such as "zero" and "precedes." In this restricted sense, arithmetic follows from a single non-logical principle concerning an equivalence of equinumerosity and one-to-one correspondence (see Demopoulos 1994; Zalta 2003). Trivially, however, there can be no derivation of Dedekind's axioms without a proof system and a way of encoding the requisite definitions. Correspondingly, much of Frege's accomplishment lies with his invention of a logic in which proofs by mathematical induction could be conducted, and the axioms could be explicitly represented. The background logic is needed to define/abstract Frege's official concept of natural number. And this is no surprise: Definition requires a language; abstraction requires apparatus.

In terms of cognition, there is an analogous point. It is not yet useful to be told that certain "raw materials" are not themselves "rich enough" to support abstraction of numerical concepts. No starting points are rich enough without the logical tools that enable abstraction. The interesting claim in this vicinity is presumably that the "raw materials" discussed in the article are not rich enough, even given plausible assumptions about the cognitive apparatus independently available for purposes of abstraction. But defending this claim would require an argument – not provided by Rips et al. – concerning the relevant cognitive apparatus, about which very little is known. Our claim is not that psychologists should posit unconscious knowledge of points (i) and (ii) referred to earlier and of Frege's logic; we have no firm views on this score. But because understanding a natural language plausibly requires derivational capacities of the kind reflected in Frege's logic, it seems rash to assume that humans do not have capacities that would let us derive natural arithmetic from analogs of (i) and (ii).

As Frege also stressed, although derivations can reveal logical relations among concepts, this does not yet tell us anything about the actual *constituents* (if any) of the relevant concepts. However, one can speculate that human learners have the representational powers required for reasoning along Fregean lines. This might involve the abilities to: determine equinumerosity via computing one-to-one correspondence (as Rips et al. note, infants do); represent an initial number, which need not be zero, in terms of prior notions (perhaps a singular/plural distinction); represent a relation of "precedence" that has a definable transitive closure; and manipulate these representations in ways corresponding to a consistent but suitably powerful fragment of Frege's (second-order) logic. In this way, one might abstract a Fregean notion of natural number and go on to derive natural analogs of the Dedekind/Peano axioms. We do not suggest that this must be the way human children acquire their concept of natural number. But such scenarios do not strike us as impossible or even especially implausible, pace Rips et al., given the alternatives.

To repeat, our point is neither that Rips et al. have it wrong, nor that the developmental psychologists have it right. But we want to caution against concluding, in the absence of empirically motivated assumptions about the abstractive/logical capacities of children, that the "raw materials" currently posited by the psychologists are inadequate starting points. One can agree that derivations are not yet to be had, while welcoming new premises.

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Learning natural numbers is conceptually different than learning counting numbers

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Abstract: How children learn number concepts reflects the conceptual and logical distinction between counting numbers, based on a same-size concept for collections of objects, and natural numbers, constructed as an algebra defined by the Peano axioms for arithmetic. Cross-cultural research illustrates the cultural specificity of counting number systems, and hence the cultural context must be taken into account.

> *Natural numbers are objects of arithmetic, but . . . natural numbers may not be objects in the original background language from which we began.*
>
> — Stewart Shapiro 1997, p. 126

Children learn about numbers in the cultural context of a number system as this is conceptualized by adults. For the latter, Rips et al. comment, "Activities such as estimating the number of objects in a collection . . . may proceed without drawing on natural number concepts. Number concepts may come into play only at a more abstract level – for example, in arithmetic" (sect. 6, para. 3). In brief, there are different logics for counting versus natural numbers, despite overlap in number names. In pre-literate societies, counting number systems are used for enumeration, but seldom for computation (Hallpike 1979); only with the natural numbers do we have a conceptual structure for computation. Even counting number systems are non-isomorphic across different cultures.

A counting number is based on two primitive concepts: (1) individuation, whereby an object can be distinguished from the collection of objects to which it belongs, and (2) matching, whereby an individuated object in a particular collection can be matched with an individuated object in another collection. The logic of matching determines an equivalence relation – call it "same size" – over an ensemble, E, of object collections as follows. For collections A and B in E, match an individuated object from A with an individuated object from B and then remove these individuated objects from their respective collections. Next, recursively apply this matching procedure to the modified collections until either A or B has no more objects. If both collections are emptied simultaneously, the collections A and B have the same size. Same size exhaustively partitions *E* into disjoint subensembles, each of which contains all (and only) collections of the same size, since it is an equivalence relation.

Next, identify a fixed reference collection – say, the twiggle collection. Let the counting number TWIGGLE be the name for the subensemble of *E* (if any) containing a collection of the same size as the twiggle collection. Apply the counting number TWIGGLE to any collection in this subensemble. This labeling procedure is not determined by the collections making up *E*; hence, we may use the counting number, TWIGGLE, without reference to *E*.

The Iqwaye of Papua New Guinea (Mimica 1988) used this matching method for a counting number to represent the quantity of males needed to raid a neighboring group. A special string of shells was matched with the collection of males to determine whether they were enough for a raid. The Iqwaye used matching in this context despite having counting numbers for enumeration. Our quantity, dozen, is similarly just a counting number designating a quantity. Its distinction from the natural numbers is shown by the sentence "I want a half dozen cookies" versus the nonsense sentence "I want a half 12 cookies." The natural number, 12, only provides the reference size for defining the counting number, dozen.

These examples illustrate that sequentiality is not necessary when defining counting numbers. Neither the string of shells nor dozen involves a sequence of counting numbers. Sequentiality derives from the between concept ("Is there a collection with size between the counting numbers X and Y?"). Recursive use of between stops with sequential counting numbers, beginning with the counting number "one," defined as the name for the size of a collection with a single, individuated object.

The logic of sequentiality leads to finite collections of counting numbers typified by "one, two, three, many" counting systems, where many has the meaning "a collection size larger than any counting number." Lest we think that counting systems of this kind are impoverished natural number systems, we need only consider the word "infinite" in Alfred Tennyson's *Timbuctoo*: "Where are the infinite ways." Here infinity refers to a size beyond the counting numbers, not the cardinality of the natural numbers. Thus, in our "one, two, three ... infinity" counting system, common usage of infinity is equivalent to many, and we only have finitely many (named) counting numbers.

Just as the logic underlying a counting number does not entail sequentiality, the logic underlying sequentiality does not entail the incorporation of a successor function, $s\ (n) = n + 1$, applied invariantly to all counting numbers. Consider the Paiela (Papua New Guinea) counting numbers, which are as follows:

> The entire count comprises 28 numbers, a compound of two units of 14 ... The count begins on the small finger ... The count then "ascends" to the thumb, the wrist, the shoulder, the head, stopping at the nose [the author only lists some of the body parts used in this sequence]... A second unit of 14 is traced [in reverse] on the other side of the body ... finishing on the small finger of the opposite hand for the 27th count. The hands are then clenched and brought together ... to signal the completion of a second 14-count unit. (Biersack 1982, p. 813)

The number names identify sequential body part pairs, with a modifier added to designate when the pair refers to the other side of the body. Thus, "one" is "pair of little fingers" and "twenty-seven" is "pair of little fingers on the other side." We can interpret the number names of the Paiela as referring to 1st pair, 2nd pair, and so on, up to the 13th pair; that is, they count sequentially by two's rather than by one's. Note that "fourteen" is "nose" even though we have a pair of nostrils, and "twenty-eight" is "clenched hands" even though we have a pair of hands. Their choices for these numbers show that the Paiela counting system is conceptually closed and not the open system entailed by the successor function.

I suggest that children initially learn the primitive concepts for the counting numbers through experience. Alone, this leads neither to a particular counting number system, nor to the natural numbers. For these, the developing child has to work out patterning in number names – and patterning in computations with the natural numbers for cultures in which the logic of the natural numbers is also part of the adult's cultural repertoire – through the child's interaction with culture-bearing adults. A child goes beyond experientially obtained primitive concepts through enculturation, the process by which cultural knowledge (in the sense of systems of ideas and concepts) is transmitted from one generation to the next.

SEVEN does not mean NATURAL NUMBER, and children know more than you think

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Abstract: Rips et al.'s critique is misplaced when it faults the induction model for not explaining the acquisition of meta-numerical knowledge: This is something the model was never meant to explain. More importantly, the critique underestimates what children know, and what they have achieved, when they learn the cardinal meanings of the number words "one" through "nine."

This commentary pertains to section 3 of the target article, where Rips et al. criticize the induction model. I argue that Rips et al.'s critique fails in two ways. First, it overstates what the induction is supposed to explain. The induction is about the acquisition of natural-number concepts, not the concept NATURAL NUMBER itself. Second, Rips et al. fail to grasp the magnitude of children's achievement in learning the cardinal meanings of a finite set of number words (e.g., the words "one" through "nine").

Forget NATURAL NUMBER, the induction is about SEVEN. The first issue to be addressed is one of terminology. There is a difference between natural-number concepts (e.g., SEVEN) and the concept NATURAL NUMBER itself. SEVEN is a mental symbol for a specific cardinality; NATURAL NUMBER is part of meta-numerical knowledge – knowledge about numbers.

When Rips et al. use the term "natural-number concepts" interchangeably with the terms "*the* natural-number concept" and "the concept NATURAL NUMBER," they blur the distinction between number concepts and meta-numerical knowledge. Similarly, when they argue that the induction model is wrong because it does not explain how children learn that numbers go on forever, or that numbers can be added in any order, what they are really saying is that the induction model does not explain the acquisition of meta-numerical knowledge.

However, the induction model was never meant to explain the acquisition of meta-numerical knowledge. The induction model explains how children acquire natural-number concepts: mental symbols for cardinalities such as 7. Specifically, we are interested in cardinalities that are (a) bigger than 4 (and thus too big for each individual to be represented in parallel), and (b) exact (and thus too fine-grained to be represented by the analog magnitude number system).

