On tau functions associated with linear systems

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ABSTRACT
Let $(-A, B, C)$ be a linear system in continuous time $t > 0$ with input and output space $\mathbf{C}$ and state space H. The function $\phi_{(x)}(t) = Ce^{-(t+2x)A}B$ determines a Hankel integral operator $\Gamma_{\phi_{(x)}}$ on $L^2((0,\infty);\mathbf{C})$; if $\Gamma_{\phi_{(x)}}$ is trace class, then the Fredholm determinant $\tau(x) = \det(I + \Gamma_{\phi_{(x)}})$ defines the tau function of $(-A, B, C)$. Such tau functions arise in Tracy and Widom's theory of matrix models, where they describe the fundamental probability distributions of random matrix theory. Dyson considered such tau functions in the inverse spectral problem for Schrödinger's equation $-f'' + uf = \lambda f$, and derived the formula for the potential $u(x) = -2\frac{d^2}{dx^2}\log\tau(x)$ in the self-adjoint scattering case *Commun. Math. Phys.* **47** (1976), 171–183. This paper introduces a operator function R_x that satisfies Lyapunov's equation $\frac{dR_x}{dx} = -AR_x - R_xA$ and $\tau(x) = \det(I + R_x)$, without assumptions of self-adjointness. When $-A$ is sectorial, and B, C are Hilbert–Schmidt, there exists a non-commutative differential ring $\mathcal{A}$ of operators in H and a differential ring homomorphism $\lfloor\ \rfloor : \mathcal{A} \to \mathbf{C}[u, u', \dots]$ such that $u = -4\lfloor A\rfloor$, which provides a substitute for the multiplication rules for Hankel operators considered by Pöppe, and McKean *Cent. Eur. J. Math.* **9** (2011), 205–243. The paper obtains conditions on $(-A, B, C)$ for Schrödinger's equation with meromorphic u to be integrable by quadratures. Special results apply to the linear systems associated with scattering u, periodic u and elliptic u. The paper constructs a family of solutions to the Kadomtsev–Petviashivili differential equations, and proves that certain families of tau functions satisfy Fay's identities.

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1. Introduction

This paper is concerned with Fredholm determinants which arise in the theory of linear systems and their application to differential equations such as the Kadomtsev–Petviashvili equation. For $\phi \in L^2((0,\infty);\mathbf{R})$, the Hankel integral operator corresponding to ϕ is Γ_ϕ where

$$\Gamma_\phi f(x) = \int_0^\infty \phi(x+y)f(y)\,dy \qquad (f \in L^2((0,\infty);\mathbf{C}). \tag{1.1}$$

Using the Laguerre system of orthogonal functions as in [60], one can express Γ_ϕ as a matrix $[\gamma_{j+k}]_{j,k=1}^{\infty}$ on ℓ^2, which has the characteristic shape of a Hankel matrix, and one can establish criteria for the operator to be bounded on $L^2((0,\infty);\mathbf{C})$. Megretski, Peller and Treil [52] determined the possible spectrum and spectral multiplicity function that can arise from a bounded and self-adjoint Hankel operator. Thus they characterized the class of bounded self-adjoint Hankel operators up to unitary equivalence. Their method involved introducing suitable linear systems on a state space H, and this motivated the approach of our paper.

Previously, Dyson [20] considered the inverse spectral problem for Schrödinger's equation $-f''+uf=\lambda f$, where $u\in C^2(\mathbf{R};\mathbf{R})$ that decays rapidly as $x\to\pm\infty$. From the asymptotic solutions, he introduced a scattering function ϕ, and considered the translations $\phi_{(x)}(y)=\phi(y+2x)$. He showed that the potential can be recovered from the scattering data by means of the formula

$$u(x)=-2\frac{d^2}{dx^2}\log\det(I+\Gamma_{\phi_{(x)}}). \tag{1.2}$$

These results were developed further by Ercolani, McKean [22] and others [36, 69, 73] to describe the inverse spectral problem for self-adjoint Schrödinger operators on $\mathbf{R}$. Remarkably, some of the methods of inverse scattering theory do not really need self-adjointness. However, a significant obstacle in this approach is that Hankel operators do not have a natural product structure, so it is unclear as to how one can fully exploit the multiplicative properties of determinants. This paper seeks to address this issue, by realizing Hankel operators from linear systems, and then introducing algebras of operators which reflect the properties of Hankel operators and their Fredholm determinants. As in [52], the Lyapunov differential equation is fundamental to the development of the theory.

Definition (i) *(Lyapunov equation).* Let H be a complex Hilbert space, known as the state space, and $\mathcal{L}(H)$ the space of bounded linear operators on H with the usual operator norm. Let $(e^{-tA})_{t\geq 0}$ be a strongly continuous (C_0) semigroup of bounded linear operators on H such that $\|e^{-tA}\|_{\mathcal{L}(H)}\leq M$ for all $t\geq 0$ and some $M<\infty$. Let $\mathcal{D}(A)$ be the domain of the generator $-A$ so that $\mathcal{D}(A)$ is itself a Hilbert space for the graph norm $\|\xi\|^2_{\mathcal{D}(A)}=\|\xi\|^2_H+\|A\xi\|^2_H$, and let $A^\dagger$ be the adjoint of A. Let $R:(0,\infty)\to\mathcal{L}(H)$ be a differentiable function. The Lyapunov equation is

$$-\frac{dR_z}{dz}=AR_z+R_zA \qquad (z>0), \tag{1.3}$$

where the right-hand side is to be interpreted as a bounded bilinear form on $\mathcal{D}(A)\times\mathcal{D}(A^\dagger)$.

(ii) *(Operator ideals).* Let $\mathcal{L}^2(H)$ be the space of Hilbert–Schmidt operators on H, and $\mathcal{L}^1(H)$ be the space of trace class operators on H, so $\mathcal{L}^1(H)=\{T:T=VW;V,W\in\mathcal{L}^2(H)\}$and let det be the Fredholm determinant defined on $\{I+T:T\in\mathcal{L}^1(H)\}$; see [67].

(iii) *(Tau function).* Suppose further that $R_x\in\mathcal{L}^1(H)$ for all $x>x_0$ for some $x_0\in\mathbf{R}$. Then the tau function is $\tau(x)=\det(I+R_x)$ for $x>x_0$.

The significant applications of this equation arises for linear systems.

Definition (i) *(Linear system).* Let H_0 be a complex separable Hilbert space which serves as the input and output spaces; let $B : H_0 \to H$ and $C : H \to H_0$ be bounded linear operators. The continuous-time linear system $(-A, B, C)$ is

$$\begin{aligned} \frac{dX}{dt} &= -AX + BU \\ Y &= CX, \\ X(0) &= 0. \end{aligned} \tag{1.4}$$

(ii) *(Scattering function).* The scattering function is $\phi(x) = Ce^{-xA}B$, which is a bounded and weakly continuous function $\phi : (0, \infty) \to \mathcal{L}(H_0)$. In control theory, the transfer function is the Laplace transform of ϕ.

(iii) *(Hankel operator).* Suppose that $\phi \in L^2((0, \infty); \mathcal{L}(H_0))$. Then the corresponding Hankel operator is Γ_ϕ on $L^2((0, \infty); H_0)$, where $\Gamma_\phi f(x) = \int_0^\infty \phi(x + y)f(y)\, dy$; see [58, 60] for boundedness criteria.

Definition *(Admissible linear system).* Let $(-A, B, C)$ be a linear system as above; suppose furthermore that the observability operator $\Theta_0 : L^2((0, \infty); H_0) \to H$ is bounded, where

$$\Theta_0 f = \int_0^\infty e^{-sA^\dagger} C^\dagger f(s)\, ds; \tag{1.5}$$

suppose that the controllability operator $\Xi_0 : L^2((0, \infty); H_0) \to H$ is also bounded, where

$$\Xi_0 f = \int_0^\infty e^{-sA} Bf(s)\, ds. \tag{1.6}$$

(i) Then $(-A, B, C)$ is an admissible linear system.

(ii) Suppose furthermore that Θ_0 and Ξ_0 belong to the ideal $\mathcal{L}^2$ of Hilbert–Schmidt operators. Then we say that $(-A, B, C)$ is $(2, 2)$-admissible.

The scattering map associates to any $(2, 2)$ admissible linear system $(-A, B, C)$ the corresponding scattering function $\phi(x) = Ce^{-xA}B$. In Corollary 2.3, we show that for a suitable addition and multiplication on the linear systems, this map is additive and multiplicative.

The inverse scattering problem involves recovering data about u from ϕ, as in (1.2). In section 2 of this paper, we analyze the existence and uniqueness problem for the Lyapunov equation, and show that for any $(2, 2)$ admissible linear system, the operator

$$R_x = \int_x^\infty e^{-tA} BCe^{-tA}\, dt \tag{1.7}$$

is trace class and gives the unique solution to (1.3) with the initial condition

$$\Big(\frac{dR_x}{dx}\Big)_{x=0} = -AR_0 - R_0A = -BC. \tag{1.8}$$

Also, $R_x \in \mathcal{L}^1(H)$ and the Fredholm determinant satisfies

$$\det(I + \lambda R_x) = \det(I + \lambda\Gamma_{\phi_{(x)}}) \qquad (x > 0, \lambda \in \mathbf{C}). \tag{1.9}$$

Definition *(Tau function)*. Given an $(2,2)$ admissible linear system $(-A, B, C)$, we define

$$\tau(x) = \det(I + R_x). \tag{1.10}$$

Using this general definition of τ, we can unify several results from the scattering theory of ordinary differential equations. Such tau functions are strongly analogous to the tau functions introduced by Jimbo, Miwa and Ueno [55, 33] to describe the isomonodromy of rational differential equations.

The Gelfand–Levitan–Marchenko equation provides the linkage between ϕ and u via R_x. Consider

$$T(x,y) + \Phi(x+y) + \mu \int_x^\infty T(x,z)\Phi(z+y)\, dz = 0 \qquad (0 < x < y) \tag{1.11}$$

where $T(x,y)$ and $\Phi(x+y)$ are $m \times m$ matrices with scalar entries. In the context of $(-A, B, C)$ we assume that $\Phi(x) = Ce^{-xA}B$ is known and aim to find $T(x,y)$. In section two, we use R_x to construct solutions to the associated Gelfand–Levitan equation (1.11), and introduce a potential

$$u(x) = -2\frac{d^2}{dx^2} \log\det(I + R_x). \tag{1.12}$$

Then we obtain a differential equation linking $\phi(x)$ to $u(x)$. In examples of interest in scattering theory, one can calculate $\det(I+\lambda R_x)$ more easily than the Hankel determinant of $\Gamma_{\phi_{(x)}}$ directly [20, 22, 47, 61], since R_x has additional properties that originate from Lyapunov's equation. In section two, we establish properties of R_x and τ for $(2,2)$ admissible linear systems, and use R_x to solve the Gelfand–Levitan equation.

Definition *(Sectorial operator)*. For $0 < \theta \leq \pi$, we introduce the sector $S_\theta = \{z \in \mathbf{C} \setminus \{0\} : |\arg z| < \theta\}$. A closed and densely defined linear operator $-A$ is sectorial [21] if there exists $\pi/2 < \theta < \pi$ such that S_θ is contained in the resolvent set of $-A$ and $|\lambda| \|(\lambda I + A)^{-1}\|_{\mathcal{L}(H)} \leq M$ for all $\lambda \in S_\theta$. Let $\mathcal{D}(A)$ be the domain of A and $\mathcal{D}(A^\infty) = \cap_{n=0}^\infty \mathcal{D}(A^n)$.

Definition *(Deformations)*. There are three basic deformations of a $(2,2)$-admissible linear system $\Sigma = (-A, B, C)$ with $H_0 = \mathbf{C}$ and $-A$ a sectorial operator, with corresponding effects on tau functions.

(i) Translation takes $\Sigma \mapsto \Sigma(a)$, where $\Sigma(a) = (-A, Be^{-aA}, Ce^{-aA})$ and $\tau(x) \mapsto \tau(x+a)$ for $a \in \Omega$. The translation operation is accounted for in the properties of Hankel operators, and is used in section two.

(ii) The Miura transform is the involution $(-A, B, C) \mapsto (-A, B, -C)$. In section three, we show that the tau functions τ_∞ for $(-A, B, C)$ and τ_0 for $(-A, B, -C)$ satisfy

$$\tau_0''(x)\tau_\infty(x) - 2\tau_0'(x)\tau_\infty'(x) + \tau_0(x)\tau_\infty''(x) = 0. \tag{1.13}$$

(iii) The Darboux addition (spectral shift) [47] maps $\Sigma \mapsto \Sigma_\zeta$ where

$$\Sigma_\zeta = (-A, (\zeta I + A)(\zeta I - A)^{-1}B, C) \qquad (-\zeta \in S_\theta) \tag{1.14}$$

has tau function τ_ζ. In section three, we use this operation to construct families of solutions to Schrödinger's equation.

$$-\psi_\zeta''(x) + u(x)\psi_\zeta(x) = -\zeta^2\psi_\zeta(x) \qquad (x > 0), \tag{1.15}$$

where u is typically complex-valued. When interpreting results, it is important to realize that there is no simple connection between the spectrum of A on H and the spectrum of Schrödinger's equation on $L^2((0,\infty);\mathbf{C})$.

Definition *(State ring).* Let Ω be a domain in $\mathbf{C}$ such that $z, w \in \Omega$ implies $z+w \in \Omega$. Suppose that $I + R_x$ is invertible for all $x \in \Omega$ and so $F_x = (I + R_x)^{-1} \in \mathcal{L}(H)$, and $F_x - I \in \mathcal{L}^1(H)$. Suppose momentarily that $A \in \mathcal{L}(H)$. We introduce an algebra $\mathcal{A}_\Sigma$ of holomorphic functions from Ω to $\mathcal{L}(H)$, which is generated by $I, A \in \mathcal{L}(H)$ and F_x $(x \in \Omega)$, and is a differential ring for the usual pointwise multiplication and differentiation d/dx over Ω as in [63]. We define a new multiplication

$$P * Q = P(AF + FA - 2FAF)Q \tag{1.16}$$

on $\mathcal{A}_\Sigma$, and a new differentiation $\partial : \mathcal{A}_\Sigma \to \mathcal{A}_\Sigma$ by

$$\partial P = A(I - 2F)P + \frac{dP}{dx} + P(I - 2F)A \tag{1.17}$$

so that $(\mathcal{A}, *, \partial)$ is a differential ring. Let $\mathcal{M}_\Omega$ be the meromorphic complex functions on Ω, with the usual pointwise operations. Then we define the bracket

$$\lfloor P \rfloor = Ce^{-xA}F_xPF_xe^{-xA}B, \tag{1.18}$$

so that $\lfloor\,.\,\rfloor : (\mathcal{A}_\Sigma, *, \partial) \to (\mathcal{M}_\Omega, \cdot, d/dx)$ is a differential ring homomorphism, so $\lfloor P * Q \rfloor = \lfloor P \rfloor \lfloor Q \rfloor$ and $(d/dx)\lfloor P \rfloor = \lfloor \partial P \rfloor$.

This algebra $(\mathcal{A}_\Sigma, *, \partial)$ is generally non commutative, is realized as an algebra of operators on the space space H of $(-A, B, C)$, and provides a substitute for the multiplication structure that is lacking in the theory of Hankel operators. Our main result is as follows.

Theorem 1.1. *Suppose that $\Sigma = (-A, B, C)$ is a $(2,2)$ admissible linear system with $-A$ sectorial for S_θ and $H_0 = \mathbf{C}$.*

(i) Then there exist $x_0 > 0$ and a solution R_x to (1.3) and (1.8) such that $\tau(x) = \det(I+R_x)$ is holomorphic and $u(x)$ is meromorphic for $x \in \Omega$ where $\Omega = \{x_0+z : z \in S_{\theta-\pi/2}\}$.

(ii) The tau functions for Σ and Σ_ζ give $\psi_\zeta(x) = e^{\zeta x}\tau_\zeta(x)/\tau(x)$ that satisfies (1.15) for $-\zeta \in S_\theta$.

(iii) There exists a differential algebra $(\mathcal{A}_\Sigma, *, \partial)$ on Ω which contains $F_x = (I + R_x)^{-1}$ and there exists a differential ring homomorphism $\lfloor\,.\,\rfloor : \mathcal{A}_\Sigma \to \mathcal{M}_\Omega$ such that $u = -4\lfloor A \rfloor$.

(iv) If A satisfies $\lfloor p(A) \rfloor = 0$ for some non-zero odd complex polynomial p, then $\mathbf{C}[u, \partial u/\partial x, \dots,]$ is a Noetherian differential ring for $\partial/\partial x$ and the standard multiplication.

(v) If $\lfloor A^{2m-1} \rfloor = 0$ for some m, then (1.15) can be integrated by quadratures.

In sections 4 and 5 we show that $\lfloor\cdot\rfloor$ maps $(-4\partial^j A)_{j=0}^\infty$ to $(u^{(j)})_{j=0}^\infty$, and $((-1)^j 2A^{2j-1})_{j=1}^\infty$ to $(f_j)_{j=1}^\infty$, where (f_j) satisfies the stationary KdV hierarchy. This provides the crucial link between algebraic properties of $\mathcal{A}_\Sigma$ for $\Sigma = (-A, B, C)$ and the spectral Schrödinger equation.

We recall that a compact Riemann surface $\mathcal{E}$ is hyperelliptic if and only if there exists a meromorphic function on $\mathcal{E}$ that has precisely two poles. In this case, there is a two-sheeted cover $\mathcal{E} \to \mathbf{P}^1$ with $2g+2$ branch points, where g is the genus of $\mathcal{E}$. The elliptic case has $g = 1$. In case (v), u is associated a meromorphic function on a hyperelliptic curve, and the potential is said to be finite-gap or algebro-geometric; see [14, 29].

Theorem 1.1 applies to scattering potentials such that $u(x) \to 0$ as $x \to \infty$. The case of periodic potentials is considered in sections 6 and 7 of this paper.

Definition *(Periodic linear system).* Let $A \in \mathcal{L}(H)$ be such that $(e^{xA})_{x\in\mathbf{R}}$ is a periodic group, and let $B \in \mathcal{L}(H_0, H)$ and $C \in \mathcal{L}(H, H_0)$ satisfy $AE + EA = BC$ for some $E \in \mathcal{L}^1(H)$. Then $\Sigma = (-A, B, C; E)$ is a uniformly periodic linear system, and $\tau(x) = \det(I + e^{xA}Ee^{xA})$ is the corresponding tau function.

In section 6 of this paper, we introduce a ring $(\mathcal{A}_\Sigma, *, \partial)$ for periodic linear systems. The periodic linear system Σ has a tau function τ and a periodic potential u, as in Hill's equation $-f'' + uf = \lambda f$. Hence Σ is associated with Hill's discriminant $\Delta(\lambda)$ and a spectral curve $\mathcal{E}$, which is typically a transcendental hyperelliptic curve of infinite genus. The Jacobi variety $\mathbf{X}$ of $\mathcal{E}$ is then an infinite dimensional complex torus.

We show that if u is an elliptic function, then there exist uniformly periodic linear systems with tau functions τ_1 and τ_0 such that $u(x) = \tau_1(x)/\tau_0(x)$. Also, we show that if Hill's equation for this u has general solution $f(x; \lambda)$ that is meromorphic in x for all but finitely many $\lambda \in \mathbf{C}$, then there exist uniformly periodic linear systems with tau function $\tau_3(x; \lambda)$ and $\tau_4(x; \lambda)$ such that $f(x; \lambda) = \tau_3(x; \lambda)/\tau_4(x; \lambda)$.

Using results of Gesztesy and Weikard [31], we prove a partial converse, namely that if Hill's equation has a general solution of the form $f(x; \lambda) = \tau_3(x; \lambda)/\tau_4(x; \lambda)$ for all $\lambda \in \mathbf{C}$, where $\tau_3(x; \lambda)$ and $\tau_4(x; \lambda)$ are the tau functions of uniformly periodic linear systems, then u is an algebro geometric potential and is associated with a hyperelliptic spectral curve $\mathcal{E}$ of finite genus.

The notion of a tau function of a linear system generalizes the classical concept of a theta function for an algebraic curve. We recall that a complete complex algebraic curves is associated with a finite dimensional Jacobian variety which has dimension determined by the genus of the curve. Riemann's theta functions may be defined on such a variety, and they satisfy some addition rules known as Fay's identities, which reflect the geometry of the underlying curve [23]. Ercolani and McKean [22] observed that Fay's identities can be deduced from secant identities which relate to the properties of Wronskians of suitably chosen functions. In this spirit, we prove that tau functions of linear systems satisfy some Wronskian identities which are counterparts of Fay's identities. Mumford [56] observed that Fay's identities give rise to nontrivial solutions of certain partial differential equations.

The Kadomtsev–Petviashvili equations describe the waves in a two dimensional dissipative medium where the scale of the propagation of the wave along the y-axis is much larger than the longitudinal scale along the x-axis. We write

$$KP \qquad \frac{\partial}{\partial x}\Big(\frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x} + 4\lambda\frac{\partial u}{\partial x} + 4\alpha\frac{\partial u}{\partial s}\Big) + 3\beta^2\frac{\partial^2 u}{\partial y^2} = 0, \tag{1.19}$$

where the $\alpha, \beta, \lambda \in \mathbf{C}$ are parameters. Krichever showed that any tau function that arises from the theta function on the Jacobi variety of a complete complex algebraic curve satisfies KP. Shiota [66] and Mulase [56] proved the converse, that if τ is the theta function of a finite-dimensional Abelian manifold, and τ gives a potential that satisfies KP, then the Abelian manifold arises as a Jacobi variety of a complex algebraic curve.

In section 9, we show that the τ functions of general linear systems under a group of deformations give rise to a set of solutions of the KP equations.

Theorem 1.2. *Let $(-A_1, B_0, C_0)$ and $(-A_2, B_0, C_0)$ be $(2,2)$ admissible linear systems with input and output spaces $\mathbf{C}$, where $A_1, A_2 \in \mathcal{L}(H)$. Let*

$$C(y;t) = C_0 e^{t(A_1^3+\lambda A_1)/\alpha - yA_1^2/\beta}, \qquad B(y;t) = e^{t(A_2^3+\lambda A_2)/\alpha + yA_2^2/\beta} B_0;$$

then with

$$R_x(y,t) = \int_x^\infty e^{-A_2 s} B(y;t) C(y;t) e^{-A_1 s} ds, \tag{1.20}$$

let

$$u(x,y,t) = -2\frac{\partial^2}{\partial x^2} \log\det(I + R_x(y,t)). \tag{1.21}$$

Then u satisfies KP in the form (1.19).

The KP differential equation is the first in a sequence of nonlinear partial differential equations known as the KP hierarchy. Shiota [66] proved that these are related by an integral involving a family of tau functions which are subject to a group of deformations involving infinitely many parameters. We introduce a family of linear systems which are subject to a group of deformations, and also show that such a family gives tau functions which make Shiota's integral vanish; this condition is known to give solutions of the KP hierarchy.

2. τ functions in terms of Lyapunov's equation and the Gelfand–Levitan equation

The following proves uniqueness of solutions of the Lyapunov equation (1.3), in a style suggested by [60, p 503]. Peller [60] discusses scattering functions that produce bounded self-adjoint Hankel operators Γ_ϕ, and their realization in terms of continuous time linear systems. He observes that in some cases one needs a bounded semigroup with unbounded generator $(-A)$. We prove the uniqueness results for bounded and strongly continuous semigroups, then specialize to holomorphic semigroups. The main application is to the Gelfand–Levitan equation, and associated determinants.

Proposition 2.1. *Let $(e^{-tA})_{t\geq 0}$ be a strongly continuous and weakly asymptotically stable semigroup on a complex Hilbert space H, so $e^{-tA}f \to 0$ weakly as $t \to \infty$ for all $f \in H$. Then*

(i) $S_t : R \mapsto e^{-tA}Re^{-tA}$ for $t \geq 0$ defines a strongly continuous semigroup on $\mathcal{L}^1(H)$, which has generator $(-L)$, with dense domain of definition $\mathcal{D}(L)$ such that

$$L(R) = AR + RA \qquad (R \in \mathcal{D}). \tag{2.1}$$

(ii) The linear operator $L : \mathcal{D}(L) \to \mathcal{L}^1(H)$ is injective, and each $R_0 \in \mathcal{D}(L)$ with $L(R_0) = X$, there exists a weakly convergent improper integral

$$R_0 = \int_0^\infty e^{-tA} X e^{-tA}\, dt. \tag{2.2}$$

(iii) Suppose moreover that $\|e^{-t_0 A}\|_{\mathcal{L}(H)} < 1$ for some $t_0 > 0$. Then $L : \mathcal{D}(L) \to \mathcal{L}^1(H)$ is surjective, the integral (2.2) converges absolutely in $\mathcal{L}^1(H)$ and R_0 gives the unique solution to $AR_0 + R_0 A = X$.

Proof. (i) First observe that by the uniform boundedness theorem, there exists M such that $\|e^{-tA}\|_{\mathcal{L}(H)} \leq M$ for all $t \geq 0$, so $(e^{-tA})_{t\geq 0}$ is uniformly bounded. Also, the adjoint semigroup $(e^{-tA^\dagger})_{t\geq 0}$ is also strongly continuous and uniformly bounded, so A and $A^\dagger$ have dense domains $\mathcal{D}(A)$ and $\mathcal{D}(A^\dagger)$ in H.

Now $\mathcal{L}^1(H) = H \hat{\otimes} H$, the projective tensor product, so for all $X \in \mathcal{L}^1(H)$, there exists a nuclear decomposition $X = \sum_{j=1}^\infty B_j C_j$ where $B_j, C_j \in H$ satisfy $\|X\|_{\mathcal{L}^1(H)} = \sum_{j=1}^\infty \|B_j\|_H \|C_j\|_H$. Then

$$S_t(X) - X = \sum_{j=1}^\infty (e^{-tA} B_j C_j e^{-tA} - B_j C_j e^{-tA}) + \sum_{j=1}^\infty (B_j C_j e^{-tA} - B_j C_j) \tag{2.3}$$

where (e^{-tA}) is bounded, $\|e^{-tA} B_j - B_j\|_H \to 0$ and $\|e^{-tA^\dagger} C_j - C_j\|_H \to 0$ as $t \to 0+$, so $\|S_t(X) - X\|_{\mathcal{L}^1(H)} \to 0$ as $t \to 0+$, so $(S_t)_{t\geq 0}$ is strongly continuous on $\mathcal{L}^1(H)$. By semigroup theory, there exists a dense linear subspace $\mathcal{D}(L)$ of $\mathcal{L}^1(H)$ such that $S_t(R)$ is differentiable at $t = 0+$ for all $R \in \mathcal{D}$, and $(d/dt)_{t=0+} S_t(R) = -AR - RA$, so the generator is $(-L)$, where $L(R) = AR + RA$.

(ii) Certainly $\mathcal{D}$ contains $\mathcal{D}(A^\dagger) \hat{\otimes} \mathcal{D}(A)$ in $\mathcal{L}^1(H) = H \hat{\otimes} H$. Choosing $f \in \mathcal{D}(A)$ and $g \in \mathcal{D}(A^\dagger)$, we find that

$$\begin{aligned} \frac{d}{dt} \langle e^{-tA} R_0 e^{-tA} f, g \rangle &= -\langle e^{-tA} (AR_0 + R_0 A) e^{-tA} f, g \rangle \\ &= -\langle e^{-tA} X e^{-tA} f, g \rangle \end{aligned} \tag{2.4}$$

a continuous function of $t > 0$, so integrating we obtain

$$\langle R_0 f, g \rangle - \langle e^{-sA} R_0 e^{-sA} f, g \rangle = \int_0^s \langle e^{-tA} X e^{-tA} f, g \rangle\, dt. \tag{2.5}$$

We extend this identity to all $f, g \in H$ by joint continuity; then we let $s \to \infty$ and observe that $R_0 : H \to H$ is trace class and hence is completely continuous, hence R_0 maps the weakly null family $(e^{-sA} f)_{s\to\infty}$ to the norm convergent family $(R_0 e^{-sA} f)_{s\to\infty}$, so $\langle e^{-sA} R_0 e^{-sA} f, g \rangle \to 0$ as $s \to \infty$, hence we have a weakly convergent improper integral

$$\langle R_0 f, g \rangle = \lim_{s\to\infty} \int_0^s \langle e^{-tA} X e^{-tA} f, g \rangle\, dt \qquad (f, g \in H). \tag{2.6}$$

(iii) The function $t \mapsto e^{-tA}Xe^{-tA}$ takes values in the separable space $\mathcal{L}^1(H)$ and is weakly continuous, hence strongly measurable, by Pettis's theorem. By considering the spectral radius, Engel and Nagel [21] show that there exist $\delta > 0$ and $M_\delta > 0$ such that $\|e^{-tA}\|_{\mathcal{L}(H)} \leq M_\delta e^{-\delta t}$ for all $t \geq 0$; hence (2.2) converges as a Bochner–Lebesgue integral with

$$\begin{aligned}\|R_x\|_{\mathcal{L}^1(H)} &\leq \int_x^\infty M_\delta^2 \|X\|_{\mathcal{L}^1(H)} e^{-2\delta t}\, dt \\ &\leq \frac{M_\delta^2}{2\delta} \|X\|_{\mathcal{L}^1(H)} e^{-2\delta x}. \end{aligned} \tag{2.7}$$

Furthermore, A is a closed linear operator and satisfies

$$\begin{aligned} A\int_x^s e^{-tA}Xe^{-tA}\, dt + \int_x^s e^{-tA}Xe^{-tA}\, dtA &= \int_x^s -\frac{d}{dt}\big(e^{-tA}Xe^{-tA}\big)\, dt \\ &= e^{-xA}Xe^{-xA} - e^{-TA}Xe^{-TA} \\ &\to e^{-xA}Xe^{-xA} \end{aligned} \tag{2.8}$$

as $s \to \infty$ where $\int_x^s e^{-tA}Xe^{-tA}dt \to R_x$; so $AR_x + R_xA = e^{-xA}Xe^{-xA}$ for all $x \geq 0$. We deduce that $x \mapsto R_x$ is a differentiable function from $(0, \infty)$ to $\mathcal{L}^1(H)$ and that the modified Lyapunov equation (1.3) holds.

□

The hypotheses (i) and (ii) are symmetrical under the adjoint $(A, R_0) \mapsto (A^\dagger, R_0^\dagger)$; however, the hypothesis (iii) is rather stringent, and will be replaced in examples by sharper conditions.

We introduce Lyapunov's equation, and the existence of solutions for suitable $(-A, B, C)$. The solution R_x is defined by a formula suggested by Heinz's theorem [8] and has properties analogous to the resolvent operator of a semigroup.

Definition *($(2,2)$ admissible linear systems).* (i) Let H be a complex Hilbert space and let $\Sigma = (-A, B, C)$ be a linear system with state space H. Suppose that there is a weakly convergent integral

$$W_c = \int_0^\infty e^{-tA}BB^\dagger e^{-tA^\dagger}\, dt \tag{2.9}$$

which defines a bounded linear operator on H; then W_c is the controllability Gramian. Suppose further that there exists a weakly convergent integral

$$W_o = \int_0^\infty e^{-tA^\dagger}C^\dagger C e^{-tA}\, dt \tag{2.10}$$

which defines a bounded linear operator on H; then W_o is the observability Gramian.

