How should firms selectively hedge? Resolving the selective hedging puzzle.

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Abstract

We provide a model of intertemporal hedging consistent with *selective hedging*, a widespread practice corroborated by recent empirical studies. We argue that the optimal hedge is a *value hedge* involving total current value of future earnings. More importantly, the hedging decision is independent of risk preferences of the firm or agent. Our closed-form solutions imply several implications for the risk management policy in a firm. In order to lock in profits a hedge increase is recommended in favorable states of nature, while in bad states the firm should decrease the hedge and wait. Our main new empirical implication is that selective hedging should be more prevalent in industries where managers are exposed to convex cash flow structures and are more likely to "value hedge" their exposures.

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1. Introduction

In this paper we provide a model of intertemporal hedging consistent with *selective hedging*, a widespread practice corroborated by recent empirical studies. Our findings indicate that the optimal hedge is a *value hedge*. A number of existing empirical findings can be explained through the idea of value hedge. In addition, we advance a number of new untested empirical predictions regarding hedging policies (see Section 6).

Contrary to textbook recommendations our results suggest that the firm should hedge selectively, decreasing the hedge when times are bad. We also indicate how to adapt our methodology to general non-linear income streams.

Our paper contributes to the existing literature on hedging by considering an intertemporal setup. Intertemporal links generate implications which are absent from prevalent models, focussing on hedging a single position with some fixed maturity T instead.[1] These simplifying assumptions (single cash flow, single maturity) are typically the cost to pay in order for the hedging problem to be mappable onto the standard portfolio choice framework of Merton (1971). However, while the Merton (1971) framework provides a good starting point, optimal hedging problems differ from investment problems. Essentially, hedging with forwards does not involve investing any funds but taking a position in a zero current value, zero sum game.

Other papers have also looked at intertemporal setups, where there is a series of forthcoming cash flows to be hedged. In a discrete-time framework, Neuberger (1999) examines hedging long-term commodity supply commitment with multiple short-term futures con-

[1] In Stulz (1984) and Ho (1984) a single period forward hedge is considered. Adler and Detemple (1988b) demonstrate the equivalence between the approach of Ho (1984) and that of Stulz (1984). Svensson and Werner (1993) assume risk aversion and provide an explanation of corporate decisions in an international context. Adler and Detemple (1988a) provide properties of optimal hedging decisions in the expected utility-maximizing sense. Duffie and Jackson (1990) and Duffie and Richardson (1991) provide single-period futures hedging solutions for several special cases including mean-variance end exponential criteria. Lioui and Poncet (1996) and (2001b) examine the effect of non-negativity constraint on wealth as well as impact of stochastic interest rates; Lioui and Poncet (2003) study general equilibrium pricing of nonredundant forward contracts.

tracts. Duffie and Stanton (1992) price continuously resettled contingent claims, which bear some similarities with continuous-time instantaneous forward contracts used here, as the current market value of such claims is always zero.

Smith and Stulz (1985) classify rationales for hedging into two categories: costs and risk aversion. If there are costs such as taxes, liquidity costs or bankruptcy costs, it is possible to assume that managers are risk-neutral. Stabilization of cash flows via hedging reduces expected costs and thus motivates hedging. Mathematically, costs "concavify" the objective function. In that case individual preferences are *not necessary* to obtain hedging behavior.

Nevertheless, Smith and Stulz (1985) devote entire section IV in their paper to hedging motivated by risk aversion of managers. Risk averse managers supply "specialized resources" and must be rewarded for bearing nondiversifiable risk. Consequently, compensation contracts must be designed, Smith and Stulz (1985) argue, so that the value of the firm increases when expected utility increases. Mathematically, manager maximizes an objective function (expected utility) which is already concave. In that case individual preferences result directly in hedging behavior.

In this paper we subscribe to hedging rationale dictated by risk aversion of managers. It is possible to add costs to our objective function (via the budget constraint) so that the firm's hedging policy is also motivated by reduction of these costs, including the special case where the manager is risk-neutral. This, however, should not change our selective hedging result because our optimal hedging policy is independent of the utility function of the shareholder. For illustration purposes, our general solution is specialized to the case of CRRA utility, non-linear income stream and mean-reverting risk modelled by an Ornstein-Uhlenbeck process.

The optimal hedging policy we obtain is consistent with empirical evidence of managerial practice known as *selective hedging*. Adam and Fernando (2006) document considerable evidence of selective hedging in gold mining industry. Similarly, Brown, Crabb and Haushalter

(2006) confirm that in gold mining industry managers decrease hedge positions when prices move against the firm. Both these papers also acknowledge that in gold mining industry timing commodity markets by hedging selectively generates economic gains which are relatively small. Meredith (2006) provides evidence that firms selectively hedge oil prices, which challenges traditional theories of hedging. Faulkender (2005) indicates that firms speculate and try to time the market rather than hedge when selecting the interest rate exposure of their new debt issuances. Fabling and Grimes (2008) measure currency hedging among New Zealand exporters. They find strong evidence of selective hedging, particularly for Australian Dollar and, to lower degree, for US Dollar exposures. Fauver and Naranjo (2010) find that poorer corporate governance and overall firm monitoring are associated with greater selective use of derivatives by managers.