The successor rule – no more, no less. Rips et al.'s conflation of numbers with meta-numerical concepts leads them to misunderstand the claims of the induction model. For example, Rips et al. read $s(n) = n+1$ as a function over the natural numbers, and argue that a child who has the concept $s(n) = n+1$ must *ipso facto* have the concept NATURAL NUMBER. Partly this is a problem with notation: We (induction theorists) have used $s(n) = n+1$ as shorthand for something much wordier. The following is what I think the child really learns.

Precursors: The child first learns the words for those cardinalities that are small enough to be represented directly, as sets of individuals. (These are learned one at a time, in order, over a period of 1–2 years.).

"one" means 1;
"two" means 2;
"three" means 3;
"four" means 4.

Induction: Then, all at once, the child learns how to assign cardinal meanings to the rest of the number words. To continue with Rips et al.'s example, a child who knows the list up to "nine" induces the following:

"five" means 1 more than 4;
"six" means 1 more than "five";
"seven" means 1 more than "six";
"eight" means 1 more than "seven";
"nine" means 1 more than "eight."

The child does not need the concept NATURAL NUMBER in order to make this induction, and the successor rule itself is represented only implicitly. The key fact, however, is that the child learns the higher number-word meanings *all at once*. That's how we know the child is applying a rule.

Rips et al. really seem to miss the import of what the child achieves when he or she learns this rule, even for a finite set of number words. A child who can represent the meaning of "seven" via this rule has acquired a way of mentally representing an exact, large quantity. This is information that simply was not representable until a set of symbols (the number words) were borrowed from outside. This is the really amazing thing about number-concept development, and it happens very early on – when the child knows only a few number words.

Even "one"-knowers won't accept a modular system. Finally, why do Rips et al. believe that a child who has learned the cardinal meanings of "one" through "nine" doesn't have natural-number concepts? One big reason seems to be that (according to Rips et al.) this child's knowledge state is equally compatible with either the natural-number system, or an alternative system such as Successor-Mod-10. A closer look at children's behavior, however, shows us that this is not true.

The original basis for the induction model was the finding that, from the time children learn what "one" means, they treat the number words as unique labels for specific cardinalities (Wynn 1992b). This is the very definition of a "one"-knower – it is a child who, when asked for "one" object, always gives only 1 and never more than that. Similarly, when a "one"-knower is asked to label a set of 1, he or she always says "one" and never other number words.

These facts mean that modular systems are off the table as a possibility by the time the child learns "one." In order for $\text{Mod}_{\text{-}10}$ or any other modular system to work, children would have to accept either multiple number words for the same cardinality, or multiple cardinalities for the same number word. In fact, however, even "one"-knowers do not accept the word "eleven" for a set of 1, or the word "one" for a set of 11.

This basic finding has been replicated many times over, not only for "one"-knowers, but also for "two"-knowers, "three"-knowers, and "four"-knowers as well (e.g., Condry & Spelke 2008; Le Corre & Carey 2007; Le Corre et al. 2006; Sarnecka & Carey, in press; Sarnecka & Gelman 2004; Sarnecka et al. 2007). Moreover, one study (Sarnecka & Gelman 2004) found that children apply this constraint even to number words whose meanings they do not yet know: "Two"- and "three"-knowers treat "five" and "six" as mutually exclusive cardinalities, even though they cannot identify the actual cardinalities 5 or 6.

Thus, Rips et al.'s critique does not hold up well to scrutiny. Where it is true (e.g., in pointing out that the induction does not yield the concept NATURAL NUMBER), it is irrelevant. And where it is relevant (e.g., in the claim that children's early number knowledge is consistent with modular systems), it is false.

Mathematical induction and its formation during childhood

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Abstract: I support Rips et al.'s critique of psychology through (1) a complementary argument about the normative, modal, constitutive nature of mathematical principles. I add two reservations about their analysis of mathematical induction, arguing (2) for constructivism against their logicism as to its interpretation and formation in childhood (Smith 2002), and (3) for Piaget's account of reasons in rule learning.

Rips et al. are right about the crevasse, with psychological research on children's quantitative representations is on one side and principles of mathematics or the other. But if children's quantitative representations are mappings from number-words to numbers and their successors ("one" represents one, "two" represents two), this could lead to any of the infinite number sequences in mathematics: for example, mod_{983}. Rips et al.'s negative argument is that "children cannot bootstrap their way from these beginnings to true math concepts" (sect. 5, para. 1), where true math includes the concept of natural number. On the other side of the crevasse, principles in the philosophy of mathematics are available. But so is the stark warning – a child sitting by a heap of peas, picking them up one by one uttering a number-word, will have access to "no bridge which leads across from the kindergarten numbers" (Frege 1979, p. 276). Rips et al.'s positive argument is that the sole way to bridge the crevasse is by means of principles.

I applaud Rips et al.'s main argument, which I support through (1) a complementary account of the nature of mathematical principles. But I also have a couple of reservations. Their bridge specifically includes principles of mathematical induction (MI) with reference to my Piaget-inspired study (Smith 2002). My reservations are about (2) the authors' interpretation of MI, and (3) reasons in rule learning.

1. Support for the argument. Regarding the authors' discussion of math principles, first, the following characteristics are worth bearing in mind:

a. Mathematical principles are *normative*: "Mathematics forms a network of norms [in that] if calculation reveals a causal connexion to you, then you are not calculating" (Wittgenstein 1978, pp. 431, 425; cf. Piaget 2006, p. 8). Note that this distinction between causality and normativity is exclusive.

b. Mathematical principles are *modal* (i.e., necessary truths): Their instantiation in experience provides occasions for their recognition. Their correct understanding depends on normative demonstration, not merely verification (Leibniz 1996, p. 85; Piaget 2006, p. 7).

c. Mathematical principles are *constitutive*: Any principle has a two-part formation covering origins and constitution. Both are required because pseudo-rationality has an empirical origin without a normatively valid constitution (Kant 1787/1933, p. B116;

Piaget 2006, p. 9). For Kant, mathematical principles are not innate (see Kant 1790/2002, sects. 8–221). Instead, they are constructed, making their debut in experience with their derivation being synthetic *a priori* (Kant 1787/1933, pp. B1, B15, B747).

This trinity supports Rips et al.'s critique of research in the psychology of mathematics. Contra (a), that mathematical principles are normative, the track-record in this research shows multiple traces of norm-denial and norm-reduction to causes, but almost nothing of norm-laden minds (Smith 2006). Contra (b), that mathematical principles are modal, the focus of this research is on truth-functionality to the exclusion of modal necessitation (Smith 2002). Contra (c), that mathematical principles are constitutive, it deals with origins to the exclusion of constitution (Smith, in press a). That is why principles are required to bridge the crevasse from the other side.

2. Interpreting mathematical induction (MI). Rips et al.'s positive recommendations specifically refer to MI principles, in relation to which they note my (Smith 2002) study in section 5.3.3. However, their discussion is open to challenge in two respects:

First, Rips et al.'s definition of MI is a modernized restatement of Poincaré's definition. The open question is its interpretation. The authors interpret MI through logicism, in that MI is really logical deduction (their argument is in Rips & Asmuth [2007]). Yet Poincaré's contrary interpretation was for constructivism (Poincaré 1905, p. 15; 1952, p. 52), under which MI is really synthetic *a priori* in its generation of novelty (Poincaré 1952, pp. 50, 162, 194). Under logicism, logical deduction and empirical induction are necessitating and universalizing respectively; for Poincaré (1905, pp. 13–14), MI combines both in a unique form of reasoning. Crucially, Piaget (1942; see Smith 2002, Ch. 3.1) was critical of logicism, and his constructivism aimed to recast Poincaré's position, notably in Inhelder and Piaget's (1963) seminal study, otherwise neglected until my replication. That is why I (Smith 2002, p. 5) quoted Poincaré's definition with some sympathy for his interpretation (cf. Smith 1999). Rips et al. state that this definition conflates universal generalization [UG] and MI. I doubt this, but reckon they should take this matter up with Poincaré and Piaget.

Next, my (2002) study was about the formation of MI in children aged 5–7 years who were investigated twice. My operationalization required serial, equal additions to two containers – in Study I, both were initially empty; in Study II, their initial content was unequal. The questions were threefold: the base equality/inequality through serial additions, universality about number, and necessity about number (Smith 2002, p. 58). This operationalization matches Poincaré's and Rips et al.'s criteria for MI. My evidence (significant in non-parametric and logistic analyses) showed these children to be successful with the first two questions, and promising as to modality (pp. 64–66, 147). Rips et al. challenge this in view of the purported conflation of MI and UG. I disagree in seeing an early formation of MI in 5–7-year-olds. (More on this, below, with regard to the second reservation.)

3. Reasons and rule-learning. Principles are rules about which four assumptions are commonly made in psychology: (i) mathematical rules are clear and exact, (ii) their applications are well-defined, (iii) their learning is typically through social experiences that (iv) are causal mechanisms responsible for rule-learning. Rips et al. seem to be committed to this quartet, but all four are contradicted in Wittgenstein's (1978) analysis of the rule-following paradox. A companion argument from Piaget's developmental epistemology leads to the same conclusion: rule-meaning is dependent on rule-use, not the other way round (Smith, in press b). Further, rule-use is regulated by reasons whose role is "to introduce new necessities into systems where they were merely implicit or remained unacknowledged" (Piaget 2006, p. 8). Reasons have origins inclusive of pseudo-necessities; they may also amount to valid necessitations.

Either way, reasons are investigable as normative facts (Smith, in press a). Returning to Rips et al.'s strictures and using my (2002, Ch. 6.3.3) study, examples include:

pseudo-necessities: "more there because you have not put the right number in"; "because it wasn't fair at the start"

recursion over number: "because if there was the same in there before and you just add another one in each box, there would be the same again"

necessitation about number: "because that's the way it has to be. From the beginning I started adding more, so there must be more."

