The Giesy–James theorem for general index p, with an application to operator ideals on the p^{th} James space

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Abstract

A theorem of Giesy and James states that c_0 is finitely representable in James' quasi-reflexive Banach space J_2. We extend this theorem to the p^{th} quasi-reflexive James space J_p for each $p \in (1, \infty)$. As an application, we obtain a new closed ideal of operators on J_p, namely the closure of the set of operators that factor through the complemented subspace $(\ell_\infty^1 \oplus \ell_\infty^2 \oplus \cdots \oplus \ell_\infty^n \oplus \cdots)_{\ell_p}$ of J_p.

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1 Introduction

As outlined in the abstract, we shall prove that c_0 is finitely representable in the p^{th} quasi-reflexive James space J_p for each $p \in (1, \infty)$ and then show how this result gives rise to a new closed ideal of operators on J_p. In order to make these statements precise, let us introduce some notation and terminology.

We denote by $\mathbb{N}_0$ and $\mathbb{N}$ the sets of non-negative and positive integers, respectively. Following Giesy and James [5], we index sequences by $\mathbb{N}_0$ and write $x(n)$ for the n^{th} element of the sequence x, where $n \in \mathbb{N}_0$. For a non-empty subset A of $\mathbb{N}_0$, we write $A = \{n_1 < n_2 < \cdots < n_k\}$ (or $A = \{n_1 < n_2 < \cdots\}$ if A is infinite) to indicate that $\{n_1, n_2, \ldots, n_k\}$ is the increasing ordering of A.

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ be the scalar field, and let $p \in (1, \infty)$. For a scalar sequence x and a finite subset $A = \{n_1 < n_2 < \cdots < n_{k+1}\}$ of $\mathbb{N}_0$ of cardinality at least two, we define

$$\nu_p(x, A) = \Big(\sum_{j=1}^{k} \big|x(n_j) - x(n_{j+1})\big|^p\Big)^{\frac{1}{p}};$$

for convenience, we let $\nu_p(x,A) = 0$ whenever $A \subseteq \mathbb{N}_0$ is empty or a singleton. Then $\nu_p(\,\cdot\,,A)$ is a seminorm on the vector space $\mathbb{K}^{\mathbb{N}_0}$ of all scalar sequences, and

$$\begin{aligned}\|x\|_{J_p} &:= \sup\bigl\{\nu_p(x,A) : A \subseteq \mathbb{N}_0,\ \operatorname{card} A < \infty\bigr\}\\ &= \sup\biggl\{\Bigl(\sum_{j=1}^{k}\bigl|x(n_j) - x(n_{j+1})\bigr|^p\Bigr)^{\frac{1}{p}} : k \in \mathbb{N},\ n_1,\ldots,n_{k+1} \in \mathbb{N}_0,\ n_1 < \cdots < n_{k+1}\biggr\}\end{aligned}$$

defines a complete norm on the subspace $J_p := \bigl\{x \in c_0 : \|x\|_{J_p} < \infty\bigr\}$, which we call the p^{th} *James space*. The sequence $(e_m)_{m=0}^{\infty}$, where $e_m \in \mathbb{K}^{\mathbb{N}_0}$ is given by

$$e_m(n) = \begin{cases} 1 & \text{if } m = n\\ 0 & \text{otherwise}\end{cases} \qquad (n \in \mathbb{N}_0),$$

forms a shrinking Schauder basis for J_p. More importantly, J_p is *quasi-reflexive* in the sense that the canonical image of J_p in its bidual has codimension one. This result, as well as the definition of J_p, is due to James [6] in the case $p = 2$; Edelstein and Mityagin [4] appear to have been the first to observe that it carries over to arbitrary $p \in (1,\infty)$.

A Banach space X is *finitely representable* in a Banach space Y if, for each finite-dimensional subspace F of X and each $\varepsilon > 0$, there is an operator $T\colon F \to Y$ such that

$$(1-\varepsilon)\|x\| \leqslant \|Tx\| \leqslant (1+\varepsilon)\|x\| \qquad (x \in F). \tag{1.1}$$

We shall in fact only consider finite representability of c_0, in which case it suffices to establish (1.1) for the finite-dimensional subspaces $F = \ell_\infty^n$, where $n \in \mathbb{N}$. Although not required, let us mention the Maurey–Pisier theorem that c_0 is finitely representable in a Banach space Y if and only if Y fails to have finite cotype (*e.g.*, see [2, Theorem 14.1]). This result shows in particular that finite representability of c_0 is an isomorphic invariant, despite the obvious dependence on the choice of norm in (1.1).

Giesy and James [5] proved that c_0 is finitely representable in J_2. Our first main result, to be proved in Section 2, extends this result to arbitrary $p \in (1,\infty)$.

Theorem 1.1. *For each $p \in (1,\infty)$, c_0 is finitely representable in J_p.*

To explain how this result leads to a new closed ideal of operators on J_p, we require some more notation. For $p \in [1,\infty)$ and a family $(X_j)_{j\in\mathbb{J}}$ of Banach spaces, we write $\bigl(\bigoplus_{j\in\mathbb{J}} X_j\bigr)_p$ for the direct sum of the X_j's in the sense of ℓ_p; that is,

$$\Bigl(\bigoplus_{j\in\mathbb{J}} X_j\Bigr)_p = \Bigl\{(x_j) : x_j \in X_j\ (j \in \mathbb{J}) \ \text{ and } \ \sum_{j\in\mathbb{J}} \|x_j\|^p < \infty\Bigr\}.$$

We shall only apply this notation in two cases, namely

$$G_p := \Bigl(\bigoplus_{n\in\mathbb{N}} \ell_\infty^n\Bigr)_p \qquad \text{and} \qquad J_p^{(\infty)} := \Bigl(\bigoplus_{n\in\mathbb{N}_0} J_p^{(n)}\Bigr)_p, \tag{1.2}$$

where $J_p^{(n)}$ denotes the subspace of J_p spanned by the first $n+1$ basis vectors $e_0, e_1, \ldots, e_n$.

Our interest in these spaces stems from the two facts that (i) J_p contains a complemented subspace isomorphic to $J_p^{(\infty)}$; and (ii) Theorem 1.1 implies that $J_p^{(\infty)}$ contains a complemented subspace isomorphic to G_p (for $p = 2$, this has already been observed by Casazza, Lin and Lohman [1, Theorem 13(i)] using the original Giesy–James theorem), and this subspace gives rise to a new closed ideal of operators on J_p, as we shall now outline.

For Banach spaces X and Y, let

$$\mathscr{G}_Y(X) = \big\{ST : T \in \mathscr{B}(X,Y),\ S \in \mathscr{B}(Y,X)\big\}$$

be the set of operators on X which factor through Y. This defines a two-sided algebraic ideal of the Banach algebra $\mathscr{B}(X)$ of bounded operators on X, provided that Y contains a complemented subspace isomorphic to $Y \oplus Y$ (which will always be the case in this paper), and hence its norm-closure, denoted by $\overline{\mathscr{G}}_Y(X)$, is a closed ideal of $\mathscr{B}(X)$.