Some empirical results document similarities between selective hedging and speculation. Speculation, however, is not "evil." Speculators incorporate new information into prices. They can do this by taking a risk they do not have but also by keeping a risk or keeping part of a risk they do have. Selective hedging is taking a view and shifting risk partially, keeping part of it.

Stulz (1996) (in addition to summarizing motives for hedging based on cost reduction) defends the practice of taking views by corporate managers. In particular, he argues that selective hedging will not violate the efficient market hypothesis. More importantly, he links position-taking to comparative advantage managers have in accessing information that is not publicly available. In a specific product market managers will have comparative advantage in predicting price levels of related inputs. In line with Stulz (1996) the simple static model of Shi (2011) confirms that by taking advantage of collected information a CEO does not need to fully hedge. By contrast, our approach uses a dynamic framework and does not require assuming any information asymmetry. We use an intertemporal continuous time setup, which is sufficiently rich to capture enough many future states of the economy. Our value-based, selective hedging couldn't be captured within a static, 2-period model. In such a model the terminal state of the economy containing the outcome to be hedged is typically converse to the current situation of the firm.

Finally, Stulz (1996) relates selective hedging to designing appropriate incentive compensation structure for managers. This is a similar problem to inducing a risk-averse manager to implement appropriate hedging policy, as already pointed out in Smith and Stulz (1985). Empirically, it has been acknowledged e.g. in Tufano (1996) that where managers have significant fraction of their own wealth tied up in the company, the larger the percentage fraction of exposure is hedged.

Interpreted from a different angle our model also fits into the ongoing discussion of the real estate crisis (see Section 7). Risks associated with price declines in the real estate sector can, to a large extent, be hedged away via trading in derivatives written on a house price index of the location. Such contracts already exist. Chicago Mercantile Exchange, for example, offers real estate options and futures.

The paper is organized as follows. In the next Section we introduce instantaneous forward contracts. In Section 3 we tackle the question of controlling long term risk with short-term instruments. General solution for optimal hedging policy is obtained and interpreted in Section 4. Section 5 provides an illustrative example of value hedge, specializing the model to CRRA utility, Ornstein-Uhlenbeck process and quadratic endowment. In Section 6 we provide comparative statics and discuss some new empirical implications. Section 7 concludes.

2. Intertemporal risk and instantaneous forwards

An observable risk x is faced by the firm. There are many examples of such risks. For example, x could measure exchange rate risk or weather risk associated with random changes of the temperature. In the case x is the temperature, it is easily observable and measurable in degrees. In this case possible values of x could also be negative, unlike rates or stock prices. Furthermore, likewise exchange rates, temperatures typically oscillate around some average value $\bar{x}$ (see Section 5). In general, however, the drift and diffusion coefficients of x are random.

Assumption 1. *Observable risk x follows an Itô process*

$$dx_t = \mu(t, Y_t)\ dt + \sigma(t, Y_t)\ dz_t \tag{1}$$

with initial value $x_0 \in \Re$, where $\mu(t, Y_t)$ and $\sigma(t, Y_t)$ (in the sequel referred to as μ_t and σ_t) are measurable and adapted functions of time t and state variables Y_t and $\{z_t\}$ is a standard Brownian motion under the original probability measure P.

We also assume that efficient market where the firm operates as price taker is populated by hedgers, investors, arbitrageurs and speculators. In particular, speculators incorporate new information into prices while arbitrageurs make pricing fair. More importantly, we assume that *instantaneous forward contracts* on x are traded in this economy. Although x is not traded, risks associated with x can be transferred using these forwards. As the firm is being offered them, these contracts complete the market. We borrow the idea of very short-term forward contracts from Breeden (1984). The structure and purpose of our framework is, however, diametrical to Breeden's intertemporal CAPM (ICAPM).[2] We use a slenderized setup to focus on manager's decisions in a firm subject to exogenous uncertainty. Instantaneous contracts allow studying hedging long term commitments with instruments of "very short" maturity. In discrete time setting a similar framework has been used by Neuberger (1999).

With a non-tradeable good the usual cost of carry arbitrage is not possible. Consequently, the usual no-arbitrage relationship between the current level of x_t and its $(t + \Delta t)$-maturity forward price $K_t^{t+\Delta t}$ will not hold, i.e. $K_t^{t+\Delta t} \neq x_t e^{r\Delta t}$, where $r > 0$ is the riskless interest rate. This is because once the arbitrage-free forward price $K_t^{t+\Delta t}$ has been agreed, the

[2] For extensions of ICAPM see e.g. Lioui and Poncet (2001a) who, unlike Breeden, use long-term forwards.

seller of the forward contract must borrow the amount $K_t^{t+\Delta t} e^{-r\Delta t}$ and immediately buy the underlying at the prevailing "spot price" x_t, so as to be prepared for delivery to occur at time $t+\Delta t$. This, however, is not possible because x_t is not traded.[3,4] Therefore, we base our model on the following assumption:

Assumption 2. *In the long run there is no risk premium associated with trading forward contracts on x.*

Our assumption[5] dictates that the equilibrium forward price $K_t^{t+\Delta t}$ is given by the expected future spot price, conditional on the information available at time t

$$K_t^{t+\Delta t} = \mathbb{E}\left[x_{t+\Delta t} \middle| \mathcal{F}_t \right]$$

Consequently, for x following an Itô process as in (1) we have

$$K_t^{t+\Delta t} = x_t + \int_t^{t+\Delta t} \mu_s \, ds \tag{2}$$

The firm chooses to pay a dividend flow c_t per unit of time and it's wealth at time t will be denoted W_t. The net cash flow, e, is subject to variations of x i.e. $e_t = e(x_t)$, where $e(\cdot)$ is some known function. The firm decides to take a position in f_t short term forwards written on x to hedge that risk. Therefore, provided that Δt is small, the budget constraint can be informally written as

$$W_{t+\Delta t} \approx W_t + (W_t r + e_t - c_t)\,\Delta t + f_t V_{t+\Delta t} \tag{3}$$

where $V_{t+\Delta t} = x_{t+\Delta t} - K_t^{t+\Delta t}$ is the payoff of a long forward or, equivalently, the value of the forward at $t+\Delta t$. To interpret (3) suppose momentarily that e_t is increasing in x_t. If

[3] For simplicity we implicitly assumed here that x has been directly expressed in some monetary unit. If x were directly tradable, then there would exist a mapping $x \mapsto f(x)$ such that $f(x)$ would give the current price of such contract. The corresponding forward price would be $K_t^{t+\Delta t} = f(x_t)\, e^{r\Delta t}$.

[4] See Geltner and Fisher (2007) who deal with analogous pricing issue in the context of forwards written on non-tradable real estate index, where arbitrage cannot actually be executed.

[5] Compare to *case 5* in Duffie and Jackson (1990).

$f_t < 0$, as we would expect, short position was taken i.e. the hedge will (partially) offset the loss on e_t due to unexpected fall in x_t. Using (2), the $(t + \Delta t)$-value of the forward written at time t can be expressed as

$$V_{t+\Delta t} = \int_t^{t+\Delta t} \sigma_s \, dz_s$$

As $V_t = 0$, we have $\int_t^{t+\Delta t} dV_s = V_{t+\Delta t}$ and thus, by letting $\Delta t \downarrow 0$, we obtain

$$dV_t = \sigma_t \, dz_t$$

This stochastic differential equation originates from the equation (1) satisfied by x. There is no drift as, by Assumption 2, there is no risk premium associated with instantaneous forward contracts. This means that there is no increasing or decreasing trend. Consequently, there is no incentive to buy and hold forward contracts. Assumption 2 can be easily relaxed, for example by using

$$dV_t = \eta_t \, dt + \sigma_t \, dz_t$$

where η_t is the drift.

3. Optimal intertemporal hedging

Intertemporal hedging of risk x can now be formulated as an infinite horizon stochastic optimal control problem over time horizon $[0, +\infty)$

$$\sup_{\{c,f\}} \mathbb{E} \left[\int_0^{\infty} e^{-\rho s} \, u(c_s) ds \right] \tag{4}$$

$$\begin{aligned} \text{s.t.} \quad dW_t &= (rW_t + e(x_t) - c_t)\, dt + f_t dV_t \\ dx_t &= \mu_t dt + \sigma_t dz_t \\ dV_t &= \sigma_t dz_t \end{aligned} \tag{5}$$

Risk averse agent maximizes the expected utility of the future dividend flow, discounted at the impatience rate $\rho > 0$. The instantaneous utility function $u(c) : [0, \infty) \to (-\infty, \infty)$ is twice continuously differentiable, increasing, strictly concave in c and admits the limiting values $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$. The agent acts in the interest of shareholders and we abstract from agency problems.

The wealth W_t can be interpreted as current level of available liquidities and acts as a state variable. The state of the system is progressively revealed through time $t \in [0, +\infty)$ with arrival of relevant information. At $t = 0$ the initial wealth is W_0 while the current risk level is x_0. Our goal is to compute the optimal dividend and hedging policy $\{c_t^*, f_t^*\}$. In other words, we seek how the "local" hedge, f_t^*, implemented with instantaneous forward contracts, can be used to hedge a long-term risk induced by x.