MI starts somewhere. Well, here is a primitive version in children aged 5–7 years. Isn't rule-learning a fallible process of construction for us all!

In conclusion, I think Rips et al. will welcome my reiteration of the normative, modal, and constitutive nature of mathematical principles as complementary to their critique of math psychology, and on reflection see promise in my two reservations as contributions to Piaget's developmental epistemology directed on an empirical account of normativity.

Precursors to number: Equivalence relations, less-than and greater-than relations, and units

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Abstract: Infants' knowledge need not have the same structure as the mature knowledge that develops from it. Fundamental to an understanding of number are concepts of equivalence and less-than and greater-than relations. These concepts, together with the concept of unit, are posited to be the starting points for the development of numerical knowledge.

The premise of the article by Rips et al. – that characterizations of initial and mature knowledge must be congruent enough that there is a viable developmental pathway linking the two – underscores the contribution a developmental perspective can make to cognitive science. The fact that there must be a developmental pathway between initial and mature knowledge, however, does not mean that initial knowledge must have the same basic structure as mature knowledge. The determination of what is the relevant precursor knowledge for a given body of mature knowledge is therefore itself an important theoretical issue.

Rips et al. begin their inquiry with a consideration of the evidence of numerical discriminations that has emerged from research with infants. They question, however, whether generalizations derived from experience with physical collections provide an adequate basis for mature natural number knowledge. Instead, they propose that infants have schematic numerical knowledge that incorporates key structural properties of a mature concept of natural number. But the chief merit of this account appears to be that it simplifies the problem of explaining how children get from initial knowledge to mature knowledge by positing a close correspondence between the two. No evidence is cited to support the attribution to infants of the schematic knowledge posited by Rips et al.

An analysis of relations between number and other mathematical concepts that may be present early in life suggests an alternative way of thinking about the developmental foundations of natural number knowledge that better fits available evidence and that does not require the close correspondence between initial and mature knowledge that Rips et al. seem to take for granted. Davydov (1975) argued that, mathematically, the concept of number presupposes concepts of equivalence, and of less-than and greater-than relations, which are more fundamental than number in that they do not depend upon a concept of number (because they apply to unenumerated continuous quantities, such as lengths, as well as to numerical quantities), whereas a concept of number does entail concepts of equivalence and less-than and greater-than relations. This conceptual asymmetry in turn suggests that, developmentally, knowledge about number might build on ideas about equivalence and inequivalence relations that have their origins in comparisons among non-numerical quantities. Sophian (2007) reviewed a wide range of empirical evidence consistent with this idea; for example, evidence that infants are sensitive to less-than and greater-than relations among non-numerical quantities in reasoning about the support relation between an object and the surface on which it rests (e.g., Baillargeon et al. 1995), and evidence that 3-year-old children pay more attention to correspondence relations between sets than to specific numerical values in selecting pictures to match stories that contains both kinds of information (Sophian et al. 1995, Experiment 1).

What distinguishes numerical cognition from thinking about non-numerical magnitudes is the use of a unit. When we state the numerical value of a quantity, what we are actually doing is describing the relation between that quantity and some unit. The unit is often an individual object, however, conceptually what is important is not its objecthood but the fact that we can measure the quantity we are trying to enumerate against it. Whether we characterize a collection of shoes as six individual shoes or as three pairs of shoes, the numerical value we give essentially represents the multiplicative relation between our chosen unit and the total quantity (cf. Sophian 2007).

Although the notion of a discrete object appears to be an important part of how infants make sense of the world from a very early age (Spelke et al. 1992), it seems unlikely that infants start out with a general mathematical concept of unit or an understanding of the relation between units and number. Unfortunately, we know next to nothing about how children do acquire the concept of unit, beyond the fact that important aspects of the concept of unit are quite challenging throughout the preschool period and beyond (Gal'perin & Georgiev 1969; Shipley & Shepperson 1990; Sophian 2002; 2007). Instructionally, it is possible to introduce units as an extension of ideas about less-than and greater-than relations (e.g., as a way of characterizing how much greater one quantity is than another; cf., Dougherty et al. 2005). Although it is not clear how closely this corresponds to the way children's knowledge develops in the absence of such instruction, the viability of instruction in units that builds on ideas about less-than and greater-than relations between unenumerated quantities demonstrates that a developmental pathway from one to the other is at least possible.

An important argument that Rips et al. give for rejecting the idea that the concept of natural number grows out of experience with physical quantities is that physical quantities are always finite, whereas the sequence of natural numbers is infinite. This objection becomes less compelling, however, if we construe children's numerical knowledge as building on their understanding of actions that are potentially infinitely repeatable (such as iterating a unit), rather than on their observations of the quantities generated by those actions, which are of course always finite. Although physical action sequences, like physical collections, can never actually be infinite, the repeatability of actions does correspond to a key insight that children express about the infinity of the number sequence – that you can go on and on (cf. Monaghan 2001).

In sum, while there must always be a developmental pathway from initial knowledge to mature knowledge, there is no guarantee that that pathway is a direct one. Consideration of possible starting points for natural number knowledge other than specifically numerical abilities led to the suggestion that numerical knowledge may develop through the bringing together of initial knowledge about equivalence relations and less-than and greater-than relations with somewhat later-emerging knowledge about units. This account is consistent with several kinds of empirical evidence, and suggests potentially fruitful directions for further research as well.

Authors' Response

Dissonances in theories of number understanding

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Abstract: Traditional theories of how children learn the positive integers start from infants' abilities in detecting the quantity of physical objects. Our target article examined this view and found no plausible accounts of such development. Most of our commentators appear to agree that no adequate developmental theory is presently available, but they attempt to hold onto a role for early enumeration. Although some defend the traditional theories, others introduce new basic quantitative abilities, new methods of transformation, or new types of end states. A survey of these proposals, however, shows that they do not succeed in bridging the gap to knowledge of the integers. We suggest that a better theory depends on starting with primitives that are inherently structural and mathematical.

By sometime in early grade school, children have enough knowledge of the positive integers to perform simple arithmetic and to recognize properties that are true of the integers. Our target article looked at the path they take to get there. Infants and many nonhuman animals are sensitive to quantitative differences: They react differently to small versus large groups of elements. Children's caregivers build on this ability in teaching them to enumerate small groups of objects – to assign to these groups the count term in their native language that denotes the appropriate number: for example, two pizzas, five cups, four trees. Developmental psychologists have naturally taken this ability as the antecedent of the children's later knowledge of the integers: two, five, four. It seems a small step from being able to enumerate small collections to being able to add two such collections by counting over them. A further step enables children to appreciate the properties of integers that support arithmetic reasoning; for example, that adding two positive integers produces a new integer larger than the first two.

Our article was meant as a skeptical analysis of the idea that there is a direct path from early numeric abilities – the abilities that underlie children's appraisal of the size of collections – to their later understanding of the positive integers.[1] **Sophian** puts this point clearly: "characterizations of initial and mature knowledge must be congruent enough that there is a viable developmental pathway linking the two." We would add that the pathway must be not only causal or motivational but also constitutive. That is, the component must be part of an initial or intermediate representation that eventuates in the final number concept.

We learned that $7 \times 8 = 56$ in part because our grade-school teacher, Ms Foos, threatened, "No snacks until you memorize the 7 row of the multiplication table." But this threat did not become part of our concept of multiplication, even though we may still become desperately hungry whenever we have to do 7×8. In the target article we argued that there might be no such constitutive pathway from the current candidates for initial number knowledge, where these candidates include mental magnitudes, object files, and internal sets. Knowledge of the integers, we think, is more likely to be imposed top-down from a mathematical schema that embodies the integers' structure.

This is such a radical thesis that we felt fairly sure that when readers got to our claim that enumerating objects may have little to do with understanding numbers, they would throw away the article with a laugh and return to more pressing business. We are grateful to the commentators for sticking with us long enough to write their thoughtful remarks.

R1. An overview of the commentaries

Because our claim is that you cannot get from *Here* to *There*, it is vulnerable in a number of places. For one thing, we may have selected the wrong *Here*. Our Figure 1 summarized one possible starting point: A current picture of infants' ability to discriminate quantity, which included parallel individuation of objects, longer-term representations of objects, and magnitude representations. As **Chiang** points out, the Figure 1 model shows that we were vision chauvinists, since the model is mainly geared toward assessing the cardinality of groups of visually presented objects. It is missing processes that infants need in order to determine the number of tones or actions in temporal sequences (Note 4 of the article mentions other patches). Our purpose, however, was not to defend this model in detail but to use it as a plausible partial theory of infants' initial quantitative skills. We also considered other possible starting points, such as the ability to place objects in groups and to form abstract representations of such groups. We argued that none of these representations provides a good conceptual basis for knowledge of the integers, as it would be difficult or impossible for children to extend them to such a concept through ordinary learning mechanisms. Some of the commentators dig in their heels and defend mental magnitudes (**Barth**; **Lourenco & Levine**) or, at least, assessments of the cardinality of small collections (**Barner**; **Le Corre**) as part of the story of number concepts. **Gelman** defends arithmetic principles as starting points – a position more in line with our own. However, other commentators propose representations that we did not consider as bases for number development. Although internal magnitudes (for example) may not be a reasonable starting point for numbers, infants might project numbers from: memory traces that embed other memory traces (**Hintzman**); neurons tuned to specific cardinalities (**Fias & Verguts**); finger counting (**Andres, Di Luca, & Pesenti** [**Andres et al.**]); spatial positions (**Fischer & Mills**); representations of sequences (**Gardiner**); and representations of relations of equivalence, less than, and greater than (**Sophian**). We have summarized the commentators' proposals about starting points in Part A of Table R1, and we examine them more closely in Section R3.