Edelstein and Mityagin [4] made the easy, but fundamental, observation that the quasi-reflexivity of J_p for $p \in (1,\infty)$ implies that the ideal $\mathscr{W}(J_p)$ of weakly compact operators has codimension one in $\mathscr{B}(J_p)$, hence is a maximal ideal. Loy and Willis [11, Open Problems 2.8] formally raised the problem of determining the structure of the lattice of closed ideals of $\mathscr{B}(J_2)$, having themselves proved that $\mathscr{K}(J_2) \subsetneq \overline{\mathscr{G}}_{\ell_2}(J_2) \subsetneq \mathscr{W}(J_2)$ and $\mathscr{S}(J_2) = \mathscr{E}(J_2) \not\supseteq \overline{\mathscr{G}}_{\ell_2}(J_2)$, where $\mathscr{S}(J_2)$ and $\mathscr{E}(J_2)$ denote the ideals of strictly singular and inessential operators, respectively (see [11, Theorem 2.7] and the text preceding it). Saksman and Tylli [13, Remark 3.9] improved the latter result by showing that $\mathscr{K}(J_2) = \mathscr{S}(J_2)$, while the third author [9, 10] generalized these results to arbitrary $p \in (1,\infty)$ and, more importantly, complemented them by showing that the lattice of closed ideals in $\mathscr{B}(J_p)$ has the following structure:

$$\begin{array}{c} \mathscr{B}(J_p) \\ | \\ \mathscr{W}(J_p) = \mathscr{G}_{J_p^{(\infty)}}(J_p) = \overline{\mathscr{G}}_{J_p^{(\infty)}}(J_p) \\ \vdots \\ \overline{\mathscr{G}}_{\ell_p}(J_p) \\ | \\ \mathscr{K}(J_p) = \mathscr{S}(J_p) = \mathscr{E}(J_p) = \mathscr{V}(J_p) \\ | \\ \{0\}, \end{array}$$

where $\mathscr{V}(J_p)$ is the ideal of completely continuous operators, the vertical lines indicate proper set-theoretic inclusion, and further closed ideals may be found only at the dotted line. In particular, $\mathscr{W}(J_p)$ is the unique maximal ideal of $\mathscr{B}(J_p)$.

The second main result of this paper, which we shall prove in Section 3, states that $\mathscr{B}(J_p)$ contains at least one other closed ideal than those listed above.

Theorem 1.2. *For each $p \in (1,\infty)$, the operator ideal $\overline{\mathscr{G}}_{G_p}(J_p)$ lies strictly between $\overline{\mathscr{G}}_{\ell_p}(J_p)$ and $\mathscr{W}(J_p)$, where G_p is the Banach space given by* (1.2).

Hence the lattice of closed ideals in $\mathscr{B}(J_p)$ has at least six distinct elements, namely

$$\{0\} \subsetneq \mathscr{K}(J_p) \subsetneq \overline{\mathscr{G}}_{\ell_p}(J_p) \subsetneq \overline{\mathscr{G}}_{G_p}(J_p) \subsetneq \mathscr{W}(J_p) \subsetneq \mathscr{B}(J_p).$$

2 Proof of Theorem 1.1

Throughout this section, we fix a number $p \in (1, \infty)$. Our aim is to prove Theorem 1.1 by modifying the proof of Giesy and James [5]. The general scheme of the proof is the same, but at several points, identities that are simple in the case $p = 2$ have to be replaced with estimations applying to other p. We follow their notation as far as possible. We show that there is a near-isometric embedding of ℓ_∞^K for each $K \in \mathbb{N}$ in the real case. It then follows easily, by standard techniques, that there is at least an isomorphic embedding in the complex case. Hence in the remainder of this section we shall assume that the scalar field is $\mathbb{R}$.

Spiky vectors play a central role in the proof. As in [5, p. 65], let

$$z_{2k} = \frac{1}{(2k)^{1/p}} \sum_{j=1}^{k} e_{2j-1} \in J_p \qquad (k \in \mathbb{N}),$$

so that z_{2k} is a unit vector with spikes in its initial k odd coordinates.

The other key ingredient is the "stretch" operator $T_n \colon J_p \to J_p$ which, for $n \in \mathbb{N}$ and $x \in J_p$, is given by $(T_n x)(kn) = x(k)$ whenever $k \in \mathbb{N}_0$ and by linear interpolation between these points. One can easily check that T_n is linear and isometric.

We use the notation $[j, k]$ for the set of integers n such that $j \leqslant n \leqslant k$.

By an inductive process, we construct, for each $K \in \mathbb{N}$, a set of K stretched spiky vectors with the parameters chosen suitably, and show that these vectors are equivalent to the usual basis of ℓ_∞^K. The inductive step is captured by the following lemma, corresponding to [5, Lemma 1].

Lemma 2.1. *Let $m \in \mathbb{N}$ and $\gamma, \varepsilon \in (0, \infty)$. Suppose that x is an element of J_p supported on the integer interval $[0, 2m-1]$ and satisfying*

$$\max_{0 \leqslant j < 2m} \bigl| x(j) - x(j+1) \bigr|^p \leqslant \frac{\gamma}{2m} \qquad \text{and} \qquad \|x\|_{J_p}^p - \nu_p\bigl(x, [0, 2m]\bigr)^p \leqslant \varepsilon. \tag{2.1}$$

For some even n, let $w = T_n x + \gamma^{1/p} z_{2mn}$. Then w is supported on the integer interval $[0, 2mn-1]$ and satisfies

$$\max_{0 \leqslant j < 2mn} |w(j) - w(j+1)|^p \leqslant \frac{\gamma}{2mn} \Bigl(1 + \frac{1}{n^{1-1/p}}\Bigr)^p \tag{2.2}$$

and

$$\|w\|_{J_p}^p - \nu_p\bigl(w, [0, 2mn]\bigr)^p \leqslant 2\varepsilon + \gamma \varphi(m, n), \tag{2.3}$$

where $\varphi(m, n) \to 0$ as $n \to \infty$ with m fixed.

We show next how Theorem 1.1 follows, and then return to the proof of Lemma 2.1.