(ii) Then we define R_x to be the bounded linear operator on H determined by the weakly convergent integral

$$R_x = \int_x^\infty e^{-tA}BCe^{-tA}\, dt. \tag{2.11}$$

(iii) Then Σ satisfying (i) is said to be balanced if $W_c = W_o$ and $\ker(W_c) = 0$.

(iv) Also, Σ satisfying (i) is said to be $(2,2)$ admissible if W_c and W_o are trace class.

(v) We introduce the scattering function $\phi(t) = Ce^{-tA}B$ and the shifted scattering function $\phi_{(x)}(t) = \phi(t+2x)$ for $x, t > 0$.

(vi) The tau function of Σ is $\tau(x) = \det(I + R_x)$.

(vii) For $0 < \delta < \pi$, we introduce the sector $S_\delta = \{z \in \mathbf{C} \setminus \{0\} : |\arg z| < \delta\}$. For $\pi/2 < \delta < \pi$, we introduce $X_\delta = \{\zeta \in S_\delta : -\zeta \in S_\delta\}$ which is an open set, symmetrical about $i\mathbf{R}$ and bounded by lines passing through 0.

Theorem 2.2. *Let $(-A, B, C)$ be a linear system such that $\|e^{-t_0 A}\|_{\mathcal{L}(H)} < 1$ for some $t_0 > 0$, and that B and C are Hilbert–Schmidt operators such that $\|B\|_{\mathcal{L}^2(H_0;H)}\|C\|_{\mathcal{L}^2(H;H_0)} \leq 1$. Suppose further that $-A$ is sectorial on S_θ for some $\pi/2 < \theta < \pi$.*

(i) Then $(-A, B, C)$ is $(2,2)$-admissible, so the trace class operators R_x give the solution to Lyapunov's equation (1.3) for $x > 0$ that satisfies the initial condition (1.8), and the solution to (1.3) with (1.8) is unique.

(ii) The function $\tau(x) = \det(I + R_x)$ is differentiable for $x \in (0, \infty)$.

(iii) Then R_z extends to a holomorphic function which satisfies (1.3) on $S_{\theta-\pi/2}$, and $R_z \to 0$ as $z \to \infty$ in $S_{\theta-\varepsilon-\pi/2}$ for all $0 < \varepsilon < \theta - \pi/2$.

Proof. (i) Since $BC \in \mathcal{L}^1(H)$, the integrand of (2.11) takes values in $\mathcal{L}^1(H)$, and we can apply Proposition 2.1(iii) to $X = BC$.

(ii) The Fredholm determinant $R \mapsto \det(I + R)$ is a continuous function on $\mathcal{L}^1(H)$. Also the integral $R_x = \int_x^\infty e^{-tA}BCe^{-tA}\,dt$ belongs to $\mathcal{D}(L)$ and gives a differentiable function of $x > 0$ with values in $\mathcal{L}^1(H)$.

(iii) By classical results of Hille, $(e^{-zA})_{z\in S_{\theta-\pi/2}}$ defines an analytic semigroup on $S_{\theta-\pi/2}$, bounded on S_ν for all $0 < \nu < \theta - \pi/2$, so we can define $R_z = e^{-zA}R_0 e^{-zA}$ and obtain an analytic solution to Lyapunov's equation. For all $0 < \varepsilon < \theta - \pi/2$, there exists M'_ε such that $\|e^{-zA}\|_{\mathcal{L}(H)} \leq M'_\varepsilon$ for all $z \in S_\delta$ where $\delta = \theta - \varepsilon - \pi/2$. Now for $z \in S_{\delta/2}$, we write $z = x/2 + (x/2 + iy)$ with $x/2 + iy \in S_\delta$ and use the bound $\|e^{-zA}\|_{\mathcal{L}(H)} \leq \|e^{-xA/2}\|_{\mathcal{L}(H)}\|e^{-(x/2+iy)A}\|_{\mathcal{L}(H)}$ to obtain $\|e^{-zA}\|_{\mathcal{L}(H)} \leq M'^2_\varepsilon \|e^{-t_0 A}\|^{x/(4t_0)}_{\mathcal{L}(H)}$, so $\|e^{-zA}\|_{\mathcal{L}(H)} \to 0$ exponentially fast as $z \to \infty$ in the sector $S_{\delta/2}$. Hence R_z is holomorphic and bounded on $S_{(\theta-\varepsilon-\pi/2)}$ and by (2.7), $R_z \to 0$ as $z \to \infty$ in $S_{(\theta-\varepsilon-\pi/2)/2}$.

$\square$

Definition Given $M, \varepsilon > 0$ and $0 < \delta < \pi/2$, let $\mathcal{R}(\delta, M, \varepsilon)$ be the space of all the holomorphic functions $\psi : S_\delta \to \mathbf{C}$ such that $e^{\varepsilon|z|}|\psi(z)| \leq M$ for all $z \in S_\delta$. Then let $\mathcal{R} = \cup_{0<M, 0<\varepsilon, 0<\delta<\pi}\mathcal{R}(\delta, M, \varepsilon)$.

We note that by Cauchy's estimates, $\mathcal{R}$ has the following properties:

(i) $\phi' \in \mathcal{R}$ for all $\phi \in \mathcal{R}$;

(ii) for all $\phi \in \mathcal{R}$, there exists $\psi \in \mathcal{R}$ such that $\psi' = \phi$;

(iii) if $\phi \in \mathcal{R}$, then $e^\phi - 1 \in \mathcal{R}$;

(iv) $\mathcal{R} + \mathbf{C}$ is an integral domain under pointwise addition and multiplication;

(v) $\mathcal{R} + \mathbf{C}$ is closed under taking of exponentials.

Hence we can therefore form the field of fractions of $\mathcal{R} + \mathbf{C}$ to obtain a differential field $\mathcal{F}$, so that every elements of $\mathcal{F}$ is meromorphic on some sector S_η.

Corollary 2.3. *The set of scattering functions ϕ which arise from the $(2,2)$ admissible linear systems as in Theorem 2.2 gives a subring of $\mathcal{R}$.*

Proof. Let $(-A_j, B_j, C_j)$ be $(2,2)$ admissible linear system with scattering function ϕ_j as in Theorem 2.2 for $j = 1, 2$. Then by Theorem 2.2(iii) the scattering functions satisfy $\phi_j \in \mathcal{R}$.

The linear system

$$\left(\begin{bmatrix} -A_1 & 0 \\ 0 & -A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}\right). \tag{2.12}$$

is $(2,2)$-admissible with scattering function $\phi_1(x) + \phi_2(x)$, as one checks by direct calculation.

The linear system

$$\big(-(A_1 \otimes I + I \otimes A_2), B_1 \otimes B_2, C_1 \otimes C_2\big). \tag{2.13}$$

is also $(2,2)$ admissible with scattering function $\phi_1(x)\phi_2(x)$. One checks that the semigroups $(e^{-tA_1} \otimes I)_{t>0}$ and likewise $(I \otimes e^{-tA_2})_{t>0}$ are strongly continuous on the tensor product Hilbert space $H \otimes H$, then $(e^{-tA_1} \otimes e^{-tA_2})_{t>0}$ is strongly continuous on $H \otimes H$ by B15 of Engel]. Also, $B_1 \otimes B_2$ and $C_1 \otimes C_2$ are Hilbert–Schmidt, as one checks by considering orthonormal bases.

$\square$

Example. Let $\Delta = -d^2/dx^2$ be the usual Laplace operator which is essentially self-adjoint and non-negative on $C_c^\infty(\mathbf{R}; \mathbf{C})$ in $L^2(\mathbf{R}; \mathbf{C})$. We introduce $A = \sqrt{I + \Delta}$ which is given by the Fourier multiplier $\mathcal{F}Af(\xi) = \sqrt{1 + \xi^2}\mathcal{F}f(\xi)$. Then (e^{-zA}) and (e^{-zA^2}) give bounded holomorphic semigroups on H, as in Theorem 2.2, on the right half-plane $\{z \in \mathbf{C} : \Re z \geq 0\}$, which is the closure of $S_{\pi/2}$. On the imaginary axis, we have unitary groups (e^{itA}) and (e^{-itA^2}). By classical results from wave equations, we can write $e^{itA} + e^{-itA} = 2\cos(tA)$ where $u(x,t) = \cos(tA)f(x)$ is given by

$$u(x,t) = \frac{1}{2}\big(f(x+t) + f(x-t)\big) + \frac{t}{2}\int_{x-t}^{x+t} f(y)\frac{J_0'(\sqrt{t^2 - (x-s)^2})}{\sqrt{t^2 - (x-s)^2}}ds \qquad (f \in C_c^\infty(\mathbf{R}; \mathbf{C})),. \tag{2.14}$$

with J_0 is Bessel's function of the first kind of order zero, and u satisfies

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} &= u(x,t) \\ u(x,0) &= f(x); \\ \frac{\partial u}{\partial t}(x,0) &= 0. \end{aligned} \tag{2.15}$$

Note that $(\exp(t(iA)^{2j-1}))$ gives a unitary group on H for $j = 0, 1, 2, \ldots$.

Definition (i) *(Block Hankel operators).* Say that $\Gamma \in \mathcal{L}(H)$ is block Hankel if there exists $1 \leq m < \infty$ such that Γ is unitarily equivalent to the block matrix $[A_{j+k-2}]_{j,k=1}^\infty$ on $\ell^2(\mathbf{C}^m)$ where A_j is a $m \times m$ complex matrix for $j = 0, 1, \ldots$.

(ii) Let $(-A, B, C)$ be a $(2,2)$ admissible linear system with input and output space H_0, where the dimension of H_0 over $\mathbf{C}$ is $m < \infty$. Then m is the number of outputs of the system, and systems with finite $m>1$ are known as MIMO for multiple input, multiple output, and give rise to block Hankel operators with $\Phi(x)=Ce^{-xA}B$; see [60].

(iii) The Gelfand–Levitan integral equation for $(-A,B,C)$ as in (ii) is

$$T(x,y)+\Phi(x+y)+\mu\int_x^\infty T(x,z)\Phi(z+y)\,dz=0\qquad (0<x<y)\tag{2.16}$$

where $T(x,y)$ and $\Phi(x+y)$ are $m\times m$ matrices with scalar entries, and $\mu\in\mathbf{C}$.

Proposition 2.4. *(i) In the notation of Theorem 2.2, there exists $x_0>0$ such that*

$$T_\mu(x,y)=-Ce^{-xA}(I+\mu R_x)^{-1}e^{-yA}B\tag{2.17}$$

satisfies the integral equation (2.16) for $x_0<x<y$ and $|\mu|<1$.

(ii) The determinant satisfies $\det(I+\mu R_x)=\det(I+\mu\Gamma_{\Phi_{(x)}})$ *and*

$$\mu\,\mathrm{trace}\,T_\mu(x,x)=\frac{d}{dx}\log\det(I+\mu R_x).\tag{2.18}$$

(iii) Suppose that $t\mapsto U(t)$ is a continuous function $[0,1]\to\mathcal{L}(H)$ such that $U(t)A=AU(t)$ and $\|U(t)\|_{\mathcal{L}(H)}\leq 1$. Then there is a family of $(2,2)$ admissible linear systems

$$\Sigma(t)=(-A,U(t)B,CU(t))\qquad (t\in[0,1]);\tag{2.19}$$

the corresponding tau function $\tau(x,t)$ is continuous for $(x,t)\in(0,\infty)\times[0,1]$.

Proof. (i) We choose x_0 so large that $e^{\delta x_0}\geq M_\delta/2\delta$, then by (2.7), we have $|\mu|\|R_x\|_{\mathcal{L}(H)}<1$ for $x>x_0$, so $I+\mu R_x$ is invertible. Substituting into the integral equation, we obtain

$$\begin{aligned}Ce^{-(x+y)A}B&-Ce^{-xA}(I+\mu R_x)^{-1}e^{-yA}B\\&-\mu Ce^{-xA}(I+\mu R_x)^{-1}\int_x^\infty e^{-zA}BCe^{-zA}\,dze^{-yA}B\\&=Ce^{-(x+y)A}B-Ce^{-xA}(I+\mu R_x)^{-1}e^{-yA}B-\mu Ce^{-xA}(I+\mu R_x)^{-1}R_xe^{-yA}B\\&=0.\end{aligned}\tag{2.20}$$

(ii) As in (1.5), the operator $\Theta_x:L^2(0,\infty)\to H$ is Hilbert–Schmidt; likewise $\Xi_x:L^2(0,\infty)\to H$ is Hilbert–Schmidt; so $(-A,B,C)$ is $(2,2)$-admissible. Hence $\Gamma_{\Phi_{(x)}}=\Theta_x^\dagger\Xi_x$ and $R_x=\Xi_x\Theta_x^\dagger$ are trace class, $(I+\mu R_x)$ is a holomorphic function of x on some sector S_δ as in Theorem 2.2 and

$$\det(I+\mu R_x)=\det(I+\mu\Xi_x\Theta_x^\dagger)=\det(I+\mu\Theta_x^\dagger\Xi_x)=\det(I+\mu\Gamma_{\Phi_{(x)}}).\tag{2.21}$$

By the Riesz functional calculus, $(I+\mu R_x)^{-1}$ is meromorphic for x in some S_δ. Correcting a typographic error in [9, p. 324], we rearrange terms and calculate the derivative

$$\begin{aligned}\mu T_\mu(x,x)&=-\mu\mathrm{trace}\Bigl(Ce^{-xA}(I+\mu R_x)^{-1}e^{-xA}B\Bigr)\\&=-\mu\mathrm{trace}(I+\mu R_x)^{-1}e^{-xA}BCe^{-xA}\\&=\mu\mathrm{trace}\Bigl((I+\mu R_x)^{-1}\frac{dR_x}{dx}\Bigr)\\&=\frac{d}{dx}\mathrm{trace}\log(I+\mu R_x).\end{aligned}\tag{2.22}$$

This identity is proved for $|\mu| < 1$ and extends by analytic continuation to the maximal domain of $T_\mu(x,x)$.

(iii) Since A commutes with $U(t)$, the domain $\mathcal{D}(A)$ is invariant under $U(t)$, and the multiplications $B \mapsto U(t)B$, $C \mapsto CU(t)$ and $e^{-xA} \mapsto U(t)e^{-xA}U(t)$ preserve the hypotheses of Theorem 2.2, so $(-A, U(t)B, CU(t))$ is $(2,2)$ admissible. By commutativity, we have

$$\tau(x,t) = \det(I + U(t)R_xU(t)), \tag{2.23}$$

which depends continuously on (x,t).

□

We refer to $\Sigma(t) = (-A, U(t)B, CU(t))$ as a deformation of Σ, and analyze particular cases below.

3. The Baker–Akhiezer function of an admissible linear system

In this section, we consider the Darboux addition rule for potentials and analyze the transformation $(-A,B,C) \mapsto (-A,B,-C)$ and the effect on the ratios and derivatives of τ functions.

Definition *(Baker–Akhiezer function).* (i) Let $(-A,B,C)$ be as in Theorem 2.2, and let

$$\Sigma_\zeta = (-A, (\zeta I + A)(\zeta I - A)^{-1}B, C) \qquad (\zeta \in \mathbf{C} \cup \{\infty\} \setminus \mathrm{Spec}(A)) \tag{3.1}$$

so that Σ_ζ defines a $(2,2)$ admissible linear systems for ζ in an open subset of $\mathbf{C} \cup \{\infty\}$ which includes $\{\zeta \in \mathbf{C} : -\zeta \in S_\theta\}$ for some $\pi/2 < \theta < \pi$. We identify Σ_∞ with $(-A,B,C)$, and Σ_0 with $(-A,B,-C)$.

(ii) Let τ_ζ be the tau function of Σ_ζ, and let the Baker–Akhiezer function for the family of linear systems be

$$\psi_\zeta(x) = \frac{\tau_\zeta(x)}{\tau_\infty(x)} \exp\bigl(\zeta x\bigr). \tag{3.2}$$

(iii) Let $\tau^*_\zeta(x) = \overline{\tau_{\bar\zeta}(\bar x)}$ as in Schwarz's reflection principle, and let

$$\Sigma^*_\zeta = (-A^\dagger, C^\dagger, B^\dagger(\zeta I + A^\dagger)(\zeta I - A^\dagger)^{-1}) \qquad (\zeta \in \mathbf{C} \cup \{\infty\} \setminus \mathrm{Spec}(A^\dagger)) \tag{3.3}$$

so $\Sigma_\zeta \mapsto \Sigma^*_\zeta$ is an involution, and Σ^*_ζ has tau function τ^*.

The following result introduces a family of solutions of Schrödinger equation corresponding to the Σ_ζ with an addition rule in the style of Darboux.

Proposition 3.1. *Let $(-A,B,C)$ be as in Theorem 2.2.*

(i) Then for $-\zeta \in S_\theta$, the linear system Σ_ζ is also $(2,2)$ admissible, and the Baker–Akhiezer function satisfies

$$-\frac{d^2}{dx^2}\psi_\zeta(x) + u_\infty(x)\psi_\zeta(x) = -\zeta^2\psi_\zeta(x). \tag{3.4}$$

(ii) There exist $h_j \in C^\infty((0,\infty);\mathbf{C})$ such that there is an asymptotic expansion

$$\psi_\zeta(x) \asymp e^{\zeta x}\Bigl(1 + \frac{h_1(x)}{\zeta} + \frac{h_2(x)}{\zeta^2} + \dots\Bigr) \tag{3.5}$$

as $\zeta \to \pm i\infty$, and the expansion is uniform for x in compact subsets of $(0,\infty)$.

Proof (i) For all $\zeta \in \mathbf{C}\backslash \mathrm{Spec}(A)$, there exists $x_0(\zeta)$ such that $\|(\zeta I+A)(\zeta I-A)^{-1}R_x\|_{\mathcal{L}^1(H)} < 1$ for all $x > x_0(\zeta)$, so that $\tau_\zeta(x)$ is continuously differentiable and non-zero as a function of $x \in (x_0(\zeta),\infty)$. In particular, suppose that $\Re\zeta < 0$, then $-\zeta \in S_\theta$ so $\zeta I - A$ is invertible. Using the R function for Σ_ζ, we write

$$\begin{aligned}\frac{\tau_\zeta(x)}{\tau_\infty(x)} &= \frac{\det\bigl(I + (\zeta I + A)(\zeta I - A)^{-1}R_x\bigr)}{\det\bigl(I + R_x\bigr)}\\ &= \frac{\det\bigl(I + (\zeta I - A)^{-1}((\zeta I - A)R_x + AR_x + R_xA)\bigr)}{\det\bigl(I + R_x\bigr)}\\ &= \frac{\det\bigl(I + R_x + (\zeta I - A)^{-1}(AR_x + R_xA)\bigr)}{\det\bigl(I + R_x\bigr)}\end{aligned} \tag{3.6}$$

so that when $AR_x + R_xA$ has rank one, the perturbing term $(\zeta I - A)^{-1}(AR_x + R_xA)$ has rank one; continuing we find

$$\begin{aligned}\frac{\tau_\zeta(x)}{\tau_\infty(x)} &= \det\bigl(I + (\zeta I - A)^{-1}e^{-xA}BCe^{-xA}(I + R_x)^{-1}\bigr)\\ &= \det\bigl(I + Ce^{-xA}(I + R_x)^{-1}(\zeta I - A)^{-1}e^{-xA}B\bigr)\\ &= 1 + Ce^{-xA}(I + R_x)^{-1}(\zeta I - A)^{-1}e^{-xA}B,\end{aligned} \tag{3.7}$$

since $B : \mathbf{C} \to H$ and $C : H \to \mathbf{C}$ have rank one. Hence

$$\begin{aligned}\psi_\zeta(x) &= \frac{\tau_\zeta(x)}{\tau_\infty(x)}\exp\bigl(\zeta x\bigr)\\ &= \exp\bigl(\zeta x\bigr) + Ce^{-xA}(I + R_x)^{-1}(\zeta I - A)^{-1}e^{-xA}B\exp\bigl(\zeta\bigr)\\ &= \exp\bigl(\zeta x\bigr) - \int_x^\infty Ce^{-xA}(I + R_x)^{-1}e^{-yA}B\exp\bigl(\zeta y\bigr)\,dy\\ &= \exp\bigl(\zeta x\bigr) + \int_x^\infty T(x,y)\exp\bigl(\zeta y\bigr)\,dy.\end{aligned} \tag{3.8}$$

Here T satisfies the Gelfand–Levitan equation, and by integrating by parts, we see that

$$\frac{\partial^2 T}{\partial x^2} - \frac{\partial^2 T}{\partial y^2} = u(x)T(x,y) \tag{3.9}$$

where $u(x) = -2\frac{d^2}{dx^2}\log\tau(x)$. Then by integrating by parts, we see that ψ_ζ satisfies Schrödinger's equation.

The solutions of the differential equation depend analytically on ζ at those points where the potential depends analytically on ζ; note that $\zeta \mapsto \tau_\zeta(x)$ is holomorphic and non zero for $\|R_x\| < 1$ and $-\zeta \in S_\theta$. Then we continue the solutions analytically to all $-\zeta$ in the sector S_θ, on which $\psi_\zeta(x)$ is holomorphic as a function of ζ for $x > 0$.

(ii) Observe that $X_\theta = S_\theta \cap (-S_\theta)$ contains $i\mathbf{R} \setminus \{0\}$. For $\zeta \in S_\theta \cap (-S_\theta)$, by (i) there exist solutions $\psi_\zeta(x)$ and $\psi_{-\zeta}(x)$ to (3.3). In particular, ψ_{ik} and $\psi_{-ik}(x)$ are solutions for $k > 0$. We integrate by parts repeatedly

$$\begin{aligned} e^{-xA}(\zeta I - A)^{-1} &= e^{-xA} \int_0^\infty e^{\zeta s} e^{-sA}\, ds \\ &= \frac{e^{-xA}}{\zeta} + \frac{Ae^{-xA}}{\zeta^2} + \ldots + \frac{A^{k-1}e^{-xA}}{\zeta^k} + \int_0^\infty \frac{A^k e^{-xA}}{\zeta^k} e^{\zeta s} e^{-sA}\, ds, \end{aligned} \tag{3.10}$$

where the integral converges by the hypothesis of Theorem 2.2. Also, (e^{-zA}) is an analytic semigroup in the sector $S_{\theta-\pi/2}$, so $\mathcal{D}(A^j)$ is a dense linear subspace of H for all $j = 1, 2, \ldots$ and $A^j e^{-xA} \in \mathcal{L}(H)$ and by Cauchy's estimates there exists $C > 0$ such that $\|A^j e^{-xA}\|_{\mathcal{L}(H)} \leq Cj!/x^j$ for all $x > 0$. So we can generate an asymptotic expansion of (3.7) with terms

$$h_j(x) = Ce^{-xA}(I + R_x)^{-1} A^{j-1} e^{-xA} B \tag{3.11}$$

which is bounded on compact subsets of $(0, \infty)$.

□

Definition *(Darboux transforms).* Let $(-A, B, C)$ be an $(2,2)$ admissible linear system with tau function $\tau_\infty(x;\mu) = \det(I + \mu R_x)$. Define the Darboux transform of $(-A, B, C)$ to be $(-A, B, -C)$ with tau function transform $\tau_0(x;\mu) = \det(I - \mu R_x)$. Let

$$v = \frac{1}{\mu}\frac{d}{dx}\log\frac{\tau_\infty}{\tau_0}, \quad w = \frac{1}{\mu}\frac{d}{dx}\log\bigl(\tau_0\tau_\infty\bigr),$$

$$u_\infty = -\frac{2}{\mu^2}\frac{d^2}{dx^2}\log\tau_\infty, \quad u_0 = -\frac{2}{\mu^2}\frac{d^2}{dx^2}\log\tau_0. \tag{3.12}$$

In the following result, we show how products and quotients of τ functions can be linked by the Gelfand–Levitan equation for 2×2 matrices, and satisfy the identities usually associated with Darboux transforms in the theory of integrable systems. Identities such as (3.18) also appear in Appendix A16 of Mehta [54].

Theorem 3.2. *Let $(-A, B, C)$ be a $(2,2)$-admissible linear system with input and output spaces $\mathbf{C}$, and let $\phi(x) = Ce^{-xA}B$.*

(i) Then there exists $\delta > 0$ such that for all $\mu \in \mathbf{C}$ such that $|\mu| < \delta$, the integral equation (2.16) with

$$T(x,y) = \begin{bmatrix} W(x,y) & V(x,y) \\ V(x,y) & W(x,y) \end{bmatrix}, \tag{3.13}$$

$$\Phi(x+y) = \begin{bmatrix} 0 & \phi(x+y) \\ \phi(x+y) & 0 \end{bmatrix} \tag{3.14}$$

has a solution such that

$$W(x,x) = \frac{d}{dx}\frac{1}{2\mu}\log\bigl(\tau_\infty(x;\mu)\tau_0(x;\mu)\bigr), \tag{3.15}$$

$$V(x,x)=\frac{d}{dx}\frac{1}{2\mu}\log\frac{\tau_\infty(x;\mu)}{\tau_0(x;\mu)}. \tag{3.16}$$

and

$$\frac{1}{2\mu}\frac{d}{dx}W(x,x)=-V(x,x)^2; \tag{3.17}$$

(ii) also Toda's equation holds in the form

$$\tau_0''\tau_\infty-2\tau_0'\tau_\infty'+\tau_0\tau_\infty''=0. \tag{3.18}$$

Proof. (i) Let

$$T_\infty(x,y)=-Ce^{-xA}(I+\mu R_x)^{-1}e^{-yA}B, \tag{3.19}$$

$$T_0(x,y)=Ce^{-xA}(I-\mu R_x)^{-1}e^{-yA}B \tag{3.20}$$

and

$$\Phi(x)=\begin{bmatrix}0&\phi(x)\\ \phi(x)&0\end{bmatrix}. \tag{3.21}$$

Now let

$$T(x,y)=\frac{1}{2}\begin{bmatrix}T_\infty+T_0&T_\infty-T_0\\ T_\infty-T_0&T_\infty+T_0\end{bmatrix} \tag{3.22}$$

so that

$$T(x,y)=-\begin{bmatrix}C&0\\ 0&C\end{bmatrix}\begin{bmatrix}e^{-xA}&0\\ 0&e^{-xA}\end{bmatrix}\begin{bmatrix}I&\mu R_x\\ \mu R_x&I\end{bmatrix}^{-1}\begin{bmatrix}e^{-yA}&0\\ 0&e^{-yA}\end{bmatrix}\begin{bmatrix}0&B\\ B&0\end{bmatrix} \tag{3.23}$$

hence T satisfies the Gelfand–Levitan equation

$$T(x,y)+\Phi(x+y)+\mu\int_x^\infty T(x,z)\Phi(z,y)\,dz=0. \tag{3.24}$$

ii) As in Proposition 2.4,

$$T_\infty(x,x)=\frac{1}{\mu}\frac{d}{dx}\log\tau_\infty(x), \tag{3.25}$$

$$T_0(x,x)=\frac{1}{\mu}\frac{d}{dx}\log\tau_0(x); \tag{3.26}$$

hence (3.18) is equivalent to the condition

$$\frac{d}{dx}T_0(x,x)+\mu\big(T_0(x,x)-T_\infty(x,x)\big)^2+\frac{d}{dx}T_\infty(x,x)=0, \tag{3.27}$$

which we now verify. The left-hand side equals

$$Ce^{-xA}\big(-A(I-\mu R_x)^{-1}-(I-\mu R_x)^{-1}\mu(AR_x+R_xA)(I-\mu R_x)^{-1}-(I-\mu R_x)^{-1}A\big)e^{-xA}B$$

$$+Ce^{-xA}\big((I-\mu R_x)^{-1}+(I+\mu R_x)^{-1}\big)e^{-xA}\mu BCe^{-xA}\big((I-\mu R_x)^{-1}+(I+\mu R_x)^{-1}\big)e^{-xA}B$$

$$+Ce^{-xA}\big(A(I+\mu R_x)^{-1}-(I+\mu R_x)^{-1}\mu(AR_x+R_xA)(I+\mu R_x)^{-1}+(I+\mu R_x)^{-1}A\big)e^{-xA}B \quad (3.28)$$

All of the terms begin with Ce^{-xA} and end with $e^{-xA}B$, and we can replace $e^{-xA}\mu BCe^{-xA}$ by $\mu(AR_x+R_xA)$ to obtain

$$\begin{aligned}(3.28) = Ce^{-xA}\Big(&-2(I-\mu R_x)^{-1}A(I-\mu R_x)^{-1}+4(I-\mu^2R_x^2)^{-1}\mu(AR_x+R_xA)(I-\mu^2R_x^2)^{-1}\\ &+2(I+\mu R_x)^{-1}A(I+\mu R_x)^{-1}\Big)e^{-xA}B\\ =0.&\end{aligned} \quad (3.29)$$

This proves (3.18), and one can easily check that (3.18) is equivalent to

$$u_0(x)=\frac{1}{\mu}\frac{dv}{dx}+v(x)^2,\quad v(x)^2=-\frac{1}{\mu}\frac{dw}{dx}. \quad (3.30)$$

The entries of T satisfy the pair of coupled integral equations

$$\begin{aligned}0&=W(x,y)+\mu\int_x^\infty V(x,s)\phi(s+y)\,ds\\ 0&=V(x,y)+\phi(x+y)+\mu\int_x^\infty W(x,s)\phi(s+y)\,ds;\end{aligned} \quad (3.31)$$

so W satisfies

$$0=-W(x,z)+\mu\int_x^\infty \phi(x+y)\phi(y+z)\,dy+\mu^2\int_x^\infty W(x,s)\int_x^\infty \phi(s+y)\phi(y+z)\,dyds, \quad (3.32)$$

which explains how $\mu^2\Gamma_\phi^2$ enters the discussion in several determinant formulas [67].

□

Definition (i) *(Darboux Addition).* For $-\zeta\in S_\theta\cup\{0\}$ we define the Darboux addition rule on $(2,2)$ admissible linear systems by $M_\zeta:(-A,B,C)\mapsto(-A,(\zeta I+A)(\zeta I-A)^{-1}B,C)$ and on potentials by

$$u_\infty\mapsto u_\zeta=u_\infty-2(\log\psi_\zeta)''. \quad (3.33)$$

(ii) Let $\mathrm{Wr}(\varphi,\psi)$ be the Wronskian of $\psi,\varphi\in C^1((0,\infty);\mathbf{C})$.

Corollary 3.3. *The set $\{M_\zeta,\ (\zeta\in X_\theta), M_0, M_\infty=I\}$ generates a group such that $M_0^2=I$, $M_\zeta M_{-\zeta}=I$ and $M_\zeta M_\eta$ corresponds to adding $-2\frac{d^2}{dx^2}\log\mathrm{Wr}(\psi_\zeta,\psi_\eta)$ to the potential.*

Proof. The definition is consistent with [22, p 484], and p. 414 of [47]. In particular, $\psi_0(x)=\tau_0(x)/\tau_\infty(x)$, and $u_0(x)=u_\infty(x)-2\frac{d^2}{dx^2}\log\psi_0(x)$, which is consistent with (3.18).