It is well known that the value function

$$J = J(t, x_t, W_t) = \max_{\{c_t,\ f_t\}} \mathbb{E}\left[\int_t^\infty e^{-\rho s}\ u(c_s) ds \middle| \mathcal{F}_t \right]$$

has the form $J(t, x, W) = e^{-\rho t} I(x, W)$ where $I = I(x_t, W_t)$ is the solution to the stationary Hamilton-Jacobi-Bellman equation

$$-\rho I + \max_{\{c,\ f\}} \mathcal{H}(c, f) = 0 \tag{6}$$

where $\mathcal{H}$ is the Hamiltonian to be maximized

$$\mathcal{H}(c, f) = u(c) + \mu I_x + (rW + e - c) I_W + \frac{\sigma^2}{2}\left(I_{xx} + 2f I_{xW} + f^2 I_{WW}\right)$$

and subscripts denote partial differentiation. First order conditions, $\mathcal{H}'_c = 0$ and $\mathcal{H}'_f = 0$, provided that $I_{WW} < 0$, give the optimal policy

$$c^* = \mathcal{I}(I_W) \qquad f^* = -\frac{I_{xW}}{I_{WW}} \tag{7}$$

where $\mathcal{I}(\cdot)$ shall denote the continuous, strictly decreasing inverse of marginal utility $u'(\cdot)$.[6,7]

[6] Note that $\mathcal{I}(\cdot)$ maps $(0, \infty)$ to itself and satisfies $\mathcal{I}(0+) = \infty$ and $\mathcal{I}(\infty) = 0$.

[7] If there is a speculative drift η present in the payoff of a long forward, the optimal hedge policy should adjust for its presence by adding $-\eta I_w \left(\sigma^2 I_{ww}\right)^{-1}$ to f^*.

Second order conditions require $\mathcal{H}''_{cc}\big|_{c=c^*} = u''(c^*) < 0$ and

$$\mathcal{H}''_{cc}\mathcal{H}''_{ff} - \left(\mathcal{H}''_{fc}\right)^2\Big|_{c=c^*} = \sigma^2 u''(c^*) I_{WW} > 0$$

which is guaranteed by $I_{WW} < 0$. The Hamilton-Jacobi-Bellman equation (6) becomes

$$-\rho I + \mathcal{U}\left(I_W\right) + \mu I_x + \left(rW + e\right) I_W + \frac{\sigma^2}{2}\left(I_{xx} - \frac{I_{xW}^2}{I_{WW}}\right) = 0 \tag{8}$$

where

$$\mathcal{U}\left(I_W\right) = u\left(\mathcal{I}\left(I_W\right)\right) - \mathcal{I}\left(I_W\right) I_W$$

Such nonlinear second order, second degree partial differential equation is in general difficult to solve analytically. The first nonlinear term, $\mathcal{U}\left(I_W\right)$, reflects the concavity of the utility function. The second nonlinear term, I_{xW}^2/I_{WW}, impacts the intertemporal behaviour and thus the way the risk is hedged.

4. Solution

It turns out that an exact transformation method will melt (8) into a second order *linear* partial differential equation of parabolic type. The latter has an analytic solution. Moreover it appears to be amenable to interpretations in terms of money flows, which is not the case of (8). The transformation interchanges the roles of independent and dependent variables. It belongs to the family of *hodograph transformations* which are well known e.g. in fluid dynamics (see Zwillinger (1989)).

First observe that the knowledge of I_W function is sufficient to establish the optimal hedging f^* by further differentiation. Let $y = I_W(x, W)$. A formal inversion of I_W gives $W = I_W^{-1}(y, x)$. Therefore, steps involve finding I_W^{-1} first, then inverting it in order to compute the optimal hedging f^*.[8] For notational simplicity let Ψ denote I_W^{-1} function i.e.

[8] See also Chow (1993) for a general method for optimal control without solving the Bellman equation. Our method is similar in the sense that solving for I_W — and not for the value function I itself — is necessary to establish the optimal policy.

$\forall x, y\ \Psi(y,x) = I_W^{-1}(y,x)$. The transformation condition is therefore

$$W = \Psi\left(I_W(x,W), x\right) \tag{9}$$

Provided that $\Psi_y \neq 0$,[9] the hodograph transformation is

$$\begin{gathered}
I_{WW} = \frac{1}{\Psi_y} \qquad I_{WWW} = -\frac{\Psi_{yy}}{\Psi_y^3} \qquad I_{xW} = -\frac{\Psi_x}{\Psi_y} \\
I_{xxW} = \frac{2\Psi_x\,\Psi_y\,\Psi_{yx} - \Psi_y^2\,\Psi_{xx} - \Psi_x^2\Psi_{yy}}{\Psi_y^3} \qquad I_{xWW} = \frac{\Psi_x\,\Psi_{yy} - \Psi_y\,\Psi_{yx}}{\Psi_y^3}
\end{gathered} \tag{10}$$

The Hamilton-Jacobi-Bellman equation (8) transforms to

$$(\rho - r)\, y\Psi_y + \mu\Psi_x + \frac{\sigma^2}{2}\Psi_{xx} = r\Psi + e\,(x) - \mathcal{I}\,(y) \tag{11}$$

In general $\rho \neq r$ and a term proportional to Ψ_y appears in (11). So far we have been silent about which boundary conditions to impose. The natural choice for our problem is to specify what happens for $t = 0$ and $t = \infty$ rather than impose stationary restrictions on marginal utility of wealth, $y = I_W$, in equation (11). A careful inspection of (11) reveals that, given dynamics followed by x, the corresponding dynamics for y is $dy = (\rho - r)\, ydt$, which has for solution

$$y = y_0 e^{(\rho - r)t} \tag{12}$$

where y_0 is a constant, equal to the marginal utility of initial wealth W_0, to be determined from boundary condition at $t = 0$. Formally, (12) acts as a change of variables, $\{y, x\} \to \{t, x\}$, transforming (11) back into it's time-dependent version

$$\Psi_t + \mu\Psi_x + \frac{\sigma^2}{2}\Psi_{xx} = r\Psi + e\,(x) - \mathcal{I}\left(y_0 e^{(\rho - r)t}\right) \tag{13}$$

where now $\Psi = \Psi\,(t, x)$. We thus obtain a *linear differential equation*, which is easy to