Proof of Theorem 1.1. With $\varepsilon > 0$ and $K \in \mathbb{N}$ given, we construct vectors $x_1, \ldots, x_K \in J_p$ with $\|x_i\|_{J_p} \geqslant 1$ for $1 \leqslant i \leqslant K$ such that $\| \sum_{i=1}^{K} \delta_i x_i \|_{J_p} \leqslant 1 + 2\varepsilon$ for all choices of

$\delta_1,\dots,\delta_K \in \{-1,1\}$. We then deduce equivalence with the usual basis of ℓ_∞^K as follows. By convexity, we have $\left\|\sum_{i=1}^K \lambda_i x_i\right\|_{J_p} \leqslant 1+2\varepsilon$ for all real λ_i with $|\lambda_i| \leqslant 1$. Suppose that $\max_{1\leqslant i\leqslant K}|\lambda_i| = |\lambda_j| = 1$. Then $\left\|\sum_{i=1}^K \lambda_i x_i - 2\lambda_j x_j\right\|_{J_p} \leqslant 1+2\varepsilon$ (the coefficient of x_j has been changed to $-\lambda_j$), so

$$\left\|\sum_{i=1}^K \lambda_i x_i\right\|_{J_p} \geqslant \|2\lambda_j x_j\|_{J_p} - \left\|\sum_{i=1}^K \lambda_i x_i - 2\lambda_j x_j\right\|_{J_p} \geqslant 2-(1+2\varepsilon) = 1-2\varepsilon.$$

Let $\varepsilon_k = \varepsilon/3^{K-k}$. At stage k, we will define $n_k \in \mathbb{N}$, $\gamma_k \in \mathbb{R}$ and $x_1^{(k)},\dots,x_k^{(k)} \in J_p$ such that the following properties hold. Firstly, $x_1^{(k)},\dots,x_k^{(k)}$ are supported on the integer interval $[0, 2m_k-1]$, where $m_k := n_1 n_2 \dots n_k$, and $\|x_i^{(k)}\|_{J_p} \geqslant 1$ for $1 \leqslant i \leqslant k$. Secondly,

$$1 \leqslant \gamma_k \leqslant 1 + \frac{\varepsilon k}{K}.$$

Thirdly, for all choices of $\delta_1,\dots,\delta_k \in \{-1,1\}$ and with $y_\delta^{(k)} := \sum_{i=1}^k \delta_i x_i^{(k)}$, we have

$$\max_{0\leqslant j<2m_k} \left|y_\delta^{(k)}(j) - y_\delta^{(k)}(j+1)\right|^p \leqslant \frac{\gamma_k}{2m_k} \tag{2.4}$$

and

$$\|y_\delta^{(k)}\|_{J_p}^p - \nu_p\big(y_\delta^{(k)}, [0,2m_k]\big)^p \leqslant \varepsilon_k. \tag{2.5}$$

By (2.4) and (2.5), we then obtain $\|y_\delta^{(k)}\|_{J_p}^p \leqslant \gamma_k + \varepsilon_k \leqslant 1+2\varepsilon \leqslant (1+2\varepsilon)^p$, from which the desired conclusion follows.

To start, take $x_1^{(1)} = z_2$ and $n_1 = \gamma_1 = 1$. Suppose now that stage $k-1$ has been completed. For a certain even integer n_k to be chosen, define

$$x_i^{(k)} = T_{n_k}(x_i^{(k-1)}) \qquad (1 \leqslant i \leqslant k-1) \qquad\text{and}\qquad x_k^{(k)} = \gamma_{k-1}^{1/p} z_{2m_k}.$$

Let $\delta_1,\dots,\delta_k \in \{-1,1\}$ be given. We may assume that $\delta_k = 1$. Apply Lemma 2.1 with $x = y_\delta^{(k-1)}$, $m = m_{k-1}$, $n = n_k$, $\varepsilon = \varepsilon_{k-1}$ and $\gamma = \gamma_{k-1}$. Then

$$w = T_{n_k}(y_\delta^{(k-1)}) + \gamma_{k-1}^{1/p} z_{2m_k} = y_\delta^{(k)},$$

hence (2.2) implies that (2.4) is satisfied with

$$\gamma_k = \gamma_{k-1}\left(1 + \frac{1}{n_k^{1-1/p}}\right)^p.$$

We choose n_k large enough to ensure that $\gamma_k \leqslant 1 + \varepsilon k/K$. By (2.3),

$$\|y_\delta^{(k)}\|_{J_p}^p - \nu_p\big(y_\delta^{(k)}, [0,2m_k]\big)^p \leqslant 2\varepsilon_{k-1} + \gamma_{k-1}\varphi(m_{k-1}, n_k).$$

Since $\varepsilon_k = 3\varepsilon_{k-1}$, to ensure (2.5), we choose n_k also to satisfy $\gamma_{k-1}\varphi(m_{k-1}, n_k) \leqslant \varepsilon_{k-1}$. $\square$

Remark 2.2. Because of the dependence of $\varphi(m,n)$ on m, it is not possible to take n_k equal to the same value n for each k, as in [5] for the case $p=2$. We shall actually see later that $\varphi(m,n)$ only depends on m when $p>2$.

Outline of proof of Lemma 2.1. Write $y = T_n x$ and $z = \gamma^{1/p} z_{2mn}$, so that $w = y + z$. Clearly, y and z are both supported on the integer interval $[0, 2mn-1]$. Also, from the definitions, we have $|z(j) - z(j+1)| = (\gamma/2mn)^{1/p}$ and

$$\big|(T_n x)(j) - (T_n x)(j+1)\big| \leqslant \frac{1}{n}\Big(\frac{\gamma}{2m}\Big)^{\frac{1}{p}} = n^{\frac{1}{p}-1}\Big(\frac{\gamma}{2mn}\Big)^{\frac{1}{p}} \qquad (0 \leqslant j < 2mn),$$

from which (2.2) follows.

The bulk of the work is the proof of (2.3). Since w is supported on the integer interval $[0, 2mn-1]$, we can find a set $A = \{a_1 < a_2 < \cdots < a_{k+1}\}$, with $a_1 = 0$ and $a_{k+1} = 2mn$, such that $\|w\|_{J_p} = \nu_p(w, A)$. The aim is to show that the whole interval acts as a reasonable substitute for this set A. This will be accomplished by four steps, summarized as follows:

$$\begin{aligned} \nu_p(w,A)^p &\leqslant \nu_p(y,A)^p + \nu_p(z,A)^p + \rho_1 && (2.6)\\ &\leqslant \nu_p\big(y, A \cup ([0,2mn] \cap n\mathbb{N}_0)\big)^p + \nu_p\big(z, A \cup ([0,2mn] \cap n\mathbb{N}_0)\big)^p + \rho_1 + \rho_2 && (2.7)\\ &\leqslant \nu_p\big(y, [0,2mn]\big)^p + \nu_p\big(z, [0,2mn]\big)^p + \rho_1 + \rho_2 && (2.8)\\ &\leqslant \nu_p\big(w, [0,2mn]\big)^p + \rho_1 + \rho_2 + \rho_3, && (2.9) \end{aligned}$$

where ρ_1, ρ_2 and ρ_3 are error terms which will emerge from the proofs. Step 1 moves from $w = y + z$ to y and z separately, and step 4 reverses this. Working with y and z separately, step 2 adjoins multiples of n to A, and step 3 adjoins all intervening integers. Because of the concepts involved, we present these four steps in the order 1, 4, 3, 2. □

Lemma 2.3. *Suppose that* a, $b > 0$. *Then* $(a+b)^p - a^p - b^p \leqslant 2^p(a^{p-1}b + ab^{p-1})$.