For $\zeta_1\neq\zeta_2$, let $\Psi(x)=Wr(\psi_{\zeta_1},\psi_{\zeta_2})/\psi_{\zeta_2}$, and observe that

$$\Psi''=\big(\zeta_2^2+u_\infty-2(\log\psi_{\zeta_1})''\big)\Psi.$$

This gives the basic composition rule for $M_{\zeta_2}M_{\zeta_1}$. The other statements follow from Proposition 3.1 and Theorem 3.2.

□

Definition (i) Let Ω be a domain in $\mathbf{C}$. A divisor δ on Ω is a function $\delta : \Omega \to \mathbf{Z}$ such that the restriction of δ to K has finite support for all compact subsets K of Ω.

(ii) For a meromorphic function f on Ω, we let $\nu(z)$ be the order of z as a zero of f, and $\nu(p)$ be the order of p as a pole. Then $(f) = \sum_z \nu(z)\delta_z - \sum_p \nu(p)\delta_p$ defines the principal divisor corresponding to f; see [65].

(iii) Let $\log_+ x = \max\{0, \log x\}$ for $x > 0$. For any meromorphic function f let

$$m(r; f) = \int_0^{2\pi} \log_+ |f(re^{i\theta})| \frac{d\theta}{2\pi} \qquad (r > 0). \tag{3.34}$$

Proposition 3.4. *Suppose that $\phi : (0, \infty) \to \mathbf{R}$ arises from a $(2, 2)$ admissible linear system. Let $\tau_\infty(x; \mu) = \det(I + \mu\Gamma_{\phi(x)})$, and $\tau_0(x; \mu) = \tau_\infty(x; -\mu)$ for all $x > 0$ and $\mu \in \mathbf{C}$.*

(i) Then $\mu \mapsto \tau_\infty(x; \mu)$ is entire, and $\overline{\tau_\infty(x; \bar\mu)} = \tau_\infty(x; \mu)$ for all $x > 0$.

(ii) Let

$$q(\mu) = \frac{\tau_\infty(0; \mu)}{\tau_0(0; \mu)}. \tag{3.35}$$

- *Then q is meromorphic, $q^*(\mu) = q(\mu)$ and $q(\mu)q(-\mu) = 1$;*
- *all the zeros and poles of q are simple and lie in $\mathbf{R} \setminus \{0\}$;*
- *the zeros (z_j) of q satisfy $\sum_{j=1}^\infty 1/|z_j| < \infty$, and*
- *$m(r; q)/r \to 0$ as $r \to \infty$.*

(iii) Conversely, let q satisfy the conclusions of (ii). Then there exists a balanced linear system $(-A, B, C)$ with input and output space $\mathbf{C}$, such that $q(\mu) = \kappa\tau_\infty(0; \mu)/\tau_0(0; \mu)$ for some constant κ.

Proof. (i) This follows from Theorem 2.2 (ii).

(ii) Since Γ_ϕ is self-adjoint and trace class, the spectrum consists of 0, together with non-zero real eigenvalues λ_j with multiplicity $\nu(\{\lambda_j\})$, and for a self-adjoint Γ_ϕ the algebraic and geometric multiplicity are equal. For a Hankel operator Γ_ϕ, the dimension of $\{\xi : \Gamma\xi = 0\}$ is either zero or infinity; so by compressing Γ_ϕ to the closure of its range, we can assume 0 is not an eigenvalue. Then the function $q(\mu) = \tau_\infty(\mu)/\tau_0(\mu)$ is meromorphic on $\mathbf{C}$, the identity $q(\mu)q(-\mu) = 1$ is trivially true, while $q = q^*$ holds by (i).

The formal difference of the zeros and poles of q on $\{\mu \in \mathbf{C} : |\mu| < \rho\}$ may be represented by the divisor

$$\begin{aligned}(q) &= \sum_{j:|1/\lambda_j|<\rho} (\nu(\{\lambda_j\})\delta_{-1/\lambda_j} - \nu(\{\lambda_j\})\delta_{1/\lambda_j}) \\ &= \sum_{j:\lambda_j\rho>1} (\nu(\lambda_j) - \nu(-\lambda_j))(\delta_{-1/\lambda_j} - \delta_{1/\lambda_j}),\end{aligned} \tag{3.36}$$

where $\nu(\{\lambda_j\}) - \nu(\{-\lambda_j\})$ belongs to $\{-1, 0, 1\}$ by a theorem of Megretskii, Peller and Treil [53]. Consider $\lambda > 0$. If $\nu(\lambda) - \nu(-\lambda) = -1$, then q has a simple zero at $-1/\lambda$ and a simple pole at $1/\lambda$; whereas if $\nu(\lambda) - \nu(-\lambda) = 1$, then q has a simple zero at $1/\lambda$ and a simple pole at $-1/\lambda$.

Also $\sum_{j=1}^\infty |\lambda_j|$ converges since Γ_ϕ is trace class, so $\sum_{j=1}^\infty 1/|z_j|$ converges. Finally, standard results on convergent infinite products from [31', Theorem 1.9] show that $m(r;q)/r\to 0$ as $r\to\infty$.

(iii) Suppose conversely that q satisfies the conclusions of (ii). Now $q(0)^2=1$ so 0 is neither a zero nor a pole of q; also z is a zero of q if and only if $p=-z$ is a pole of q, so the divisor of q is

$$(q)=\sum_z \delta_z-\delta_{-z}, \tag{3.37}$$

where we have summed over the set of zeros z of q. Now $z\in\mathbf{R}\setminus\{0\}$ for all the zeros, and the z have no point of accumulation on $\mathbf{R}$ since q is meromorphic, so we can list the zeros in a (possibly finite) sequence $(z_j)_{j=1}^\infty$ such that $(|z_j|)_{j=1}^\infty$ is strictly increasing. By hypothesis $\sum_{j=1}^\infty 1/|z_j|$ converges.

Let $\lambda_j=-1/z_j$ and define $\nu:\mathbf{R}\to\mathbf{Z}$ by $\nu(\lambda_j)=1$ for $j=1,2,\dots,$ and $\nu(\lambda)=0$ otherwise. Then there exists a self-adjoint Hankel operator Γ on $L^2((0,\infty);\mathbf{C})$ with spectral multiplicity function ν and spectrum $\{\lambda_j: j=1,2,\dots\}\cup\{0\}$. Since $\sum_{j=1}^\infty|\lambda_j|$ converges, Γ is trace class and has a Fredholm determinant, so

$$\frac{\det(I+\mu\Gamma)}{\det(I-\mu\Gamma)}=\prod_{j=1}^\infty\frac{1-\mu/z_j}{1+\mu/z_j} \tag{3.38}$$

is meromorphic with only simple zeros at z_j and simple poles at $-z_j$. Hence the product in (3.38) has the same divisor as f, and we deduce that there exists an entire function f such that $q(\mu)e^{-f(\mu)}$ equals the product in (3.38). The next step is to eliminate this f by using elementary Nevanlinna theory.

Let $n(r)=\sharp\{z_j:|z_j|\leq r\}$ be the counting function for zeros, or equivalently poles of q, and then let $N(r;q)=\int_0^r n(s)s^{-1}ds$. By standard results, we have convergent expressions

$$\sum_{j=1}^\infty\frac{1}{|z_j|}=\int_0^\infty\frac{n(r)dr}{r^2}, \tag{3.39}$$

from which we deduce that $n(r)/r\to 0$ and $N(r;q)/r\to 0$ as $r\to\infty$; so the Nevanlinna characteristic satisfies $N(r;q)+m(r;q)=o(r)$ as $r\to\infty$. By [31', Theorem 1.9], we deduce that $f(\mu)$ is a polynomial of degee zero, namely a constant.

By Theorem 1.1 of [53], Γ may be realized by a balanced linear system $(-A,B,C)$ with input and output space $\mathbf{C}$, where $A\in\mathcal{L}(H)$ bounded, and $\Gamma=\Gamma_\phi$ as in (1.1), where $\phi(t)=Ce^{-tA}B$.

□

We now show how a Schrödinger differential equation of scattering type gives rise to an admissible linear system as in Theorem 3.2; this justifies the terminology 'scattering function' as applied to ϕ. In section 4 of [9], we realized the scattering data from Schrödinger's equation from a linear system with unbounded A; in the following result, we realize the data with a linear system with $A \in \mathcal{L}(H)$ .The differential equation $-f''+(v'+v^2)f = k^2 f$ may be written as

$$\frac{d}{dx}\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} v & ik \\ ik & -v \end{bmatrix}\begin{bmatrix} f \\ g \end{bmatrix} \tag{3.40}$$

where we suppose that $v \in C_c^\infty(\mathbf{R};\mathbf{R})$ for simplicity. For $\lambda = k^2 > 0$, there exists a solution $f(x) = \psi(x;k)$ such that

$$\psi(x;k) \asymp \begin{cases} e^{-ikx} + s_{21}(k)e^{ikx}, & \text{as } x \to \infty; \\ \bar{s}_{11}(k)e^{-ikx}, & \text{as } x \to -\infty. \end{cases} \tag{3.41}$$

There may also be a discrete spectrum $\lambda_j = -\kappa_j^2$ with $\kappa_n \geq \ldots \geq \kappa_1 > 0$, where each κ_j^2 is associated to a real eigenfunction $\psi(x; -\kappa_j^2)$ called a bound state that is asymptotic to $c(-\kappa_j^2)e^{-\kappa_j x}$ as $x \to \infty$, where $c(-\kappa_j^2)$ is normalized by taking $\int_{-\infty}^\infty \psi(x; -\kappa_j^2)^2 dx = 1$. The discrete spectrum gives rise to a linear system with finite dimensional state space, as we discuss in Proposition 4.5. Note that $-\kappa_j - \kappa_\ell < 0$ for all j, ℓ.

So we suppose for the moment that the discrete spectrum is absent, and the inverse spectral problem is to recover v from $s_{21}(k)$ and $s_{11}(k)$, up to equivalence. We aim to introduce an admissible linear system from the spectral data and determine the potential.

As in [22], the scattering matrix is

$$S(k) = \begin{bmatrix} s_{11}(k) & -\bar{s}_{21}(k) \\ s_{21}(k) & \bar{s}_{11}(k) \end{bmatrix} \in SU(2) \tag{3.42}$$

where $s_{21}(k)$ is called the reflection coefficient and $s_{11}(k)$ is the transmission coefficient for $k \in \mathbf{R}$. We suppose that $s_{21}(-k) = \overline{s_{21}(k)}$, and that $s_{21} \in L^2(\mathbf{R};\mathbf{C})$ is absolutely continuous with $s_{21}' \in L^2(\mathbf{R};\mathbf{C})$. Let

$$a_n = \frac{(-1)^n}{2\pi}\int_{-\infty}^\infty s_{21}(k)\frac{(1/2+ik)^n}{(1/2-ik)^{n+1}}dk, \tag{3.43}$$

Proposition 3.5. *Suppose that (3.41) has no bound states, and suppose that $(a_n)_{n=0}^\infty$ satisfy*

$$\limsup_{n\to\infty} |a_n|^{1/n} < 1/3. \tag{3.44}$$

Then there exists a $(2,2)$ admissible linear system $(-A, B, C)$ with $H_0 = \mathbf{C}$ and bounded A, B, C such that the potential $u_0 = v' + v^2$ of $-f'' + u_0 f = \lambda f$ is determined by the τ function of $(-A, B, -C)$.

Proof. We introduce the scattering function

$$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^\infty s_{21}(k)e^{ixk}\, dk \qquad (x > 0), \tag{3.45}$$

so that $\phi \in L^2((0,\infty);\mathbf{R})$ and $x\phi(x) \in L^2((0,\infty);\mathbf{R})$; hence Γ_ϕ determines a self-adjoint and Hilbert–Schmidt operator on $L^2((0,\infty);\mathbf{C})$. We now introduce a linear system which realizes this ϕ.

Let $L_n(x) = e^x(d/dx)^n(x^ne^{-x})/n!$ be the Laguerre polynomial of order zero and degree n; then the n^{th} Laguerre function is

$$e^{-x/2}L_n(x) = \frac{1}{2\pi i}\int_{C(1,\delta)} e^{-xz/2}\frac{(1+z)^n}{(1-z)^n}\frac{dz}{1-z}, \tag{3.46}$$

where $C(1,\delta)$ is the circle of centre 1 and radius $0<\delta<1$ in $\mathbf{C}$. This is a special case of a formula of Tricomi, and follows from Cauchy's integral formula. Then $(e^{-x/2}L_n(x))_{n=0}^\infty$ gives an orthonormal basis for $L^2(0,\infty)$. By Plancherel's formula we find that

$$a_n = \int_0^\infty \phi(x)e^{-x/2}L_n(x)dx \qquad (n=0,1,2,\dots) \tag{3.47}$$

are the coefficients of ϕ with respect to $(e^{-x/2}L_n(x))_{n=0}^\infty$ and we introduce

$$b(z) = \sum_{n=0}^\infty a_n\frac{(1+z)^n}{(1-z)^n} \qquad (z\in C(1,\delta)); \tag{3.48}$$

by the hypothesis, $\limsup_{n\to\infty}|a_n|^{1/n} = \rho < 1/3$ so the series for $b(z)$ converges for all $z\in\mathbf{C}$ outside of the disc $D((1+\rho^2)/(1-\rho^2); 2\rho/(1-\rho^2))$. Hence can choose $0<\delta<1$ so $(2+\delta)\limsup_{n\to\infty}|a_n|^{1/n}/\delta < 1$ so that this series converges absolutely and uniformly on $C(1,\delta)$.

We parametrize $C(1,\delta)$ by $z=1+\delta e^{i\theta}$ for $\theta\in[0,2\pi]$, and introduce the Hilbert space $H = L^2(C(1,\delta);d\theta;\mathbf{C})$. Then we introduce bounded linear operators

$$\begin{aligned} A: H\to H:&\quad f(z)\mapsto zf(z)/2\\ B:\mathbf{C}\to H:&\quad \beta\mapsto \beta b(z)\\ C: H\to\mathbf{C}:&\quad g\mapsto \frac{1}{2\pi i}\int_{C(1,\delta)} g(z)\frac{dz}{1-z}. \end{aligned} \tag{3.49}$$

Thus $(-A,B,C)$ gives a linear system with state space H, input and output space $\mathbf{C}$.

From the orthogonal series expansion $\phi(t)=\sum_{n=0}^\infty a_ne^{-t/2}L_n(t)$, we note that

$$\phi(t) = \frac{1}{2\pi i}\int_{C(1,\delta)}\sum_{n=0}^\infty \frac{a_n(1+z)^ne^{-zt/2}dz}{(1-z)^{n+1}} = Ce^{-tA}B. \tag{3.50}$$

Also, the corresponding R operator has integral kernel

$$R_x = \int_x^\infty e^{-tA}BCe^{-tA}dt \leftrightarrow \frac{e^{-xz/2-xw/2}b(z)}{z+w}. \tag{3.51}$$

We observe that $\Re z\geq 1-\delta$ for all z on $C(1,\delta)$; hence $\|e^{-tA}\|_{\mathcal{L}(H)}\leq e^{-t(1-\delta)/2}$; so $(-A,B,C)$ is $(2,2)$ admissible by Theorem 2.2(i).

For all $\phi \in L^2((0,\infty);\mathbf{R})$ we have $(a_n)_{n=0}^\infty \in \ell^2$ and hence $b(iy)/(1-iy)$ gives a function in $L^2(\mathbf{R};\mathbf{C})$. Hence by deforming (3.48) to an integral along the imaginary axis $\Re z = 0$, we can obtain an L^2 Plancherel integral

$$\phi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ity}\frac{b(iy)dy}{1-iy}. \tag{3.52}$$

However, the operation of multiplication by iy is clearly unbounded on this deformed contour.

We now recover u from $(-A,B,C)$ via the Gelfand–Levitan equation. The kernel

$$T_0(x,y) = Ce^{-xA}(I-R_x)^{-1}e^{-yA}B \qquad (x_0 < x < y) \tag{3.53}$$

as in (2.17) gives the unique solution of the integral equation

$$-T_0(x,y) + \phi(x+y) + \int_x^\infty T_0(x,y)\phi(y+z)dz = 0 \qquad (x_0 < x < y) \tag{3.54}$$

for some $x_0 > 0$. Then one checks that

$$\frac{\partial^2 T_0}{\partial x^2} - \frac{\partial^2 T_0}{\partial y^2} - q(x)T_0(x,y) = 0, \tag{3.55}$$

where $q(x) = -2(d/dx)T_0(x,x)$. Also, one has $T_0(x,x) = (d/dx)\log\det(I-R_x)$, so $q(x) = -2(d^2/dx^2)\log\tau(x)$. Given this partial equation for $T_0(x,y)$ one can check that

$$f(x;k) = e^{ikx} + \int_x^\infty e^{iky}T_0(x,y)dy \tag{3.56}$$

satisfies $-f''(x;k) + q(x)f(x;k) = k^2 f(x;k)$. Hence we can identify $q(x)$ with $u_0(x) = v'(x) + v(x)^2$, and thus obtain a solution of the inverse spectral problem.

□

4 The state ring associated with an admissible linear system

Gelfand and Dikii [26] considered the algebra $\mathcal{A}_u = \mathbf{C}[u, u', u'', \dots]$ of complex polynomials in a smooth potential u and its derivatives. They showed that if u satisfies the stationary higher order KdV equations (5.1), then $\mathcal{A}_u$ is a Noetherian ring [6] and the associated Schrödinger equation is integrable by quadratures; see [14, 63]. In this section, we introduce an analogue $\mathcal{A}_\Sigma$ for an admissible linear system. In the subsequent section, we link this to the result from [26].

We introduce these state rings in this section, and develop a calculus for R_x which is the counterpart of Pöppe's functional calculus for Hankel operators from [61, 62]. As we see in subsequent sections, our theory of state rings has wider scope for generalization.

Definition *(Differential rings).* (i) Let $\mathcal{R}$ be a ring with ideal $\mathcal{J}$, and let $\partial : \mathcal{R} \to \mathcal{R}$ be a derivation. Then $\mathcal{R}_\mathcal{J} = \{r \in \mathcal{R} : \partial(r) \in \mathcal{J}\}$ gives a subring of $\mathcal{R}$, the ring of constants relative to $\mathcal{J}$. When $\mathcal{R}$ is an algebra over $\mathbf{C}$ and $\mathcal{J} = 0$, we call $\mathcal{R}_0$ the constants; see [63].

(ii) Let H and H_0 be separable complex Hilbert spaces, let $\mathcal{L}(H)$ be the ring of bounded linear operators on H. Let $\mathcal{S}$ be a subring of $C^\infty((0,\infty);\mathcal{L}(H))$ and $\mathcal{B}$ be a subring of $C^\infty((0,\infty);\mathcal{L}(H_0))$; that is we suppose that each $T\in\mathcal{S}$ is a differentiable function of $x\in(0,\infty)$ as we indicate by writing T_x; we suppose further that $dT_x/dx\in\mathcal{S}$, and that $(d/dx)(ST)=(dS/dx)T+S(dT/dx)$. Then $\mathcal{S}$ is a differential ring. When $I\in\mathcal{S}$ and $\theta\in\mathbf{C}$, we identify θI with θ to simplify notation.

Definition *(State ring of a linear system).* Let $(-A,B,C)$ be a linear system such that $A\in\mathcal{L}(H)$. Suppose that:
(i) $\mathcal{S}$ is a differential subring of $C^\infty((0,\infty);\mathcal{L}(H))$;
(ii) I,A and BC are constant elements of $\mathcal{S}$;
(iii) e^{-xA}, R_x and $F_x=(I+R_x)^{-1}$ belong to $\mathcal{S}$.
Then $\mathcal{S}$ is a state ring for $(-A,B,C)$.

Lemma 4.1. *Suppose that $(-A,B,C)$ is a linear system with bounded A and that R_x gives a solution of Lyapunov's equation (1.3) such that $I+R_x$ is invertible for $x>0$ with inverse F_x.*

Then the free associative algebra $\mathcal{S}$ generated by I,R_0,A,F_0,e^{-xA},R_x and F_x is a state ring for $(-A,B,C)$ on $(0,\infty)$. For all $t>0$, there exists a ring homomorphism $S_t:\mathcal{S}\to\mathcal{S}$ given by $S_t: G(x)\mapsto G(x+t)$ such that S_t commutes with d/dx

Proof. We can regard $\mathcal{S}$ as a subring of $C_b((0,\infty),\mathcal{L}(H)))$, so the multiplication is well defined. Then we note that $BC=AR_0+R_0A$ belongs to $\mathcal{S}$, as required. We also note that $(d/dx)e^{-xA}=-Ae^{-xA}$ and that Lyapunov's equation (1.3) gives

$$\frac{d}{dx}(I+R_x)^{-1}=(I+R_x)^{-1}(AR_x+R_xA)(I+R_x)^{-1}, \tag{4.1}$$

which implies

$$\frac{dF_x}{dx}=AF_x+F_xA-2F_xAF_x. \tag{4.2}$$

with the initial condition

$$AF_0+F_0A-2F_0AF_0=F_0BCF_0. \tag{4.3}$$

Hence $\mathcal{S}$ is a differential ring.

We can map $I\mapsto I$, $e^{-xA}\mapsto e^{-(x+t)A}$, $R_0\mapsto e^{-tA}R_0e^{-tA}$, $R_x\mapsto e^{-tA}R_xe^{-tA}$ and $F_x\mapsto (I+e^{-tA}R_xe^{-tA})^{-1}$, and thus produce a ring homomorphism $G(x)\mapsto G(x+t)$ which satisfies $(d/dx)S_tG(x)=G'(x+t)=S_t(d/dx)G(x)$.

□

Definition *(Products and brackets).* (i) Given a state ring $\mathcal{S}$ for $(-A,B,C)$, and let $\mathcal{B}$ be any differential ring of functions from $(0,\infty)\to\mathcal{L}(H_0)$. Let

$$\mathcal{A}_\Sigma=\mathrm{span}_{\mathbf{C}}\{A^{n_1},A^{n_1}F_xA^{n_2}\dots F_xA^{n_r}:n_j\in\mathbf{N}\}. \tag{4.4}$$

(ii) On $\mathcal{S}$ we introduce the associative product $*$ by

$$P*Q=P(AF+FA-2FAF)Q, \tag{4.5}$$

which is distributive over the standard addition, and the derivation $\partial : \mathcal{S} \to \mathcal{S}$ by

$$\partial P = A(I - 2F)P + \frac{dP}{dx} + P(I - 2F)A, \tag{4.6}$$

(iii) Let $\lfloor \cdot \rfloor : \mathcal{S} \to \mathcal{B}$ be the linear map

$$\lfloor Y \rfloor = Ce^{-xA} F_x Y F_x e^{-xA} B \qquad (Y \in \mathbf{S}), \tag{4.7}$$

so that $x \mapsto \lfloor Y \rfloor$ is a function $(x_0, x_1) \to \mathcal{L}(H_0)$.

Proposition 4.2. *Then $(\mathcal{A}_\Sigma, *, \partial)$ is a differential ring, and there is a homomorphism of differential rings $(\mathcal{A}_\Sigma, *, \partial) \to (\mathcal{B}, \cdot, d/dx)$ given by $P \mapsto \lfloor P \rfloor$.*

Proof. The basic observation is that $dF/dx = AF + FA - 2FAF$, so one can check that

$$\partial(P * Q) = (\partial P) * Q + P * (\partial Q); \tag{4.8}$$

hence $(\mathcal{S}, *, \partial)$ is a differential ring.

We can multiply elements in $\mathcal{S}$ by concatenating words and taking linear combinations. Since all words in $\mathcal{A}_\Sigma$ begin and end with A, we obtain words of the required form, hence $\mathcal{A}_\Sigma$ is a subring. To differentiate a word in $\mathcal{A}_\Sigma$ we add words in which we successively replace each F_x by $AF_x + F_x A - 2F_x AF_x$, giving a linear combination of words of the required form.

From the definition of R_x, we have $AR_x + R_x A = e^{-xA} BCe^{-xA}$, and hence

$$F_x e^{-xA} BCe^{-xA} F_x = AF_x + F_x A - 2F_x AF_x, \tag{4.9}$$

which implies

$$\begin{aligned} \lfloor P \rfloor \lfloor Q \rfloor &= Ce^{-xA} F_x P F_x e^{-xA} BCe^{-xA} F_x Q F_x e^{-xA} B \\ &= Ce^{-xA} F_x P (AF_x + F_x A - 2F_x AF_x) Q F_x e^{-xA} B \\ &= \lfloor P(AF_x + F_x A - 2F_x AF_x) Q \rfloor \\ &= \lfloor P * Q \rfloor. \end{aligned} \tag{4.10}$$

Moreover, the first and last terms in $\lfloor P \rfloor$ have derivatives

$$\frac{d}{dx} Ce^{-xA} F_x = Ce^{-xA} F_x A(I - 2F_x), \qquad \frac{d}{dx} F_x e^{-xA} B = (I - 2F_x) AF_x e^{-xA} B, \tag{4.11}$$

which implies (3.7).

The bracket operation satisfies

$$\frac{d}{dx} \lfloor P \rfloor = \Big\lfloor A(I - 2F_x)P + \frac{dP}{dx} + P(I - 2F_x)A \Big\rfloor = \lfloor \partial P \rfloor. \tag{4.12}$$

$\square$

For $x_0 \geq 0$ and $0 < \phi < \pi$, let $S_\delta^{x_0}$ be the translated sector $S_\delta^{x_0} = \{z = x_0 + w : w \in \mathbf{C} \setminus \{0\}; |\arg w| < \delta\}$ and let $H^\infty(S_\delta^{x_0})$ the the bounded holomorphic complex functions on

$S_\delta^{x_0}$. Then let $H_\infty^\infty = \cup_{x_0>0} H^\infty(S_\delta^{x_0})$ be the algebra of complex functions which are bounded on some translated sector $S_\delta^{x_0}$, with the usual pointwise multiplication.

Theorem 4.3. *Let $(-A, B, C)$ be a $(2,2)$-admissible linear system with $H_0 = \mathbf{C}$ as in Theorem 2.2, so (e^{-zA}) for $z \in S_\phi^0$ is a bounded holomorphic semigroup on H. Let $\Theta_0 = \{P \in \mathcal{A}_\Sigma : \lfloor P \rfloor = 0\}$.*

(i) Then $(\mathcal{A}_\Sigma, *, \partial)$ is a differential ring with bracket $\lfloor \cdot \rfloor$;

(ii) there is a homomorphism of differential rings $\lfloor \cdot \rfloor : (\mathcal{A}_\Sigma, *, \partial) \to (H_\infty^\infty, \cdot, d/dz)$;

(iii) Θ_0 is a differential ideal in $(\mathcal{A}_\Sigma, *, \partial)$ such that $\mathcal{A}_\Sigma/\Theta_0$ is a commutative differential ring, and an integral domain.

Proof (i) In this case A is possibly unbounded as an operator, so we use the holomorphic semigroup to ensure that products and brackets are well defined. We observe that $\mathcal{A}_\Sigma$ has a grading $\mathcal{A}_\Sigma = \oplus_{n=1}^\infty A_n$, where A_n is the span of the elements that have total degree n when viewed as products of A and F. For $X_n \in A_n$ and $Y_m \in A_m$, we have $X_n * Y_m \in A_{n+m+2} \oplus A_{n+m+3}$ and $\partial X_n \in A_{n+1} \oplus A_{n+2}$.

Also we have $A^k e^{-zA} \in \mathcal{L}(H)$ for all $z \in S_\phi^0$ and $\|A^k e^{-zA}\|_{\mathcal{L}(H)} \to 0$ as $z \to \infty$ in S_ϕ^0; hence $R_z A^k \to 0$ and $A^k R_z \to 0$ in $\mathcal{L}(H)$ as $z \to \infty$ in S_ϕ^0. Hence there exists an increasing positive sequence $(x_k)_{k=0}^\infty$ such that $A^k F_z - A^k \in \mathcal{L}(H)$ for all $z \in S_\phi^{x_k}$ and $A^k F_z - A^k \to 0$ in $\mathcal{L}(H)$ as $z \to \infty$ in $S_\phi^{x_k}$. Let $X_n \in A_n$ and consider a typical summand $AF_z A^k F_z \dots A$ in X_n; we replace each factor like $A^k F_z$ by the sum of $A^k(F_z - I)$ and A^k where $k \leq n$; then we observe that there in an initial factor Ce^{-zA} and a final factor $e^{-zA}B$ in $\lfloor X_n \rfloor$; hence $\lfloor X_n \rfloor$ determines an element of $H^\infty(S_\phi^{x_n})$.

(ii) We can identify H_∞^∞ with the algebraic direct limit $H_\infty^\infty = \lim_{n\to\infty} H^\infty(S_\phi^{x_0+n})$. By the principle of isolated zeros, the multiplication on H_∞^∞ is consistently defined, and H_∞^∞ is an integral domain. Now each $f \in H_\infty^\infty$ gives $f \in H^\infty(S_\phi^{x_0})$ so $f' \in H^\infty(S_\phi^{x_0+1})$ by Cauchy's estimates, so $f' \in H^\infty - \infty$. From (i) we deduce that $\lfloor \cdot \rfloor : \oplus_{n=1}^\infty A_n \to \cup_{n=1}^\infty H^\infty(S_\phi^{x_n})$ is well-defined and the bracket is multiplicative with respect to $*$, and behaves naturally with respect to differentiation.

(iii) We check that $\lfloor . \rfloor$ is a trace on $(\mathcal{A}_\Sigma, *, \partial)$, by computing

$$\begin{aligned}
\lfloor P * Q \rfloor &= \operatorname{trace}\big(Ce^{-xA}FPFe^{-xA}BCe^{-xA}FQFe^{-xA}B\big) \\
&= \operatorname{trace}\big(Ce^{-xA}FQFe^{-xA}BCe^{-xA}FPFe^{-xA}B\big) \\
&= \lfloor Q * P \rfloor. \qquad (4.13)
\end{aligned}$$

Hence Θ_0 contains all the commutators $P*Q-Q*P$, and Θ_0 is the kernel of the homomorphism $\lfloor \cdot \rfloor$, hence is an ideal for $*$. Also, we observe that for all $Q \in \Theta_0$, we have $\partial Q \in \Theta_0$ since $\lfloor \partial Q \rfloor = (d/dx)\lfloor Q \rfloor = 0$. Hence Θ_0 is a differential ideal which contains the commutator subspace of $(\mathcal{A}_\Sigma, *)$, so $\mathcal{A}_\Sigma/\Theta_0$ is a commutative algebra. Also, ∂ determines a unique derivation $\bar\partial$ on $\mathcal{S}/\Theta_0$ by $\bar\partial Q = \partial Q + \Theta_0$ for all $Q \in \mathcal{S}$; hence $\mathcal{A}_\Sigma/\Theta_0$ is a differential algebra.

We can identify $\mathcal{A}_\Sigma/\Theta_0$ with a subalgebra of H_∞^∞, which is an integral domain.