[9] The case $\Psi_y = 0$ would correspond to the situation in which the wealth W would not change when the marginal utility of wealth, I_W, changes.

interpret and solve. First recall that, by assumption, $\Psi = W$ i.e. the unknown function Ψ represents the level of wealth. Therefore, the left hand side of (13) describes the variation of wealth W with respect to the change in underlying variables $\{t, x\}$. Less formally, it can be interpreted as the expected instantaneous increase of wealth per unit of time. By abusing notation, we could be tempted re-writing (13) into the easily interpretable equation

$$\frac{E[dW]}{dt} = rW + e - c^*$$

The expected variation of wealth must therefore be equal to the sum of three components:

1. Capital appreciation at the riskless interest rate r;
2. Contribution of the incoming earnings flow e;
3. Outgoing optimal dividend payment flow c^*.

Hence, what remains of capital appreciation rW and the entering flow e after dividends c^* have been paid, is going to increase the wealth W. If at some point the dividend flow c^* exceeds the entering flow $rW + e$, the overall contribution will be negative, thus lowering the accumulated wealth W.

In the particular case of equality between the rate of intertemporal impatience ρ and the interest rate r, Ψ_y would not be present on the left hand side of (13), thus indicating that the level of marginal utility, y, remains constant, i.e. $y = y_0$ by virtue of (12). In such case the dividend payment c^*, a function of y only, is also constant in time. Finally, we can check for the agreement of measure units: the entering and outgoing flows e and c^*, as well as capitalization contribution rW, are all expressed in the same monetary unit per unit of time.

By the Feynman-Kac theorem (see Karatzas and Shreve (1988)), the *unique* solution to the PDE (13) with appropriate boundary condition imposed at T i.e. $\Psi(T, x) = W(x_T)$

admits the stochastic representation

$$\Psi(t, x_t) = \mathbb{E}\left[e^{-r(T-t)} \left\{ W(x_T) + \int_t^T e^{r(T-s)} \left[\mathcal{I}(y_s) - e(x_s) \right] ds \right\} \middle| \mathcal{F}_t \right]$$

In the context of our infinite horizon problem we impose the following transversality condition

$$\lim_{T \to \infty} \mathbb{E}\left[e^{-r(T-t)} W(x_T) \middle| \mathcal{F}_t \right] = 0 \qquad P\text{-a.s.}$$

i.e. taking into account discounting, it is expected that the wealth will not "cumulate" at the terminal "instant" $T = \infty$. Therefore, the unique *solution* to our problem, given x_t, is given by

$$\Psi(t, x_t) = \mathbb{E}\left[\int_t^\infty e^{-r(s-t)} \left[\mathcal{I}\left(y_0 e^{(\rho - r)s} \right) - e(x_s) \right] ds \middle| \mathcal{F}_t \right] \tag{14}$$

where the constant y_0 can be determined from the initial condition

$$\Psi(0, x_0) = W_0 \tag{15}$$

Once the function $\Psi = \Psi(t, x_t)$ and the constant y_0 are computed, the *optimal hedging and dividend policy* $\{c^*, f^*\}$ obtains from the first order condition (7). Using (10) and (12) we obtain

$$c^* = \mathcal{I}\left(y_0 e^{(\rho - r)t} \right) \tag{16}$$

$$f^* = \Psi_x \tag{17}$$

The optimal dividend policy c^* is thus a deterministic function of time, while the optimal position in short-term forward contracts reflects the exposure of wealth Ψ to changes in risk x, as measured by the partial derivative of Ψ w.r.t. x.

We thus obtained a *value hedge*. To see this notice that the wealth function Ψ, as given by (14), expresses current wealth as the difference between expected, discounted future payments minus expected, discounted future earnings. Inspection of (14) reveals that cumulated expected dividend payments do not depend on x_t. Therefore we must have

$$f^* = -\frac{\partial}{\partial x_t}\mathbb{E}\left[\int_t^\infty e^{-r(s-t)}e\left(x_s\right)ds\middle|\mathcal{F}_t\right] \tag{18}$$

The *value of all future earnings* is taken into account. This "construction" provides a way to understand why f^* is not just a local hedge of current earnings flow $e\left(x_t\right)$. Optimal hedging policy f^* provides a way to ensure that the total exposure (and not just local exposure of current earnings) is intertemporal and optimally managed by the firm in order to maximize shareholder's satisfaction from dividend stream they expect to receive.