Proof. With no loss of generality, we may assume that $a \geqslant b$. Writing $b/a = t$, we see that the stated inequality is equivalent to $(1+t)^p - 1 - t^p \leqslant 2^p(t + t^{p-1})$ for $0 < t \leqslant 1$. For such t, since the function $t \mapsto (1+t)^p$ is convex and $t = (1-t)\cdot 0 + t \cdot 1$, we have $(1+t)^p \leqslant (1-t)\cdot 1 + 2^p t$, hence $(1+t)^p - 1 \leqslant (2^p - 1)t$, which of course implies the required inequality. □

Remark 2.4. The estimation in Lemma 2.3 is quite adequate for our purposes. In fact, the best constant on the right-hand side of the inequality is p for $2 \leqslant p \leqslant 3$, and $2^{p-1} - 1$ otherwise [7].

Step 1: Proof of (2.6). Write $\ell_i = a_{i+1} - a_i$, so that $\sum_{i=1}^{k} \ell_i = 2mn$. Then we have, by definition,

$$\big|y(a_i) - y(a_{i+1})\big| \leqslant \frac{\ell_i}{n}\Big(\frac{\gamma}{2m}\Big)^{\frac{1}{p}} \qquad \text{and} \qquad \big|z(a_i) - z(a_{i+1})\big| \leqslant \Big(\frac{\gamma}{2mn}\Big)^{\frac{1}{p}} \qquad (1 \leqslant i \leqslant k).$$

Lemma 2.3 implies that $\nu_p(y+z,A)^p - \nu_p(y,A)^p - \nu_p(z,A)^p \leqslant 2^p s$, where

$$\begin{aligned}
s &:= \sum_{i=1}^{k}\Big(\big|y(a_i)-y(a_{i+1})\big|^{p-1}\big|z(a_i)-z(a_{i+1})\big| + \big|y(a_i)-y(a_{i+1})\big|\,\big|z(a_i)-z(a_{i+1})\big|^{p-1}\Big)\\
&\leqslant \sum_{i=1}^{k}\bigg(\Big(\frac{\ell_i}{n}\Big)^{p-1}\Big(\frac{\gamma}{2m}\Big)^{1-\frac{1}{p}}\Big(\frac{\gamma}{2mn}\Big)^{\frac{1}{p}} + \frac{\ell_i}{n}\Big(\frac{\gamma}{2m}\Big)^{\frac{1}{p}}\Big(\frac{\gamma}{2mn}\Big)^{1-\frac{1}{p}}\bigg)\\
&= \gamma\bigg(\frac{\sum_{i=1}^{k}\ell_i^{p-1}}{2mn^{p-1+1/p}} + \frac{1}{n^{1-1/p}}\bigg),
\end{aligned}$$

since $\sum_{i=1}^{k}\ell_i = 2mn$. For $1 < p \leqslant 2$, we have $\ell_i^{p-1} \leqslant \ell_i$, hence

$$s \leqslant \gamma\Big(\frac{1}{n^{p-2+1/p}} + \frac{1}{n^{1-1/p}}\Big),$$

whereas for $p > 2$, $\sum_{i=1}^{k}\ell_i^{p-1} \leqslant \big(\sum_{i=1}^{k}\ell_i\big)^{p-1} = (2mn)^{p-1}$, so that

$$s \leqslant \gamma\Big(\frac{(2m)^{p-2}}{n^{1/p}} + \frac{1}{n^{1-1/p}}\Big).$$

Multiplying these upper bounds on s by 2^p, we conclude that (2.6) is satisfied with

$$\rho_1 = \begin{cases} 2^p\gamma\Big(\dfrac{1}{n^{p-2+1/p}} + \dfrac{1}{n^{1-1/p}}\Big) & \text{for } 1 < p \leqslant 2,\\[2ex] 2^p\gamma\Big(\dfrac{(2m)^{p-2}}{n^{1/p}} + \dfrac{1}{n^{1-1/p}}\Big) & \text{for } p > 2. \end{cases}$$

□

Step 4: Proof of (2.9), with $\rho_3 = \gamma/n^{p-1}$. Letting

$$s_\ell = \sum_{j=\ell n}^{(\ell+1)n-1}\Big(\big|w(j)-w(j+1)\big|^p - \big|y(j)-y(j+1)\big|^p - \big|z(j)-z(j+1)\big|^p\Big),$$

we can write

$$\nu_p\big(w,[0,2mn]\big)^p - \nu_p\big(y,[0,2mn]\big)^p - \nu_p\big(z,[0,2mn]\big)^p = \sum_{\ell=0}^{2m-1} s_\ell.$$

Our claim is that this quantity is at least $-\gamma/n^{p-1}$.

To verify this, fix integers $\ell \in [0, 2m-1]$ and $j \in [\ell n, (\ell+1)n-1]$. Then $z(j) - z(j+1)$ is alternately $\pm c$, where $c := (\gamma/2mn)^{1/p}$, while $y(j) - y(j+1) = \frac{1}{n}\big(x(\ell) - x(\ell+1)\big)$, and by assumption $d_\ell := \frac{1}{n}\big|x(\ell) - x(\ell+1)\big| \leqslant \frac{1}{n}(\gamma/2m)^{1/p}$. Since $c > d_\ell$, we see that $\big|w(j) - w(j+1)\big|$ is alternately $c + d_\ell$ and $c - d_\ell$. Hence, as n is even,

$$s_\ell = \frac{n}{2}\big((c+d_\ell)^p + (c-d_\ell)^p - 2d_\ell^p - 2c^p\big). \tag{2.10}$$

By convexity of the function $t \mapsto t^p$, we have $(c+d_\ell)^p + (c-d_\ell)^p \geqslant 2c^p$. Therefore $s_\ell \geqslant -nd_\ell^p \geqslant -\gamma/2mn^{p-1}$, so $\sum_{\ell=0}^{2m-1} s_\ell \geqslant -\gamma/n^{p-1}$, as required. □

Remark 2.5. Equation (2.10) shows that $s_\ell = 0$ for $p = 2$, and in fact one can prove that $s_\ell \geqslant 0$ whenever $p \geqslant 2$, thus rendering the error term ρ_3 superfluous for such p.

We now come to Step 3, which is really the heart of the method, and it is the one where it is essential to work with $\nu_p(\,\cdot\,,\,\cdot\,)^p$ rather than $\nu_p(\,\cdot\,,\,\cdot\,)$ itself. We shall adjoin all intervening integers to the set $A \cup ([0, 2mn] \cap n\mathbb{N}_0)$. This has the effect of reducing $\nu_p(y,\,\cdot\,)^p$, but the reduction is more than offset by an increase in $\nu_p(z,\,\cdot\,)^p$.