A second point of logical vulnerability in our account is the transition from earlier to later numerical concepts. We are claiming that current theories have difficulty explaining this transition, but many commentators think that we adopt too limited a view of how kids get from *Here* to *There*. Part B of Table R1 includes those commentators who defend current views of the bridge to the positive integers and those who advance new types of connection.

Table R1. *A breakdown of the Commentaries according to their view of where revision or defense is needed in theories of acquisition of the positive integers*

Commentator	Innovation/Critique
A. Bases for Numerical Concepts	
Barth	Defense of magnitudes
Le Corre	Defense of cardinality/ magnitudes
Barner	Defense of cardinality
Lourenco & Levine	Defense of cardinality
Chiang	Magnitudes for small arrays
Gelman	Counting and arithmetic principles
Gardiner	Sequences
Sophian	Relations of ≡, <, and >
Fias & Verguts	Neurons tuned to cardinality
Fischer & Mills	Spatial position
Andres et al.	Finger counting
Hintzman	Recursively structured memory traces
B. Bridges from Early to Mature Number Concepts	
Sarnecka	Defense of induction to a limited domain
Carey	Defense of bootstrapping
Martins-Mourao & Karmiloff-Smith	Part-whole structure
Muldoon et al.	Disengagement from objects
Mix	Integration of multiple sources of information
Núñez	Integration of multiple metaphors
Halberda & Feigenson	Sets
Gordon	Sets
De Cruz	Social/Cultural factors
Read	Social/Cultural factors
Cowan	Social/Cultural factors
Lakoff	Brain processes
Cohen Kadosh & Walsh	Brain processes
C. End States	
Noël et al.	Critique of innate number schemas
Morris & Masnick	Algorithmic vs. probabilistic concepts of number
Smith	Rules vs. rule use
Hodes	Numeral-individual vs. metalinguistic concepts
Pietroski & Lidz	Peano arithmetic vs. Frege arithmetic
Decock	Peano arithmetic vs. Frege arithmetic

Some authors stick to versions of the traditional theory in which integer concepts arise from mappings between the first few count terms ("one," "two," "three,"... "ten") and the cardinality of small collections, possibly supplemented by other mechanisms (**Carey**; **Mix**; **Muldoon, Lewis, & Freeman** [**Muldoon et al.**]; **Sarnecka**). Other authors implicate the role of additional structures, such as part-whole relations (**Martins-Mourao & Karmiloff-Smith**), empirical experience with objects in sets (**Gordon**; **Halberda & Feigenson**), motion along a path, object construction and collection, and measurement (**Núñez**). A few commentaries emphasize general factors that got short shrift in our article: the role of schooling and other social or cultural influences (**Cowan**; **De Cruz**; **Read**) and brain processes (**Cohen Kadosh & Walsh**; **Lakoff**). As **Le Corre** and Núñez note, related questions about bridging arise within our own theory when it comes to explaining how children achieve the correct schema for a math domain. We will have more to say about bridging in section R4.

Our argument could also go wrong in describing the nature of adults' number concepts. If instead of arriving at *There*, children arrive at *Elsewhere*, then the problems we raised may be irrelevant. We assumed that children eventually attain a concept (i.e., mental representation) of the positive integers as distinct individuals that obey the Dedekind-Peano axioms – what **Hodes** calls numerical-individual concepts. But several commentators (see Table R1C) question this assumption, suggesting instead that even adults may have no more than probabilistically defined number concepts (**Morris & Masnick**) or metalinguistic concepts of numerals (Hodes). Others doubt our conjecture that adults' number concepts are imposed top-down by schemas reflecting the axioms (**Noël, Grégoire, Meert, & Seron** [**Noël et al.**]). Perhaps such concepts are use-based (**Smith**) or reflect an underlying notion of equivalence between sets through one-to-one mapping of their elements (**Decock**; **Pietroski & Lidz**). We discuss these possibilities in section R5.

The pigeonholing in Table R1 is one way to organize the commentaries, but it obviously does not capture their full import. Many commentators' points fall into more than one of our categories. For example, if you believe that adults' representations of the positive integers involve equivalence relations on sets of objects, then you presumably think that the starting point for understanding numbers is the ability to represent sets, and that the route from early to mature representations takes sets into equivalence classes of these sets. The likely *There* carries implications for *Here* and for the route between them. Our purpose is only to indicate the relative emphasis of the critiques, and we have located them in Table R1 at the position where we believe their points are most telling. Within each of the three groups in Table R1, we have also ordered the commentaries roughly from those taking more conservative approaches (i.e., defending current theories) to those adopting novel proposals. This ordering suffers from the usual problem of having to collapse dimensions, and we hope readers won't take our listing as more than an outline.

R2. Clarifications

Before we go through the three substantive groups of commentaries in Table R1, we would first like to deal with some misperceptions about our project. These are all interrelated and are likely due to the sketchiness of the theory we tentatively offered in section 5. Several commentators (**Mix**; **Muldoon et al.**; **Noël et al.**; **Sophian**) take us to believe that any developmental account (or theory of learning) of the positive integers is futile; instead, knowledge of the positive integers is innate, much like knowledge of universal grammar. Other commentators (**Carey**; Mix) incorrectly attribute to us a view in which knowledge of the integers is based on verbal counting or enumerating objects. A third group (**Barner**; **Cowan**; **Morris & Masnick**; Muldoon et al.) seem to think we are evading a proper developmental account by uncritically accepting proposals from mathematicians and philosophers about the integers and by unfairly criticizing psychological theories if they fail to conform to these proposals.

R2.1. Assumptions about development and innateness

Some commentators take us to be advocating a nativist view of the integers, and they argue that anyone who has seriously studied the development of number concepts knows that children's progress is painfully slow and complicated, a struggle that is inconsistent with the triggering of an innate schema. We are not claiming, however, that people are born with an innate schema for the positive integers. Although this is possible, our approach is consistent with children gradually acquiring such a schema from more primitive components. In fact, we thought we were pretty explicit about this: "[Children] therefore do not start out with a schema for natural numbers in the sense in which an undergraduate who has just learned the axioms of set theory has a schema for set theory. Instead, children gradually acquire the information they need to understand the meanings of numbers" (target article, sect. 5.2, para. 1). Our negative thesis is that children cannot project these meanings from their experience grouping or enumerating physical objects, but this does not mean they do not learn them at all. We welcome developmental theories of how children learn the positive integers, but these theories won't get off the ground if they start with the wrong components.

Of course, something has to be innate in order for children to construct number schemas. In section 5.2 we suggested as starting points the concepts of a mapping and of uniqueness (among other concepts), because these would allow children to capture the structure of the positive integers (a unique initial element, a unique new element following each given element, and so forth). Children could also use these primitives to frame other types of structure, such as circular lists, partial orders, or finite linear lists (as **Le Corre** observes), but they could choose the correct structure through top-down, inference-to-the-best-explanation for arithmetic properties. Our suggestion could be completely off base – to our knowledge, no existing developmental evidence bears on it – but, contrary to **Noël et al.**, it does not commit us to denying the gradual evolution of number concepts either in cultural history or in children. Again, we thought we were explicit about this: "In addition to the built-in aspects, however, children must still assemble the schematic or structural information that is specific to a domain of mathematics. . . . We typically expect children to acquire abilities such as these in a measured way that depends in part on their exposure to the key information" (sect. 5.2, para. 3).

R2.2. Assumptions about counting, enumerating, and modeling

Nor are we claiming, as **Mix** believes we are, that children learn the integers via knowledge of the numeral system (what we called "advanced counting"). As we mentioned in section 3.2.2, (para. 2) "we think it more likely that children learn an underlying set of principles that facilitates both advanced counting and the concept of natural number." Along related lines, **Carey** defends her theory of math learning by claiming that:

> the mastery of simple counting is a necessary prerequisite for the mastery of complex counting, which Rips et al. agree is likely to be a necessary part of acquiring the math schema of natural number. The mastery of simple counting draws on the resources of Figure 1 (plus others), and, in this sense, these structures are all part of the precursors of natural number.

However, we don't go along with this trajectory or with this description of our views. **Carey** misdescribes our account by ignoring our definitions of simple and advanced counting in section 1.1. Simple counting "consists of just reciting the number sequence to some fixed numeral, for example, 'ten' or 'one hundred'" (sect. 1.1, para. 2), and no one's theory implicates the Figure 1 components in this kind of recitation. Second, "advanced counting is the ability to get from any numeral 'n' to its successor '$n + 1$' in some system of numerals for the natural numbers" (sect. 1.1, para. 2). Although simple counting may be necessary for advanced counting, neither draws on the components of Figure 1. Third, as we just noted, our view does not commit us to holding that advanced counting is a necessary part of acquiring the integers. In short, there is no connection in our account between the enumeration skills exhibited in Figure 1 and children's later knowledge of numbers. This is a good thing, because as far as we can see, no such connection exists.

Carey also believes that "the big mistake Rips et al. make is methodological: They miss the fact that modeling activities can give placeholder structures meaning, even if in the end the structures involved in these modeling processes, such as the schemata of Figure 1, are part of an acquisition ladder that is not essential to the conceptual role constructed." Carey's "placeholder structure" is initially the simple count list we just mentioned (e.g., the sequence of numerals from "one" to "nine"), and the critical gap in her theory is that she provides no explanation of how enriching this structure by modeling (or other means) can give rise to the correct structure of the integers. The trouble is that the initial placeholder structure is a finite list, whereas the structure of the integers is a (particular type of) infinite list. Using the finite list to model the number of objects in a collection, for example, might motivate the search for alternative models, but it does not necessarily provide the right one. Of course, once you have the right structure – the structure of the positive integers – further modeling can be useful in establishing applications, as we describe in section R6. But the enhancement that these applications provide is not part of the meaning of the integers, as that meaning is already inherent in the structure. Numbers obviously have many uses, from figuring out the distance to Omaha, to calculating change from a $20 bill, to labeling house addresses. All these uses enrich the structure of the numbers, but they do not determine their meaning.