Moreover, and most importantly, the value hedge appears to be *independent of risk preferences*, as neither $u\left(\cdot\right)$ nor the inverse of it's derivative $\mathcal{I}\left(\cdot\right)$ enter the expression (18). This feature is reminiscent of the fact that there is no speculative motive associated with holding instantaneous forwards. The consumption decision (16) is thus *separated* from hedging decision (18).

Finally, we note that all results in this section can be obtained and verified using martingale approach of Cox and Huang (1989).[10] The alternative is then to transform (4) and (5) into a static optimization problem, involving (4) and the static budget constraint (15). Obtaining y_0 amounts then to solving for the Lagrange multiplier associated with (15). The martingale method yields the solution (14) and optimal hedging (16) and (17). The linear differential equation (13) results as an intermediate "by-product" through an application of the Feynman-Kac theorem.

5. An illustrative example

Consider the following example. A business is exposed to a mean-reverting risk x. Within the neighborhood of the mean (set by $\bar{x}$) the net income is assumed to increase with the risk level x. To further specify this feature we assume that the net earnings of this firm

[10] A more rigorous proof is available from authors upon request.

can be approximated by a quadratic function $e(x) = x^2$. The risk variable x follows a mean-reverting pattern described by the Ornstein-Uhlenbeck process

$$dx_t = \alpha(\bar{x} - x_t)\ dt + \sigma\ dz_t$$

with initial value $x_0 \in \Re$, mean-reverting value $\bar{x} \in \Re$, force of reversion $\alpha > 0$ and volatility $\sigma > 0$ equal to some given constants. For CRRA utility $u(c) = c^\gamma/\gamma$ with $\gamma < 1, \gamma \neq 0$ the inverse of $u'(c) = c^{\gamma-1}$ is $\mathcal{I}(y) = y^{1/(\gamma-1)}$.

Our specification can be refined, for example, to a firm exposed to interest rate or exchange rate risk, to a realtor exposed to house price risk or to a food retail business exposed to temperature risk. In particular, parameters (e.g. α, σ and $\bar{x}$) can be adjusted so that the risk variable has very low probability (or, on the contrary, is allowed) to become negative (e.g. temperature). The quadratic cash flow structure is the simplest non-trivial, non-linear convex structure. It often has a meaningful interpretation. For the temperature risk example, profits can increase at both cold weather and hot weather extremes (e.g. firm sells more cold drinks and ice cream in summer while "solid" food sales may increase during winter months etc.).[11]

In our setup from Section 3, provided that the risk aversion is strong enough i.e. $\gamma < \frac{\rho}{r}$, the wealth function (14) turns out to be

$$\Psi(t, x_t) = \frac{\gamma - 1}{r\gamma - \rho} y_0^{1/(\gamma-1)} \exp\left\{\frac{\rho - r}{\gamma - 1} t\right\} - \mathcal{E}(x_t) \tag{19}$$

where $\mathcal{E}(x_t)$ is the value of the expected endowment at time t, given by

$$\mathcal{E}(x) = \frac{1}{2\alpha + r}(x - \bar{x})^2 + \frac{2\bar{x}}{\alpha + r}(x - \bar{x}) + \frac{1}{r}\left(\bar{x}^2 + \frac{\sigma^2}{2\alpha + r}\right) \tag{20}$$

We notice that $\Psi(t, x)$ is separable in t and x. Moreover, endowment $\mathcal{E}(x)$ turns out to be

[11] For background information on pricing weather derivatives see Geman (1999). Our example provides an answer to the question of how to optimally use such forward contracts.

a quadratic function of the distance from the mean level $x - \bar{x}$.

Computation of $\mathcal{E}$ involves the second conditional central moment of x. In our example this computation is particularly straightforward as the assumed non-linear cash flow structure is one of the simplest possible i.e. quadratic. This required intermediate component can thus be computed using conditional mean and variance of an Ornstein-Uhlenbeck process

$$\begin{aligned}\mathbb{E}\left[x_s^2 \middle| \mathcal{F}_t\right] &= \operatorname{var}\left(x_s | \mathcal{F}_t\right) + \left(\mathbb{E}\left[x_s | \mathcal{F}_t\right]\right)^2 \\ &= \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(s-t)}\right) + \left[\bar{x} + \left(x_t - \bar{x}\right) e^{-\alpha(s-t)}\right]^2\end{aligned}$$

For cash flow shapes which can be approximated by a polynomial expression in x, the analytic solution method will yield closed-form expressions, similar to (19). Consider a "cubic" flow: $e(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, where a_i are some constants. In this case calculation of the expectation in (14) would require using the third conditional moment $\mathbb{E}\left[x_s^3 \middle| \mathcal{F}_t\right]$, which can be easily obtained from the characteristic function of the normal distribution.

This suggests that our method will apply to almost *any shape of cash flow* function $e(\cdot)$, provided that the latter can be suitably approximated by a polynomial. For a polynomial of degree n, the procedure would require extracting from the characteristic function and including in expectation calculations all central moments up to degree n.