Lemma 2.6. *Suppose that $t \geqslant 1$. Then $t^p - t \leqslant (t-1)(t+1)^{p-1}$.*

Proof. For $1 < p \leqslant 2$, we have $t^{p-1} \leqslant t$, hence $t^p - t \leqslant t^p - t^{p-1} = (t-1)t^{p-1}$, which is stronger than the stated inequality. For $p \geqslant 2$, we use the convexity of the function $t \mapsto t^{p-1}$. Since

$$t = \frac{t-1}{t}(t+1) + \frac{1}{t}\cdot 1,$$

we have

$$t^{p-1} \leqslant \frac{t-1}{t}(t+1)^{p-1} + \frac{1}{t},$$

which is again stronger than the stated inequality. □

Step 3: Proof of (2.8). Let $B = A \cup ([0, 2mn] \cap n\mathbb{N}_0)$. Our aim is to prove that

$$\delta := \nu_p\big(y, [0, 2mn]\big)^p + \nu_p\big(z, [0, 2mn]\big)^p - \nu_p(y, B)^p - \nu_p(z, B)^p$$

is non-negative. Writing $B = \{b_1 < b_2 < \cdots < b_{h+1}\}$, we have $\delta = \sum_{j=1}^{h}\big(\Delta_j(y) + \Delta_j(z)\big)$, where

$$\Delta_j(y) = \left(\sum_{i=b_j}^{b_{j+1}-1} \big|y(i) - y(i+1)\big|^p\right) - \big|y(b_j) - y(b_{j+1})\big|^p$$

and $\Delta_j(z)$ is defined similarly. Hence it suffices to prove that $\Delta_j(y) + \Delta_j(z) \geqslant 0$ for each integer $j \in [1, h]$.

The definition of B shows that b_j and b_{j+1} both belong to an interval of the form $[rn, (r+1)n]$ for some $r \in \mathbb{N}_0$. As in the proof of Step 4, this implies that

$$\big|y(i) - y(i+1)\big| = d_r \qquad (b_j \leqslant i < b_{j+1}) \qquad \text{and} \qquad \big|y(b_j) - y(b_{j+1})\big| = \ell_j d_r,$$

where $d_r := \frac{1}{n}\big|x(r) - x(r+1)\big| \leqslant \frac{1}{n}(\gamma/2m)^{1/p}$ and $\ell_j := b_{j+1} - b_j$, so $\Delta_j(y) = (\ell_j - \ell_j^p)d_r^p$. Meanwhile, $\big|z(i) - z(i+1)\big| = c$ for each i, where $c := (\gamma/2mn)^{1/p}$, and $\big|z(b_j) - z(b_{j+1})\big|$ equals 0 if ℓ_j is even and c if ℓ_j is odd, thus in both cases $\Delta_j(z) \geqslant (\ell_j - 1)c^p$.

Now if $\ell_j \leqslant n - 1$, we find

$$\Delta_j(y) + \Delta_j(z) \geqslant (\ell_j - \ell_j^p)d_r^p + (\ell_j - 1)c^p \geqslant \big((\ell_j - \ell_j^p) + (\ell_j - 1)n^{p-1}\big)d_r^p$$

because $c^p \geqslant n^{p-1}d_r^p$. Since $n \geqslant \ell_j + 1$, Lemma 2.6 gives $(\ell_j - 1)n^{p-1} \geqslant \ell_j^p - \ell_j$, hence $\Delta_j(y) + \Delta_j(z) \geqslant 0$, as required. Otherwise $\ell_j = n$, which is assumed even, so that $\Delta_j(z) = nc^p$, and

$$\Delta_j(y) + \Delta_j(z) = (n - n^p)d_r^p + nc^p \geqslant (n - n^p + n^p)d_r^p = nd_r^p > 0. \qquad \square$$

Finally, we reach Step 2 where multiples of n are adjoined to the set A. We require two lemmas, the first of which describes the effect on $\nu_p(\,\cdot\,,A)^p$ of substituting new end points in A, while the second considers the effect of filling in gaps in A.

Lemma 2.7. *Consider integers $\ell \geqslant 3$ and $0 \leqslant c \leqslant b_1 < b_2 < \cdots < b_\ell \leqslant c'$, let $B = \{b_1, b_2, \ldots, b_\ell\}$ and $C = \{c, b_2, \ldots, b_{\ell-1}, c'\}$, and suppose that $v \in J_p$ satisfies*

$$v(c) \leqslant v(b_1) \leqslant v(b_j) \leqslant v(b_\ell) \leqslant v(c') \quad \text{or} \quad v(c) \geqslant v(b_1) \geqslant v(b_j) \geqslant v(b_\ell) \geqslant v(c') \tag{2.11}$$

for $1 < j < \ell$. Then

$$\nu_p\big(v, \{b_1, b_\ell\}\big)^p - \nu_p(v, B)^p \leqslant \nu_p\big(v, \{c, c'\}\big)^p - \nu_p(v, C)^p.$$

Proof. We consider only the case where the first set of inequalities in (2.11) is satisfied; the other case is similar. We replace the end points of B one at a time. Let $D = \{c, b_2, \ldots, b_{\ell-1}, b_\ell\}$. In the sum under consideration, $r := v(b_2) - v(b_1)$ is replaced with $s := v(b_2) - v(c)$, and both are non-negative, so $\nu_p(v, D)^p - \nu_p(v, B)^p = s^p - r^p$. Differentiation shows that the function $t \mapsto (s+t)^p - (r+t)^p$ is increasing on $[0, \infty)$ because $s \geqslant r$, and hence $s^p - r^p \leqslant (s+t)^p - (r+t)^p$ for each $t \geqslant 0$. Taking $t := v(b_\ell) - v(b_2)$, we obtain $s + t = v(b_\ell) - v(c)$ and $r + t = v(b_\ell) - v(b_1)$, so

$$\nu_p(v, D)^p - \nu_p(v, B)^p \leqslant \nu_p\big(v, \{c, b_\ell\}\big)^p - \nu_p\big(v, \{b_1, b_\ell\}\big)^p.$$

A similar argument with $r := v(b_\ell) - v(b_{\ell-1})$, $s := v(c') - v(b_{\ell-1})$ and $t := v(b_{\ell-1}) - v(c)$ shows that

$$\nu_p(v, C)^p - \nu_p(v, D)^p \leqslant \nu_p\big(v, \{c, c'\}\big)^p - \nu_p\big(v, \{c, b_\ell\}\big)^p.$$

Adding these two inequalities, we conclude that

$$\nu_p(v, C)^p - \nu_p(v, B)^p \leqslant \nu_p\big(v, \{c, c'\}\big)^p - \nu_p\big(v, \{b_1, b_\ell\}\big)^p,$$

from which our statement follows. □

Lemma 2.8. *Let $\ell \in \mathbb{N}$, and suppose that $C_1, \ldots, C_\ell$ and $D_1, \ldots, D_\ell$ are finite subsets of $\mathbb{N}_0$ with $\min C_j = \min D_j =: m_j$ and $\max C_j = \max D_j =: m_j'$, where $m_j' \leqslant m_{j+1}$ for each j. Suppose further that $E_1, \ldots, E_{\ell-1}$ are finite subsets of $\mathbb{N}_0$ such that $\min E_j = m_j'$ and $\max E_j = m_{j+1}$ for each j (so E_j is between $C_j \cup D_j$ and $C_{j+1} \cup D_{j+1}$), and let $E_\ell = \{m_\ell'\}$. Then*

$$\sum_{j=1}^{\ell} \big(\nu_p(v, D_j)^p - \nu_p(v, C_j)^p\big) = \nu_p\Big(v, \bigcup_{j=1}^{\ell} (D_j \cup E_j)\Big)^p - \nu_p\Big(v, \bigcup_{j=1}^{\ell} (C_j \cup E_j)\Big)^p \qquad (v \in J_p).$$