R2.3. Assumptions about mathematics and psychology

In describing what children eventually need to know about the integers, we relied on the standard Dedekind-Peano analysis. Mathematics and philosophy of math are helpful in this respect by providing a clear description of the properties of mathematical systems. Unlike **Morris & Masnick**, we think it extremely unlikely that adults could possess the intuitions and abilities they have in arithmetic unless they possess a concept of the integers that satisfies the requirements of analyses like Dedekind's or Frege's. No one, we hope, would set out to investigate kids' concept of prime numbers without taking into account the relevant mathematical definition (i.e., a positive integer evenly divisible only by itself and 1), and it is unclear why the same should not be true for kids' concepts of the positive integers themselves. We tried to emphasize, however, that this does not mean these concepts are identical to their formal mathematical definitions. What the axioms impose are constraints that a representation of the integers has to satisfy.

Moreover, we do not believe our use of these analyses violates any methodological strictures from the psychological side. **Barner** writes,

> Rips et al.'s general thesis is that knowledge of natural number is defined in terms of an inferential system, and that therefore developmentalists should focus their efforts on evaluating how children come to manipulate numbers as syntactic objects, independent of their particular denotations. Stipulating that this particular knowledge should act as a metric of competence, however, is entirely arbitrary, and unprecedented in developmental psychology.

But we are not claiming that representations of numbers have a purely syntactic role. What we are suggesting is that rival theories may be wrong in supposing that the referents of numerals are sets of sets of physical objects (e.g., that "three" represents the set of all three-membered sets of physical objects), not that they have no referents at all. We also see nothing arbitrary in thinking that people's representations of the integers have to ensure that they function correctly in arithmetic.[2] **Cowan** similarly suggests that relying on the philosophy of mathematics is unpsychological because philosophers disagree among themselves about math foundations and provide definitions that are not "operationalizable."[3] However, although philosophers of mathematics differ (just as psychologists do) about the underlying nature of mathematics, we know of no serious disagreement about the correctness of the Dedekind-Peano axioms. We are also unconvinced that "operationalizing" mathematically or philosophically inspired theories is any more problematic than doing the same for other psychological theories. In short, although we are taking a novel approach to the development of math, the approach does not violate any well-grounded methodological principles within cognitive or developmental psychology.

R3. Bases for numerical concepts

We proposed that psychologists might be looking in the wrong place for the origins of knowledge of the integers. Instead of beginning with representations geared for physical objects, children may start from primitives that are more inherently mathematical. However, many of our commentators believe that we have not considered carefully enough the full range of possible initial representations. These commentaries seem to cluster in roughly three groups: (1) those that defend the traditional proposals about children's starting points, (2) those that take a structural approach closer to our own, and (3) those that propose novel representations.

R3.1. Magnitudes and cardinalities

As we predicted, several commentators think we are wrong to doubt that enumerating objects plays a direct role in number acquisition, and they have attempted to provide evidence for such a connection. Adults spend a lot of time getting children to connect the numeral "2" to pictures of pairs of pizzas, "4" to quartets of trees, and so on, with the hope that this will advance the children's knowledge of numbers. Assertions about cardinality are preschool teachers' most frequent use of numbers (e.g., "There are five new blocks in the block area"), far more frequent than use in calculations (Ehrlich 2007). How could it be that such efforts make no contact with later math knowledge?

Traditional theories assume that infants represent the cardinality of groups of objects by means of continuous internal magnitudes, discrete representations of individual objects, internal sets of these objects, or some combination of these. Debates among developmentalists concern what role (if any) each of these representations plays in promoting later math concepts, and traces of this debate appear in some of the commentaries. For example, Le Corre and Carey (2007) believe that magnitudes play no role in acquiring principles for enumerating objects, including Principle (3), repeated here:

(3) For any count word "n," the next count word "$s(n)$" in the count sequence refers to the cardinality $(n + 1)$ obtained by adding one element to collections whose cardinality is denoted by "n."

Barth, however, thinks Le Corre and Carey's (2007) evidence is not decisive, and Carey and Le Corre themselves allow a role for magnitudes in determining the relative cardinality of collections containing more than six items (see also Le Corre 2005). Because we are skeptical about the role that all such representations play in understanding the integers, we won't dwell on the debate about magnitudes in particular. For reasons that we give in section 3.2.2 of our article (and in more detail in Rips et al. 2006; 2008), even if magnitudes are involved in acquiring Principle (3), this principle is not sufficient for understanding the integers. Similarly, Le Corre's (2005) experiment shows at most that magnitudes help children grasp the relative cardinality of small collections, not the relative ordering of integers. We know that 3,491 is less than 3,492, but this is far beyond the reach of the magnitude system. Although magnitudes might conceivably be a catalyst for learning integer order, they are not constitutive unless there is a reasonable story about how people transform magnitudes into integers.[4]

Our point about cardinality is that children's ability to enumerate two pizzas or to use sentences like "Two pizzas are on the table" does not require the children to use "two" to refer to an integer, as such sentences are expressible without reference to numbers. This is the point that logical paraphrases, such as Proposition (4) in our article, drive home. Let us call the use of numerals to refer to small cardinalities of physical objects "*c*-use" and the use of numerals to refer to integers as "*i*-use." Then our point is that the ability to engage in *c*-use does not determine whether children are capable of *i*-use. It is possible that mastering *c*-use in phrases such as "two pizzas" is causally necessary for acquiring the concept of the integer two, but we know of no compelling evidence that this is so. The fact that *c*-use emerges before *i*-use in children (as **Barner** and **Le Corre** assert) is not good evidence on this point, as the prior appearance of the former could be due to *c*-use's greater practicality, simplicity, concreteness, frequency in caregivers' speech, and many other factors. Much the same is true for the prior historical appearance of terms for cardinalities. As **Read** points out, the two can coexist in the same culture as independent systems.

Is there any empirical evidence for a causal-constitutive link between the use of numerals for cardinalities and their use for integers? **Lourenco & Levine** call our attention to several recent studies that suggest that early understanding or exposure to *c*-uses may influence children's later math abilities. Two of these studies (Ehrlich 2007 and Klibanoff et al. 2006) show that the amount of time preschool teachers talk about numbers (predominantly in *c*-uses) influences their students' math improvement during that preschool year. Two others (Denton & West 2002 and Duncan et al. 2007) analyze large-scale studies linking children's math skills when they enter kindergarten to their later math performance in grade school. None of these studies, however, establishes that *c*-use causally influences *i*-use (nor were they designed to do so). The preschool studies are powerless in this respect because the investigators assessed the preschoolers' math improvement largely in terms of *c*-uses; *i*-uses were never tested. In Klibanoff et al., a particular task asked students to "point to the one that has more" for a display containing seven dots and five dots; another showed students a card with the numeral "2" on it and asked, "Which one of these goes with this one?" for a display containing one to four dots. A third task asked children to "point to four" for a display containing two to five objects. Although one additional task asked for a calculation, it was also phrased in terms of *c*-use, rather than *i*-use ("Johnny has one apple, and his mommy gives him one more. Point to how many apples Johnny has now," where the display contained pictures of one to four apples). Ehrlich used a standardized test of math achievement (Ginsburg & Baroody 2004), but according to Klibanoff et al., this test had a high degree of overlap with their own.[5] What these studies show, then, is that teachers' number talk influences pupils' *c*-performance, not necessarily their understanding of integers.

The large-scale math studies are no more helpful in showing that *c*-use directly affects *i*-use. Initial performance in these studies comes from beginning kindergartners (not preschoolers) whose *i*-abilities may already be in place, and the investigators do not attempt to break out the types of math skills that influence later performance. In fact, Duncan et al. (2007) report that math achievement at the beginning of kindergarten is not only correlated with later math skills, but also has a nearly equal correlation with later language skills, such as reading. This suggests that whatever the initial math achievement tests are tapping, it is likely to be considerably more general than mathematics.

R3.2. Structures and principles

We think psychologists should take more seriously the idea that the meaning of an integer is its position within the appropriate structure, and we are therefore sympathetic to the commentators who see structures or principles as primary. Like **Gardiner**, for example, we believe sequences are fundamental, but of course, the sequence of integers differs in its properties from the sequence of notes in a musical scale (with which Gardiner is concerned), letters in the alphabet, or days of the week. Appreciating that one element follows another is not sufficient for understanding the integers, but at least it is possible to see how adding further constraints could lead to the right destination.

Along somewhat similar lines, **Muldoon et al.** and **Sophian** suggest that concepts of equivalence and nonequivalence (along with the concept of a unit) may provide a developmental basis for the integers. The notion presumably is that one-to-one equivalence between sets provides a criterion for whether the sets have the same number of elements, an idea sometimes called "Hume's principle." We ignored this possibility in our article mainly because of Gelman and Gallistel's arguments that children do not use Hume's principle in determining equality (i.e., same cardinality); instead, they enumerate both sets to see whether they contain the same total:

> Young children, however, seem clearly to prefer (if not require) that decisions about the equivalence or nonequivalence of numerosities be based on the identity or nonidentity of their numerical representations rather than on the possibility of establishing a one-to-one correspondence between them. (Gelman & Gallistel 1978, p. 163)

According to this view, equivalence depends on cardinality rather than the other way round.