□

Remarks 4.4 (i) Pöppe [61, 62] introduced a linear functional $\lceil . \rceil$ on Fredholm kernels $K(x,y)$ on $L^2(0,\infty)$ by $\lceil K \rceil = K(0,0)$. In particular, let K, G, H, L be integral operators on $L^2(0,\infty)$

that have smooth kernels of compact support, let $\Gamma = \Gamma_{\phi_{(x)}}$ have kernel $\phi(s+t+2x)$, let $\Gamma' = \frac{d}{dx}\Gamma$ and $G = \Gamma_{\psi_{(x)}}$ be another Hankel operator; then the trace satisfies

$$\lceil \Gamma \rceil = -\frac{d}{dx}\mathrm{trace}\,\Gamma \tag{4.14}$$

$$\lceil \Gamma K G \rceil = -\frac{1}{2}\frac{d}{dx}\mathrm{trace}\,\Gamma K G \tag{4.15}$$

$$\lceil (I+\Gamma)^{-1}\Gamma \rceil = -\mathrm{trace}\big((I+\Gamma)^{-1}\Gamma'\big), \tag{4.16}$$

$$\lceil K\Gamma \rceil \lceil GL \rceil = -\frac{1}{2}\lceil K(\Gamma' G + \Gamma G')L \rceil, \tag{4.17}$$

where (4.17) is known as the product formula. The easiest way to prove (4.15)-(4.18) is to observe that $\Gamma' G + \Gamma G'$ is the integral operator with kernel $-2\phi_{(x)}(s)\psi_{(x)}(t)$, which has rank one. These ideas were subsequently revived by McKean [48]. Our formulas (4.5) and (4.6) incorporate a similar idea, and are the basis of the proof of Proposition 4.2. The results we obtain appear to be more general than those of Pöppe, and extend to periodic linear systems.

(ii) Mumford [57] considers the ring R_1 of complex functions that are holomorphic on some neighbourhood of zero, and the ring $R_2 = C_c^\infty(\mathbf{R};\mathbf{C})$ of compactly supported smooth functions; Mulase [56] considers the differential ring $R_3 = \mathbf{C}[[x]]$ of formal complex power series; McKean and van Moerbeke [50] consider the ring $R_4 = C^\infty(\mathbf{T};\mathbf{C})$ of smooth periodic function; then R_j is a differential rings with respect to $D = d/dx$, and one can form the rings $R_j\{D\} = \{\sum_{k=-\infty}^n a_k(x)D^j : n < \infty; a_k \in R_j, -\infty; k \leq n\}$ of pseudo differential operators. In this paper we use a differential ring $(\mathcal{A}, *, \partial)$ of operators on state space.

In the literature on inverse scattering, as in [4], the operator R_x appears implicitly in various formulas, especially in the special case in which A has finite rank and may be represented by a matrix. The following result extends a special case of the Sylvester–Rosenblum theorem [8]. The formula (4.18) resembles the expressions used to obtain soliton solutions of KdV, as in [35, (14.12.11)].

For the remainder of this section, we let A be a $n \times n$ complex matrix with eigenvalues λ_j $(j = 1, \dots, m)$ with geometric multiplicity n_j such that $\lambda_j + \lambda_k \neq 0$ for all $j, k \in \{1, \dots, m\}$; let $\mathbf{K} = \mathbf{C}(e^{-\lambda_1 t}, \dots, e^{-\lambda_m t}, t)$. Also, let $B \in \mathbf{C}^{n\times 1}$ and $C \in \mathbf{C}^{1\times n}$.

Proposition 4.5. *(i) There exists a solution R_t to Lyapunov's equation (1.3) with initial condition BC, such that the entries of R_t belong to $\mathbf{K}$, and $\tau(t) \in \mathbf{K}$;*

(ii) $\phi \in \mathbf{K}$ satisfies a linear differential equation with constant coefficients.

(iii) Suppose further that all the eigenvalues of A are simple. Then there exists an invertible matrix S such that $S^{-1}B = (b_j)_{j=1}^n \in \mathbf{C}^{n\times 1}$ and $CS = (c_j)_{j=1}^n \in \mathbf{C}^{1\times n}$ and the tau function is given by

$$\begin{aligned}\tau(t;\mu) =& 1 + \sum_{j=1}^n \frac{b_j c_j e^{-2\lambda_j t}}{2\lambda_j}\\ &+ \sum_{(j,k),(m,p):j\neq m; k\neq p} (-1)^{j+k+m+p} \frac{b_j b_m c_k c_p e^{-(\lambda_j+\lambda_k+\lambda_m+\lambda_p)t}}{(\lambda_j+\lambda_m)(\lambda_k+\lambda_p)} + \dots\end{aligned}$$

$$+\prod_{j=1}^{n}\frac{b_jc_j}{2\lambda_j}\prod_{1\leq j<k\leq n}\frac{(\lambda_j-\lambda_k)^2}{(\lambda_j+\lambda_k)^2}e^{-2\sum_{j=1}^{n}\lambda_j t}. \tag{4.18}$$

Proof.(i) By the hypothesis, we can introduce a chain of circles Σ that go once round each λ_j in the positive sense and have all the points $-\lambda_k$ in their exterior. Then by [8], the matrix

$$R_0=\frac{-1}{2\pi i}\int_\Sigma (A+\lambda I)^{-1}BC(A-\lambda I)^{-1}d\lambda \tag{4.19}$$

gives a solution to the equation $-AR_0-R_0A=-BC$. To see this, one considers $(A+\lambda I)R_0+R_0(A-\lambda I)$ and then uses the calculus of residues.

By the Riesz functional calculus, we also have

$$e^{-tA}=\frac{1}{2\pi i}\int_\Sigma \big(\lambda I-A\big)^{-1}e^{-t\lambda}d\lambda; \tag{4.20}$$

hence by Cauchy's residue theorem, there exist complex polynomials p_j and q_j, and integers $m_j\geq 0$ such that

$$e^{-tA}=\sum_{j=1}^{m}q_j(t)e^{-t\lambda_j}p_j(A), \tag{4.21}$$

where $q_j(t)$ is constant if the corresponding eigenvalue is simple. We let $R_t=e^{-tA}R_0e^{-tA}$, which gives a solution to Lyapunov's equation with initial condition $-BC$. From (4.21), we see that all the entries of R_t belong to $\mathbf{K}$. By the Laplace expansion of the determinant, we see that all entries of $\tau(t)=\det(I+R_t)$ also belong to $\mathbf{K}$.

(ii) We take $\phi(t)=Ce^{-tA}B\in\mathbf{K}$ by (4.21). Also, we introduce the characteristic polynomial of $(-A)$ by $\det(\lambda I+A)=\sum_{j=0}^{n}a_j\lambda^j$. Then by the Cayley–Hamilton theorem.

$$\sum_{j=0}^{n}a_j\frac{d^j\phi(t)}{dt^j}=0. \tag{4.22}$$

(iii) We recall Cauchy's determinant formula. For x_r and y_s complex numbers such that $x_ry_s\neq 1$, the determinant satisfies

$$\det\Big[\frac{1}{1-x_jy_k}\Big]_{j,k=1}^{n}=\frac{\prod_{1\leq j<k\leq n}(x_j-x_k)\prod_{1\leq m<p\leq n}(y_m-y_p)}{\prod_{1\leq r,s\leq n}(1-x_ry_s)}. \tag{4.23}$$

There exists an invertible matrix S such that SAS^{-1} is the $n\times n$ diagonal matrix $D=\operatorname{diag}(\lambda_1,\dots,\lambda_n)$, and we observe that

$$R_t=\Big[\frac{b_jc_ke^{-(\lambda_j+\lambda_k)t}}{\lambda_j+\lambda_k}\Big]_{j,k=1}^{n} \tag{4.24}$$

satisfies $\frac{d}{dt}R_t = -[b_jc_ke^{-(\lambda_j+\lambda_k)t}]_{j,k=1}^n$ and $-DR_t - R_tD = -[b_jc_ke^{-(\lambda_j+\lambda_k)t}]_{j,k=1}^n$; so R_t gives a solution of the Lyapunov equation with generator $-D$ and initial condition given by the rank-one matrix $-S^{-1}BCS = -[b_jc_k]_{j,k=1}^n$. Hence the tau function is given by $\tau(t;\mu) = \det(I+\mu R_t)$ for this matrix, and there is an expansion

$$\det\Bigl[\delta_{jk} + \frac{\mu b_jc_ke^{-(\lambda_j+\lambda_k)x}}{\lambda_j+\lambda_k}\Bigr]_{j,k=1}^n = \sum_{\sigma\subseteq\{1,\dots,n\}} \mu^{\sharp\sigma} \det\Bigl[\frac{b_jc_ke^{-\lambda_jx-\lambda_kx}}{\lambda_j+\lambda_k}\Bigr]_{j,k\in\sigma} \tag{4.25}$$

in which each subset σ of $\{1,\dots,n\}$ of order $\sharp\sigma$, contributes a minor indexed by $j,k\in\sigma$. Letting $x_r = \lambda_r$ and $y_r = -1/\lambda_r$ in the Cauchy determinant formula, we obtain the identity

$$\det\Bigl[\frac{b_jc_ke^{-\lambda_jx-\lambda_kx}}{\lambda_j+\lambda_k}\Bigr]_{j,k\in\sigma} = \prod_{j\in\sigma}\frac{b_jc_je^{-2\lambda_jx}}{2\lambda_j}\prod_{j,k\in\sigma:j\neq k}\frac{\lambda_j-\lambda_k}{\lambda_j+\lambda_k}. \tag{4.26}$$

□

In the next three sections, we give significant examples of differential rings associated with linear systems.

5. The diagonal Green's function and the stationary KdV hierarchy

In this section, we obtain properties of $\mathcal{A}_\Sigma$ in terms of A. Thus we obtain some sufficient conditions for some differential equations to be integrable. Throughout this section, we suppose that the hypotheses of Theorem 4.3 are in force, so the any finite set of elements of $\mathcal{A}_\Sigma$ are holomorphic functions on a some sector Ω containing (x_0,∞) for some $x_0\geq 0$. We do not generally require u to be real valued, although in Theorem 5.2(iv) we impose this further condition so that we can compare our results with the classical spectral theory for the Schrödinger equation on the real line.

Definition *(Stationary KdV hierarchy).* (i) Let $f_0 = 1$ and $f_1 = (1/2)u$. Then the KdV recursion formula is

$$4\frac{d}{dx}f_{m+1}(x) = 4f_1(x)\frac{d}{dx}f_m(x) + 4\frac{d}{dx}\bigl(f_1(x)f_m(x)\bigr) - \frac{d^3}{dx^3}f_m(x). \tag{5.1}$$

(ii) Let $g_k = f_k'$ for $k=1,2,\dots$.

(iii) If u satisfies $f_m = 0$ for all m greater than or equal to some m_0, then u satisfies the stationary KdV hierarchy and is said to be an algebro-geometric (finite gap) potential; see [29,14] .

(iv) Suppose that $u(x)\to 0$ as $x\to\infty$, and likewise for all the partial derivatives $\partial^\ell u/\partial x^\ell$; suppose further that $f_j(x)\to 0$ as $x\to 0$ as $x\to\infty$ for all $j=1,2,\dots$. Then we say that the f_j are homogeneous solutions of the KdV hierarchy, and write $\hat f_j$ for f_j to indicate that the system of differential equations (5.1) has no arbitrary constants of integration.

Proposition 5.1. *Let $\mathcal{A}_\Sigma$ be as in Theorem 4.3. Then $f_m = (-1)^m2\lfloor A^{2m-1}\rfloor$ satisfies the stationary KdV hierarchy (Novikov's equations), since*

$$4\frac{d}{dx}\lfloor A^{2m+3}\rfloor = \frac{d^3}{dx^3}\lfloor A^{2m+1}\rfloor + 8\Bigl(\frac{d}{dx}\lfloor A\rfloor\Bigr)\lfloor A^{2m+1}\rfloor + 16\lfloor A\rfloor\Bigl(\frac{d}{dx}\lfloor A^{2m+1}\rfloor\Bigr). \tag{5.2}$$

Proof. (i) We have the basic identities

$$\lfloor A(I-2F)A(I-2F)X\rfloor = \lfloor A^2X\rfloor - 2\lfloor A\rfloor\lfloor X\rfloor; \tag{5.3}$$

$$-2A(AF+FA-2FAF) = A(I-2F)A(I-2F) - A^2 \tag{5.4}$$

and their mirror images. Hence

$$\frac{d}{dx}\lfloor A^{2m+1}\rfloor = \lfloor A(I-2F)A^{2m+1} + A^{2m+1}(I-2F)A\rfloor, \tag{5.5}$$

so

$$\begin{aligned}
\frac{d^2}{dx^2}\lfloor A^{2m+1}\rfloor &= \lfloor A(I-2F)A(I-2F)A^{2m+1} + 2A(I-2F)A^{2m+1}(I-2F)A \\
&\quad + A^{2m+1}(I-2F)A(I-2F)A \\
&\quad - 2A(AF+AF-2FAF)A^{2m+1} - 2A^{2m+1}(AF+FA-2FAF)A\rfloor \\
&= \lfloor A(I-2F)A(I-2F)A^{2m+1} + 2A(I-2F)A^{2m+1}(I-2F)A \\
&\quad + A^{2m+1}(I-2F)A(I-2F)A \\
&\quad + A(I-2F)A(I-2F)A^{2m+1} - A^{2m+1} + A^{2m+3}(I-2F)A(I-2F)A - A^{2m+3}\rfloor \\
&= 2\lfloor A(I-2F)A^{2m+1}(I-2F)A\rfloor - 2\lfloor A^{2m+3}\rfloor \\
&= 2\lfloor A(I-2F)A(I-2F)A^{2m+1}\rfloor - 2\lfloor A^{2m+3}\rfloor \\
&\quad + \lfloor A(I-2F)A(I-2F)A^{2m+1}\rfloor + 2\lfloor A^{2m+1}(I-2F)A(I-2F)A\rfloor \\
&= 2\lfloor A(I-2F)A^{2m+1}(I-2F)A\rfloor + 2\lfloor A^{2m+3}\rfloor \\
&\quad - 4\lfloor A^{2m+1}\rfloor\lfloor A\rfloor - 4\lfloor A\rfloor\lfloor A^{2m+1}\rfloor.
\end{aligned} \tag{5.6}$$

Now we differentiate the first summand of the final term

$$\begin{aligned}
&\frac{d}{dx}2\lfloor A(I-2F)A^{2m+1}(I-2F)A\rfloor \\
&= 2\lfloor A(I-2F)A(I-2F)A^{2m+1}(I-2F)A\rfloor + 2\lfloor A(I-2F)A^{2m+1}(I-2F)A(I-2F)A\rfloor \\
&\quad - 4\lfloor A(AF+FA-2FAF)A^{2m+1}(I-2F)A\rfloor - 4\lfloor A(I-2F)A^{2m+1}(AF+FA-2FAF)A\rfloor \\
&= 2\lfloor A(I-2F)A(I-2F)A^{2m+1}(I-2F)A\rfloor + 2\lfloor A(I-2F)A^{2m+1}(I-2F)A(I-2F)A\rfloor \\
&\quad + 2\lfloor A(I-2F)A(I-2F)A^{2m+1}(I-2F)A\rfloor - 2\lfloor A^{2m+3}(I-2F)A\rfloor \\
&\quad + 2\lfloor A(I-2F)A^{2m+1}(I-2F)A(I-2F)A\rfloor - 2\lfloor A(I-2F)A^{2m+3}\rfloor \\
&= 4\lfloor A(I-2F)A(I-2F)A^{2m+1}(I-2F)A\rfloor + 4\lfloor A(I-2F)A^{2m+1}(I-2F)A(I-2F)A\rfloor \\
&\quad - 2\lfloor A(I-2F)A^{2m+3} + A^{2m+3}(I-2F)A\rfloor \\
&= -8\lfloor A\rfloor\lfloor A^{2m+1}(I-2F)A\rfloor + 4\lfloor A^{2m+3}(I-2F)A\rfloor \\
&\quad - 8\lfloor A\rfloor\lfloor A(I-2F)A^{2m+1}\rfloor + 4\lfloor A(I-2F)A^{2m+3}\rfloor - 2\frac{d}{dx}\lfloor A^{2m+3}\rfloor \\
&= -8\lfloor A\rfloor\lfloor A(I-2F)A^{2m+1} + A^{2m+1}(I-2F)A\rfloor \\
&\quad + 4\lfloor A(I-2F)A^{2m+3} + A^{2m+3}(I-2F)A\rfloor - 2\frac{d}{dx}\lfloor A^{2m+3}\rfloor \\
&= -8\lfloor A\rfloor\frac{d}{dx}\lfloor A^{2m+1}\rfloor + 2\frac{d}{dx}\lfloor A^{2m+3}\rfloor;
\end{aligned} \tag{5.7}$$

hence

$$\frac{d^3}{dx^3}\lfloor A^{2m+1}\rfloor = -8\lfloor A\rfloor\frac{d}{dx}\lfloor A^{2m+1}\rfloor + 4\frac{d}{dx}\lfloor A^{2m+3}\rfloor - 8\frac{d}{dx}\Big(\lfloor A\rfloor\lfloor A^{2m+1}\rfloor\Big); \tag{5.8}$$

which gives the stated result.

□

Definition *(Diagonal Greens function).* Let $(-A, B, C)$ be as in Theorem 2.2. Then the diagonal Green's function is $g_0(x;\zeta)/\sqrt{\zeta}$ where

$$g_0(x;\zeta) = (1/2) + \lfloor A(\zeta I - A^2)^{-1}\rfloor. \tag{5.9}$$

The notation $g_0(x;\zeta)$ is chosen to indicate a generating function and also the diagonal of a Green's function; now we explain the latter connection. Let $\mathbf{C}_+ = \{\lambda \in \mathbf{C} : \Im\lambda > 0\}$ be the open upper half plane.

Theorem 5.2. *Let $(-A, B, C)$ be as in Theorem 2.2.*

(i) Then $g_0(x;\zeta)$ is bounded and continuously differentiable in x and has a unique asymptotic expansion depending on the odd powers of A,

$$g_0(x;\zeta) \asymp \frac{1}{2} + \frac{\lfloor A\rfloor}{\zeta} + \frac{\lfloor A^3\rfloor}{\zeta^2} + \frac{\lfloor A^5\rfloor}{\zeta^3} + \dots \qquad (\zeta \to -\infty); \tag{5.10}$$

(ii) $g_0(x;\zeta)$ satisfies Drach's equation

$$\frac{d^3 g_0}{dx^3} = 4(u+\zeta)\frac{dg_0}{dx} + 2\frac{du}{dx}g_0 \qquad (x > x_0; -\zeta > \omega); \tag{5.11}$$

(iii) there exists $x_1 > 0$ such that

$$\psi_\pm(x,\zeta) = \sqrt{g_0(x,-\zeta)}\exp\Big(\mp\sqrt{-\zeta}\int_{x_1}^{x}\frac{dy}{2g_0(y;-\zeta)}\Big) \tag{5.12}$$

satisfies

$$-\psi_\pm''(x;\zeta) + u(x)\psi_\pm(x,\zeta) = \zeta\psi_\pm(x;\zeta) \qquad (x > x_1, \zeta > \omega). \tag{5.13}$$

(iv) Suppose that u is a continuous real function that is bounded below, and that $\psi_\pm$ from (iii) satisfy $\psi_+(x;\zeta) \in L^2((0,\infty);\mathbf{C})$ and $\psi_-(x;\zeta) \in L^2((-\infty,0);\mathbf{C})$ for all $\zeta \in \mathbf{C}_+$. Then $L = -\frac{d^2}{dx^2} + u(x)$ defines an essentially self-adjoint operator in $L^2(\mathbf{R};\mathbf{C})$, and the Greens function $G(x,y;\zeta)$ which represents $(\zeta I - L)^{-1}$ has a diagonal that satisfies

$$G(x,x;\zeta) = \frac{g_0(x;-\zeta)}{\sqrt{-\zeta}}. \tag{5.14}$$

Proof. (i) Let $\pi - \theta < \arg\lambda < \theta$, so λ and $-\lambda$ both lie in S_θ, hence $\zeta = \lambda^2$ satisfies $2\pi - 2\theta < \arg\zeta < 2\theta$, so ζ lies close to $(-\infty, 0)$. Then $\zeta I - A^2$ is invertible and $|\zeta|\|(\zeta I - A^2)^{-1}\|_{\mathcal{L}(H)} \leq M$. The function

$$g_0(x;\zeta) = \frac{1}{2} + Ce^{-xA}(I+R_x)^{-1}A(\zeta I - A^2)^{-1}(I+R_x)^{-1}e^{-xA}B \qquad (x > 0) \tag{5.15}$$

is well defined by Theorem 2.2(iii).

To obtain the asymptotic expansion, we note that $e^{-xA}(I+R_x)^{-1}$ and $(I+R_x)e^{-xA}$ involve the factor e^{-xA}, where (e^{-zA}) is a holomorphic semigroup on $S_{\theta-\pi/2}$. Hence $A^{2j+1}e^{-xA}\in\mathcal{L}(H)$ and by Cauchy's estimates there exist $x_0, M_0>0$ such that $\|A^{2j+1}e^{-xA}\|_{\mathcal{L}(H)}\leq M_0(2j+1)!$ for all $x\geq x_0>0$. As in Proposition 3.1, we have an asymptotic expansion of

$$\begin{aligned}e^{-zA}\big((\lambda I-A)^{-1}-(\lambda I+A)^{-1}\big)&=-e^{-zA}\int_0^\infty e^{\lambda s}e^{-sA}\,ds-e^{-zA}\int_0^\infty e^{-\lambda s}e^{-sA}\,ds\\&=e^{-zA}\Big(\frac{A}{\lambda^2}+\frac{A^3}{\lambda^4}+\dots+\frac{A^{2j-1}}{\lambda^{2j}}\Big)\\&\quad+\frac{e^{-zA}}{\lambda^{2j+1}}\int_0^\infty A^{2j+1}e^{-sA}(e^{s\lambda}-e^{-\lambda s})\,ds,\end{aligned}\tag{5.16}$$

in which all the summands are in $\mathcal{L}(H)$ due to the factor e^{-zA} for $z\in S_{\theta-\pi/2}$. Hence

$$Ce^{-xA}e^{-xA}(I+R_x)^{-1}\int_0^\infty A^{2j+1}e^{-sA}(e^{s\lambda}-e^{-s\lambda})\,ds(I+R_x)^{-1}e^{-xA}B\to 0\qquad(x>0)\tag{5.17}$$

as $\lambda\to i\infty$, or equivalently $\zeta\to-\infty$, so

$$g_0(x,\zeta)=\frac{1}{2}+Ce^{-xA}(I+R_x)^{-1}\Big(\frac{A}{\zeta}+\frac{A^3}{\zeta^2}+\dots+\frac{A^{2j-1}}{\zeta^j}\Big)(I+R_x)^{-1}e^{-xA}B+O\Big(\frac{1}{\zeta^{j+1}}\Big).\tag{5.18}$$

This gives the asymptotic series; generally, the series is not convergent since the implied constants in the term $O(\zeta^{-(j+1)})$ involve $(2j+1)!$.

(ii) From Proposition 5.1 we have

$$\begin{aligned}4\frac{d}{dx}\sum_{m=0}^\infty\frac{\lfloor A^{2m+3}\rfloor}{\zeta^{m+1}}&=\frac{d^3}{dx^3}\sum_{m=0}^\infty\frac{\lfloor A^{2m+1}\rfloor}{\zeta^{m+1}}\\&\quad+8\Big(\frac{d}{dx}\lfloor A\rfloor\Big)\sum_{m=0}^\infty\frac{\lfloor A^{2m+1}\rfloor}{\zeta^{m+1}}+16\lfloor A\rfloor\frac{d}{dx}\sum_{m=0}^\infty\frac{\lfloor A^{2m+1}\rfloor}{\zeta^{m+1}};\end{aligned}\tag{5.19}$$

the required result follows on rearranging.

Conversely, suppose that g_0 as defined in (5.9) has an asymptotic expansion with coefficients in $C^\infty((0,\infty);\mathbf{C})$ as $\zeta\to-\infty$ and that $g_0(x;\zeta)$ satisfies (5.11). Then the coefficients of ζ^{-j} satisfy a recurrence relation which is equivalent to the systems of differential equations (5.1).

The asymptotic expansion is unique in the following sense. Suppose momentarily that $t\mapsto\lfloor Ae^{-tA^2}\rfloor$ is bounded and repeatedly differentiable on $(0,\infty)$, with $M,\omega>0$ such that $|\lfloor Ae^{-tA^2}\rfloor|\leq Me^{\omega t}$ for $t>0$, and that there is a Maclaurin expansion

$$\lfloor Ae^{-tA^2}\rfloor=\lfloor A\rfloor-\lfloor A^3\rfloor t+\frac{\lfloor A^5\rfloor t^2}{2!}-\dots+O(t^k)\tag{5.20}$$

on some neighbourhood of $0+$. Then by Watson's Lemma, the integral $\int_0^\infty\lfloor Ae^{-tA^2}\rfloor e^{t\zeta}\,dt$ has an asymptotic expansion as $\zeta\to-\infty$, where the coefficients give the formula (5.10).

(iii) Since $(e^{-tA})_{t>0}$ is a contraction semigroup on H, we have $\mathcal{D}(A^2) \subseteq \mathcal{D}(A)$ and $\|Af\|_H^2 \leq 2\|A^2 f\|_H \|f\|_H$ for all $f \in \mathcal{D}(A^2)$ by the Hardy-Littlewood-Landau inequality, so $\|\zeta f + A^2 f\|_H \geq \sqrt{\zeta}\|Af\|_H$ for $\zeta > 0$. We deduce that $A^2 - 2A + \zeta I$ is invertible for $\zeta > 9$ and generally for all $\zeta \in \mathbf{C}$ such that $\Re\zeta$ is sufficiently large. By Proposition 5.1 and the multiplicative property of the bracket, we have

$$\frac{1}{2g_0(x;-\zeta)} = 1 + \lfloor 2A(\zeta I + A^2 - 2A)^{-1} \rfloor, \tag{5.21}$$

and we observe that $g_0(x;-\zeta) \to 1/2$ as $x \to \infty$, so there exists $x_1 > 0$ such that $g_0(x,-\zeta) > 0$ for all $x > x_1$ and the differential equation integrates to

$$g_0 \frac{d^2 g_0}{dx^2} - \frac{1}{2}\Big(\frac{dg_0}{dx}\Big)^2 = 2(u-\zeta)g_0^2 + \frac{\zeta}{2}. \tag{5.22}$$

So we can define $\psi(x;\zeta)$ as in (5.13), and then one verifies the differential equation for $\psi(x;\zeta)$ by using (5.22).

(iv) By a theorem of Weyl [32, 10.1.4], L is of limit point type at $\pm\infty$, and there exist nontrivial solutions $\psi_\pm(x;\zeta)$ to $-\psi_\pm''(x;\zeta) + u(x)\psi_\pm(x;\zeta) = \zeta\psi_\pm(x;\zeta)$ such that $\psi_+(x;\zeta) \in L^2(0,\infty)$ and $\psi_-(x;\zeta) \in L^2(-\infty,0)$, and these are unique up to constant multiples. Also the inverse operator $(-\zeta I + L)^{-1}$ may be represented as an integral operator in $L^2(\mathbf{R};\mathbf{C})$ with kernel $G(x,y;\zeta)$, which has diagonal

$$G(x,x;\zeta) = \frac{\psi_+(x;\zeta)\psi_-(x;\zeta)}{\mathrm{Wr}(\psi_+(\,;\zeta),\psi_-(\,;\zeta))} \qquad (\Im\zeta > 0), \tag{5.23}$$

Given $\psi_\mp$ as in (iii), we can compute $\psi_+(x;\zeta)\psi_-(x;\zeta) = g_0(x;-\zeta)$ and their Wronskian is $\mathrm{Wr}(\psi_+,\psi_-) = \sqrt{-\zeta}$, hence the result.

□

Remarks 5.3 (i) The importance of the diagonal Greens function is emphasized in [30]. Gesztesy and Holden [29, Lemma 1.6.1] obtain an asymptotic expansion of the diagonal $G(x,x;\zeta)$ which is consistent with Theorem 5.2(i). Under conditions discussed in (5.48), we have similar asymptotics as $-\zeta \to \infty$.

(ii) Let ψ and φ be solutions of $-f'' + uf = \zeta f$ for some $u \in C^2(\mathbf{R};\mathbf{C})$ such that $\mathrm{Wr}(\psi,\varphi)^2 = -\zeta$, and then let $g(x) = \psi(x)\varphi(x)$. By differentiating, one checks that $gg'' - (g')^2/2 - 2(u-\zeta)g^2 = C$ for some constant C, and one can evaluate $C = -\mathrm{Wr}(\psi,\varphi)^2/2$. Hence

$$\rho(x) = \sqrt{g(x)}\exp\Big(\pm\sqrt{-\zeta}\int_0^x \frac{dy}{2g(y)}\Big) \tag{5.24}$$

gives a solution of $-\rho'' + u\rho = \zeta\rho$.

(iii) Drach observed that one can start with the differential equation (5.11), and produce the solutions (5.24); see [14]. He showed that Schrödinger's equation is integrable by quadratures, if and only if (5.11) can be integrated by quadratures for typical values of ζ, and Brezhnev translated his results into the modern theory of finite gap integration [14]. Having established integrability of Schrödinger's equation by quadratures, one can introduce the hyperelliptic spectral curve $\mathcal{E}$ with $g < \infty$ and proceed to express the solution in terms of the Baker–Akhiezer function. Hence one can integrate the equation and express the solution in terms of the Riemann's theta function on the Jacobian of $\mathcal{E}$, as in [43]. Our presentation follows Drach's; one advantage is that we can deal with rather degenerate solutions of differential equations, such as occur in the theory of solitons, and can evade technicalities regarding special points on Jacobians of compact Riemann surfaces.

Proposition 5.4. *(i) In the context of Theorem 4.3, suppose that there exists a non-zero odd complex polynomial $p_0(X)$ such that $\lfloor p_0(A)\rfloor = 0$. Then $\mathbf{C}[u, du/dx, \dots,]$ is a Noetherian differential ring for d/dx and the usual multiplication.*

(ii) In particular, (i) holds when A^2 is algebraic.

Proof. (i) It follows from Proposition 5.1 that $\lfloor A^{2j-1}\rfloor = c_j u^{(2j)} + P_j(u, u', \dots, u^{(2j-1)})$ where P is a complex polynomial, and $c_j \neq 0$. Adding multiples of such identities, and using the hypotheses, we deduce that there exists m such that

$$\frac{d^{2m}u}{dx^{2m}} = Q_{2m}\Big(u, \frac{du}{dx}, \dots, \frac{d^{2m-1}u}{dx^{2m-1}}\Big), \tag{5.25}$$

where Q_{2m} is a complex polynomial which is determined by p_0 and the f_j. By repeatedly differentiating this identity, and substituting back, one can obtain polynomials Q_n such that $u^{(n)} = Q_n(u, du/dx, \dots, d^{n-1}u/dx^{n-1})$, for all $n \geq 2m$. Hence $\mathbf{C}[u, du/dx, \dots, d^{2m-1}u/dx^{2m-1}]$ gives all of $\mathbf{C}[u, du/dx, \dots,]$.