Proof. Clearly, we have

$$\nu_p\Big(v, \bigcup_{j=1}^{\ell} (C_j \cup E_j)\Big)^p = \sum_{j=1}^{\ell} \nu_p(v, C_j)^p + \sum_{j=1}^{\ell} \nu_p(v, E_j)^p,$$

which together with the corresponding formula for $\nu_p\big(v, \bigcup_{j=1}^{\ell} (D_j \cup E_j)\big)^p$ gives the result.
□

Step 2: Proof of (2.7), with $\rho_2 = 2\varepsilon$. Let $N = [0, 2mn] \cap n\mathbb{N}_0$. The effect on z of adjoining elements to the set $A = \{a_1 < \cdots < a_{k+1}\}$ is easily seen. Let $\ell_i = a_{i+1} - a_i$ for $1 \leqslant i \leqslant k$. As in the proof of Step 3 above, $\bigl|z(a_i) - z(a_{i+1})\bigr|$ is $c := (\gamma/2mn)^{1/p}$ if ℓ_i is odd, and 0 if ℓ_i is even. If ℓ_i is odd and new points are inserted between a_i and a_{i+1}, then at least one of the new intervals, say $[b_j, b_{j+1}]$, has odd length, so $\bigl|z(b_j) - z(b_{j+1})\bigr| = c$. Hence

$$\nu_p(z, A) \leqslant \nu_p(z, A \cup N).$$

We shall now prove the corresponding inequality for y, just with an error term added on the right-hand side. Recall that $a_1 = 0$ and $a_{k+1} = 2mn$. Note that if there is some $b \in N$ such that $a_i < b < a_{i+1}$ for some i and either $y(b) < \min\bigl\{y(a_i), y(a_{i+1})\bigr\}$ or $y(b) > \max\bigl\{y(a_i), y(a_{i+1})\bigr\}$, then $\bigl|y(a_i) - y(b)\bigr|^p + \bigl|y(b) - y(a_{i+1})\bigr|^p > \bigl|y(a_i) - y(a_{i+1})\bigr|^p$. Hence we may adjoin any such points b to the set A, thereby increasing $\nu_p(y, A)$ without changing $A \cup N$; we still use the notation $A = \{a_1 < \cdots < a_{k+1}\}$ for the augmented set.

Let the intervals $[a_i, a_{i+1}]$ $(1 \leqslant i \leqslant k)$ that contain at least one multiple of n be relabelled $[b_j, b_j']$ $(1 \leqslant j \leqslant h)$ and ordered increasingly; that is, $b_1 < b_1' \leqslant b_2 < b_2' \leqslant \cdots \leqslant b_h < b_h'$. Note that $b_1 = a_1 = 0$ and $b_h' = a_{k+1} = 2mn$, and that b_j' may or may not be equal to b_{j+1} for $1 \leqslant j \leqslant h-1$. Then, with $B_j := \bigl([b_j, b_j'] \cap n\mathbb{N}\bigr) \cup \{b_j, b_j'\}$ for $1 \leqslant j \leqslant h$, we obtain

$$\nu_p(y, A)^p - \nu_p(y, A \cup N)^p = \sum_{j=1}^{h} \Bigl(\nu_p\bigl(y, \{b_j, b_j'\}\bigr)^p - \nu_p(y, B_j)^p\Bigr). \tag{2.12}$$

Let $c_j = \max\bigl([0, b_j] \cap n\mathbb{N}_0\bigr)$ and $c_j' = \min\bigl([b_j', \infty) \cap n\mathbb{N}_0\bigr)$, and let $C_j = [c_j, c_j'] \cap n\mathbb{N}_0$. Then $c_1 = 0$ and $c_h' = 2mn$, and we have

$$\nu_p\bigl(y, \{b_j, b_j'\}\bigr)^p - \nu_p(y, B_j)^p \leqslant \nu_p\bigl(y, \{c_j, c_j'\}\bigr)^p - \nu_p(y, C_j)^p \qquad (1 \leqslant j \leqslant h) \tag{2.13}$$

by Lemma 2.7, which applies because the augmentation of the set A carried out in the previous paragraph ensures that y satisfies the hypothesis (2.11). Indeed, it is clear that $y(b)$ lies between $y(b_j)$ and $y(b_j')$ for each $b \in B_j$. To check the remaining inequalities concerning the values of y at the points c_j and c_j', let us for definiteness consider the case where $y(b_j) \leqslant y(b_j')$ and explain why $y(c_j) \leqslant y(b_j)$; the other cases are similar. The inequality is obvious if $c_j = b_j$. Otherwise we write $c_j = rn$, where $r \in \mathbb{N}_0$, and note that $(r+1)n > b_j$ by the definition of c_j. Since $[b_j, b_j'] \cap n\mathbb{N}_0 \neq \emptyset$, we conclude that $(r+1)n \leqslant b_j'$, so the augmentation of A implies that $y(b_j) \leqslant y\bigl((r+1)n\bigr) \leqslant y(b_j')$. Now recall that $y = T_n x$, so $y(b_j)$ is found by interpolation between $x(r) = y(rn) = y(c_j)$ and $x(r+1) = y\bigl((r+1)n\bigr)$. Since $y\bigl((r+1)n\bigr) \geqslant y(b_j)$, we must therefore have $y(c_j) \leqslant y(b_j)$, as required.

We next seek to invoke Lemma 2.8 with the sets $\{c_j, c_j'\}$ playing the role of the D_j's. To do so, we require some more notation. Let $c_0 = c_0' = 0$, $C_0 = \{0\}$, $c_{h+1} = c_{h+1}' = 2mn$ and $C_{h+1} = \{2mn\}$. Then clearly $\min\{c_j, c_j'\} = \min C_j = c_j$ and $\max\{c_j, c_j'\} = \max C_j = c_j'$ for each integer $j \in [0, h+1]$, but $c_j' \leqslant c_{j+1}$ need not be satisfied for each $j \in [1, h]$. It is, however, true that $c_j' \leqslant c_{j+2}$ for each $j \in [0, h-1]$ because the interval $[b_{j+1}, b_{j+1}']$ contains a multiple of n. Hence, taking $E_j = [c_j', c_{j+2}] \cap n\mathbb{N}_0$ for $0 \leqslant j \leqslant h-1$ and letting

$E_h = E_{h+1} = \{2mn\}$, we can apply Lemma 2.8 for even and odd indices j separately. We observe that $C_j \cup E_j = [c_j, c_{j+2}] \cap n\mathbb{N}_0$ for $0 \leqslant j \leqslant h-1$, so $\bigcup_{j\in\Gamma_r}(C_j \cup E_j) = N$ for $r \in \{0,1\}$, where Γ_0 and Γ_1 denote the sets of even and odd integers in $[0, h+1]$, respectively. Thus Lemma 2.8 gives

$$\sum_{j\in\Gamma_r}\Bigl(\nu_p\bigl(y,\{c_j,c_j'\}\bigr)^p - \nu_p(y,C_j)^p\Bigr) = \nu_p\Bigl(y, \bigcup_{j\in\Gamma_r}(\{c_j,c_j'\}\cup E_j)\Bigr)^p - \nu_p(y,N)^p. \tag{2.14}$$