From initial conditions $\{x_0, W_0\}$, the constant y_0 can be obtained explicitly from (15) and (19) as

$$y_0 = \left\{\frac{r\gamma - \rho}{\gamma - 1}\left[W_0 + \mathcal{E}\left(x_0\right)\right]\right\}^{\gamma - 1}$$

which completes the computation of the wealth function. The optimal dividend policy is given by

$$c_t^* = \left[y_0 e^{(\rho - r)t}\right]^{\frac{1}{\gamma - 1}} \tag{21}$$

and is decreasing (increasing) in time if propensity to consume is strong $\rho > r$ (weak $\rho < r$).

The optimal hedging turns out to be

$$f_t^* = -\mathcal{E}'(x_t) = -\left[\frac{2\bar{x}}{\alpha + r} + \frac{2}{2\alpha + r}(x_t - \bar{x})\right] \tag{22}$$

Clearly, f_t^* is not equal to the local exposure $e'(x_t) = 2x_t$. Moreover, the last term contains the distance $x_t - \bar{x}$ separating the current level x_t from the parity level $\bar{x}$. In fact we can distinguish two components:

1. **Perpetual component**, $-\frac{2\bar{x}}{\alpha+r}$, linked to the parity level $\bar{x}$. This component is constant and always negative (forward sale).
2. **Instantaneous component**, $-\frac{2}{2\alpha+r}(x_t - \bar{x})$, the sign of which depends on the distance of x_t from the parity level $\bar{x}$.

The presence of the second, instantaneous component suggests the following interpretation:

- If the situation is favorable i.e. x_t is *above* the parity level $\bar{x}$, an *additional forward sale* is recommended in order to lock in profits;
- If the situation is bad i.e. x_t is *below* the parity level $\bar{x}$, the manager should *decrease the hedge and wait* until the level of x rises.

In a sense, the manager is effectively taking the other side of the trade in bad times, selectively decreasing the hedge. Conversely, in good times we know times will *not* remain good. Because of mean-reversion economic conditions are expected to worsen and the manager should therefore enhance the hedge.

The expected endowment $\mathcal{E}(x)$ is a parabola which attains a minimum at $x_- < 0$ (provided that $\bar{x} > 0$), such that

$$x_- = -\frac{\alpha}{\alpha + r}\bar{x}$$

Optimal hedging policy f_t^* as expressed by (22) is therefore *negative* for $x_t > x_-$. "Typical" situations in our example occur when x_t is located towards it's mean level $\bar{x} > 0$ i.e. x_t has value in the area where the probability density of the Ornstein-Uhlenbeck process is maximal. So whenever a "typical" situation occurs (and "typical" situations will happen most of the time in our example) the optimal hedge f_t^* is a short position (forward sale) in instantaneous forward contracts, as expected. In "extreme" situations the hedge can become *positive* (forward purchase) i.e. $f_t^* > 0$ which is akin to speculation. This occurs for extremely low values of the state variable x, when $x_t < x_-$.

Note also that the minimum expected endowment is always positive i.e.

$$\mathcal{E}\left(x_-\right) = \frac{1}{r}\left[\left(\frac{\alpha\bar{x}}{r+\alpha}\right)^2 + \frac{\sigma^2}{2\alpha+r}\right] > 0 \tag{23}$$

A particular feature of the Ornstein-Uhlenbeck process is that its conditional variance does not depend on the current level of x_t. As a consequence, and somewhat perversely, the volatility σ will not appear in the expression (22) for optimal hedging f^*.[12] To see why this will occur, observe that the volatility σ enters only the constant term in the expression (20) giving $\mathcal{E}(x)$. However, increased volatility σ will be beneficial to the agent through a sort of "convexity effect," rising the level of minimum expected endowment (23).

6. Comparative statics and new empirical implications

Our example suggests that corporations should hedge in good times to lock in profits and not to hedge in bad times but wait for profits to re-emerge. If in our example we assumed e.g. a geometric Brownian motion with positive drift, we could still see this result. However, under such regime profits would always be expected to increase on average. In a sense, we would always be in bad times. That is, the result would be to decrease the hedge and wait

[12]It is easy to show that this would not be the case if we employed the familiar geometric Brownian motion in place of Ornstein-Uhlenbeck process, i.e. the volatility parameter σ would appear in the analogue of expression (22) giving the optimal hedging f^*. We leave this verification as an exercise for the reader.

for a portion of profits to emerge later, because of exponential increasing trend. A mean-reverting process illustrates our point much better. Our model then yields recommendations not only for "bad times" but also for "good times."

We collect the set of new testable implications which can be derived from our framework in Tables below. Empirical examination of these implications may provide new insights into the nature and motives of hedging.

Topics	**New Testable Implications**
1. What is hedged? Long term vs short term.	Managers acting optimally will use available short term instruments to hedge all expected future cash flows (long term, *value hedge*), not just the short term exposure.
2. Industry sector.	Selective hedging should be more prevalent in industries where managers are exposed to convex cash flow structures.
3. Risk aversion.	Level of value hedge independent of risk preferences of the manager.

For the mean reverting case specification from the previous section it is helpful to derive some comparative statics results. These yield some more new empirical implications.