Since u is meromorphic on the domain Ω, the algebra $\mathbf{C}[u, du/dx, \dots, d^{2m-1}u/\partial x^{2m-1}]$ is an integral domain, and we have shown it to be closed under differentiation. By mapping $X_j \mapsto d^j u/dx^j$ for $j = 0, \dots, 2m-1$, we obtain a short exact sequence of algebra homomorphisms

$$0 \to J \to \mathbf{C}[X_0, \dots, X_{2m-1}] \to \mathbf{C}\Big[u, \frac{du}{dx}, \dots, \frac{d^{2m-1}u}{dx^{2m-1}}\Big] \to 0, \tag{5.26}$$

where the ideal J is prime. Hence $\mathbf{C}[u, du/dx, \dots, d^{2m-1}u/dx^{2m-1}]$ is finitely generated as an algebra. Also, the $\mathbf{C}[X_0, \dots, X_{2m-1}]/J$ is naturally isomorphic as an algebra to the co ordinate ring $\mathbf{C}[V]$, where V is the affine variety $\{z \in \mathbf{C}^{2m} : f(z) = 0, \forall f \in J\}$.

(ii) If A^2 is algebraic, then there exists a monic complex polynomial such that $p(A^2) = 0$, hence $Ap(A^2) = 0$ and (i) applies.

□

Proposition 5.5. *Suppose that $\lfloor A^{2n+1}\rfloor = 0$ for some $n \geq 0$.*

(i) Then $u = -4\lfloor A\rfloor$ is finite gap;

(ii) $-d^2/dx^2 + u$ commutes with a differential operator of odd order;

(iii) Schrödinger's equation can be integrated by quadratures over $\mathbf{C}[u, du/dx, \dots,]$.

(iv) The image of $(A, \partial A, \dots, \partial^{2n}A) \in \mathcal{A}_\Sigma^{2n+1}$ under $\lfloor\cdot\rfloor$ equals $(u, u', \dots, u^{(2n)})$, where $(u.u', \dots, u^{(2n)})$ satisfy a system of polynomial equations which determine a complex algebraic variety.

Proof. (i) Note that the hypothesis does not change if we apply the translation $(-A, B, C) \mapsto (-A, e^{-tA}B, C)$ so that $u(x) \mapsto u(x+t)$.Then $u = -4\lfloor A \rfloor$ satisfies

$$\begin{aligned} f_1 &= (1/2)u + (1/2)c_1 = -2\lfloor A \rfloor + (1/2)c_1; \\ f_2 &= -(1/8)u'' + (3/8)u^2 + (c_1/4)u + (c_2/2) - c_1/8 = 2\lfloor A^3 \rfloor - c_1 \lfloor A \rfloor + (c_2/2) - c_1/8, \end{aligned} \tag{5.27}$$

Under the hypotheses of Theorem 4.3, all of the c_j vanish since u and its derivatives converge to zero as $x \to \infty$, so the f_j are homogeneous solutions of KdV hierarchy. By Proposition 5.1, $f_m(x) = 0$ for all $m > n$.

(ii) With $L = -d^2/dx^2 + u$, we have an ordinary differential operator

$$P_{2n+1} = \sum_{j=0}^{n} \Bigl(f_{n-j} \frac{d}{dx} - 2^{-1} \frac{df_{n-j}}{dx} \Bigr) (-L)^j \tag{5.28}$$

so that

$$[L, P_{2n+1}] = 2f'_{n+1}. \tag{5.29}$$

(iii) By Propositions 5.1 (ii) and 5.4, we have a Noetherian differential ring

$$\mathbf{C}[u, du/dx, \dots, ; \zeta; g, g, g'']. \tag{5.30}$$

We aim to prove that we can obtain all elements of this ring by quadratures over $\mathbf{C}[u, du/dx, \dots,]$, and thereby solve Schrödinger's equation. We observe that $2\zeta^n g_0(x; \zeta)$ is a monic polynomial of degree n in ζ. Changing notation, we introduce $F(x; \lambda) = \sum_{\ell=0}^{n} f_{n-\ell}(x)\lambda^\ell$ where the coefficients of F, as a monic polynomial of degree n in λ, are polynomials in u and its spatial derivatives, since $f_j \in \mathcal{A}$. Hence

$$p(\lambda) = \frac{1}{2} F(x; \lambda) \frac{\partial^2 F(x; \lambda)}{\partial x^2} - \frac{1}{4} \Bigl(\frac{\partial F(x; \lambda)}{\partial x} \Bigr)^2 - (u(x) - \lambda) F(x; \lambda)^2 \tag{5.31}$$

is a polynomial of degree $2n+1$ in λ, with coefficients in $\mathcal{A}$, which is actually independent of x, so $p(\lambda) \in \mathbf{C}[\lambda]$. Now we introduce

$$\psi_{\pm}(x; \lambda) = \sqrt{F(x; \lambda)} \exp\Bigl(\pm \mu \int^x \frac{d\xi}{F(\xi; \lambda)} \Bigr), \tag{5.32}$$

which satisfies

$$\frac{\psi''_\lambda}{\psi_\lambda} = \frac{1}{F(x; \lambda)^2} \bigl((u - \lambda) F(x; \lambda)^2 + p(\lambda) + \mu^2 \bigr); \tag{5.33}$$

hence ψ_λ gives the solution to Schrödinger's equation $\psi''_{\pm}(x; \lambda) = (u - \lambda)\psi''_{\pm}(x; \lambda)$ when (λ, μ) lies on

$$\mathcal{E} = \{ (\lambda, \mu) : \mu^2 = -p(\lambda) \}. \tag{5.34}$$

(iv) This follows from an argument of Mumford [57]. We introduce the polynomials in λ:

$$U(\lambda; x) = F; \quad W(\lambda; x) = (\lambda - u(x))F + \frac{1}{2} \frac{\partial^2 F}{\partial x^2}; \quad V(\lambda; x) = \frac{i}{2} \frac{\partial F}{\partial x} \tag{5.35}$$

which have degrees n, $n+1$ and $n-1$ respectively, for typical x, and the coefficients are given by the f_j and their partial derivatives with respect to x. Then we have a monic polynomial

$$p(\lambda) = U(\lambda; x)W(\lambda; x) + V(\lambda; x)^2 \tag{5.36}$$

of degree $2n+1$, where the leading terms are

$$U = \lambda^n + \lambda^{n-1} f_1 + \dots, \quad V = (i/2) f_1' \lambda^{n-1} + \dots, \quad W = \lambda^{n+1} + (f_1 - u)\lambda^n + \dots. \tag{5.37}$$

Then $\mathcal{E}$ gives a hyperelliptic curve over $\mathbf{C}$ of genus $g \leq n$. The Picard group $\mathrm{Pic}(\mathcal{E})$ is the set of divisors on $\mathcal{E}$, modulo linear equivalence as in [65]. Then the Jacobian J is the subgroup of $\mathrm{Pic}(\mathcal{E})$ consisting of equivalence classes that have degree zero; this group J gives an Abelian variety which may be determined algebraically by arguments presented in [57]. There is a Riemann theta function ϑ which is entire and quasi-periodic on J, and has zero set $\theta_0 = \{z \in J : \vartheta(z) = 0\}$. When we replace $u(x)$ by translation to $u(x+t)$, we make a corresponding automorphism $p(\lambda) \mapsto p(\lambda; t)$, $U(\lambda; x) \mapsto U(\lambda; x, t)$, etc. One can then differentiate with respect to the parameter t, and thus introduce a vector field D_∞ on J.

One can also introduce a vector field D_θ on J and the meromorphic function $\wp(z) = D_\infty^2 \log \vartheta$ on $J \setminus \theta_0$. Comparing our calculations with those of Mumford, we obtain $D_\infty = (i/2)\partial/\partial x$ and $4\wp + d = u/2$ for some constant d. Thus we can regard x as a coordinate for motion along the vector field determined by D_∞ in the tangent space to J, and $u = 8\wp + 2d$ as an extension of u to a function on $J \setminus \theta_0$. Likewise, the functions f_j, g_k of (5.1) may be extended to complex functions on $J \setminus \theta_0$ give rise to an embedding $J \setminus \theta_0 \to \mathbf{C}^{2g+1}$ by $z \mapsto (\wp, D_\infty \wp, \dots, D_\infty^{2n} \wp)$. By analogy, in the context of Theorem 5.3 we can regard $(A, \partial A, \dots, \partial^{2n} A) \in \mathcal{A}_\Sigma^{2n+1}$ as coordinates for the Jacobian. See [49, article 9].

$\square$

Example 5.6. Suppose that A satisfies $\lfloor a_3 A^3 + a_1 A \rfloor = 0$, which is the first non trivial case of Proposition 5.4. Then $a_3 f_2 - a_1 f_1 = 0$, so

$$a_3\big(-u''/16 + 3u^2/16 + c_1 u/8 + c_2/4 - c_1/16\big) - (a_1/4)u - (a_1/4)c_1 = 0. \tag{5.38}$$

Hence we can identify V with a curve $X_1^2 = p(X_0)$ where p is a cubic; see [49]. In the context of Theorem 4.3, we have $u(x) \to 0$ as $x \to \infty$ along with u', u'', so the solution has the form

$$u(x) = \frac{2a_1}{a_3} \mathrm{sech}^2 \sqrt{\frac{-a_1}{4a_3}}(x + \gamma), \tag{5.39}$$

with constant $\gamma \in \mathbf{C}$, as is familiar from the theory of solitons. However, (5.37) also has solutions in terms of Weierstrass's elliptic function $\wp$, and in section 7 we construct linear systems with potential $\wp$.

(ii) For $g > 1$, does there exist a linear system such that the corresponding potential is the $\wp$ function for a hyperelliptic curve of genus g? We obtain a partial solution of this in section 7, by using the Schottky–Klein prime function.

Proposition 5.7. *Suppose that $g_0(x_j,\zeta_j)=0$ and $\frac{\partial g_0}{\partial\zeta}(x_j,\zeta_j)\neq 0$ for some $x_j>x_0$ and nonzero $\zeta_j\in\mathbf{C}\setminus\mathrm{Spec}(A^2)$.*

(i) Then there exist $\varepsilon>0$ and a differentiable family of solutions of (5.13) which are parametrized by an arc $\{\mu_j(t):x_j-\varepsilon<t<x_j+\varepsilon)\}$ passing through ζ_j such that $\psi(t,\mu_j(t))=0$;

(ii) Dubrovin's equation holds

$$\frac{d\mu_j}{dx}(t)=\frac{\pm\sqrt{\zeta}}{\frac{\partial g_0}{\partial\zeta}(t,\mu_j(t))}. \tag{5.40}$$

Proof. (i) First we make an observation about the zeros of $g_0(x,\zeta)$. The exponential matrix

$$\exp\Bigl(t\begin{bmatrix}0&1&0\\0&0&1\\0&4\zeta&0\end{bmatrix}\Bigr) \tag{5.41}$$

has entries that are entire functions of ζ of order ρ, where $\rho\leq 1/2$. We deduce that the general solution $g(x,\zeta)$ of (5.11) may be written as an entire function of ζ of order $\rho\leq 1/2$, so there exist functions $g_1(x)$ and $\mu_j(x)$ such that

$$g(x,\zeta)=g_1(x)(\zeta-\mu_0(x))\prod_{j=1}^{\infty}\Bigl(1-\frac{\zeta}{\mu_j(x)}\Bigr). \tag{5.42}$$

We can, of course, divide this by any function of ζ and still have a solution of the linear differential equation; in particular, we obtain $g_0(x,\zeta)$ in this way.

Let $x_j>x_0$ and suppose that $g_0(x_j,\zeta_j)=0$ where $\zeta_j\in\mathbf{C}\setminus\mathrm{Spec}(A^2)$ so that $\zeta\mapsto g_0(x_j,\zeta)$ is holomorphic. By hypothesis, we have $\frac{\partial g_0}{\partial\zeta}(x_j,\zeta_j)\neq 0$, which rules out the possibility of a multiple zero at ζ_j in the factorization (). Now we apply the implicit function theorem to the formula $g_0(t,\zeta)=0$, noting that $\zeta\mapsto g_0(x_j,\zeta)$ has a simple zero at $\zeta=\zeta_j$. By Rouché's theorem and the calculus of residues, there exist $\varepsilon_1,\varepsilon_2>0$ such that the contour integral

$$\mu_j(t)=\frac{1}{2\pi i}\int_{C(\zeta_j,\varepsilon_1)}\frac{z\frac{\partial g_0}{\partial\zeta}(t,z)dz}{g_0(t,z)} \tag{5.43}$$

determines a continuously differentiable function $\mu_j:(x_j-\varepsilon_2,x_j+\varepsilon_2)\to\mathbf{C}$ which satisfies $\mu_j(x_j)=\zeta_j$ and

$$\frac{\partial g_0}{\partial x}(t,\mu_j(t))+\frac{\partial g_0}{\partial\zeta}(t,\mu_j(t))\frac{d\mu_j}{dx}(t)=0. \tag{5.44}$$

(ii) By (5.22), we have $\frac{\partial g_0}{\partial x}(x_j,\zeta_j)^2=\zeta\neq 0$ hence Dubrovin's equation holds. Also $d\mu_j/dx\neq 0$, so μ_j determines a differentiable arc in $\mathbf{C}\setminus\mathrm{Spec}(A^2)$. Then for $\zeta=\mu_j(t)$ on this arc, $\psi_+(x,\mu_j(t))\psi_-(x,\mu_j(t))=g_0(x,\mu_j(t))$ vanishes at $x=t$, so one of the solutions $\psi_\pm(x;\mu_j(t))$ satisfies $\psi_\pm(t;\mu_j(t)))=0$ for all $t\in(x_j-\varepsilon_3,x_j+\varepsilon_3)$ for some $0<\varepsilon_3<\varepsilon_1$.

If $\zeta_j\in\mathbf{R}$ and $g_0(x,\zeta)$ is real-valued for all $(x,\zeta)\in\mathbf{R}^2$ in a neighbourhood of (x_j,ζ_j), then we can choose the arc $\{\mu_j(t):t\in(x_j-\varepsilon_3,x_j+\varepsilon_3)\}$ to be an open subinterval of $\mathbf{R}$.

$\square$

Definition *(Characteristic function).* For $(-A, B, C)$ as in Theorem 2.2 let

$$\varphi(\lambda; x) = \int_0^\lambda \frac{ig_0(x; -\zeta)}{\sqrt{-\zeta}}\, d\zeta, \tag{5.45}$$

and define the characteristic function by

$$\Delta(\lambda; x) = 2\cos\varphi(\lambda; x) \qquad (2\pi - 2\theta < \arg\lambda < 2\theta). \tag{5.46}$$

Gesztesy and Simon [30] have developed another approach to $G(x, x; \zeta)$ for self-adjoint Schrödinger operators which involves the xi function and Krein's spectral shift; see also [34]. We adopt their terminology from [30, (6.8)] in the following context. To understand the sign conventions, it helps to bear in mind that, for L self-adjoint, the diagonal Greens function $G(x, x; \zeta)$ is purely imaginary for ζ in the essential spectrum of L.

Suppose that (λ_j) is a non-zero complex sequence such that
(i) $(\Re\lambda_j)_{j=0}^\infty$ is increasing;
(ii) $\sum_{j=0}^\infty 1/(1 + |\Re\lambda_j|^\alpha)$ converges for all $\alpha > 1/2$;
(iii) $\sum_{n=1}^\infty |\lambda_{2n-1} - \lambda_{2n}|$ converges.
By (ii), we can introduce the functions

$$F_e(\lambda) = \prod_{j=0}^\infty \Bigl(1 - \frac{\lambda}{\lambda_{2j}}\Bigr), \quad F_o(\lambda) = \prod_{j=1}^\infty \Bigl(1 - \frac{\lambda}{\lambda_{2j-1}}\Bigr), \tag{5.47}$$

which are entire and of order $\rho \le 1/2$. Then $F_0(\lambda)/F_e(\lambda)$ is meromorphic with only simple zeros and poles, so

$$\mathcal{E} = \Bigl\{(z, \lambda) : z^2 = \frac{F_o(\lambda)}{F_e(\lambda)}\Bigr\} \tag{5.48}$$

determines a hyperelliptic curve of genus $g \le \infty$. There exists a homology basis of $\mathcal{E}$ which includes loops α_j around $[\lambda_{2j-1}, \lambda_{2j}]$. Let $\mu_j(x)$ lie on α_j.

Definition *(Discretely dominated).* Then g_0 is discretely dominated if there exist such data, with only the μ_j depending upon x and

$$g_0(x, -\zeta) = \frac{1}{2\sqrt{\lambda_0 - \zeta}} \prod_{j=1}^g \frac{\mu_j(x) - \zeta}{\sqrt{(\lambda_{2j} - \zeta)(\lambda_{2j-1} - \zeta)}} \qquad (\Im\zeta > 0). \tag{5.49}$$

Gesztesy and Simon provide several examples of real potentials such that g_0 is discretely dominated; our constants are consistent with their Example 3.2. The $\mu_j(x)$ are referred to as Dirichlet eigenvalues, or tied eigenvalues, while the differential equation (5.40) was derived by Dubrovin for real finite-gap periodic potentials. Given the general form (5.49), the further analysis reduces to the system of coupled differential equations for $d\mu_j(x)/dx$. When g_0 is discretely dominated, its properties are most easily understood in terms of conformal mapping.

In the following, we define the square root function by $s(\lambda) = \sqrt{\lambda} = |\lambda|^{1/2}\exp(i\arg(\lambda)/2)$, where $-\pi < \arg\lambda \leq \pi$ is the principal value of the argument, and recall that $h^*(\lambda, x) = \overline{h(\bar\lambda, \bar x)}$.

Proposition 5.8. *(i) Suppose that $(\lfloor A^{2j-1}\rfloor)_{j=1}^\infty$ is a sequence of real functions of $x \in \mathbf{R}$. Then $\varphi^*(\lambda; x) = \varphi(\lambda; x)$.*

(ii) Suppose that g_0 is discretely dominated with $\lambda_0 = 0$ and λ_j real, and suppose that that $\Delta(\lambda; x)$ is real for all $\lambda \in \mathbf{R}$. Then $\lambda \mapsto \varphi(\lambda; x)$ gives a conformal map of the upper half plane onto a slit domain in the first quadrant with possible vertical slits at $n\pi$, for $n \in \mathbf{N}$, and $\lambda \mapsto \Delta(\lambda; x)$ is entire.

Proof. (i) By the uniqueness of asymptotic expansions in Theorem 5.2(i), we have $g_0^*(x;\zeta) = g_0(x;\zeta)$, and since $s^*(\zeta) = s(\zeta)$, we deduce that $\varphi^*(\lambda; x) = \varphi(\lambda; x)$ for $x \in \mathbf{R}$ and $\lambda \in \mathbf{C} \setminus \mathbf{R}$. At this stage, we do not claim that φ is continuous across the real axis.

(ii) Suppose that $\lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_{2g}$ and $\lambda_{2j-1} < \mu_j < \lambda_{2j}$ for $j = 1, \ldots, g$; then for $c_0 \geq 0$, let

$$\varphi(\lambda) = \int_0^\lambda \frac{1}{2\sqrt{\zeta}} \prod_{j=1}^g \frac{\mu_j - \zeta}{\sqrt{(\lambda_{2j} - \zeta)(\lambda_{2j-1} - \zeta)}} d\zeta + ic_0. \tag{5.50}$$

Then φ is holomorphic for $\{\lambda \in \mathbf{C} : \Im\lambda > 0\}$ with $\varphi(\lambda)/\sqrt{\lambda} \to 1$ as $|\lambda| \to \infty$; the image of $\lambda \in (-\infty, 0)$ is the subinterval $(ic_0, i\infty)$ of the imaginary axis; whereas the image of $(0, \infty)$ consists of horizontal line segments running from left to right, interspersed by vertical lines running upwards then downwards, and the horizontal line segment together run towards ∞. For all sufficiently large $c_0 > 0$, the image of $\{\lambda \in \mathbf{C} : \Im\lambda > 0\}$ is a domain Ω in the first quadrant, with boundary consisting of horizontal and vertical line segments. This is a degenerate case of the Schwarz–Christoffel map, in which the triple $\lambda_{2j-1} < \mu_j < \lambda_{2j}$ corresponds to a degenerate triangle on the edge of Ω; two of the vertices of the degenerate triangle may coincide, giving a slit.

Now suppose $c_0 = 0$, and observe that $\Delta(\lambda; x)$ is real, if and only if either $\varphi(\lambda; x) \in \mathbf{R}$ or there exists $n \in \mathbf{Z}$ such that $\varphi(\lambda; x) - n\pi \in i\mathbf{R}$. We deduce that the only possible slits are at $\Re\varphi(\lambda; \pi) = n\pi$ for some $n \in \mathbf{N}$, that all slits are vertical, and each starts and finishes at the same point on the real axis.

We apply the Schwarz reflection principle. By (i), $\Delta^*(\lambda; x) = \Delta(\lambda; x)$, so we need to check continuity across $\lambda \in \mathbf{R}$ Whereas $\sqrt{\zeta}$ is discontinuous across $(-\infty, 0)$, we only need deal with $\cos\sqrt{\zeta}$, which is continuous. More precisely, by (i), the function $\varphi(\lambda; x)$ satisfies $\varphi(\lambda; x) = \varphi^*(\lambda; x)$, and $g(x; -\zeta)$ and $s(-\zeta)^2$ are continuous across $(-\infty, 0)$, hence $\varphi(\lambda; x)^2$ is holomorphic across $(-\infty, 0)$, so $\Delta(\lambda; x)$ is holomorphic across $(-\infty; 0)$; likewise, $\Delta(\lambda; x)$ is holomorphic across the spectral bands $[\lambda_{2j}, \lambda_{2j+1}]$. As λ approaches a spectral gap $(\lambda_{2j-1}, \lambda_{2j})$ the image $\varphi(\lambda; x)$ approaches a slit on the boundary of Ω; now $\cos(k\pi \pm iy) = (-1)^k \cosh y$, so $\Delta(\lambda; x)$ takes the same value, irrespective of which side λ approaches from, hence $\Delta(\lambda; x)$ is continuous across $\mathbf{R}$ and defines an entire function.

□

6. The differential ring of a periodic linear system

In this section we obtain analogues of Theorem 4.3 for periodic groups. First we formulate the notion of a periodic linear system, where we take section 4 as our guide. We show that the corresponding τ functions have properties analogous to those in Theorem 4.3. For periodic and meromorphic u, the differential equation $-\psi''+u\psi=\lambda\psi$ is known as the complex Hill's equation. We show how periodic linear systems appear in the Floquet solutions, and obtain a counterpart of Proposition 5.5. Previous authors [22, 28 37, 73] explored the connection between Hill's equation and scattering solutions of Schrödinger's equation on the line. In the current paper, we show how Lyapunov's equation is the basis for some analogies. In section 7 we consider particular examples, which exhibit subtle effects.

For periodic linear systems, the defining integral for R_x in Proposition 2.1 does not converge, and the contour integral for R_0 in Proposition 4.5 is inapplicable; nevertheless, we can adapt a result of Bhatia, Dacis and McIntosh discussed in [8] and otherwise construct R_x satisfying (1.3).

Lemma 6.1. *Let $B\in\mathcal{L}^1(H)$ and $C\in\mathcal{L}(H)$, and let $(e^{-tA})_{t\in\mathbf{R}}$ be a bounded and strongly continuous group of operators on H.*

(i) The space $\mathcal{D}(A^\infty)$ is dense.

(ii) Suppose that the spectrum of A does not intersect the spectrum of $-A$. Then there exists $E\in\mathcal{L}^1(H)$ such that $R_x=e^{xA}Ee^{xA}$ gives a solution to the Lyapunov equation $-\frac{d}{dx}R_x=AR_x+R_xA$ such that $AR_0+R_0A=BC$ and R_x is trace class for all $x\in\mathbf{R}$.

(iii) Suppose that the range of E is contained in $\mathcal{D}(A^\infty)$ and $A^kE\in\mathcal{L}^1(H)$ for all k. Then

$$\tau_\zeta(x)=\det\bigl(I+(\zeta I+A)(\zeta I-A)^{-1}e^{xA}Ee^{xA}\bigr)\qquad (x\in\mathbf{R})\tag{6.1}$$

has an asymptotic expansion in powers of ζ^{-j} as $\zeta\to\pm\infty$.

(iv) Suppose further that (e^{xA}) is periodic with period 2π. Then the spectrum of A is contained in $i\mathbf{Z}$ and the coefficients in the asymptotic expansion are periodic with period π.

Proof. (i) By standard results [21], A^2 generates an analytic semigroup

$$e^{tA^2}=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}e^{-s^2/4t}e^{sA}\,ds\qquad (t>0).\tag{6.2}$$

The domains of the powers of A satisfy $\mathcal{D}(A)\supseteq\mathcal{D}(A^2)$, and $\mathcal{D}(A^\infty)$ is dense.

(ii) The main problem is to find E such that $EA+AE=BC$. By a theorem of Sz.-Nagy, the group (e^{-tA}) is similar to a group of unitaries, so there exists an invertible operator S and a unitary group $(U_t)_{t\in\mathbf{R}}$ such that $e^{-tA}=SU_tS^{-1}$. Hence the spectrum of A lies on $i\mathbf{R}$ and is a closed subset. By hypothesis, there exists $\delta>0$ such that the spectra of A and $-A$ are separated by δ and $\sigma(A)\cup\sigma(-A)$ does not intersect $[-i\delta,i\delta]$. By Plancherel's theorem, we can construct $f\in L^1(\mathbf{R};\mathbf{C})$ such that $\hat f(\xi)=1/\xi$ for all $\xi\in\mathbf{R}$ such that $|\xi|\geq\delta$. Then the integral

$$E=\int_{-\infty}^{\infty}e^{-xA}BCe^{-xA}f(x)dx\tag{6.3}$$

has a weakly continuous integrand in $\mathcal{L}^1(H)$, and is absolutely convergent with

$$\Vert E\Vert_{\mathcal{L}^1(H)}\leq\int_{-\infty}^{\infty}\Vert B\Vert_{\mathcal{L}^1(H)}\Vert C\Vert_{\mathcal{L}(H)}M^2|f(x)|\,dx\tag{6.4}$$

hence E is trace class. Using the spectral representation of U_t, one can show that $AE+EA=BC$. Next we introduce $R_x = e^{-xA}Ee^{-xA}$ which gives a one parameter family of trace class operators such that $-\frac{dR_x}{dx} = AR_x + R_xA$. On verifies this identity on $\mathcal{D}(A)$ and then observes that both sides are trace class.

(iii) For all $\zeta \in \mathbf{R}$ we can invert $\zeta I - A$, and there is an asymptotic expansion

$$(\zeta I + A)(\zeta I - A)^{-1}Ee^{2xA} = \Bigl(I + \frac{2A}{\zeta} + \frac{2A^2}{\zeta^2} + \dots\Bigr)Ee^{2xA} \tag{6.5}$$

valid as $\zeta \to \pm\infty$. The result follows.

(iv) If (e^{xA}) is periodic with period 2π, then $(e^{xA})_{x\in\mathbf{R}}$ is bounded, and the spectrum of A is contained in $i\mathbf{Z}$.

□

Example 6.2. Let $H = L^2(\mathbf{R}/2\pi\mathbf{Z}; d\theta/(2\pi))$ and $A : e^{in\theta} \mapsto i(1+|n|)e^{in\theta}$, so (e^{tA}) is a 2π periodic strongly continuous unitary group. Also, $(e^{itA})_{t>0}$ gives a strongly continuous contraction semigroup $e^{itA} : \sum_{n\in\mathbf{Z}} a_n e^{in\theta} \mapsto e^{-t}\sum_{n\in\mathbf{Z}} a_n e^{-|n|t}e^{in\theta}$. For comparison, the Poisson semigroup is $P_r : \sum_{n\in\mathbf{Z}} a_n e^{in\theta} \mapsto \sum_{n\in\mathbf{Z}} a_n r^{|n|}e^{in\theta}$, so $e^{itA} = e^{-t}P_{e^{-t}}$ for $t>0$.

Definition *(Periodic linear system).* (i) Let $(e^{-xA})_{x\in\mathbf{R}}$ be a strongly continuous group of operators on H such that $e^{2\pi A} = I$ and A is invertible. Suppose further that E is trace class operators on H, and that $B : H_0 \to H$ and $C : H \to H_0$ are bounded linear operators, such that $AE + EA = BC$ and either B is trace class, or B and C are Hilbert–Schmidt. Then $\Sigma_\infty = (-A, B, C; E)$ is a periodic linear system with input and output space H_0 and state space H. (Unlike in Theorem 2.2, we generally take $H = H_0$ so the linear system has infinitely many inputs and outputs.)

(ii) Moreover, if $(e^{-xA})_{x\in\mathbf{R}}$ is uniformly continuous, or equivalently $A \in \mathcal{L}(H)$, we say that $(-A, B, C; E)$ is a uniform periodic linear system.

(iii) The τ function of Σ_∞ is $\tau(x) = \det(I + e^{-xA}Ee^{-xA})$; then let $u(x) = -2\frac{d^2}{dx^2}\log\tau_\infty(x)$ be the potential.

(iv) Let $\Phi(x) = Ce^{-xA}B$ be the operator scattering function so that $\phi(x) = \operatorname{trace}\Phi(x)$ is the (scalar) scattering function.

(v) Let $R_x = e^{-xA}Ee^{-xA}$, then we introduce $F_x = (I + e^{-xA}Ee^{-xA})^{-1}$.

(vi) Let $\mathrm{Spec}(A)$ be the spectrum of A as an operator in H, let $\mathbf{P} = \mathbf{C} \cup \{\infty\}$ be the Riemann sphere and introduce the periodic linear system

$$\Sigma_\lambda = \bigl(-A, (\lambda I + A)(\lambda I - A)^{-1}B, C; (\lambda I + A)(\lambda I - A)^{-1}E\bigr) \qquad (\lambda \in \mathbf{P} \setminus \mathrm{Spec}\ (A)) \tag{6.6}$$

and its accompanying tau function τ_λ.

(vii) We also introduce the (non commutative) algebra $\mathcal{S} = \mathbf{C}\{I, A, BC, F_x\}$, and then let $\mathcal{A}$ be the subring of $\mathcal{S}$ spanned by A^{n_1} and by the ordered products $A^{n_1}FA^{n_2}\dots FA^{n_r}$ for $n_j \in \mathbf{N}$.

Definition *(Bracket and $*$ product).* (i) As in Lemma 4.1, we introduce on $\mathcal{S}$ the product $*$ and derivation ∂ by

$$P * Q = P(AF + FA - 2FAF)Q, \qquad \partial P = A(I-2F)P + \frac{dP}{dx} + P(I-2F)A. \tag{6.7}$$

(ii) We also introduce the bracket $\lfloor\,.\,\rfloor : \mathcal{S} \to \mathcal{M}(\mathcal{L}^1(H))$:

$$\lfloor P \rfloor = Ce^{-xA}FPFe^{-xA}B. \tag{6.8}$$

(iii) Let $\lfloor \mathcal{A} \rfloor = \{\lfloor P \rfloor : P \in \mathcal{A}\}$ and $\Theta = \{P \in \mathcal{A} : \mathrm{trace}\lfloor P \rfloor = 0\}$, which is a linear subspace, and not necessarily a ring. Then $\mathcal{A}/\Theta = \{\mathrm{trace}\lfloor P \rfloor : P \in \mathcal{A}\}$, so that $\mathcal{A}/\Theta$ is analogous to the differential ring generated by the potential u.