Since $y = T_n x$ and $N = [0, 2mn] \cap n\mathbb{N}_0$, we have $\nu_p(y,N)^p = \nu_p\bigl(x,[0,2m]\bigr)^p \geqslant \|x\|_{J_p}^p - \varepsilon$ by (2.1), while $\nu_p\bigl(y,\bigcup_{j\in\Gamma_r}(\{c_j,c_j'\}\cup E_j)\bigr)^p \leqslant \|y\|_{J_p}^p = \|x\|_{J_p}^p$. Hence the sum on the left-hand side of (2.14) is no greater than ε, so adding the two cases ($r=0$ and $r=1$) and using (2.12) and (2.13), we conclude that

$$\begin{aligned}\nu_p(y,A)^p - \nu_p(y,A\cup N)^p &\leqslant \sum_{j=1}^{h}\Bigl(\nu_p\bigl(y,\{c_j,c_j'\}\bigr)^p - \nu_p(y,C_j)^p\Bigr)\\ &= \sum_{j=0}^{h+1}\Bigl(\nu_p\bigl(y,\{c_j,c_j'\}\bigr)^p - \nu_p(y,C_j)^p\Bigr) \leqslant 2\varepsilon. \qquad \square\end{aligned}$$

Completion of the proof of Lemma 2.1. With the four steps completed, it is clear that Lemma 2.1 holds with

$$\varphi(m,n) = 2^p\Bigl(\psi(m,n) + \frac{1}{n^{1-1/p}}\Bigr) + \frac{1}{n^{p-1}},$$

where

$$\psi(m,n) = \begin{cases} \dfrac{1}{n^{p-2+1/p}} & \text{for } 1 < p \leqslant 2\\[2ex] \dfrac{(2m)^{p-2}}{n^{1/p}} & \text{for } p > 2.\end{cases}$$

Note that m does not appear in the case $p \leqslant 2$, and that $p - 2 + 1/p > 0$, so in both cases $\varphi(m,n) \to 0$ as $n \to \infty$ with m fixed. $\square$

3 Proof of Theorem 1.2

We begin with an elementary observation which is tailored to reduce Theorem 1.2 to the statement that the Banach spaces G_p and $J_p^{(\infty)}$ given by (1.2) are non-isomorphic. A closely related result can be found in [12, Proposition 5.3.8].

Lemma 3.1. *Let X, Y and Z be Banach spaces satisfying:*

(i) *X contains a complemented subspace isomorphic to Y;*

(ii) *Y contains a complemented subspace isomorphic to Z;*

(iii) *$Y \cong Y \oplus Y$ and $Z \cong Z \oplus Z$.*

Then $\overline{\mathscr{G}}_Z(X) \subseteq \overline{\mathscr{G}}_Y(X)$, with equality if and only if $Z \cong Y$.

Proof. The inclusion $\overline{\mathscr{G}}_Z(X) \subseteq \overline{\mathscr{G}}_Y(X)$ is clear, as is the equality of these two ideals in the case where $Z \cong Y$.

Conversely, suppose that $\overline{\mathscr{G}}_Z(X) = \overline{\mathscr{G}}_Y(X)$, and let P be a projection on X with $P(X) \cong Y$. Clearly P factors through Y, so P belongs to $\overline{\mathscr{G}}_Z(X)$ by the assumption. It then follows from standard results that Z contains a complemented subspace isomorphic to Y (*e.g.*, see [9, Proposition 3.4 and Lemma 3.6(ii)] for details), and therefore Y and Z are isomorphic by the Pełczyński decomposition method. $\square$

We shall next record the facts required to invoke Lemma 3.1 in the proof of Theorem 1.2.

Lemma 3.2. *For each $p \in (1, \infty)$,*

(i) *G_p contains a complemented subspace isomorphic to ℓ_p;*

(ii) *$J_p^{(\infty)}$ contains a complemented subspace isomorphic to G_p;*

(iii) *J_p contains a complemented subspace isomorphic to $J_p^{(\infty)}$;*

(iv) *$\ell_p \cong \ell_p \oplus \ell_p$, $G_p \cong G_p \oplus G_p$ and $J_p^{(\infty)} \cong J_p^{(\infty)} \oplus J_p^{(\infty)}$.*

Proof. All but one of these results are well known. The exception is (ii) which, however, follows from Theorem 1.1 in exactly the same way as the corresponding result for $p = 2$ is deduced from the original Giesy–James theorem in [1, Theorem 13(i)].

References for the other statements are as follows; (i) and the first part of (iv) are obvious, while (iii) and the remaining two parts of (iv) follow from [4, Lemmas 5 and 6]. (A key condition appears to be missing in the statement of [4, Lemma 5], though, namely that the sequence denoted by ν is unbounded.) $\square$

Remark 3.3. Let X and Y be Banach spaces. An operator $T\colon X \to Y$ is *bounded below* by $\varepsilon > 0$ if $\|Tx\| \geqslant \varepsilon\|x\|$ for each $x \in X$. In this case T is an isomorphism onto its image, and the inverse operator has norm at most ε^{-1}, so in particular the Banach–Mazur distance d_{BM} between the domain X and the image $T(X)$ of T satisfies

$$d_{\mathrm{BM}}\bigl(X, T(X)\bigr) \leqslant \frac{\|T\|}{\varepsilon}.$$

Now suppose that X is a closed subspace of Y and that $T\colon X \to Y$ is linear and satisfies

$$\|x - Tx\| \leqslant \eta\|x\| \qquad (x \in X)$$

for some $\eta \in (0, 1)$. Then we have $(1-\eta)\|x\| \leqslant \|Tx\| \leqslant (1+\eta)\|x\|$ for each $x \in X$, so by the previous paragraph T is an isomorphism onto its image, and

$$d_{\mathrm{BM}}\bigl(X, T(X)\bigr) \leqslant \frac{1+\eta}{1-\eta}.$$

Definition 3.4. Let F be a finite-dimensional Banach space. The *unconditional basis constant* of a basis $\mathbf{b} = \{b_1, \ldots, b_n\}$ for F is given by

$$K_{\mathbf{b}} := \sup\biggl\{\Bigl\|\sum_{j=1}^{n} \alpha_j\beta_j b_j\Bigr\| : \alpha_j, \beta_j \in \mathbb{K},\ |\alpha_j| \leqslant 1\ (j = 1, \ldots, n),\ \Bigl\|\sum_{j=1}^{n} \beta_j b_j\Bigr\| \leqslant 1\biggr\}.$$

The infimum of the unconditional basis constants of all possible bases for F is the *unconditional constant* of F; we denote it by $\mathrm{uc}(F)$.