Of course, as **Gelman** reminds us, her theory has always insisted that principles of arithmetic underlie children's ability to determine cardinality. In particular, one-one matching plays a role in enumerating sets, since children have to match numerals to the objects in the sets in order to get a correct total. This is Gelman and Gallistel's (1978) "One-One Principle." However, Gelman and Gallistel believe that this numeral-to-object matching is special – isolated from more general reasoning about equality:

> From a logical point of view, the child's procedure for deciding numerical equivalence depends on the fact that the numerosities of both sets can be placed in a relation of one-to-one equivalence with the same set of counting tags. But the child does not normally take cognizance of the transitivity of one-to-one correspondence. He ignores or is indifferent to the fact that the cardinal numerons [i.e., mental count tags] representing two equally numerous sets are identical precisely because both sets have been placed in one-to-one correspondence with a count sequence that terminates with that cardinal numeron. (Gelman & Gallistel 1978, pp. 198–99)

It is possible, however, that **Muldoon et al.** and **Sophian** are right and that Gelman and Gallistel's conclusion should be revisited. Both **Decock** and **Pietroski & Lidz** point out that recent work in the foundations of mathematics shows that Hume's principle can serve as the basis for arithmetic (e.g., as an alternative to the Dedekind-Peano axioms), and an early appreciation of this principle could thus provide the right starting place for later mathematics. This is a tempting point of view, and we return to it in section R5.

R3.3. New cognitive starting points

A third set of commentators in Table R1-A propose a role for initial representations that we did not consider in our article and that may have properties closer to those of the integers. All these representations, however, still lack some of the integers' essential properties and therefore require additional learning to take up the slack. Some of these representations are simply variations on magnitudes – for example, neurons tuned approximately to specific cardinalities (**Fias & Verguts**) or spatial associations of numbers, similar to a number line (**Fischer & Mills**). Neither is able to represent the precise value of an integer but only the value plus or minus some amount. The problem with these representations is, in this respect, the same as that for magnitude formulations: There is no obvious method to get from them to mature integer concepts. Fias & Verguts assume that language is able to produce this shift, but they don't explain how this happens or why this proposal does not succumb to the difficulties pointed out by Gelman and Butterworth (2005), Laurence and Margolis (2005), and us.

Other commentators propose starting points that are genuinely different from magnitudes. For example, **Hintzman** suggests that retrieval of a memory trace creates a further memory trace that includes the first as a component; retrieval of this new trace creates, in turn, a third trace that embeds the first two, and so on. This structure is similar to the recursive representation for "3" that we presented as a tree diagram in section 3.2.1 (see Fig. 2) and to Dedekind's (1888/1963) famous attempt to prove that infinite systems exist. However, aspects of long-term memory impose limits on people's ability to resolve these degrees of embedding (as Hintzman acknowledges), whereas there is no such limit on our ability to deal with the positive integers. We can think about 3,491, but in doing so we do not represent it as a memory trace 3,491 levels deep. So children would still need a distinct representation of the integers and a means to learn this representation from the memory traces. Similarly, finger counting (see **Andres et al.**) would be a fine basis for the integers if only people had infinite fingers.

R4. Bridges from early to mature numerical concepts

Mental magnitudes, object files, internal sets, number lines, embedded memory traces, fingers, and other proposed representations have some properties in common with the integers, and some not in common. If children had a way to learn the integers from these representations, then there would be no need for the top-down approach that we believe is more likely. However, our review of proposals about such learning turned up no possibilities that did not appear to pull a rabbit out of a hat, where the rabbit in question is usually the infinite number or the discreteness of the integers. The typical strategy in these proposals is that if a representation lacks an essential integer property, then combine it with another representation that has that property. Mental magnitudes are not discrete, for example, but there are infinitely many of them. The initial members of the simple count list ("one," "two," . . ., "ten") are not infinite, but they are discrete. So perhaps by combining magnitudes with simple counting, children could learn the integers. This combination strategy is tricky, however, because the offending properties do not automatically disappear. If you combine "one" with a fuzzy magnitude, "two" with a larger, fuzzier magnitude, and so on, why don't terms like "four" end up meaning several, rather than exactly four? When you reach the end of your (small finite) count list, what gets paired with the next fuzzy magnitude? "Lots"?

Our commentators adopt a variety of responses to this problem. One is to retrench, for example, by claiming that the learning proposal deals only with "one" through "nine," not with the entire set of positive integers. A second response is to continue to apply the combination strategy by adding further components. A third is to recruit some external factor, such as schooling or neural mechanisms, to reinforce the right properties and to squelch the wrong ones. Let us see if any of these maneuvers helps.

R4.1. Retrenchment

We were hoping for an answer to the question, "How do children learn the positive integers?" where this is the set of numbers whose characteristics we defined in section 1.1 of the target article. We sometimes phrased this question, "How do children learn the concept of the positive integers (or the concept POSITIVE INTEGER)?" meaning a mental representation of the relevant set (i.e., the set of all and only the positive integers). This question perfectly parallels other questions that developmental psychologists ask about children's learning, such as how they learn the concept of cows or tables (or the concept COW or TABLE). In criticizing proposed learning procedures for the integers, we pointed to properties that these procedures failed to capture (e.g., the fact that there are infinitely many integers), and we concluded that the procedures were unable to give a correct answer to our question.

Sarnecka, however, believes we have misdirected these criticisms, as the learning procedures were never meant to handle the concepts we singled out. According to Sarnecka, properties of the positive integers – such as the property of there being infinitely many of them, or the property that for any two integers, x and y, $x + y = y + x$ – are parts of "meta-numerical knowledge," and children learn them by other means. The procedures in question only learn the meanings of specific integers, such as 7, which Sarnecka believes are cardinalities. Although these procedures capture concepts of particular positive integers ("numerical knowledge"), they are not supposed to capture the concept of the positive integers ("meta-numerical knowledge"), and Sarnecka thinks we have run the two together. This defense, however, is exactly like the claim that when psychologists investigate how children learn the concept of cows (or the concept COW) – for example, how children learn that all cows have four stomachs – they are actually investigating "meta-bovine" knowledge. By implication, this is secondary knowledge, and what the psychologists should be studying is how children learn the concept of Bossie or Buttercup, which is object-level bovine knowledge.

There is obviously a distinction between concepts of individual items like 7 or Bossie, and concepts of categories, like integers or cows, and it is fine to take as one's mission the study of concepts of individual integers (though we do not recommend trying to study all of them one at a time). So what is the scope of the learning procedure that **Sarnecka** is proposing? A first approximation is "we are interested in cardinalities that are (a) bigger than 4 (and thus too big for each individual to be represented in parallel), and (b) exact (and thus too fine-grained to be represented by the analog magnitude system)," as Sarnecka writes. This sounds as if the learning procedure is good for all integers greater than 4, which is surely how most people read the original proposals (e.g., Carey 2004; Carey & Sarnecka 2006). The arguments in section 3 of our target article (given in more detail in Rips et al. 2006; 2008) show, however, that the procedure is useless outside the list of numerals the child currently knows (say, "one" through "nine"). And, in fact, Sarnecka now states that what the induction procedure actually yields is: "'five' means 1 more than 4; 'six' means 1 more than 'five'; [. . .]; 'nine' means 1 more than 'eight.'" Full stop. As we've already mentioned, for small numerals like these and for the cardinality uses with which Sarnecka is concerned (e.g., "seven pencils"), no conclusions can be reached about whether children have representations of the corresponding integers (even the concept SEVEN). (See sect. 3.3 of our article for our reasoning about this.) Thus, Sarnecka's hypothesis fails on its own terms. That is why general statements about the integers are important: They are more likely to require concepts of the integers themselves, since they explicitly quantify over the integers. **Carey** asserts that the bootstrapping process that gets children to "nine" is "only one of several that eventually result in the capacity for representing natural number," but she provides no details on what these further procedures might be. We don't mean to disparage children's achievement in getting from "four" to "nine" (or investigators' achievement in explaining how they do it), but this leaves unanswered our original question of how children learn the concept of the positive integers.[6]

Martins-Mourao & Karmiloff-Smith's proposal is similar to Carey and Sarnecka's (2006) theory in supposing that children learn the integers by coordinating methods for enumerating collections with their knowledge of the numerals. (For Martins-Mourao & Karmiloff-Smith, the latter knowledge includes understanding the part-whole relations that govern the numerals' place-value system.) We agree that children eventually have to make these connections, but we are waiting for an account of how the learning takes place. The systems are supposed to develop in parallel, but it is unclear why this parallel development converges on the correct structure (the structure of the positive integers), rather than on something quite bizarre: for example, a structure that is like the integers up to the last term in the child's current count list – such as, "nine" – and then circles around from "ten" to "eleven" onward through to "nineteen" and then back to "ten" (see Fig. 1 of Rips et al. 2008).

R4.2. Numbers by combination and abstraction

We have no doubt that children reach the integers by a complex route. But merely acknowledging this complexity (or the host of possible influences on this process) should not be confused with a theory. It is possible that the integers emerge "from within a complex, ever-shifting, multifactorial system" (to use **Mix**'s phrase), but so far we have not seen even a single persuasive theory of how this emergence takes place. Suppose one type of precursor representation has properties p_1, p_2, p_3, and p_4, and a second type of precursor has properties p_3, p_4, p_5, and p_6. We might try to combine the properties of the two representations to get something that neither can provide. For example, we could try taking the intersection of the property sets (i.e., p_3 and p_4) in the hope that intersection will eliminate possibly offending properties (e.g., p_1 might be fuzziness in the case of magnitudes). However, if each representation's resources were initially insufficient (e.g., because magnitudes lack discreteness), intersecting their properties will not provide the missing ingredients. If instead we take the union of their properties (i.e., p_1-p_6), we are including those same offending properties that disqualified them as number representations in the first place (e.g., fuzziness, again). Apparently, we require a smarter combination process than simple intersection or union, but what is it?