Theorem 6.3. *Let $(-A, B, C; E)$ be a uniform periodic linear system.*

(i) Then $\tau_\lambda(x)$ is holomorphic except at finitely many singularities; so $x \mapsto \tau_\lambda(x)$ is entire, while $\lambda \mapsto \tau_\lambda(x)$ is holomorphic on $\mathbf{P} \setminus \mathrm{Spec}(A)$;

(ii) there is a homomorphism of complex differential rings $(\mathcal{A}, *, \partial) \to \mathbf{M_C}(\mathcal{L}(H_0))$ given by $X \mapsto \lfloor X \rfloor$;

(iii) the potential u is meromorphic and π-periodic on $\mathbf{C}$ and belongs to $\mathcal{A}/\Theta$.

(iv) Also, let $T(x, y) = -Ce^{-xA}F_xe^{-yA}B$. Then

$$\frac{\partial^2}{\partial x^2}T(x,y) - \frac{\partial^2}{\partial y^2}T(x,y) = -2\Big(\frac{d}{dx}T(x,x)\Big)T(x,y), \tag{6.9}$$

and $u(x) = -2\frac{d}{dx}\mathrm{trace}\, T(x, x)$.

Proof. (i) First we show that A is an algebraic operator. By periodicity, the group $(e^{-xA})_{x\in\mathbf{R}}$ is bounded and hence by Sz.-Nagy's theorem, e^{xA} is similar to a unitary group on H, so A is similar to a skew symmetric operator. By uniform continuity, A is bounded, and hence has spectrum contained in $\{-iN, \dots, iN\}$ for some integer N; see [21]. Consequently, there exists a monic polynomial p such that $p(A) = 0$. As in Proposition 4.5, $Ce^{-xA}B$ satisfies a linear differential equation with constant coefficients.

Hence A is an invertible algebraic operator, so as in (4.21), A^{-1} is a polynomial in A and $(\lambda I + A)(\lambda I - A)^{-1} \in \mathcal{S}$ for all λ in the resolvent set of A. Observe that $(\lambda I + A)(\lambda I - A)^{-1}$ is a polynomial in A with coefficients that are rational functions of λ, and holomorphic except when λ is in the spectrum of A; in particular it is holomorphic on $\{\lambda : |\lambda| < 1\} \cup \{\lambda : |\lambda| > \|A\|\}$.

We also introduce polynomials p_j for each point in the spectrum of A such that $p_j(ik) = \delta_{jk}$ for $k = -N, \dots, N$, and since A is similar to a skew operator, we deduce that

$$e^{-xA} = \sum_{j=-N; j\neq 0}^{N} p_j(A)e^{-ijx}. \tag{6.10}$$

Hence τ_λ is a holomorphic function of λ, except at $\lambda \in \mathrm{Spec}(A)$, which is a finite set.

(ii) First we check that $(\mathcal{S}, *, \partial)$ is a complex differential ring for $(-A, B, C; E)$ and for Σ_λ. By (4.19), the operator E belongs to $\mathcal{S}$ and hence by (6.10) $R_x = e^{-xA}Ee^{-xA}$ also belongs to $\mathcal{S}$. Hence we have

$$\frac{d}{dx}R_x = -e^{-xA}AEe^{-xA} - e^{-xA}EAe^{-xA} = -e^{-xA}BCe^{-xA}. \tag{6.11}$$

By the Riesz theory of compact operators, F_x is a meromorphic operator-valued function of x, and so $AF+FA-2FAF=Fe^{-xA}BCe^{-xA}F$, hence

$$\frac{dF}{dx}=AF+FA-2FAF; \tag{6.12}$$

so that

$$\partial(P*Q)=(\partial P)*Q+P*(\partial Q); \tag{6.13}$$

thus $(\mathcal{S},*,\partial)$ is a differential ring. Moreover $\lfloor\,.\,\rfloor:(\mathcal{S},*,\partial)\to(\mathcal{S},\cdot,d/dx)$ is a homomorphism of differential rings, in the sense that

$$\frac{d}{dx}\lfloor P\rfloor=\lfloor\partial P\rfloor,\quad \lfloor P\rfloor\lfloor Q\rfloor=\lfloor P*Q\rfloor. \tag{6.14}$$

Note that $\mathcal{A}$ is a subring of $\mathcal{S}$, and hence $\lfloor\mathcal{A}\rfloor$ is also a differential ring.

(iii) Since e^{-xA} is an entire operator function, we deduce that τ_∞ is entire, and π periodic since $\tau_\infty(x)=\det(I+e^{2xA}E)$ and $e^{2\pi A}=I$. When $\tau_\infty(x)\neq 0$, we have

$$\begin{aligned}\frac{d}{dx}\log\det(I+e^{-xA}Ee^{-xA})&=-\mathrm{trace}\big((I+e^{-xA}Ee^{-xA})^{-1}e^{-xA}(AE+EA)e^{-xA}\big)\\&=-\mathrm{trace}\big((I+e^{-xA}Ee^{-xA})^{-1}e^{-xA}BCe^{-xA}\big)\\&=-\mathrm{trace}\big(Ce^{-xA}(I+e^{-xA}Ee^{-xA})^{-1}e^{-xA}B\big)\\&=-\mathrm{trace}\big(Ce^{-xA}Fe^{-xA}B\big),\end{aligned} \tag{6.15}$$

and hence

$$\begin{aligned}u&=-2\frac{d^2}{dx^2}\log\det(I+e^{-xA}Ee^{-xA})\\&=-4\mathrm{trace}\,Ce^{-xA}FAFe^{-xA}B\\&=-4\mathrm{trace}\,\lfloor A\rfloor;\end{aligned} \tag{6.16}$$

so u belongs to $\mathcal{A}_0=\{\mathrm{trace}\lfloor P\rfloor : P\in\mathcal{A}\}$. Likewise, the derivatives $u^{(j)}$ belong to $\mathbf{A}_0$ since $\lfloor\mathcal{A}\rfloor$ is a differential ring.

(iv) By repeated differentiation, we have

$$\begin{aligned}\frac{\partial^2}{\partial x^2}T(x,y)&=-Ce^{-xA}\big(FA^2-4FAFA-4FA^2F+8FAFAF\big)e^{-yA}B,\\ \frac{\partial^2}{\partial y^2}T(x,y)&=-Ce^{-xA}\big(FA^2\big)e^{-yA}B\end{aligned} \tag{6.17}$$

while the Lyapunov equation gives

$$\begin{aligned}-2\Big(\frac{d}{dx}T(x,x)\Big)T(x,y)&=4Ce^{-xA}FAFc^{-xA}BCe^{-xA}Fe^{-yA}B\\&=4Ce^{-xA}FA(AF+FA-2FAF)e^{-yA}B,\end{aligned} \tag{6.18}$$

hence the result. In [12, Lemma 7.2] we obtained a variant of this formula via an integral equation in the style of Gelfand–Levitan, under special commutativity conditions.

□

Definition *(Solutions of Hill's equation).* (i) Let $u \in C^2(\mathbf{R};\mathbf{C})$ be π-periodic, and consider Hill's equation in the form

$$\frac{d}{dx}\begin{bmatrix}\psi\\ \psi'\end{bmatrix} = \begin{bmatrix}0 & 1\\ u(x)-\lambda & 0\end{bmatrix}\begin{bmatrix}\psi\\ \psi'\end{bmatrix}. \tag{6.19}$$

Let $F_\lambda(x)$ be the fundamental solution matrix, and $\Delta(\lambda) = \mathrm{trace}(F_\lambda(\pi))$ the discriminant.

(ii) The multiplier curve is $\mathcal{E} = \{\mathbf{p} = (\lambda,\rho) : \rho^2 - \Delta(\lambda)\rho + 1 = 0\}$, as in [22].

(iii) A Floquet solution consists of a nonzero function f such that $-f''(x) + q(x)f(x) = \lambda f(x)$ and $f(x+\pi) = \rho f(x)$ for some λ. We call λ the eigenvalue and ρ the multiplier, and (λ,ρ) lies on $\mathcal{E}$.

(iv) In particular, when $\rho = 1$, we say that f is periodic, and when $\rho = -1$, we say that f is anti periodic. Then $(\lambda, \pm 1)$ is a branch point on $\mathcal{E}$.

(v) Suppose that u is real-valued; then $L = -d^2/dx^2 + u$ is essentially self-adjoint. The Bloch spectrum of Schrödinger's operator consists of those $\lambda \in \mathbf{C}$ such that there exists a nontrivial and bounded solution f of $-f'' + uf = \lambda f$. Real potentials are said to belong to the same spectral equivalence class if their multiplier curves are equal. See [47, 50].

(vi) The Dirichlet eigenvalues $(\mu_j)_{j=1}^\infty$ are the μ such that

$$-y''(x;\mu) + u(x)y(x,\mu) = \mu y(x;\mu)$$

$$y(0;\mu) = 0 = y(\pi;\mu) \tag{6.20}$$

has a nontrivial solution. Replacing $u(x)$ by $u(x+t)$ we obtain the Dirichlet eigenvalues $(\mu_j(t))_{j=1}^\infty$.

The following result is related to results from Brett's thesis [13], and relates to the case in which the Gelfand–Levitan equation is scalar-valued, as in Proposition 3.1 and [27]. Given a periodic linear system $(-A, B, C)$, we introduce

$$S_x = \int_0^x e^{-tA}BCe^{tA}\,dt, \qquad V_x = \int_0^x e^{tA}BCe^{tA}\,dt \tag{6.21}$$

which satisfy the Lyapunov equation

$$\frac{d}{dx}\begin{bmatrix}V & 0\\ 0 & S\end{bmatrix} = \begin{bmatrix}A & 0\\ 0 & -A\end{bmatrix}\begin{bmatrix}V & 0\\ 0 & S\end{bmatrix} + \begin{bmatrix}V & 0\\ 0 & S\end{bmatrix}\begin{bmatrix}A & 0\\ 0 & A\end{bmatrix} + \begin{bmatrix}BC & 0\\ 0 & BC\end{bmatrix}. \tag{6.22}$$

Let $W_x = V_x - S_x$ and

$$L(x,y) = -Ce^{xA}(I + W_x)^{-1}(e^{yA} - e^{-yA})B; \tag{6.23}$$

then let $Z_x = V_x + S_x$ and

$$K(x,y) = -Ce^{xA}(I + Z_x)^{-1}(e^{yA} + e^{-yA})B. \tag{6.24}$$

Proposition 6.4. *Let $(-A, B, C)$ be a periodic linear system with $H_0 = \mathbf{C}$ such that $\phi(x) = Ce^{xA}B$ is even and suppose there exist $E, E_- \in \mathcal{L}^1(H)$ such that $AE + EA = BC$ and $-AE_- + E_-A = BC$.*

(i) Then the potential $w(x) = -2\frac{d^2}{dx^2}\log\det(I + W_x)$ is periodic, and

$$\varphi(x) = \frac{\sin kx}{k} + \int_0^x L(x, y)\frac{\sin ky}{k}dy \tag{6.25}$$

satisfies

$$\begin{aligned} -\varphi''(x) + w(x)\varphi(x) &= k^2\varphi(x) \\ \varphi(0) = 0, \qquad &\varphi'(0) = 1. \end{aligned} \tag{6.26}$$

(ii) Also the potential $u(x) = -2\frac{d^2}{dx^2}\log\det(I + Z_x)$ is periodic and

$$\psi(x) = \cos kx + \int_0^x K(x, y)\cos ky\, dy \tag{6.27}$$

satisfies

$$\begin{aligned} -\psi''(x) + u(x)\psi(x) &= k^2\psi(x) \\ \psi(0) = 1, \qquad &\psi'(0) = -2CB. \end{aligned} \tag{6.28}$$

Proof. (i) We have

$$\det(I + W_x) = \det(I + e^{xA}Ee^{xA} - e^{-xA}E_-e^{xA} - E + E_-) \tag{6.29}$$

which is periodic. Also, $L(x, 0) = 0$ and

$$L(x, x) = -\frac{d}{dx}\log\det(I + W_x). \tag{6.30}$$

One can verify that

$$\phi(x + y) - \phi(x - y) + L(x, y) + \int_0^x L(x, t)\big(\phi(t + y) - \phi(t - y)\big)\, dt = 0. \tag{6.31}$$

and with $w(x) = 2\frac{d}{dx}L(x, x)$, we deduce that

$$\frac{\partial^2 L}{\partial x^2} - \frac{\partial^2 L}{\partial y^2} = w(x)L(x, y) \tag{6.32}$$

and by manipulating the Gelfand–Levitan equation, one deduces the differential equation for φ.

(iii) We have

$$\det(I + Z_x) = \det(I + e^{xA}Ee^{xA} + e^{-xA}E_-e^{xA} - E - E_-), \tag{6.33}$$

which is periodic, with $\frac{\partial}{\partial y}K(x,0)=0$ and

$$K(x,x)=-\frac{d}{dx}\log\det(I+Z_x). \tag{6.34}$$

Then we verify that

$$\phi(x+y)+\phi(x-y)+K(x,y)+\int_0^x K(x,t)\big(\phi(t+y)+\phi(t-y)\big)\,dt=0. \tag{6.35}$$

and with $u(x)=2\frac{d}{dx}K(x,x)$, we have

$$\frac{\partial^2 K}{\partial x^2}-\frac{\partial^2 K}{\partial y^2}=u(x)K(x,y); \tag{6.36}$$

then by manipulating the Gelfand–Levitan equation, one deduces the differential equation for ψ.

□

Given Proposition 6.4, it is tempting to seek a version of Proposition 6.1 for periodic linear systems, and try to express the general solution of Hill's equation in terms of quotients of tau functions of periodic linear systems. The following result indicates the restriction that must be imposed upon u for such a representation to be valid.

Proposition 6.5. *Suppose that $\Sigma=(-A,B,C;E)$ is a uniformly periodic linear system with $H_0=H$ such that for all but finitely many $\lambda\in\mathbf{C}$, Hill's equation has a pair of linearly independent solutions of the form*

$$\psi_\lambda(x)=e^{\nu x}\prod_{j=1}^{n}\frac{\tau_{\zeta_j}(x-a_j)}{\tau_{\eta_j}(x-b_j)} \tag{6.37}$$

for some $\nu,a_j,b_j,\zeta_j,\eta_k$ depending upon λ .

(i) Then u is a Picard potential, in the sense that Hill's equation has a meromorphic general solution for all but finitely many $\lambda\in\mathbf{C}$;

(ii) u is finite gap as a potential for Hill's equation, and there exists a differential operator P_{2g+1} of order $2g+1$ such that

$$P_{2g+1}^2=\prod_{j=0}^{2g}(L-\lambda_j) \tag{6.38}$$

for some $\lambda_j\in\mathbf{C}$ such that $\Delta^2(\lambda_j)=4$, and the diagonal Greens function g_0 is discretely dominated and satisfies (5.49).

(iii) Suppose further that u is real-valued. Then the Bloch spectrum of L is associated with a hyperelliptic algebraic complex curve

$$\mathcal{E}_0=\Big\{(\mu,\lambda)\in\mathbf{C}^2:\mu^2=\prod_{j=0}^{2g}(\lambda-\lambda_j^o)\Big\}\cup\big\{(\infty,\infty)\big\} \tag{6.39}$$

where λ_j^o a simple real zero of $\Delta(\lambda)^2-4=0$, and such that the $\lambda\in\sigma_B$ give real points (μ,λ) on $\mathcal{E}_0$.

Proof. (i) In this case $x\mapsto \det(I+e^{-xA}Ee^{-xA})$ is entire and hence ψ_λ is meromorphic; hence there exists a meromorphic fundamental system of solutions on $\mathbf{C}$. Since τ is periodic, we have $\phi_\lambda(x+\pi)=e^{\nu\pi}\phi_\lambda(x)$, so that ψ_λ is a Floquet solution with Floquet exponent $\pi\nu$. By choosing a pair of solutions with Floquet exponents $\pm\pi\nu(\lambda)$, we obtain a fundamental solution matrix such that $\Delta(\lambda)=2\cosh\pi\nu(\lambda)$.

(ii) This is a particular case of the Burchnall–Chaundry theorem, and the required P_{2g+1} is given in the proof of Proposition 5.5. See [29, Theorem 4.1] for details. Also, for periodic $u\in C^2(\mathbf{R},\mathbf{C})$, Gesztesy and Weikard [29] obtain the formula (4.31) where the λ_j are the periodic eigenvalues which correspond to periodic solutions of (6.19). The $\mu_j(x)$ are referred to as Dirichlet eigenvalues, or tied eigenvalues.

(iii) The set of all λ that give $\rho=\pm1$ is called the periodic spectrum. Such a u has periodic spectrum $\lambda_0<\lambda_1\leq\lambda_2<\lambda_3\leq\dots$, where for λ_{2j} there exists a periodic eigenfunction, while for λ_{2j+1} there exists an anti periodic eigenfunction. We have $\sigma_B=\cup_{k=0}^{\infty}[\lambda_{2k},\lambda_{2k+1}]$, where many of these intervals are abutting. For real x, we deduce that just as $\psi_\lambda(x)$ in (6.19) has parameters ν,a_j,η_j,b_j and η_j for $j=1,\dots,n$ and gives a solution of Hill's equation with λ, the complex conjugate $\overline{\psi_\lambda(x)}$ has parameters $\bar\nu,\bar a_j,\bar\zeta_j,\bar b_j$ and $\bar\eta_j$ for $j=1,\dots,n$ and hence gives a solution of Hill's equation with $\bar\lambda$. We select ψ_λ with Floquet exponent $\pi\nu$, and $\phi_{\bar\lambda}$ with Floquet exponent $\pi\bar\nu$, and observe that $\overline{\psi_{\bar\lambda}}(x)=\psi_\lambda(x)$ as they both solve Hill's equation with λ and have the same Floquet exponent $\pi\nu$.

Note that $\tau_\infty(x)$ is real and non zero for real x, since $x\mapsto u_\infty(x):\mathbf{R}\to\mathbf{R}$ is twice continuously differentiable, and such that Hill's equation has two linearly independent Floquet solutions $\psi_\lambda(x)$ for all but finitely many ζ. Then Gesztesy and Weikard proved [31, Theorem 4.1] that $u_\infty(x)$ is finite gap, and hence that the discriminant equation $\Delta(\lambda)^2-4=0$ has only finitely many zeros that are simple, namely $\lambda_0^o<\dots<\lambda_{2g}^o$. Hence the Bloch spectrum is $\cup_{j=0}^{g-1}[\lambda_{2j}^0,\lambda_{2j+1}^o]\cup[\lambda_{2g}^o,\infty)$ and each point in the interior of this union of real intervals corresponds to a pair of real points on the curve $\mathcal{E}$.

□

Example 6.6. Let $u\in C^2(\mathbf{R};\mathbf{R})$ be a finite gap Hill's potential. Here the fundamental solution matrix is real for all $\lambda\in\mathbf{R}$, hence $\Delta(\lambda)\in\mathbf{R}$ for all $\lambda\in\mathbf{R}$. Hochstadt observed that the simple periodic spectrum (λ_j^o) determines the double periodic spectrum and the nontrivial roots λ_j' of $\Delta'(\lambda)=0$, so we can write

$$\frac{\Delta'(\lambda)}{\sqrt{4-\Delta(\lambda)^2}}=c_3\frac{\prod_{j=1}^{g}(\lambda-\lambda_j')}{\sqrt{\prod_{j=0}^{2g}(\lambda-\lambda_j^o)}}. \tag{6.40}$$

McKean and van Moerbecke [50] proved that the set of potentials with simple periodic spectrum $\lambda_0^0<\dots<\lambda_{2g}^o$ gives a 2^g sheeted cover of the cell $[\lambda_1^0,\lambda_2^o]\times\dots\times[\lambda_{2g-1}^o,\lambda_{2g}^o]$ in which the Dirichlet spectrum lives. We select one such potential q such that $\mu_j=\lambda_j'$. Then

$$\varphi(\lambda)=\cos^{-1}\frac{\Delta(\lambda)}{2}=-\int_0^\lambda\frac{\Delta'(\zeta)d\zeta}{\sqrt{4-\Delta(\zeta)^2}} \tag{6.41}$$

gives a conformal mapping from the upper half plane $\{\lambda : \Im\lambda > 0\}$ to a slit domain, as in Proposition 5.8.

7. Linear systems on the complex torus, and the hyperelliptic prime function

In this section, we show that there are significant examples of periodic linear systems which satisfy the hypotheses of the previous section. In order to give explicit formulas, we do not seek the greatest generality in the presentation. We start with genus one, for which our results are most complete, and then progress to hyperelliptic cases.

Definition *(Elliptic functions).* (i) For $\omega_1, \omega_2 \in \mathbf{C} \setminus \{0\}$ with $\Im(\omega_2/\omega_1) > 0$ uppose let $\Lambda = \mathbf{Z}2\omega_1 + \mathbf{Z}2w_2$ be a lattice, and let $\mathcal{T} = \mathbf{C}/\Lambda$ be the corresponding torus. A meromorphic function on $\mathbf{C}$ is elliptic (of the first kind) if it is doubly periodic with respect to Λ. A meromorphic function is elliptic of the second kind if there exist multipliers $\rho_j \in \mathbf{C}$ such that $f(z + 2\omega_j) = \rho_j f(z)$; so that f is quasi-periodic with respect to Λ. A meromorphic function is elliptic of the third kind if there exist $a_j, b_j \in \mathbf{C}$ for $j = 1, 2$ such that $f(z+2\omega_j) = e^{a_j z + b_j} f(z)$. See [43, 49].

(ii) Let $\omega \in \mathbf{C}$ have $\Im\omega > 0$; then Jacobi's elliptic theta function is

$$\vartheta_1(x \mid \omega) = i \sum_{n=-\infty}^{\infty} (-1)^n e^{(2n-1)\pi i x + (n+1/2)^2 \pi i \omega} \qquad (x \in \mathbf{C}), \tag{7.1}$$

which is elliptic of the third kind on with respect to $\mathbf{Z} + \omega\mathbf{Z}$.

Lemma 7.1. *(i) Let δ be a positive divisor on $\mathbf{C}/\Lambda$. Then there exists a uniformly periodic linear system with tau function τ, where τ is elliptic of the third kind with zero divisor δ.*

(ii) Let η be a divisor on $\mathbf{C}/\Lambda$ of degree zero. Then there exists a pair of uniformly periodic linear systems with tau functions τ_0 and τ_1 such that τ_1/τ_0 is elliptic of the second kind with η the divisor of its poles and zeros.

(iii) Let $\gamma = \sum_{j=1}^n \delta_{a_j} - \sum_{j=1}^n \delta_{b_j}$ be a divisor of degree zero, where $\sum_{j=1}^n (a_j - b_j) \in \Lambda$. Then there exists a pair of uniformly periodic linear systems with tau functions τ_1 and τ_0 such that τ_1/τ_0 is elliptic of the first kind with γ the divisor of the poles and zeros.

Proof (i) We introduce $\mathrm{q} = e^{\pi i \omega}$ and introduce Jacobi's elliptic function of the third kind by the product

$$\vartheta_3(x) = \prod_{n=1}^{\infty} (1 - \mathrm{q}^{2n}) \prod_{n=1}^{\infty} (1 + 2\mathrm{q}^{2n-1} \cos 2\pi x + \mathrm{q}^{4n-2}). \tag{7.2}$$

Observing that

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mathrm{q}^{2n-1} \begin{bmatrix} \cos 2\pi x & \sin 2\pi x \\ -\sin 2\pi x & \cos 2\pi x \end{bmatrix} \right) = 1 + 2\mathrm{q}^{2n-1} \cos 2\pi x + \mathrm{q}^{4n-2}, \tag{7.3}$$

one introduces

$$A_n = C_n = \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix}, \quad B_n = 2E_n = \begin{bmatrix} \mathrm{q}^{2n-1} & 0 \\ 0 & \mathrm{q}^{2n-1} \end{bmatrix} \tag{7.4}$$

so that $A_nE_n + E_nA_n = B_nC_n$. By forming the direct sum $\bigoplus_{n=1}^{\infty}(A_n, B_n, C_n; E_n)$ can easily construct a linear system $(-A, B, C; E)$ with $H = H_0 = \oplus_{n-1}^{\infty}\mathbf{C}^{2\times 1}$ that generates a uniform periodic semigroup and with tau function $\vartheta_3(x)/c$ for $c = \prod_{n=1}^{\infty}(1 - \mathrm{q}^{2n})$.

We then introduce $\vartheta_1(x) = -ie^{\pi ix}\mathrm{q}^{1/4}\vartheta_3(x + \omega/2 + 1/2)$. By simple manipulations of (7.22), one shows that ϑ_1 is entire and elliptic of the third kind, and from the product formula it is evident that ϑ_1 has a simple zero at $x = 0$ and no others in the fundamental cell of $\mathbf{C}/\Lambda$. In section 7 of [11] we likewise obtained a uniform periodic linear system with tau function $\vartheta_1(x)$.

Given a positive divisor $\delta = \sum_{j=1}^{n}\delta_{a_j}$, we can form block diagonal sums, and obtain a periodic linear system with tau function $\tau_0(x) = \prod_{j=1}^{n}\vartheta_1(x - a_j)$, so τ_0 is elliptic of the third kind with simple zeros at $a_1, \dots, a_j$ and the points congruent to these with respect to Λ.

(ii) Given $\eta = \sum_j \delta_{a_j} - \sum_k \delta_{b_k}$, we introduce periodic linear systems with tau functions τ_1 and τ_0 as in (i) so that

$$\frac{\tau_0(x)}{\tau_1(x)} = \prod_{j=1}^{n}\frac{\vartheta_1(x - a_j)}{\vartheta_1(x - b_j)} \tag{7.5}$$

is elliptic of the second kind with zeros $a_1, \dots, a_n$ and poles $b_1, \dots, b_n$ listed according to multiplicity modulo Λ.

(iii) In the particular case of (ii) in which $\sum_{j=1}^{n}(a_j - b_j) \in \Lambda$, then τ_0/τ_1 is elliptic of the first kind, as in Abel's theorem.

□

Proposition 7.2. *Suppose that u is elliptic of the first kind, and that Hill's equation $-\psi''(x) + u(x)\psi(x) = \lambda\psi(x)$ has a meromorphic fundamental system of solutions for some $\lambda \in \mathbf{C}$. Then Hill's equation has a nontrivial elliptic solution of the second kind, which may be expressed as a quotient of tau functions that arise from uniformly periodic linear systems and systems with finite-dimensional state space.*

Proof. The first part is conventionally attributed to Picard. Let V_λ be the vector space of meromorphic solutions of Hill's equation, and suppose that V_λ has dimension two. Observe that the monodromy operators $T_j : \psi(z) \mapsto \psi(z + 2\omega_j)$ are commuting operators such that $T_j(V_\lambda) \subseteq V_\lambda$ for $j = 1, 2$ since u is elliptic, so we can take Λ to be the group generated by T_1 and T_2. Then T_1 and T_2 have a common eigenvector, which gives an elliptic solution of the second kind. (Furthermore, if T_1 or T_2 has distinct eigenvalues as an operator on V_λ, then there exists a fundamental system of elliptic functions of the second kind; in particular, this happens for T_1 when $\Delta(\lambda)^2 - 4 \neq 0$.)

Let ψ be a solution that is elliptic of the second kind hence has the form

$$\psi(x) = e^{bx+c}\frac{\prod_{j=1}^{n}\vartheta_1(x - a_j)}{\prod_{j=1}^{n}\vartheta_1(x - b_j)}. \tag{7.6}$$

The initial exponential factor e^{bx+c} is the tau function of a linear system with state space $\mathbf{C}$, while the quotient of ϑ_1 functions can be realized from a pair of uniformly periodic linear systems by Lemma 7.1.

□

Example 7.3. *(Lamé's equation).* (i) By [49, page 132] there exists a constant e_2 such that

$$\wp(x) = e_2 + \Bigl(\frac{\vartheta_1'(0)\vartheta_3(x)}{\vartheta_1(x)\vartheta_3(0)}\Bigr)^2, \tag{7.7}$$

where Weierstrass's function $\wp$ is meromorphic and doubly periodic with respect to $\mathbf{Z} + \omega\mathbf{Z}$.

The fundamental example of a finite gap elliptic differential equation is Lamé's equation. Let $(X, Z) = (\wp(x), \wp'(x))$ and $(Y, W) = (\wp(y), \wp'(y))$ be points on the elliptic curve $\mathcal{T} = \{(X, Z) : Z^2 = 4X^3 - g_2X - g_3\} \cup \{(\infty, \infty)\}$, where $g_2^3 - 27g_3^2 \neq 0$, with Klein's invariant $J = g_2^3/(g_2^3 - 27g_3^2)$, and $\mathbf{K} = \mathbf{C}(X)[Z]$ is the elliptic function field. Then Lamé's equation is

$$-\frac{d^2}{dx^2}\psi(x) + \ell(\ell+1)\wp(x)\psi(x) = \lambda\psi(x) \tag{7.8}$$

and we write $\Psi(X) = \psi(x)$ to convert between coordinates on the curve and the torus. See [44] for a detailed discussion of various forms of the solutions.

To solve the case $\ell = 1$, we introduce

$$\psi_2(x, \alpha) = -2q^{1/4}e^{(\zeta(\alpha)-2\alpha\eta_1/\pi)x}\frac{\vartheta_1(x-\alpha)}{\vartheta_1(\alpha)\vartheta_1(x)}\prod_{n=1}^{\infty}(1-q^{2n})^3, \tag{7.9}$$

which satisfies Lamé's equation with $\lambda = -\wp(\alpha)$ and is such that $\alpha \mapsto \psi_2(x, \alpha)$ is doubly periodic and meromorphic, and $x \mapsto \psi_2(x, \alpha)$ is elliptic of the second kind; moreover $\psi_2(x, \alpha)\psi_2(-x, \alpha) = \wp(\alpha) - \wp(x)$. By Lemma 7.1, $\psi_2(x, \alpha)$ can be expressed as a quotient of tau functions from periodic linear systems.