Result 1. Sensitivity of the optimal hedge (22) to changes in parity level $\bar{x}$ is given by

$$\frac{\partial f_t^*}{\partial \bar{x}} = 2\left(\frac{1}{2\alpha + r} - \frac{1}{\alpha + r}\right) < 0 \ .$$

Therefore, when the parity level increases $\Delta\bar{x} > 0$ and $f_t^* < 0$ we should expect f_t^* to become larger in absolute value (more negative) i.e. the value hedge to increase. Furthermore, this sensitivity is a constant independent of the parity level $\bar{x}$ and risk level x_t.

Result 2. Sensitivity of the optimal hedge (22) to changes in the force of mean reversion α is given by

$$\frac{\partial f_t^*}{\partial \alpha} = \frac{4(x_t - \bar{x})}{(2\alpha + r)^2} + \frac{2\bar{x}}{(\alpha + r)^2} \ . \tag{24}$$

This expression is positive for higher levels of the risk level x_t but becomes negative for

$$x_t < \frac{\bar{x}\left(r^2 - 2\alpha^2\right)}{2(\alpha + r)^2} \ .$$

This suggests that when the force of mean reversion α increases $\Delta\alpha > 0$ and $f_t^* < 0$ we should expect f_t^* to increase and become closer to zero i.e. the value hedge to decrease in good times. Conversely, when the force of mean reversion α increases in extreme bad times where the firm "speculates" ($f_t^* > 0$), the firm should increase the hedge i.e. reduce "speculation" closer to zero or make f_t^* negative (start hedging) in anticipation of stronger prospects to return to normal sooner.

Result 3. Sensitivity of the optimal hedge (22) to changes in the level of interest r is given by

$$\frac{\partial f_t^*}{\partial r} = \frac{2(x_t - \bar{x})}{(2\alpha + r)^2} + \frac{2\bar{x}}{(\alpha + r)^2} \ .$$

This expression is structurally resembling the sensitivity to changes in the force of mean reversion α (compare to equation (24)). However, it is numerically different and becomes negative when

$$x_t < -\frac{\bar{x}\alpha\left(2r + 3\alpha\right)}{(\alpha + r)^2} \ .$$

This suggests that we should expect the value hedge to decrease in good times as a response to the interest rate rising $\Delta r > 0$. Conversely, in (extreme) bad times the firm should (reduce "speculation") start/increase hedging if interest rates rise. This is consistent with observation that in our intertemporal setup hedging is a "device" which allows to lock in profits.

Industry sectors ***where the risk dimension exhibits mean-reversion***	
Topics	**New Testable Implications**
1. Volatility.	Selective hedging independent of volatility.
2. State of the economy. Speculation.	In economic downturn optimal hedging akin to speculation, hedging decreases (exposure increases) in anticipation of the economy returning to normal later.
3. Parity level $\bar{x}$.	Hedging increases when the long-term parity level increases.
4. Force of mean reversion α	In good times hedging decreases in response of decline to normal expected sooner; In bad times hedging increases (replacing "speculation") if improvements closer on horizon.
5. Level of interest rates r	Decrease of hedging in good times & increase of hedging in bad times following an interest rate increase.

Furthermore, another implication which can be tested is whether the payout policy in value hedging regimes is (or is not) dependent on the risk aversion of the manager. Our model suggests that the optimal payout policy is dependent in exponential fashion on the risk aversion parameter (γ in equation (21)), while the optimal hedging is not dependent on risk preferences (no γ in equation (22)).

7. Concluding remarks

We established the optimal hedging strategy for instantaneous forward contracts to hedge a continuum of exposures. The optimal control appears to be a value hedge involving total current value of future earnings. Our results hold for general risk preferences as well as risk process and endowment flow specification. More importantly, hedging decision is independent of risk preferences of the firm or agent. Moreover, hedging and dividend (consumption) decisions separate.

We suggest a method to easily specialize our general result to any cash flow profile. In the example provided, we study mean-reverting risk following an Ornstein-Uhlenbeck process, coupled with quadratic cash flow profile. Extensions and practical implementations of our result to such tractable situations as other Gaussian processes or richer cash flow structures are straightforward.

More importantly, our special case suggests that the usual textbook recommendation to fully hedge any cash flow to come will fail in our setup. Optimal policy is not static and hedging should be either increased or decreased according to whether the current state of nature is in "good" or "bad" zone, respectively. In other words, firms should hedge selectively.

An interesting issue is whether every firm that follows this advice would make money from its hedging policy? In our setup hedging will decrease variability of hedged cash flows (and improve utility) while not increasing the money flow on average, because every transaction in the forward market is, by construction, a zero sum game. However, our findings are consistent with recommendations to avoid loss given in Working (1962). The advice is to hedge when prices are expected to decline.

In our model we assume a properly incentivized risk-averse manager. However, another possibility is to interpret our setup as an individual's problem. An investor might be exposed to some non-tradable risks. A realtor's income, for example, may depend on the level of house price. The realtor could hedge exposure using housing derivatives.

Finally, our optimal hedging policy provides many new empirical implications and is consistent with existing empirical evidence. In particular, value hedging approach provides justification for *selective hedging*, a widespread practice documented in several recent empirical studies.

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