The intersection idea is quite similar to the possibility that children obtain the integers by abstracting over initial representations. By abstracting over the bad properties and retaining the good ones, you might get a representation more in line with the integers. For example, if you start with collections of three physical objects and abstract over them, then you might end up with a representation of three. **Muldoon et al.** suggest that children "disengage" numbers from small collections by reflecting on "how the accuracy of the counting procedure both identifies ordinal relations and determines the validity of the cardinal representation of the set." But this idea is very similar to the theory that children learn the integers via Principle (3), and it is susceptible to the same objections. Some additional process must be involved in acquiring the integers. What process could this be?

A related issue puzzles us about **Núñez**'s and **Lakoff**'s commentaries (and about Lakoff & Núñez 2000). The integers are supposed to be formed from

> the isomorphic structure of four different source domains of primary experiences.... Such isomorphism provides structural correspondences across the source domains of four different grounding metaphors, yielding equivalent numerical results. This understanding is not about a mere phrasing of operations on physical objects, it is about the abstracted numbers. (Quoted from Núñez's commentary)

But we do not understand why piling on metaphors is helpful, especially since there are many competing isomorphisms among complex domains. This does not bother **Núñez** because he believes that clashing inferences are commonplace in the history of mathematics and are sometimes resolved and sometimes not. For children, according to Núñez, "Similar, although more subtle, top-down dynamics must be accounted for in explaining the child's 'acquisition' of the concept natural number." Just what might these be?

For **Halberda & Feigenson**, the fundamental bridging concepts are sets and the membership relation that binds individuals to sets. **Gordon** also proposes that "the indirect process of trying to figure out the nature of objects as objects and individuals within sets might be the catalyst for arriving at these principles in the relatively informal manner that we also acquire knowledge of other domains of folk-science." In our article, we considered a related possibility – proposed by Carey and Sarnecka (2006) – that internal sets might play a role in learning the integers. According to this approach, children first link the count terms "one," "two," and "three" to set-like representations, such as {*a*}, {*a,b*}, {*a,b,c*}, and then infer that increases in set size correlate with later terms in the count list (i.e., Principle [3]).[7] Carey and Sarnecka (2006), Halberda & Feigenson, and Gordon might be right that these internal sets are better representations for one, two, and three than are magnitudes or object files; and Harberda & Feigenson are careful to note that the notion of a set to which they are appealing is not simply perceptual grouping. However, these sets do not help in learning the full range of integers for the reasons we spelled out in section 3 of the target article.

As **Halberda & Feigenson** acknowledge, "limits on parallel individuation or on working memory would therefore limit these set representations to small numbers of items," and we lack an adequate explanation of how children could extend them to a full representation of the integers. Of course, investigators could adopt a more full-blooded notion of set, but these more complex formulations run directly into the issue raised by **Hodes**:

> Perhaps this emergence of set-theoretic concepts [is] built upon some sort of 'implicit conception' [...] of set-hood with older roots; but such a conception would also be sophisticated. In places [...], [Rips et al.] seem to suggest that reputable psychologists believe children to possess these concepts (or their precursor conceptions)! I recommend that psychologists concern themselves with the concept of being three-membered only when investigating the psychology of professional mathematicians. (Hodes's commentary)

R4.3. External factors

Cohen Kadosh & Walsh and **Lakoff** suggest that brain processes help explain how children get from preliminary representations to representations for the integers, but we don't see how transposing the problem from mind to brain helps solve it. The very same difficulties about learning that we discussed in our target article and in the preceding subsection would seem to remain, whether the mechanisms in question are cognitive or neurological ones. According to Cohen Kadosh & Walsh, although "[Rips et al.] suggest why either magnitude *or* language might not play a role in the formulation of natural number, they do not explain why general magnitude *and* language cannot play an integrative role in shaping the understanding of natural numbers" (emphasis in the original). Yet we discuss exactly that possibility (in sect. 3.2.2) in connection with Spelke's (2000) similar proposal. Nor are we the only ones who have questioned this idea; see Laurence and Margolis (2005) and Leslie et al. (2007).

A suggestion that may be more helpful is that children learn the integers through external guidance by parents and teachers and through interaction with external symbols, such as the Arabic numerals. And, of course, mathematical knowledge does propagate through cultural channels, not only from adults to children but also from adults to other adults (De Millo et al. 1979). By the time children have a good grip on the integers, they may well be in kindergarten or early grade school; so school is a likely source of their knowledge. Thus, **Cowan**, **De Cruz**, **Mix**, **Read**, and others are probably correct in thinking that schooling, parent-child interactions, and other social-cultural factors are part of the story. As these commentators would agree, however, children cannot absorb this information unless they have enough relevant background to understand it. There are no current proposals about what this background is, which cultural inputs are effective, and how the inputs transform the background.

R5. End states

Older children and adults can correctly identify arithmetic facts (e.g., that $4 - 3 = 8{,}238 - 8{,}237$) that are out of range for internal magnitudes, object files, working-memory object representations, internal sets, internal number lines, finger counting, embedded memory traces, and similar representations. Of course, in dealing with these facts, people sometimes make mistakes, due to the many factors that can interfere with their calculations. Nevertheless, people usually agree about arithmetic facts, and this is very unlikely to be entirely a result of chance or external influences (e.g., calculators, charts and diagrams, or teachers and mathematicians), though these external factors sometimes play a role. People's typically correct answers to arithmetic problems therefore require psychological explanation.

R5.1. Probabilistic and diagrammatic representations

We think the likely explanation of adults' arithmetic abilities is that they know the structure of the integers and the operations this structure supports. Perhaps people represent this structure in a non-propositional format, such as an internal diagram or model, as **Morris & Masnick** suggest, but we know of no diagrammatic representations that can handle the integers' infinite extent. Mental models are inadequate in this respect, because, as Johnson-Laird notes:

> Mental models can contain only a finite number of entities, but we can reason about infinite quantities and sets of infinite size such as the natural numbers. There is accordingly a distinction between naïve or intuitive reasoning, which is directly based on mental models, and mathematical reasoning, which relies on other mechanisms. (Johnson-Laird 1983, p. 444)

Morris & Masnick also think that magnitudes could explain adults' arithmetic, as long as we are willing to accept a probabilistic rather than an algorithmic notion of the integers. However, outputs of the magnitude system are not just noisy or approximate. Their noise increases with the size of the represented cardinality. It is very unlikely that such a system could represent 8,238 as greater than 8,237 with greater than chance frequency, even after many trials. On any one trial, the likelihood that the system could represent $4 - 3$ as equal to $8{,}238 - 8{,}237$ is zero if each of these numbers is a magnitude. This isn't anything like adult performance.

R5.2. Rule use

Like most cognitive psychologists, we assume that adults have internal principles or rules that govern arithmetic operations. As **Smith** mentions, philosophers have challenged the idea that these internal rules alone can establish the correctness of their operations (e.g., Kripke 1982; Wittgenstein 1958). To account for the correctness (normativity and necessity) of math, we need to appeal to factors other than a bare description of an individual's mental rules, and this idea has sometimes motivated the view that internal representations and rules are irrelevant to knowledge of mathematics: "As a body of knowledge, mathematics is not something I know, you know, or any individual knows: It is a part of our culture, our *collective* possession" (Wilder 1950/1998, p. 188, emphasis in the original text). Smith, however, does not seem to accept such a thoroughgoing cultural approach. According to him, "rule-meaning is dependent on rule-use. . . . Further, rule-use is regulated by reasons" and "reasons are investigable as normative facts." Examples of these investigable reasons are children's statements that certain facts "must be the case" or "have to be true." We agree about the psychological importance of justifications of this kind, but we don't see how they evade the skeptical challenges that Smith is trying to address. Taking reasons, rather than rules, as fundamental leaves exactly the same problem about the correctness of the reasons. Children's statements that certain mathematical facts must be true (or necessarily follow) do not certify their own correctness any more than do descriptions of rules.

R5.3. Numerals

The integer schema that we are proposing captures the structure of the integers, but it also captures the structure of the numerals for those integers because the two are isomorphic. We called knowledge of the numerals "advanced counting" and noted that it provides a better model for the integers than groupings of physical objects. As we mentioned earlier, though, we did not propose that children learn the integers through advanced counting. However, **Hodes** raises a related possibility – one that we did not consider – that most people never get beyond advanced counting. He doubts that even middle-school students have singular concepts for the integers, and notes that the claim that they possess such concepts "would have to distinguish such possession from possession of corresponding metalinguistic concepts; for example, of Arabic numerals." Do older children and adults think that "three" in a sentence like "Three is less than four" refers to a numeral rather than a number? If so, then advanced counting might be the final stage in most people's conception of the integers. We know of no research on this matter, but middle-school children learn several different numeral systems, for example, Roman numerals, Arabic numerals, scientific notation, and sometimes others. One pre-algebra text (Wilcox 1990) explicitly compares Chinese, Roman, Greek, and Arabic numerals. Unless the metalinguistic concept in question can encompass, for example, "6," "VI," and "6×10^0," then the simplest assumption might be that older children have a concept of six that each of these numerals represents.[8]

R5.4. Frege arithmetic

In proposing that children and adults have a schema for the positive integers, we were not assuming that this schema takes exactly the shape of the Dedekind-Peano axioms. As we said in the target article, "we are not claiming that the Dedekind-Peano axioms are the only ones that are sufficient for producing the natural numbers or that they are the most cognitively plausible for the job" (sect. 5.3.4). We do believe, however, that the integer schema has to conform to these axioms in order to provide a reasonable basis for people's arithmetical reasoning. Both **Decock** and **Pietroski & Lidz** point out, however, that there is a route to such knowledge via Hume's principle (there are the same number of *F*'s as *G*'s if the *F*'s and *G*'s can be paired one-to-one), and such a route might be more psychologically plausible than the more direct specification stated in the axioms. Hume's principle, however, is not sufficient for specifying the integers. It is perfectly consistent, for example, with systems containing only a finite set of numbers and with systems containing cardinals beyond the natural numbers. To get the natural numbers (or the positive integers), you also need some additional definitions (as Pietroski & Lidz note), including the following key definition of the natural numbers: n is a natural number if and only if n is a member of the successor series beginning with 0 (see, e.g., Heck [2000] or Zalta [2008] for a formal definition; further definitions are required for 0 and for the successor relation). In fact, several of the Dedekind-Peano axioms, including mathematical induction, follow from the definition (just given) of natural number alone, without the need for Hume's principle (see Rips & Asmuth [2007] for a derivation).