(ii) The Lamé example is fundamental, since several elementary examples can be derived from it. In each of the following, γ and the potential u are meromorphic functions on a Riemann surface $\mathcal{E}$ and ψ satisfies the addition rule

$$\psi(x+y) = \frac{\psi'(x)\psi(y) - \psi(x)\psi'(y)}{\gamma(x) - \gamma(y)}. \tag{7.10}$$

and each entry can be obtained from periodic linear systems by taking a limit of the real or imaginary period to infinity.

$$\begin{array}{ccccc}
\mathcal{E} & u(x) & \psi(x) & \gamma(x) & \tau(x) \\
\mathbf{P}^1 & g(g+1)/x^2 & (g+1)/x & -(g+1)/x^2 & x^{g(g+1)/2} \\
\mathbf{C}/\pi\mathbf{Z} & g(g+1)\mathrm{cosec}^2 x & (g+1)\cot x & -(g+1)\mathrm{cosec}^2 x & (\sin x)^{g(g+1)/2} \\
\mathbf{C}/\pi i\mathbf{Z} & g(g+1)\mathrm{cosech}^2 x & (g+1)\coth x & -(g+1)\mathrm{cosech}^2 x & (\sinh x)^{g(g+1)/2} \\
\mathbf{C}/\Lambda & 2\wp(x \mid \Lambda) & \psi_2(x, \alpha) & -\wp(x \mid \Lambda) & \vartheta_1(x \mid \Lambda)
\end{array} \tag{7.11}$$

Lemma 7.1 extends to symmetric compact Riemann surfaces of genus $g \geq 2$ via the Schottky–Klein function ϖ, which depends upon the Schottky's model for $\mathcal{E}$. In Schottky's model, $\mathcal{E}$ arises as the quotient of $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$ under the action of a discrete subgroup of $PSL(2, \mathbf{C})$. A Riemann surface $\mathcal{E}$ is said to be symmetric if there exists an anti-conformal involution $\varphi : \mathcal{E} \to \mathcal{E}$. Whereas the defining formula for ϖ does not seem well adapted for computation, Crowdy *et al* [5,17, 18,70] have recently devised algorithms which facilitate computing ϖ in geometrical contexts which arise in applied mathematics.

Example 7.4 *(Schottky's model).* To each $T \in PSL(2, \mathbf{C})$, we associate the Möbius transformation

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \mapsto \frac{az + b}{cz + d}. \tag{7.12}$$

In particular, each $T \in PSL(2, \mathbf{R})$ gives a Möbius transformation which preserves $\mathbf{R} \cup \{\infty\}$, and is said to be Fuchsian. Let $\hat{\gamma}_0$ be the unit circle $C(0, 1)$, and let $\hat{\gamma}_j$ for $j = 1, \dots, g$ be Jordan curves inside $\hat{\gamma}_0$ which are mutually exterior to one another. Let φ be the anti conformal involution of $\mathbf{C}_\infty$ consisting of reflection in $\hat{\gamma}_0$, as in $\varphi(z) = 1/\bar{z}$ then we obtain $\hat{\gamma}_{-j}$ by reflecting $\hat{\gamma}_j$ in $\hat{\gamma}_0$ via φ so $\hat{\gamma}_{-j}$ are Jordan curves in the exterior of $\hat{\gamma}_0$; then $\hat{\gamma}_{\pm 1}, \dots, \hat{\gamma}_{\pm g}$ are mutually exterior. The fundamental region is the set $\mathcal{F}$ which is exterior to all the $2g$ Jordan curves $\hat{\gamma}_{\pm j}$ for $j = 1, \dots, g$, so $\mathcal{F}$ has $2g$ holes, and comes equipped with the anti conformal involution φ. Let $\hat{T}_j \in PSL(2, \mathbf{C})$ map γ_j to $\hat{\gamma}_{-j}$ and the exterior of $\hat{\gamma}_j$ with respect to $\mathbf{C}_\infty$ onto the interior of $\hat{\gamma}_{-j}$. Then we introduce the group Γ which is generated by $\hat{T}_j$ for $j = 1, \dots, g$. Let $D_0 = D(0, 1)$ be the interior of $\hat{\gamma}_0$ and D_j the interior of $\hat{\gamma}_j$ for $j - 0, \dots, g$; then let $D_\Gamma = D_0 \setminus \cup_{j=1}^g D_j$, so that D_Γ is a set with g holes. Then we can form the Schottky double of D_Γ by attaching another copy of D_Γ to itself along the curves γ_j for $j = 0, \dots, g$ to form a symmetric Riemann surface $\mathcal{E}$ with genus g.

Now we separate off the identity transformation, and pair up all other transformations with their inverses, so there is a partition $\Gamma = \{I\} \cup \bigcup_{k=1}^\infty \{T_k, T_k^{-1}\}$, where $T_k(z) = (a_k z + b_k)/(c_k z + d_k)$. By classical results summarized in [3], there exists a singular set E_0 such that Γ acts properly discontinuously on $\mathbf{C}_\infty \setminus E_0$, the set E_0 is perfect and nowhere dense, and E_0 has Lebesgue area measure zero. Given any relatively compact domain D' contained in $\mathbf{C}_\infty \setminus E_0$, there exist only finitely many points $-d_k/c_k \in D'$; hence there exists a nonempty domain $D'' \subset D'$ such that D'' does not contain $-d_k/c_k$ or ∞.

Definition *(Prime function).* Suppose momentarily that the following product converges for some $z, \zeta \in D''$; then the product defines the Schottky–Klein prime function of $\mathcal{E}$ by

$$\varpi(z, \zeta) = (z - \zeta) \prod_{k=1}^\infty \frac{z - T_k(\zeta)}{z - T_k(z)} \frac{\zeta - T_k(z)}{\zeta - T_k(\zeta)}. \tag{7.13}$$

Each factor in this product is a cross-ratio, hence does not change when T_k is replaced by T_k^{-1}. We write $T_{-k} = T_k^{-1}$ for $k = 1, 2, \dots$. See [7, Chapter 7] for other functional relations.

Proposition 7.5. *Suppose that Γ is a non elementary Fuchsian group such that the limit set is a proper subset of $\mathbf{R}$, and let $\zeta_0 \in \mathcal{F}$ have $\Im\zeta_0 > 0$. Then there exists a sequence of uniformly periodic linear systems Σ_k with tau functions τ_k, and a constant c_1 such that*

$$\frac{\varpi(e^{i\theta}, \zeta_0)}{\varpi(e^{i\phi}, \zeta_0)} = c_1 \prod_{k=-\infty}^\infty \frac{\tau_k(\theta)}{\tau_k(\phi)}. \tag{7.14}$$

Proof. Burnside [15] showed that for Γ such a Fuchsian group, the series $\sum_k |c_k z + d_k|^{-2}$ converges for all $\Im z > 0$; he referred to such groups as groups of the first class, a terminology which is no longer current. The set of limit points of $(T_k(\zeta_0))_{k=-\infty}^{\infty}$ is contained in the limit set of Γ, hence is a proper subset of $\mathbf{R}$. Akaza considered the property

$$\sum_{k=1}^{\infty} \frac{1}{|c_k|^2} < \infty \tag{7.15}$$

and showed that this is equivalent to the absolute and uniform convergence of various Poincare series of dimension (-2), and this property holds in the present context. Hence the series

$$\sum_k \Bigl(\frac{1}{T_k(s) - z} - \frac{1}{T_k(s) - \zeta} \Bigr) T_k'(s) \tag{7.16}$$

converges for all $s \in D''$, and $\zeta, z \in \hat{\gamma}_0$, as in Theorem A of [3]. Upon integrating (7.16), and noting Appendix A of [5] we obtain

$$\prod_{T \in \Gamma} \frac{T(\zeta_0) - z}{T(\zeta_0) - \zeta} = C \frac{\varpi(z, \zeta_0)}{\varpi(\zeta, \zeta_0)}, \tag{7.17}$$

for some constant C; we write this as a convergent product

$$C \frac{\varpi(z, \zeta_0)}{\varpi(\zeta, \zeta_0)} = \prod_{k=1}^{\infty} \frac{(a_k \zeta_0 + b_k) - (c_k \zeta_0 + d_k) z}{(a_k \zeta_0 + b_k) - (c_k \zeta_0 + d_k) \zeta}. \tag{7.18}$$

Now we introduce a periodic linear system for each k, as specified by the matrices

$$A_k = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad E_k = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{bmatrix}, \tag{7.19}$$

with coefficients to be determined. We have

$$\tau_k(\theta) = \det\bigl(I + e^{\theta A} E\bigr) = 1 + (\alpha_k \beta_k - \gamma_k \delta_k) + (\alpha_k + \delta_k) \cos\theta + (\gamma_k - \beta_k) \sin\theta, \tag{7.20}$$

so we choose complex α_k and β_k to solve the quadratic equation

$$\frac{a_k \zeta_0 + b_k}{c_k \zeta_0 + d_k} = \frac{1 - \alpha_k^2 - \beta_k^2}{c_k \zeta_0 + d_k} - \alpha_k + i\beta_k, \tag{7.21}$$

(for instance, one can choose $\beta_k = i\alpha_k$), then let

$$\delta_k = -\alpha_k - (c_k \zeta_0 + d_k), \qquad \gamma_k = \beta_k - i(c_k \zeta_0 + d_k). \tag{7.22}$$

Let $z = e^{i\theta}$ and $\zeta = e^{i\phi}$ so z and ζ lie on the circle $\hat{\gamma}_0$, which lies inside $\mathcal{F}$, and the product (7.18) converges. The identity (7.14) follows from our choice of A_k and E_k.

$\square$

Remarks 7.6 (i) Lemma 7.1 is a particular case of Proposition 7.5. When $g = 1$ and $\hat\gamma_1 = C(0, \mathrm{q})$ for some $0 < \mathrm{q} < 1$, the concentric circles $\hat\gamma_1$ and $\hat\gamma_{-1}$ bound an annulus, from which one can construct a model of the torus $\mathbf{C}/(\mathbf{Z}2\omega_1 + \mathbf{Z}2\omega_2)$ where $\omega_1 = \log 1/\mathrm{q}$ and $\omega_2 = \pi i$, by identifying points on the inner and outer circles as in [48, p 48]. Here the Schottky–Klein function for the complex torus is

$$\varpi(\zeta, z) = (\zeta - z)\prod_{n=1}^{\infty}\frac{(1 - \mathrm{q}^2(\zeta/z + z/\zeta) + \mathrm{q}^{4n})}{(1 - \mathrm{q}^{2n})^2}, \tag{7.23}$$

which strongly resembles the formula for ϑ_3 as given in (7.2), and the analogy with the product for ϑ_1 is closer still; see [48 p135], and [18]. By contrast, for $g \geq 2$, the universal cover of $\mathcal{E}$ is the hyperbolic upper half plane $\mathbf{C}_+$ rather than $\mathbf{C}$, and Γ is a non abelian group.

(ii) The function $z \mapsto \varpi(z, \zeta)$ is holomorphic and has a first order zero at $z = \zeta$, and no other zeros in $\mathcal{F}$; hence $\varpi(z, \zeta)$ can be used to build meromorphic functions on $\mathcal{E}$ with zeros and poles, subject to Abel's theorem. Baker [7] describes the functional equation of ϖ, viewing $z \mapsto \varpi(z, \zeta)$ as an automorphic function with respect to the action of Γ.

(iii) By Koebe's retrosection theorem, any compact Riemann surface can be uniformized by a Schottky group, as in the preceding discussion, where Γ is some discrete subgroup of $PSL(2, \mathbf{C})$. In Proposition 7.5, we have chosen Γ to be a Fuchsian group so as to simplify the geometry and to identify a class of Schottky groups for which the product (7.13) converges; in this case, one can choose the $\hat\gamma_j$ to be circles with centres on $\mathbf{R}$. The Riemann surface $\mathcal{E}$ is not to be confused with $\mathbf{C}_+/\Gamma$. There are alternative definitions of ϖ which avoid the infinite product (7.13); see [5] for an existence theorem for ϖ which is based on potential theory.

(iv) Fay and Mumford [23, 57] introduce ϖ via Fay's prime form, and their theory leads naturally to theta functions on the Jacobi variety of $\mathcal{E}$. Let $\mathcal{E}$ be a symmetric compact Riemann surface of genus g. Then $\mathcal{E}$ has a prime form E, and we suppose that $\varphi : \mathcal{E} \to \mathcal{E}$ satisfies

$$E(\varphi(\zeta), \varphi(z)) = \overline{E(\zeta, z)}. \tag{7.24}$$

Then the Schottky–Klein function $\varpi : \mathcal{E} \times \mathcal{E} \to \mathbf{C}$ satisfies

$$E(\zeta, z) = \frac{\varpi(\zeta, z)}{\sqrt{d\zeta dz}}, \tag{7.25}$$

and the functional relations

$$\varpi(\zeta, z) = -\varpi(z, \zeta), \quad \overline{\varpi(1/\bar\zeta, 1/\bar z)} = \frac{-\varpi(\zeta, z)}{\zeta z}. \tag{7.26}$$

Example 7.7. The following configuration was studied by Burnside [16]. Let $1 < e_1 < e_2 < \dots < e_{2g-1} < e_{2g}$, and let $\hat\gamma_{-j}$ be the circle with centre $(e_{2j-1} + e_{2j})/2$ and radius $(e_{2j} - e_{2j-1})/2$; then we let $\hat\gamma_0$ be the circle with centre 0 and radius 1, and let $\hat\gamma_j$ be the inversion of $\hat\gamma_{-j}$ in $\hat\gamma_0$ for $j = 1, \dots, g$. This gives us $2g + 1$ circles with centres on $\mathbf{R}$ such that

$\hat{\gamma}_{\pm j}$ for $j = 1, \dots, g$ are $2g$ circles with mutually disjoint interiors, and we can construct the Schottky surface $\mathcal{E}$ as above. Baker [7] shows that the functions

$$X(\zeta) = \Bigl(\frac{\varpi(\zeta, -1)\varpi(1, \zeta_0)}{\varpi(\zeta, 1)\varpi(-1, \zeta_0)}\Bigr)^2,$$

$$Y(\zeta) = \frac{\varpi(\zeta, -1)\varpi(1, \zeta_0)}{\varpi(\zeta, 1)\varpi(-1, \zeta_0)} \prod_{j=1}^{g} \frac{\varpi(\zeta, e_{2j-1})\varpi(e_{2j}, \zeta_0)}{\varpi(\zeta, e_{2j})\varpi(e_{2j-1}, \zeta_0)} \tag{7.27}$$

are meromorphic and invariant under the action of Γ, so $Y(T(\zeta)) = Y(\zeta)$ for all $T \in \Gamma$. Hence X defines a rational function of degree two on $\mathcal{E} \to \mathbf{C}_\infty$, with double pole at $\zeta = 1$ and a double zero at $\zeta = -1$, hence $\mathcal{E}$ is hyperelliptic. To identify the corresponding curve, let $\lambda_j = X(e_j)$, then $Z = X \prod_{j=1}^{g}(X - \lambda_{2j-1})$. Baker [7] shows that $\mathcal{E}$ is the closure of the complex curve

$$\Bigl\{(X, Z) \in \mathbf{C}^2 : Z^2 = X \prod_{j=1}^{2g}(X - \lambda_j)\Bigr\}. \tag{7.28}$$

Renumbering the λ_j if needs be, we can consider $(\lambda_{2j-1}, \lambda_{2j})$ as consecutive gaps. By Abel's theorem, there exists a discrete subgroup Λ of $\mathbf{C}^g$ of full rank such that the map

$$J : \delta_\zeta - \delta_{\zeta_0} \mapsto \Bigl(\int_{\zeta_0}^{\zeta} \frac{X^{j-1}dX}{Z}\Bigr)_{j=1}^{g} \in \mathbf{C}^g \qquad (\mathrm{mod} \quad \Lambda) \tag{7.29}$$

extends via $\mathbf{Z}$-linear combinations and defines a surjective group homomorphism from the divisors on $\mathcal{E}$ of degree zero onto the quotient group $\mathbf{C}^g/\Lambda$. Thus one can identify the Picard variety with $\mathbf{C}^g/\Lambda$ via the Jacobian map J, which is an isomorphism of abelian groups.

Marcenko showed that the set of finite gap potentials is norm dense in $L^2([0, \pi]; \mathbf{R})$; see[25] for more on spectral gaps for L^2 potentials.

8. Kadomtsev–Petviashvili differential equations

For a meromorphic complex function $u(x, y, s)$, the KP equation is

$$\frac{\partial}{\partial x}\Bigl(\frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x} + 4\lambda\frac{\partial u}{\partial x} + 4\alpha\frac{\partial u}{\partial s}\Bigr) + 3\beta^2\frac{\partial^2 u}{\partial y^2} = 0, \tag{8.1}$$

where the $\alpha, \beta, \lambda \in \mathbf{C}$ are parameters. In this section, we use $(2, 2)$-admissible linear systems, one can produce solutions to the KP equations via scattering functions and the Gelfand–Levitan equation. Zakharov and Shabat [72] considered the associated scattering function ϕ, which we take to be a meromorphic complex function $\phi(x, y, z, t)$ that satisfies the linear KP equations

$$\alpha\frac{\partial \phi}{\partial t} + \frac{\partial^3 \phi}{\partial x^3} + \frac{\partial^3 \phi}{\partial z^3} + \lambda\Bigl(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z}\Bigr) = 0 \tag{8.2}$$

and

$$\beta\frac{\partial \phi}{\partial y} + \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial z^2} = 0. \tag{8.3}$$

Definition *(GL equation for KP).* The appropriate version of the Gelfand–Levitan equation for the linear KP equations is

$$\phi(x,z;y,t)+K(x,z;y,t)+\int_x^\infty K(x,s;y,t)\phi(s,z;y,t)ds=0\qquad (x<z). \tag{8.4}$$

For a solution $K(x,z;y,t)$, define the potential by

$$u(x;y,t)=-2\frac{d}{dx}K(x,x;y,t). \tag{8.5}$$

Remark. Note that in comparison with [72], our potential has an extra minus sign. We regard (x,z) as the main variables, (x,x) as the diagonal and (y,t) as parameters which describe the deformation of solutions of the integral equation. We write $\frac{\partial}{\partial x}$ to indicate differentiation with respect to the first variable, $\frac{\partial}{\partial z}$ to indicate differentiation with respect to the second variable, and $\frac{d}{dx}$ to indicate differentiation along the diagonal.

We shall introduce a suitable family of admissible linear systems such that u arises as their tau function, and solve the Gelfand–Levitan equation in a similar way to section 2.

Definition. Given $(2,2)$ admissible linear systems $(-A_1,B_0,C_0)$ and $(-A_2,B_0,C_0)$ as in Theorem 2.2 with $A_1,A_2\in\mathcal{L}(H)$. We write $(-A_1,-A_2;B_0,C_0)$ for brevity, which is not to be confused with the notation (A,B,C,D) which is used as shorthand for the colligation matrix of a linear system. Let

$$C(y;t)=C_0e^{t(A_1^3+\lambda A_1)/\alpha-yA_1^2/\beta} \tag{8.6}$$

$$B(y;t)=e^{t(A_2^3+\lambda A_2)/\alpha+yA_2^2/\beta}B_0. \tag{8.7}$$

Then let

$$\phi(x,z;y;t)=C(y;t)e^{-xA_1}e^{-zA_2}B(y;t), \tag{8.8}$$

$$R_x=R_x(y,t)=\int_x^\infty e^{-A_2s}B(y;t)C(y;t)e^{-A_1s}ds, \tag{8.9}$$

and

$$K(x,z;y;t)=-C(y;t)e^{-xA_1}(I+R_x)^{-1}e^{-zA_2}B(y;t). \tag{8.10}$$

The signs before A_1^2 and A_2^2 in (8.6) and (8.7) are purposefully different. We do not assume that A_1 and A_2 commute, so that ϕ really does depend upon y in general. When $A_1=A_2$, the formula (8.9) reduces to our usual R operator in the style (1.7), Proposition 8.1 reduces to Proposition 2.4, and ψ reduces to a Hankel type kernel $\psi(x,z;y,t)=\phi(x+z;t)$, independent of y.

Zakharov and Shabat used this method for finite rank A_1 and A_2 to produce soliton solutions of KP. For solutions in the style of Proposition 4.5, see [35, Proposition 14.12]. In order to ensure that various products and brackets are well defined, we have imposed the condition $A_1,A_2\in\mathcal{L}(H)$; some of the results hold under less stringent conditions.

Proposition 8.1. *(i) Then $\phi(x,z;y;t)$ satisfies the scattering equations (8.2) and (8.3) for the KP equation.*

(ii) $K(x,z;y;t)$ satisfies the integral equation (8.4) and

$$K(x,x;y;t) = \frac{d}{dx}\log\det(I+R_x). \tag{8.11}$$

(iii) there exists x_0 such that $K(x,z;y;t)$ and the corresponding u from (8.5) satisfy

$$\frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial z^2} + \beta\frac{\partial K}{\partial y} = u(x;y;t)K(x,z;y;t) \qquad (x_0 < x < z). \tag{8.12}$$

Proof. (i) Since the operators are all bounded, the functions are differentiable and one can verify the differential equations, without assuming that A_1 and A_2 commute.

(ii) The linear system

$$\hat\Sigma = \left(\begin{bmatrix} -A_1 & 0 \\ 0 & -A_2 \end{bmatrix}, \begin{bmatrix} B_0 & 0 \\ 0 & B(y;t) \end{bmatrix}, \begin{bmatrix} 0 & -C_0 \\ C(y;t) & 0 \end{bmatrix}\right) \tag{8.13}$$

is $(2,2)$ admissible and by Theorem 2.2 the corresponding $\hat R_x$ operator

$$\hat R_x = \begin{bmatrix} 0 & -\int_x^\infty e^{-sA_1}B_0C_0e^{-sA_2}ds \\ \int_x^\infty e^{-sA_2}B(y;t)C(y;t)e^{-sA_1}ds & 0 \end{bmatrix}, \tag{8.14}$$

is trace class as in Proposition 2.4. Hence

$$\begin{aligned}
\phi(x,z;y,t) &+ K(x,z;y,t) + \int_x^\infty K(x,s;y,t)\phi(s,z;y,t)ds \\
&= C(y;t)e^{-xA_1}e^{-zA_2}B(y;t) - C(y;t)e^{-xA_1}(I+R_x)^{-1}e^{-zA_2}B(y;t) \\
&\quad - C(y;t)e^{-xA_1}(I+R_x)^{-1}\int_x^\infty e^{-sA_2}B(y;t)C(y;t)e^{-sA_1}e^{-zA_2}B(y;t)ds \\
&= C(y;t)e^{-xA_1}\Big(I-(I+R_x)^{-1}-(I+R_x)^{-1}R_x\Big)e^{-zA_2}B(y;t) \\
&= 0,
\end{aligned} \tag{8.15}$$

as in the proof of Proposition 2.4. One then verifies the determinant identity (8.11), which involves the bottom left entry of $\hat R_x$ satisfying the asymmetric Lyapunov equation

$$\frac{d}{dx}R_x = -A_2R_x - R_xA_1 = -e^{-xA_2}B(y;t)C(y;t)e^{-xA_1} \qquad (x>0). \tag{8.16}$$

(iii) The solution of the integral equation is unique for large enough x since $\|e^{-xA_1}\|\to 0$ and $\|e^{-xA_2}\|\to 0$ exponentially fast as $x\to\infty$; hence $\Psi(x;z;y;t)\to 0$ exponentially fast as $x\to\infty$. Using the scattering equation (8.2), one shows by differentiating (8.4) repeatedly that

$$\frac{\partial^2}{\partial x^2}K(x,z;y;t) - \frac{\partial^2}{\partial z^2}K(x,z;y;t) + \beta\frac{\partial}{\partial y}K(x,z;y;t)$$

and $u(x;y;t)K(x,z;y;t)$ both satisfy the equation which appears when (8.4) is multiplied by $u(x;y;t)$, and so by uniqueness are equal.

$\square$

Theorem 8.2. *The potential u associated with $\tau(x,y,t)$ from $(-A_1,-A_2;B(y;t),C(y;t))$ satisfies the KP equation (8.1).*

The proof involves a calculation which extends the results of sections 4 and 5, and we split this into two Lemmas.

Definition *(Product and bracket).* In the notation of Proposition 8.1, let $F_x=(I+R_x)^{-1}$. Let $\mathcal{B}$ be any differential ring of functions from $(0,\infty)\to\mathcal{L}(H_0)$, then let

$$\mathcal{A}=\mathrm{span}_{\mathbf{C}}\bigl\{A_1^{n_1}A_2^{m_1}A_1^{p_1},A_1^{n_1}A_2^{m_1}A_1^{p_1}F_xA_1^{n_2}\dots F_xA_1^{n_r}A_2^{m_r}A_1^{p_r}:n_j,m_j,p_j\in\mathbf{Z}_+\bigr\}\tag{8.17}$$

be the algebra generated by I,A_1,A_2 and F. On $\mathcal{A}$ we introduce the associative product $*$ by

$$P*Q=P(A_1F+FA_2-F(A_1+A_2)F)Q,\tag{8.18}$$

which is distributive over the standard addition, and the derivation $\partial:\mathcal{A}\to\mathcal{A}$ by

$$\partial P=(A_2-(A_1+A_2)F)P+\frac{dP}{dx}+P(A_1-F(A_1+A_2)),\tag{8.19}$$

Then let the bracket $\lfloor\cdot\rfloor:\mathcal{A}\to\mathcal{B}$ be the linear map

$$\lfloor Y\rfloor=Ce^{-xA_1}F_xYF_xe^{-xA_2}B\qquad(Y\in\mathcal{A}).\tag{8.20}$$

Lemma 8.3. *Then $(\mathcal{A},*,\partial)$ is a differential ring, and the bracket gives a homomorphism of differential rings $\lfloor\cdot\rfloor:(\mathcal{A},*,\partial)\to(\mathcal{B},\cdot,d/dx)$.*

Proof. The basic observation is that $dF/dx=A_1F+FA_2-F(A_1+A_2)F$, so one can check that

$$\begin{aligned}\frac{d}{dx}\bigl(A_1F+FA_2-F(A_1+A_2)F\bigr)&=\bigl(A_1-F(A_1+A_2)\bigr)(\bigl(A_1F+FA_2-F(A_1+A_2)F\bigr)\\&\quad+\bigl(A_1F+FA_2-F(A_1+A_2)F\bigr)\bigl(A_2-(A_1+A_2)F\bigr)\end{aligned}\tag{8.21}$$

so that

$$\begin{aligned}\frac{d}{dx}P\bigl(A_1F+FA_2-F(A_1+A_2)F\bigr)Q&\\=\frac{dP}{dx}\bigl(A_1F+FA_2-F(A_1+A_2)F\bigr)Q&\\+P\bigl(A_1-F(A_1+A_2)\bigr)\bigl(A_1F+FA_2-F(A_1+A_2)F\bigr)Q&\\+P\bigl(A_1F+FA_2-F(A_1+A_2)F\bigr)\bigl(A_2-(A_1+A_2)F\bigr)Q&\\+P\bigl(A_1F+FA_2-F(A_1+A_2)F\bigr)\frac{dQ}{dx},&\end{aligned}\tag{8.22}$$

so by adding the terms at either end of this expression, one shows that

$$\partial(P*Q)=(\partial P)*Q+P*(\partial Q).\tag{8.23}$$

Now we consider the bracket, and find from Lyapunov's equation that

$$\begin{aligned}\lfloor P\rfloor\lfloor Q\rfloor &= Ce^{-xA_1}FPFe^{-xA_2}BCe^{-xA_1}FQFe^{-xA_2}B\\ &= Ce^{-xA_1}FPF(A_2S+SA_1)FQFe^{-xA_2}B\\ &= Ce^{-xA_1}FP(A_1F+FA_2)-F(A_1+A_2)F)QFe^{-xA_2}B\\ &= \lfloor P*Q\rfloor; \end{aligned}\tag{8.24}$$

and

$$\begin{aligned}\frac{d}{dx}\lfloor P\rfloor &= \frac{d}{dx}Ce^{-xA_1}FPFe^{-xA_2}B\\ &= Ce^{-xA_1}F\Big(\big(A_2-(A_1+A_2)F\big)P+\frac{dP}{dx}+P\big(A_1-F(A_1+A_2)\big)\Big)Fe^{-xA_2}B\\ &= \lfloor \partial P\rfloor. \end{aligned}\tag{8.25}$$

□

With the usual operator multiplication, let $\mathcal{A}_0$ be the subalgebra of $\mathcal{A}$ that is generated by I, A_1, and A_2. Let $\mathbf{J}$ be the ideal in $\mathcal{A}$ generated by F; then the powers $\mathbf{J}^n$ give a decreasing chain of ideals such that $\cap_{n=1}^{\infty}\mathbf{J}^n=\{0\}$; any product including n factors of F belongs to $\mathbf{J}^n$. Now $\mathcal{A}/\mathbf{J}$ is isomorphic as an algebra to $\mathcal{A}_0$, and $\partial(\mathbf{J}^n)\subseteq\mathbf{J}^n$, so there are induced maps $\tilde{\partial}:(\mathbf{J}^n/\mathbf{J}^{n+1})\to(\mathbf{J}^n/\mathbf{J}^{n+1})$; in particular $\tilde{\partial}:\mathcal{A}/\mathbf{J}\to\mathcal{A}/\mathbf{J}$ may be identified with $P\mapsto A_2P+PA_1$ on $\mathcal{A}_0$. Thus we may regard $\mathcal{A}$ as a graded algebra consisting of polynomials in F with a (noncommutative) algebra $\mathcal{A}_0$ of coefficients. We regard ∂ as the sum of the derivation d/dx, the multiplications $X\mapsto A_2X+XA_1$ which typically preserve the degree, and the multiplications $X\mapsto -(A_1+A_2)FX$ and $X\mapsto -XF(A_1+A_2)$ which can raise the degree by one at most.