The attraction of Hume's principle (and Frege's theorem deriving the axioms of arithmetic from this principle and the additional definitions) is presumably that it starts with cardinality (same number of *F*'s as *G*'s), rather than starting with a characterization of the natural numbers. In a developmental context, we could take this to mean that children could begin with one-to-one matching between collections as indicating that the collections have the same number of elements. However, as we noted earlier, Gelman and Gallistel's (1978) evidence on this score is not encouraging. **Decock** believes that we shouldn't take this evidence too seriously, as it may be biased by the training Western kids receive in enumerating objects. Maybe so. But even if we set aside the empirical evidence, children's concept of same number via one-one matching is going to play a limited role in their understanding of the natural numbers. Although Decock suggests that "it seems more promising to start with a psychological study of one-to-one correspondence [. . .] than with Peano's mathematical induction or commutativity," one-to-one correspondence does not allow an end run around mathematical induction. Even children for whom "same number" is defined in terms of one-one matching are going to have to learn mathematical induction (or an equivalent) in some other way.[9] In sum, it is possible that children start with a concept of cardinality as defined by one-one matching, learn further definitions of natural number, zero, and successor; and then deduce the correct properties of these numbers. However, this route does not give cardinality the central role that developmentalists imagine.

Of course, **Pietroski & Lidz** are right that what is deducible from what depends on the background logic available to children, but this doesn't affect the current point. Four-year-old logicists who start down the road of Frege's theorem armed with Hume's principle and full second-order logic still won't be able to deduce the definition for natural number, and we lack any account of how they learn it. Moreover, none of the theories that we examine in our article holds that children deduce the defining properties of the integers; assumptions about children's logical sophistication are of no help to them. All assume that children acquire this information by some form of empirical induction. That's the process we don't understand.

R6. Closing comments

In its instructions for replies, *BBS* explains that the commentaries will create an expectation among readers that "the other shoe will drop" for each of the commentators' points. The purpose of the authors' response is to drop those shoes. We thus find ourselves having to release 31 shoes, calculating conservatively just one pair of shoes per commentary. We have been throwing down shoes as fast as we can, but we apologize for any mismatched or undropped items. In surveying this array of often stylish footwear, we find that most commentators seem to agree with us that psychologists do not have a complete and convincing story about how children get from their early quantitative abilities to their later number concepts; but the commentators differ on the remedy. The solutions on offer here include beefing up infants' concepts for dealing with quantity, widening the range of factors that could lead them to more mature concepts, revising the nature of the mature concepts to put them within easier reach, or some combination of these strategies.

The purpose of most of these proposals is to preserve the link between early enumeration of physical objects and later concepts of the integers. The intuition is strong that enumeration must play some constitutive role in children's understanding of number, and the commentators see us as too quick to discard it. For the reasons given in sections R3 through R5 of this Response, however, we do not find the commentators' efforts to rescue enumeration convincing. The fancier methods that some commentators propose for early enumeration help reduce the difficulties with previous methods, but they don't eliminate them. The new learning procedures that other commentators advance to bridge between enumeration and number are too vague to be helpful explanations. And some of the alternative concepts of the integers give up too many of the properties that adults recognize and use in arithmetic.

Still, what is the connection between early enumeration and knowledge of the integers? As **Carey** suggests, enumeration might be important in providing children with a target for modeling efforts. According to our view, however, knowledge of the integers precedes this modeling. Suppose children have in mind a set of possible structures that they have constructed from primitive relations (e.g., x follows y). The structures might include linear ones with a finite number of elements (useful for understanding the alphabet, for example); circular lists (useful for understanding the days of the week); linear structures with an initial element but no final element; linear ones with no initial and no final element; and so on. Once children have the notion of a linear list with an initial element and no final one, they have the concept of the positive integers. But what purposes could this concept serve? The children's experience in enumerating collections suggests a correlation between the structure of the count terms and the structure of the cardinality of these collections, but it does not settle the question of which structure is the right one. However, evidence from several sources may increase the likelihood that the integer sequence is correct. Such sources may include many of those suggested by the commentators: feedback from preliminary experiments; hints from teachers and parents; and facts about the internal structure of the numerals. The numerals denote the elements of the integer sequence, and the elements have enumeration as one of their uses.

As important as it is to study such applications, the gist of our positive proposal is that we also need to look directly at the integer sequence's internal properties. There are already many elegant experiments on principles of enumeration and principles of arithmetic. Where are the studies of the integers' properties?

NOTES

1. In our article we actually used the term "natural number" rather than "positive integer," but since the natural numbers are usually assumed to start with 0 rather than 1, "positive integer" is more precise. In this reply, we occasionally use "integer" for short, but we always mean the positive integers only.

2. Although we are not championing a purely "syntactic" approach to positive integers, it is worth noticing that the type of inferential theory **Barner** has in mind is hardly "unprecedented in developmental psychology." There is a clear tradition of such theories in the area of reasoning, running from Inhelder and Piaget (1964) to Osherson (1974) and Braine and O'Brien (1998).

3. Cowan also believes that philosophers of mathematics are uninterested in "the development of the individual or even in explaining the history of mathematics." But, although this might be true in some quarters, there are many counterexamples. For philosophical studies of the history of mathematics, see Kitcher (1983), Lakatos (1976), and Tappenden (2005), to name just a few. For studies of people's knowledge of math, see Giaquinto (2001), Heck (2000), Maddy (2007), and, of course, the commentaries from philosophers in this issue.

4. We also find somewhat puzzling the nature of **Le Corre**'s claim. Magnitudes are not supposed to be part of the acquisition of Principle (3), but Principle (3) suffices to determine the relative ordering of collections, for example, that six fish are less than ten fish. So why are magnitudes causally necessary for the latter judgments? Le Corre (2005) thinks that children may simply be unaware of this implication of Principle (3), but this is hard to believe since relative ordering would seem to be (3)'s main business. A more likely explanation, we suspect, is that Le Corre's test for whether children knew the principle – which was whether they could tender six objects in response to the request "Give me six" – was too lax, classifying children as understanding Principle (3) who in fact had a less general form of the principle.

5. Moreover, Ehrlich's (2007) results suggest that, although c-uses dominate teachers' number talk, they do not seem to have improved math test scores much more than other number activities, such as naming number symbols, rote counting, or even nominal use of numbers in addresses and phone numbers, when adjustment is made for floor effects on their overall frequency. Ehrlich also included in her study a questionnaire for parents about home number activities. The results from this questionnaire found that only one activity seemed to have a significant effect on standardized test scores: "mentioned number facts, such as '$1 + 1 = 2$' or '$4 - 2 = 2$.'" The direction of causality is uncertain here, as Ehrlich notes, as superior students may have been more likely to elicit these facts from their parents, but it is of interest that this item is one of the few that reflects i- rather than c-use. Moreover, a regression analysis showed that when these home activities are taken into account, the effect of teacher talk drops out (though this conclusion may depend on the sample of children for whom parent questionnaires were available).

6. There is, however, a relevant difference between learning the concept BOSSIE and learning the concept SEVEN. **Sarnecka** is assuming that the meaning of SEVEN is a cardinality and that it possesses this meaning in isolation from the meanings of other integers, such as SIX or EIGHT. We doubt that this is the best theory of integer meanings, however (see sect. 2 of the target article). Instead, SEVEN may be a function of its relative position in the integer sequence, and if so, SEVEN may depend on INTEGER in a way that BOSSIE does not depend on COW. We also disagree with Sarnecka's assertion that "modular systems are off the table as a possibility by the time the child learns 'one.'" If this were true, then it would be impossible for children to learn such systems at all, and we very much doubt this is the case, given their ability to learn other circular lists, such as the days of the week, months of the year, terms for musical pitches, and so on (see Rips et al. 2008 for more on this point).

7. No set theorist would use, for example, $\{a, b\}$ as a representation for 2, since this representation does not guarantee that a and b are distinct. The usual proposal for the cardinal 2 in set theory is $\{\emptyset,\{\emptyset\}\}$, where $\emptyset$ is the empty set. The $\{a, b\}$ representation could be a problem for psychological accounts, depending on how a and b are interpreted. For example, if a and b are variables that can be bound to any entities, then the theory needs additional apparatus to ensure they are different. But we won't be fussy here and will simply assume that a and b denote distinct physical objects.

8. Hodes also believes that our argument about learning the integers versus learning a modular system is irrelevant because modular counting "is not a way of counting on which Principle (3) would be true." We suspect this apparent disagreement is simply due to an ambiguity about what is meant by "count sequence" in (3). Our point about (3) is that at the time the child is supposed to discover it (around age 4), his or her count sequence consists of only a short initial subsequence of the integers (e.g., "one" through "nine"). Beyond "nine," "$s(n)$" is completely unknown. Hence, at that point, (3) can't fix the meaning of the full set of count terms for the integers.

9. It is true, too, that one-one correspondence is also involved in the statement of the Dedekind-Peano axioms, since these axioms specify the successor function as one-to-one. Therefore, whether children start with the axioms themselves or with Hume's principle and the auxiliary definitions, one-one functions will be in on the ground floor. This supports **Decock**'s suggestion about the developmental importance of one-one relations. Our present point, however, is that one-one functions are not sufficient to derive the positive integers.

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Letters "a" and "r" appearing before authors' initials refer to target article and response, respectively.

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