Lemma 8.4. *The function*

$$w=4\alpha\frac{\partial u}{\partial t}+4\lambda\frac{\partial u}{\partial x}+\frac{\partial^3u}{\partial x^3}-6u\frac{\partial u}{\partial x}\tag{8.26}$$

is the image under the bracket $\lfloor\cdot\rfloor$ of the sum of terms

$$\begin{aligned}&6A_1^4-12A_2^2+6A_2^4\\ &+12(A_1+A_2)FA_2(A_1-A_2)^2-12(A_1^2-A_2^2)A_1F(A_1+A_2)\\ &+6(A_1^2+2A_2A_1)+A_2)^2F(A_1^2-A_2^2)-6(A_1^2-A_2^2)F(A_1^2+2A_2A_1+A_2^2)\\ &+12(A_1^2-A_2^2)F(A_1+A_2)F(A_1+A_2)-12(A_1+A_2)F(A_1+A_2)F(A_1^2-A_2^2). \end{aligned}\tag{8.27}$$

Proof. We introduce the ordered products, in which powers of A_2 powers occur before powers of A_1,

$$A^{(1)}=A_1+A_2,\quad A^{(2)}=A_1^2+2A_2A_1+A_2^2,\quad A^{(3)}=A_1^3+3A_2A_1^2+3A_2^2A_1+A_2^3,\dots\tag{8.28}$$

and the coefficients are as in Pascal's triangle. It suffices to compute

$$W = 4\alpha\partial_t U + 4\lambda\partial U + \partial^3 U - 3U * \partial U - 3\partial U * U, \tag{8.29}$$

since $w = \lfloor W \rfloor$ by Lemma 8.3. The terms from (8.29) are given in the following multiplication table (8.30). Starting with $U = -2A^{(1)}$, we have derivatives

$$\begin{aligned}
\partial U &= -2A^{(2)} + 4A^{(1)}FA^{(1)}, \\
\partial^2 U &= -2A^{(3)} + 6A^{(2)}FA^{(1)} + 6A^{(1)}FA^{(2)} - 12A^{(1)}FA^{(1)}FA^{(1)}, \\
\partial^3 U &= -2A^{(4)} + 8A^{(3)}FA^{(1)} + 8A^{(1)}FA^{(3)} + 12A^{(2)}FA^{(2)} \\
&\quad - 24A^{(1)}FA^{(1)}FA^{(2)} - 24A^{(1)}FA^{(2)}FA^{(1)} - 24A^{(2)}FA^{(1)}FA^{(1)} \\
&\quad + 48A^{(1)}FA^{(1)}FA^{(1)}FA^{(1)};
\end{aligned} \tag{8.30}$$

these products exhibit a high degree of symmetry. The only proof known to the authors is applying ∂ repeatedly, then patiently multiplying out and gathering the various products. To respect the symmetry of terms, we use

$$\begin{aligned}
U * \partial U + \partial U * U &= 4A^{(1)}FA_2A^{(2)} + A^{(2)}A_1FA^{(1)} + A^{(2)}FA_2A^{(1)} + A^{(1)}A_1FA^{(2)} \\
&\quad - 8A^{(2)}FA^{(1)}FA^{(1)} - 8A^{(1)}FA^{(2)}FA^{(1)} - 8A^{(1)}FA^{(1)}FA^{(2)} \\
&\quad - 4(A_1^2 - A_2^2)FA^{(1)}FA^{(1)} + 4A^{(1)}FA^{(1)}F(A_1^2 - A_2^2) \\
&\quad + 16^{(1)}FA^{(1)}FA^{(1)}FA^{(1)}.
\end{aligned} \tag{8.31}$$

We likewise introduce

$$\begin{aligned}
\alpha\partial_t U &= 2(A_2^3 + \lambda A_2)(A_1 + A_2) + 2(A_1 + A_2)(A_1^3 + \lambda A_1) \\
&\quad - 2(A_1^3 + A_2^3 + \lambda A_1 + \lambda A_2)F(A_1 + A_2) \\
&\quad\quad - 2(A_1 + A_2)F(A_1^3 + A_2^3 + \lambda A_1 + \lambda A_2)
\end{aligned} \tag{8.32}$$

and

$$\begin{aligned}
\beta^2\partial_y^2 U &= -2(A_1^5 + A_2A_1^4 - 2A_2^2A_1^3 - 2A_2^3A_2^2 + A_2^4A_1 + A_2^5) \\
&\quad + 2(A_1 + A_2)F(A_1^4 - 2A_1^2A_2^2 + A_2^4) + 2(A_1^4 - 2A_1^2A_2^2 + A_2^4)F(A_1 + A_2) \\
&\quad + 4(A_1^2 - A_2^2)F(A_1^3 + A_2A_1^2 - A_2^2A_1 - A_2^3) \\
&\quad + 4(A_1^3 + A_2A_1^2 - A_2^2A_1 - A_2^3)F(A_1^2 - A_2^2) \\
&\quad - 4(A_1^2 - A_2^2)F(A_1 + A_2)F(A_1^2 - A_2^2) \\
&\quad - 4(A_1 + A_2)F(A_1^2 - A_2^2)F(A_1^2 - A_2^2) \\
&\quad - 4(A_1^2 - A_2^2)F(A_1^2 - A_2^2)F(A_1 + A_2).
\end{aligned} \tag{8.33}$$

Then one checks that W reduces to the combination (8.29).

□

Proof of Theorem 8.2 Let $\Theta = \{X \in \mathcal{A} : \lfloor X \rfloor = 0\}$ and observe that Θ contains the commutator subspace spanned by $Q * P - P * Q$. The final two terms in W are of degree two in F, which would give terms of degree three in ∂W, which do not appear in the formula for $\partial_y^2 U$. Hence we replace them by terms of degree one, before differentiating; or equivalently, we show that $4\beta^2 \partial_y^2 U + \partial W$ belongs to Θ. We have

$$\begin{aligned}
0 &= \lfloor A_1^2 - A_2^2 \rfloor \lfloor A_1 + A_2 \rfloor - \lfloor A_1 + A_2 \rfloor \lfloor A_1^2 - A_2^2 \rfloor \\
&= \lfloor (A_1^2 - A_2^2)(A_1 F + F A_2 - F(A_1 + A_2)F)(A_1 + A_2) \rfloor \\
&\quad - \lfloor (A_1 + A_2)(A_1 F + F A_1 - F(A_1 + A_2)F(A_1^2 - A_2^2) \rfloor \\
&= \lfloor (A_1^2 - A_2^2)(A_1 F + F A_2)(A_1 + A_2) - (A_1 + A_2)(A_1 F + F A_2)(A_1^2 - A_2^2) \rfloor \\
&\quad - \lfloor (A_1^2 - A_2^2)F(A_1 + A_2)F(A_1 + A_2) - (A_1 + A_2)F(A_1 + A_2)F(A_1^2 - A_2^2) \rfloor . \qquad (8.34)
\end{aligned}$$

So when we replace the final two terms of degree two by terms such as $(A_1^2 - A_2^2)(A_1 F + F A_2)(A_1 + A_2)$, we obtain the following collection of terms of degree one

$$\begin{aligned}
&12(A_1 + A_2)F A_2 (A_1^2 - A_2^2) - 12(A_1^2 - A_2^2)A_1 F(A_1 + A_2) \\
&+6(A_1^2 + 2A_2 A_1 + A_2^2)F(A_1^2 - A_2^2) - 6(A_1^2 - A_2^2)F(A_1^2 + 2A_2 A_1 + A_2^2) \\
&+6(A_1^2 + 2A_2 A_1) + A_2)^2 F(A_1^2 - A_2^2) - 6(A_1^2 - A_2^2)F(A_1^2 + 2A_2 A_1 + A_2^2) \\
&+12(A_1^2 - A_2^2)A_1 F(A_1 + A_2) + 12(A_1^2 - A_2^2)F A_2 (A_1 + A_2) \\
&-12(A_1 + A_2)A_1 F(A_1^2 - A_2^2) - 12(A_1 + A_2)F A_2 (A_1^2 - A_2^2) \\
&= -12(A_1^2 - A_2^2)F(A_1^2 - A_2^2). \qquad (8.35)
\end{aligned}$$

Now we compute

$$\begin{aligned}
&\partial\big(6A_1^4 - 12A_2^2 + 6A_2^4 - 12(A_1^2 - A_2^2)F(A_1^2 - A_2^2)\big) \\
&\quad = 6A_2 A_1^4 - 12A_2^3 A_1^2 + 6A_2^5 + 6A_1^5 - 12A_2^2 A_1^3 + 6A_2^4 A_1 \\
&\qquad - (A_1 + A_2)F(6A_1^4 - 12A_2^2 A_1^2 + 6A_2^2) - (6A_1^4 - 12A_2^2 A_1^2 - A_2^4)F(A_1 + A_2) \\
&\qquad - 12A_2(A_1^2 - A_2^2)F(A_1^2 - A_2^2) - 12(A_1^2 - A_2^2)F(A_1^2 - A_2^2)A_1 \\
&\qquad + 12(A_1 + A_2)F(A_1^2 - A_2^2)F(A_1^2 - A_2^2) + 12(A_1^2 - A_2^2)F(A_1^2 - A_2^2)F(A_1 + A_2) \\
&\qquad - 12(A_1^2 - A_2^2)(A_1 F + F A_2 - F(A_1 + A_2)F)(A_1^2 - A_2^2) \\
&\quad = 6A_1^5 + 6A_2 A_1^4 - 12A_2^2 A_1^3 - 12A_2^3 A_1^3 + 6A_2 A_1^4 + 6A_1^5 \\
&\qquad - 6(A_1 + A_2)F(A_1^4 - 2A_2^2 A_1^2 + A_1^4) - 6(A_1^4 - 2A_2^2 A_1^2 + A_1^4)F(A_1 + A_2) \\
&\qquad + (-12(A_2 A_1^2 + 12A_2^3 - 12A_1^3 + 12A_2^2 A_1)F(A_1^2 - A_2^2) \\
&\qquad + (A_1^2 - A_2^2)F(-12A_1^3 + 12A_2^2 A_1 - 12A_2 A_1^2 + 12A_1^3) \\
&\qquad + 12(A_1 + A_2)F(A_1^2 - A_2^2)F(A_1^2 - A_2^2) + 12(A_1^2 - A_2^2)F(A_1^2 - A_2^2)F(A_1 + A_2) \\
&\qquad + 12(A_1^2 - A_2^2)F(A_1 + A_2)F(A_1^2 - A_2^2). \qquad (8.36)
\end{aligned}$$

By comparing this with (8.27) we obtain the result.

$\square$

9. The Baker–Akhiezer function for KP

In this final section, we obtain solutions to the time dependent Schrödinger equation in the form of quotients of tau functions for a family of admissible linear systems. When u satisfies KP, one can choose w so that the operators

$$\beta\frac{\partial}{\partial y}+L=\beta\frac{\partial}{\partial y}-\frac{\partial^2}{\partial x^2}+u(x;y,t) \tag{9.1}$$

and

$$\frac{\partial}{\partial t}+M=\frac{\partial}{\partial t}+\frac{\partial^3}{\partial x^3}-\frac{3}{2}u(x;y,t)\frac{\partial}{\partial x}-\frac{3}{4}\frac{\partial u}{\partial x}-3\alpha w(x;y,t) \tag{9.2}$$

commute. In the following result, we obtain an explicit form for a common eigenfunction for both these operators, so

$$\big(\beta\frac{\partial}{\partial y}+L\big)\psi_\zeta=0=\big(\frac{\partial}{\partial t}+M\big)\psi_\zeta. \tag{9.3}$$

By analogy with (3.2), we call a particular family of solutions the Baker–Akhiezer function.

Definition *(Baker–Akhiezer function).* Consider the linear system $(-A_1,A_2;B(y,t),C(y,t))$ from Theorem 8.2, with spectral parameter ζ, and $R_x=R_x(y,t)$ as in (8.9). Then the Baker–Akhiezer function is

$$\psi_\zeta(x;y,t)=e^{\zeta x-\zeta^2 y/\beta-\zeta^3 t}\frac{\det(I+R_x(\zeta I+A_1)(\zeta I-A_2)^{-1})}{\det(I+R_x)}, \tag{9.4}$$

defined on $\zeta\in\mathbf{C}\setminus\mathrm{Spec}(A_2)$.

Proposition 9.1. *Let* $\alpha w(x;y,t)=\frac{\partial K}{2\partial y}(x,x;y,t)$.

(i) Then $\beta\frac{\partial}{\partial y}+L$ *and* $\frac{\partial}{\partial t}+M$ *commute;*

(ii) The Baker–Akhiezer function ψ_ζ *is meromorphic in* $(x,y,t)\in\mathbf{C}^3$*;*

(iii) also ψ_ζ *satisfies*

$$-\frac{\partial^2\psi_\zeta(x;y,t)}{\partial x^2}+u(x;y,t)\psi_\zeta(x;y,t)+\beta\frac{\partial\psi_\zeta(x;y,t)}{\partial y}=0, \tag{9.5}$$

(iv) and satisfies

$$\frac{\partial\psi_\zeta}{\partial t}+\frac{\partial^3\psi_\zeta}{\partial x^3}-\frac{3}{2}u(x;y,t)\frac{\partial\psi_\zeta}{\partial x}-\frac{3}{4}\frac{\partial u}{\partial x}\psi_\zeta(x;y,t)-3w(x;y,t)\psi_\zeta(x;y,t)=0. \tag{9.6}$$

Proof (i) We have

$$\alpha\frac{\partial w}{\partial x}(x;y,t)=\frac{1}{2}\frac{\partial}{\partial y}\frac{dK}{dx}(x,x;y,t)=\frac{1}{4}\frac{\partial u}{\partial y}(x;y,t), \tag{9.7}$$

which is what one needs to make the operators commute.

(ii) The function $(x, y, t) \mapsto R_x$ is entire from $\mathbf{C}^3 \to \mathcal{L}^1(H)$, hence

$$\det(I + (\zeta I + A_1)R_x(\zeta I - A_2)^{-1}) \tag{9.8}$$

is entire and ψ_ζ is a quotient of entire functions, hence is meromorphic.

(iii) By some simple manipulations of the determinants, we have

$$\begin{aligned}\psi_\zeta(x; y, t) &= e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t} \frac{\det(I + ((\zeta I - A_2)R_x + (A_2 R_x + R_x A_1)(\zeta I - A_2)^{-1})}{\det(I + R_x)} \\ &= e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t} \frac{\det(I + R_x + (\zeta I - A_2)^{-1}(A_2 R_x + R_x A_1))}{\det(I + R_x)} \\ &= e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t} \det(I + (I + R_x)^{-1}(\zeta I - A_2)^{-1}(A_2 R_x + R_x A_1));\end{aligned} \tag{9.9}$$

so by Lyapunov's equation, we have

$$\begin{aligned}\psi_\zeta(x; y, t) &= e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t} \det(I + (I + R_x)^{-1}(\zeta I - A_2)^{-1} e^{-xA_2} B(y, t)C(y, t)e^{-xA_1}) \\ &= e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t}\Big(1 + \mathrm{trace}(I + R_x)^{-1}(\zeta I - A_2)^{-1} e^{-xA_2} B(y, t)C(y, t)e^{-xA_1}\Big) \\ &= e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t}\Big(1 + C(y, t)e^{-xA_1}(I + R_x)^{-1}(\zeta I - A_2)^{-1} e^{-xA_2} B(y, t)\Big)\end{aligned} \tag{9.10}$$

since $B(y, t)C(y, t)$ has rank one; then we write this as

$$\begin{aligned}\psi_\zeta(x; y, t) &= e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t}\Big(1 - C(y, t)e^{-xA_1}(I + R_x)^{-1} \int_x^\infty e^{-zA_2} e^{\zeta(z-x)}\, dz B(y, t)\Big) \\ &= e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t}\Big(1 + \int_x^\infty K(x, z; y, t)e^{\zeta(z-x)}\, dz\Big).\end{aligned} \tag{9.11}$$

Now we calculate

$$\frac{\partial \psi_\zeta}{\partial y} = -(\zeta^2/\beta)e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t} + \int_x^\infty \frac{\partial}{\partial y} K(x, z, y, t)e^{\zeta z - \zeta^2 y/\beta - \zeta^3 t}\, dz \tag{9.12}$$

and

$$\frac{\partial \psi_\zeta}{\partial x} = \zeta e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t} - e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t} K(x, x; y, t) + \int_x^\infty \frac{\partial}{\partial x} K(x, z; y, t)e^{\zeta z - \zeta^2 y/\beta - \zeta^3 t}\, dz \tag{9.13}$$

hence

$$\begin{aligned}\frac{\partial^2 \psi_\zeta}{\partial x^2} &= \zeta^2 e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t} - \zeta e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t} K(x, x; y, t) + e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t} \frac{d}{dx} K(x, x; y, t) \\ &\quad - \frac{\partial}{\partial x} K(x, x; y, t)e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t} + \int_x^\infty \frac{\partial^2}{\partial x^2} K(x, z; y, t)e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t}\, dz.\end{aligned} \tag{9.14}$$

Integrating by parts, we obtain

$$\begin{aligned}
&\int_x^\infty \frac{\partial^2}{\partial z^2} K(x,z;y,t) e^{\zeta z - \zeta^2 y/\beta - \zeta^3 t} dz \\
&\quad = -\frac{\partial}{\partial z} K(x,x;y,t) e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t} - \zeta \int_x^\infty \frac{\partial}{\partial z} K(x,z;y,t) e^{\zeta z - \zeta^2 y/\beta - \zeta^3 t}\, dz \\
&\quad = -\frac{\partial}{\partial z} K(x,x;y,t) e^{\zeta z - \zeta^2 y/\beta - \zeta^3 t} + K(x,x;y,t) e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t} \\
&\qquad + \zeta^2 \int_x^\infty K(x,z;y,t) e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t}\, dz;
\end{aligned} \tag{9.15}$$

recalling the (8.10), and the definition of $u(x;y,t)$, we deduce the differential equation for ψ_ζ.

(iv) One starts with

$$\psi_\zeta(x,y,t) = e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t} + \int_x^\infty e^{\zeta z - \zeta^2 y/\beta - \zeta^3 t} K(x,z;y,t) dz. \tag{9.16}$$

Then by manipulating the Gelfand–Levitan equation, one deduces (9.6).

□

Let (ζ_j) be a sequence of distinct complex numbers and $(\psi_{\zeta_j})_{j=1}^\infty$ a corresponding sequence of distinct solutions of the pair of equations (9.4) and (9.5), where u is as in (8.5) and fixed. Then, taking derivatives in the x-variable, one forms the Wronskian

$$\Delta_n = \mathrm{Wr}\big(\psi_{\zeta_1}, \dots, \psi_{\zeta_n}\big) \tag{9.17}$$

and introduces a sequence of new potentials and new Baker–Akhiezer functions by

$$u_n(x,y,t) = u(x,y,t) - 2\frac{\partial^2}{\partial x^2} \log \Delta_n(x,y,t) \tag{9.18}$$

$$\Psi_n(x,y,t) = \frac{\Delta_{n+1}(x,y,t)}{\Delta_n(x,y,t)}. \tag{9.19}$$

Corollary 9.2. (Matveev) *Then Ψ_n and u_n satisfy (9.5), and the corresponding (9.6).*

Proof. Matveev [46] showed that this follows from Proposition 9.1 by direct calculation.

□

Remarks 9.3 (i) Corollary 9.2 enables us to generate a sequence of solutions (u_n) of the KP equation. If all the (ψ_{ζ_j}) belong to a differential field $(\mathcal{F}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial t})$, then (u_n) and $(\Psi_n)_{n=1}^\infty$ also belong to $\mathcal{F}$. The case in which $u = 0$ and $\psi_\zeta(x;y,t) = e^{\zeta x - \zeta^2 y/\beta - \zeta^3 t}$ gives soliton solutions to KP.

(ii) If the potential u, which appears as a coefficient of L in Proposition 9.1 does not depend upon y, then we can reduce Proposition 9.1(i) to a Lax equation $\frac{\partial L}{\partial t} = [L, M]$. The results of this section are applicable even when the determinant quotient (9.4) indeed depends upon y.

(iii) Krichever and Novikov [41, 42] consider Baker–Akhiezer functions $\psi_\zeta(x,t)$ that are meromorphic with respect to the spectral parameter ζ and produce examples based upon quasi-periodic theta functions; in particular, the function $\psi_2(x,\zeta)$ of section 7 is meromorphic; see [39, 32]. Proposition 9.1 does not assert that $\zeta \mapsto \tau_\zeta(x,t)$ is meromorphic on $\mathbf{C}$ and the case when the state space H has infinite dimension is problematic. In the rest of this section we circumvent this problem by introducing infinitely many time variables, and acting on the linear systems with a type of infinite dimensional Lie group. However, Segal and Wilson [64, Proposition 6.11] have identified τ_ζ functions that are meromorphic.

Definition (i) For fixed $A_1, A_2 \in \mathcal{L}(H)$, let $\mathbf{\Sigma}_{A_1,A_2}$ be the set of $\Sigma = (-A_1,-A_2;B,C)$ that give a $(2,2)$ admissible linear system, where $B:\mathbf{C}\to H$ and $C: H\to\mathbf{C}$ vary. Let $\mathbf{C}^\infty_\eta = \{(a_j)_{j=0}^\infty \in \mathbf{C}^\infty : \limsup_{j\to\infty}|a_j|^{1/j}\leq \eta\}$ be the space of coefficients of complex power series with radius of convergence greater than or equal to $1/\eta$, which may be identified with the algebra of holomorphic functions $D(0,\eta)\to\mathbf{C}$. Let

$$V(t) = \exp\Bigl(\sum_{j=1}^\infty t_j A_2^j\Bigr), \qquad W(t) = \exp\Bigl(-\sum_{j=1}^\infty t_j(-A_1)^j\Bigr) \qquad (t=(t_j)\in \mathbf{C}_0^\infty). \tag{9.20}$$

and extend to $t\in\mathbf{C}^\infty_\eta$ with $\eta>0$ when the series converge absolutely. There is an action ρ of $\mathbf{C}_0^\infty$ on $\mathbf{\Sigma}_{A_1,A_2}$ which is given by

$$\rho(t): \bigl(-A_1,-A_2,B,C\bigr)\mapsto\bigl(-A_1,-A_2,V(t)B,CW(t)\bigr) \quad (t\in\mathbf{C}_0^\infty). \tag{9.21}$$

(ii) *(Tau function).* The tau function of the right-hand side of (9.21) is defined to be

$$\tau(x;t) = \det\Bigl(I+V(t)R_xW(t)\Bigr) \qquad (t=(t_j)_{j=1}^\infty) \tag{9.22}$$

where $R_x=\int_x^\infty e^{-vA_2}BCe^{-vA_1}\,dv$.

(iii) *(Spectral shift).* We introduce $[s]=(s^j/j)_{j=1}^\infty$ so that for sufficiently small $|s|$, ρ extends to

$$\rho([s]): \bigl(-A_1,-A_2,B,C\bigr)\mapsto\bigl(-A_1,-A_2,(I-sA_2)^{-1}B, C(I+sA_1)\bigr), \tag{9.23}$$

then choose $\zeta=1/s$ with $\zeta\in\mathbf{C}\setminus\operatorname{Spec}(A_2)$ so that the spectral shift is

$$\bigl(-A_1,-A_2,B,C\bigr)\mapsto\bigl(-A_1,-A_2,(I-A_2/\zeta)^{-1}B, C(I+A_1/\zeta)\bigr). \tag{9.24}$$

Then we define the Baker–Akhiezer function by

$$\psi_\zeta(x;t)=\exp\Bigl(x\zeta+\sum_{j=1}^\infty \zeta^j t_j\Bigr)\frac{\tau(x;t+[1/\zeta])}{\tau(x;t)} \qquad (t=(t_j)_{j=1}^\infty). \tag{9.25}$$

This matches with the definition used in Proposition 9.1 when we choose

$$(t_1,t_2,t_3,t_4,\dots) = (t\lambda/\alpha, y/\beta, t/\alpha, 0,\dots). \tag{9.26}$$

(iv) *(Sato's integral)*. The Sato integral is

$$\int_{|\zeta|=r} \tau\big(x; t+[1/\zeta]\big)\tau\big(x; t'-[1/\zeta]\big)\exp\Big(\sum_{j=1}^{\infty}(t_j-t'_j)\zeta^j\Big)\, d\zeta \qquad (t=(t_j)_{j=1}^{\infty}, t'=(t'_j)_{j=1}^{\infty}\in \mathbf{C}_0^{\infty}). \tag{9.27}$$

Theorem 9.5. *Suppose that $A_1, A_2 \in \mathcal{L}(H)$ be as in Theorem 2.2. Then Sato's integral vanishes identically for all $r > \max\{\|A_1\|, \|A_2\|\}$.*

Proof We consider the integral

$$S(t,y) = \int_{|\zeta|=r} \exp\Big(-2\sum_{j=1}^{\infty}\zeta^j y_j\Big)\frac{\tau(x,t+y+[1/\zeta])\tau(x,t-y-[1/\zeta])}{\tau(x,t+y)\tau(x,t-y)}d\zeta \tag{9.28}$$

which as in (9.8) we can write as

$$\begin{aligned}\int_{|\zeta|=r}&\Big(1+Ce^{-xA_2}W(t+y)(I+V(t+y)R_xW(t+y))^{-1}V(t+y)e^{-xA_2}(\zeta I-A_2)^{-1}B\Big)\\ &\times\Big(1-C(\zeta I+A_1)^{-1}e^{-xA_1}W(t-y)(I+V(t-y)R_xW(t-y))^{-1}V(t-y)e^{-xA_2}B\Big)\\ &\times\exp\Big(-2\sum_{j=1}^{\infty}\zeta^j y_j\Big)\, d\zeta \end{aligned}\tag{9.29}$$

which we split as a sum of four terms: first we have

$$\int_{|\zeta|=r}\exp\Big(-2\sum_{j=1}^{\infty}\zeta^j y_j\Big)d\zeta = 0, \tag{9.30}$$

by Cauchy's theorem; the second is

$$\begin{aligned}\int_{|\zeta|=r}&\exp\Big(-2\sum_{j=1}^{\infty}\zeta^j y_j\Big)Ce^{-xA_2}W(t+y)(I+V(t+y)R_xW(t+y))^{-1}V(t+y)e^{-xA_2}(\zeta I-A_2)^{-1}Bd\zeta\\ &=2\pi i Ce^{-xA_1}W(t+y))(I+V(t+y)R_xW(t+y))^{-1}V(t+y)e^{-xA_2}\exp\Big(-2\sum_{j=1}^{\infty}y_jA_2^j\Big)B\\ &=2\pi i Ce^{-xA_1}W(t+y))(I+V(t+y)R_xW(t+y))^{-1}V(t-y)e^{-xA_2}B; \end{aligned}\tag{9.31}$$

by the residue theorem; the third is

$$\begin{aligned}-\int_{|\zeta|=r}&\exp\Big(-2\sum_{j=1}^{\infty}\zeta^j y_j\Big)C(\zeta I+A_1)^{-1}e^{-xA_1}W(t-y)\\ &\times(I+V(t-y)R_xW(t-y))^{-1}V(t-y)e^{-xA_2}B\, d\zeta\\ &=-Ce^{-xA_1}\exp\Big(-2\sum_{j=1}^{\infty}(-A_1)^jy_j\Big)W(t-y)(I+V(t-y)R_xW(t-y))^{-1}V(t-y)e^{-xA_2}B\\ &=-Ce^{-xA_1}W(t+y)(I+V(t-y)R_xW(t-y))^{-1}V(t-y)e^{-xA_2}B \end{aligned}\tag{9.32}$$

likewise; and finally

$$-\int_{|\zeta|=r}\exp\Big(-2\sum_{j=1}^{\infty}\zeta^j y_j\Big)Ce^{-xA_2}W(t+y)(I+V(t+y)R_xU(t+y))^{-1}V(t+y)e^{-xA_2}(\zeta I-A_2)^{-1}$$
$$\times BC(\zeta I+A_1)^{-1}e^{-xA_1}W(t-y)(I+V(t-y)R_xW(t-y))^{-1}V(t-y)e^{-xA_2}B\,d\zeta \tag{9.33}$$

which involves

$$J(y)=\int_{|\zeta|=r}\exp\Big(-2\sum_{j=1}^{\infty}\zeta^j y_j\Big)(\zeta I-A_2)^{-1}BC(\zeta I+A_1)^{-1}d\zeta. \tag{9.34}$$

This integral resembles (4.40), and likewise gives a solution to a type of Lyapunov equation. Now

$$\begin{aligned}-\frac{1}{2}\frac{\partial J}{\partial y_1}+JA_1&=\int_{|\zeta|=r}\exp\Big(-2\sum_{j=1}^{\infty}\zeta^j y_j\Big)(\zeta I-A_2)^{-1}BCd\zeta\\&=2\pi iV(-2y)BC\end{aligned} \tag{9.35}$$

and

$$\begin{aligned}-\frac{1}{2}\frac{\partial J}{\partial y_1}-A_2J&=\int_{|\zeta|=r}\exp\Big(-2\sum_{j=1}^{\infty}\zeta^j y_j\Big)(\zeta I-A_2)^{-1}BCd\zeta\\&=2\pi iBCW(2y)\end{aligned} \tag{9.36}$$

so by subtracting and applying the residue theorem, we have

$$A_2J+JA_1=2\pi i(V(-2y)BC-BCW(2y)). \tag{9.37}$$

Then we introduce $J_0=2\pi i(V(-2y)R_0-R_0W(2y))$, which satisfies

$$\begin{aligned}A_2J_0+J_0A_1&=2\pi iV(-2y)(A_2R_0+R_0A_1)-2\pi i(A_2R_0+R_0A_1)W(2y)\\&=2\pi i(V(-2y)BC-BCW(2y))=A_2J+JA_1,\end{aligned} \tag{9.38}$$

by Lyapunov's equation. By the uniqueness of solution of this equation, we deduce that

$$J(y)=J_0=2\pi i\big(V(-2y)R_0-R_0W(2y)\big). \tag{9.39}$$

Then, combining the terms (9.30), (9.31), (9.32) and (9.33) via (9.39), we have

$$\begin{aligned}S(y,t)&=2\pi iCW(t+y)e^{-xA_1}\Big((I+V(t+y)R_xW(t+y))^{-1}-(I+V(t-y)R_xW(t-y))^{-1}\\&\quad-(I+V(t+y)R_xW(t+y))^{-1}V(t+y)e^{-xA_1}Je^{-xA_2}W(t-y)\\&\quad\times(I+V(t-y)R_xW(t-y))^{-1}\Big)V(t-y)e^{-xA_2}B\\&=2\pi iCW(t+y)e^{-xA_1}(I+V(t+y)R_xW(t+y))^{-1}\Big((I+V(t-y)R_xW(t-y))\\&\quad-(I+V(t-y)R_xW(t-y))+(V(t+y)R_xW(t+y)-V(t-y)R_xW(t-y))\Big)\\&\quad\times(I+V(t-y)R_xW(t-y))^{-1}V(t-y)e^{-xA_2}B\\&=0,\end{aligned} \tag{9.40}$$

as required.

$\square$

Corollary 9.6. *(i) For $s_0, s_1, s_2, s_3 \in \mathbf{C}$, let $\sigma_{jk} = (s_j - s_k)\tau(x; t + [s_j] + [s_k])$. Then Fay's identity holds*

$$\sigma_{0,1}\sigma_{2,3} - \sigma_{0,2}\sigma_{1,3} + \sigma_{0,3}\sigma_{1,2} = 0; \tag{9.41}$$

(ii) for the Wronskian with derivatives in the x-variable, the differential form of Fay's identity holds

$$\mathrm{Wr}\big(\tau\big(x; t + [s_1]\big), \tau\big(x; t + [s_2]\big)\big) = \frac{s_1 - s_2}{s_1 s_2} \det \begin{bmatrix} \tau(x; t) & \tau(x; t + [s_2]) \\ \tau(x; t + [s_1]) & \tau(x; t + [s_1] + [s_2]) \end{bmatrix}; \tag{9.42}$$

(iii) the second-order differential identity holds

$$\frac{\partial^2}{\partial\zeta\partial x} \log \tau\big(x; t + [1/\zeta]\big) = 1 - \frac{\tau(x; t)\tau(x; t + 2[1/\zeta])}{\tau(x; t + [1/\zeta])^2}. \tag{9.43}$$

Proof. (i) See [66] and [2]. We have written the result in the style of a Plücker relation.

(ii) See [66] and [2]. This spectral addition rule has a similar style to the Toda equation (3.18).

(iii) We divide (ii) by $\tau(x; t+[s_1])\tau(x; t+[s_2])$ so as to obtain $\frac{\partial}{\partial x} \log \tau(x; t+[s_2])/\tau(x; t+[s_1])$ on the left-hand side; then we differentiate with respect to s_2, thus obtaining

$$\Big(\frac{\partial^2}{\partial s_2 \partial x}\Big)_{s_2 = s_1} \log \tau(x; t + [s_2]) = -\frac{1}{s_1^2}\Big(1 - \frac{\tau(x; t)\tau(x; t + 2[s_1])}{\tau(x; t + [s_1])^2}\Big); \tag{9.44}$$

then we change variables to $s_1 = 1/\zeta$. This resembles the proof in [57, II 3.124].

$\square$

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