It is easy to verify that, for Banach spaces E and F of the same finite dimension, we have

$$\mathrm{uc}(E) \leqslant d_{\mathrm{BM}}(E, F)\,\mathrm{uc}(F). \tag{3.1}$$

Definition 3.5. (Dubinsky, Pełczyński and Rosenthal [3, Definition 3.1].) Let $C \in [1, \infty)$. A Banach space X has *local unconditional structure* (or *l.u.st.* for short) with *constant* at most C if each finite-dimensional subspace of X is contained in some larger finite-dimensional subspace F of X with $\mathrm{uc}(F) \leqslant C$.

A Banach space with an unconditional basis has l.u.st. This applies in particular to G_p. On the other hand, Johnson and Tzafriri [8, Corollary 2] have shown that no quasi-reflexive Banach space has l.u.st. We shall use this result to prove that $J_p^{(\infty)}$ does not have l.u.st.

We begin with a generalization of the above-mentioned fact that every Banach space with an unconditional basis has l.u.st. This result is probably well-known to specialists, but as we have been unable to locate a reference, we include a proof.

Lemma 3.6. *Let X be a Banach space with a Schauder basis $(b_n)_{n\in\mathbb{N}_0}$, and let $C \in [1, \infty)$. Suppose that X contains a sequence $(F_n)_{n\in\mathbb{N}_0}$ of finite-dimensional subspaces satisfying*

$$b_0, b_1, \ldots, b_n \in F_n \qquad \textit{and} \qquad \mathrm{uc}(F_n) \leqslant C \qquad (n \in \mathbb{N}_0). \tag{3.2}$$

Then X has l.u.st. with constant at most $C + \delta$ for each $\delta > 0$.

Proof. Take $\varepsilon \in (0, \frac{1}{2})$ such that $C/(1-2\varepsilon) < C+\delta$, and let E be a k-dimensional subspace of X for some $k \in \mathbb{N}$. Approximation of each vector of an Auerbach basis for E shows that, for each $\eta > 0$, there is $M \in \mathbb{N}_0$ such that

$$\|x - P_m x\| \leqslant \eta\|x\| \qquad (m \geqslant M,\ x \in E), \tag{3.3}$$

where P_m denotes the m^{th} basis projection associated with $(b_n)_{n\in\mathbb{N}_0}$. Applying this conclusion with $\eta > 0$ chosen such that

$$\frac{\eta\sqrt{k}}{1-\eta} \leqslant \frac{\varepsilon}{1-\varepsilon}, \tag{3.4}$$

we obtain by Remark 3.3 that the operator $U\colon x \mapsto P_M x,\ E \to P_M(E)$, is an isomorphism with $\|U\| \leqslant 1 + \eta$ and $\|U^{-1}\| \leqslant (1 - \eta)^{-1}$.

Since $U(E) = P_M(E) \subseteq \operatorname{span}\{b_0, b_1, \ldots, b_M\} \subseteq F_M$ and $\dim U(E) = k$, we can find a projection Q on F_M such that $Q(F_M) = U(E)$ and $\|Q\| \leqslant \sqrt{k}$ by the Kadec–Snobar theorem (*e.g.*, see [2, Theorem 4.18]). The operator $T\colon x \mapsto x - Qx + U^{-1}Qx,\ F_M \to X$, then satisfies

$$\|x - Tx\| = \|Qx - U^{-1}Qx\| = \|P_M U^{-1}Qx - U^{-1}Qx\| \leqslant \eta\|U^{-1}Qx\| \leqslant \eta\|U^{-1}\|\,\|Q\|\,\|x\|,$$

where the penultimate estimate follows from (3.3), and hence $\|x - Tx\| \leqslant \varepsilon(1-\varepsilon)^{-1}\|x\|$ for each $x \in F_M$ by (3.4). Since $\varepsilon(1-\varepsilon)^{-1} < 1$, Remark 3.3 implies that T is an isomorphism onto its image, and

$$d_{\mathrm{BM}}\bigl(F_M, T(F_M)\bigr) \leqslant \frac{1 + \varepsilon(1-\varepsilon)^{-1}}{1 - \varepsilon(1-\varepsilon)^{-1}} = \frac{1}{1-2\varepsilon},$$

so $\operatorname{uc}\bigl(T(F_M)\bigr) \leqslant C/(1-2\varepsilon) \leqslant C + \delta$ by (3.1).

The conclusion now follows because $E \subseteq T(F_M)$. Indeed, for each $x \in E$, $y := Ux$ belongs to F_M and satisfies $Qy = y$, so that

$$T(F_M) \ni Ty = y - Qy + U^{-1}y = x,$$

as desired. (In fact, an easy dimension argument shows that $T(F_M) = \ker Q + E$.) $\square$

Proposition 3.7. *The Banach space $J_p^{(\infty)}$ does not have l.u.st. for any $p \in (1,\infty)$.*

Proof. Assume towards a contradiction that $J_p^{(\infty)}$ has l.u.st. with constant at most $C \geqslant 1$ for some $p \in (1,\infty)$, and let $n \in \mathbb{N}_0$. Denote by $\iota_n\colon J_p^{(n)} \to J_p^{(\infty)}$ and $\rho_n\colon J_p^{(\infty)} \to J_p^{(n)}$ the canonical n^{th} coordinate embedding and projection, respectively, and let $j_n\colon J_p^{(n)} \to J_p$ be the natural inclusion operator. By assumption, $\iota_n(J_p^{(n)})$ is contained in some finite-dimensional subspace F_n of $J_p^{(\infty)}$ with $\operatorname{uc}(F_n) \leqslant C$.

Let $R_n\colon J_p \to J_p$ be the $(n+2)$-fold right shift given by $R_n e_k = e_{n+k+2}$ for each $k \in \mathbb{N}_0$. This defines an operator of norm $2^{1/p}$ on J_p, and R_n is bounded below by 1. Lemma 3.2(iii) implies that there are operators $U \in \mathscr{B}(J_p^{(\infty)}, J_p)$ and $V \in \mathscr{B}(J_p, J_p^{(\infty)})$ such that $VU = I_{J_p^{(\infty)}}$; we may clearly suppose that V has norm 1. We shall now consider the operator $S_n := j_n\rho_n + R_nU(I_{J_p^{(\infty)}} - \iota_n\rho_n) \in \mathscr{B}(J_p^{(\infty)}, J_p)$. The obvious norm estimates show that $\|S_n\| \leqslant 1 + 2^{1/p}\,\|U\|$. To prove that S_n is bounded below by 1, let $x \in J_p^{(\infty)}$ and $\varepsilon > 0$ be given. Introducing $y := (I_{J_p^{(\infty)}} - \iota_n\rho_n)x \in J_p^{(\infty)}$, we obtain

$$\|x\|^p_{J_p^{(\infty)}} = \|\rho_n x\|^p_{J_p} + \|y\|^p_{J_p^{(\infty)}} = \|j_n\rho_n x\|^p_{J_p} + \|VUy\|^p_{J_p^{(\infty)}} \leqslant \|j_n\rho_n x\|^p_{J_p} + \|R_nUy\|^p_{J_p} \tag{3.5}$$

because $\|V\| = 1$ and R_n is bounded below by 1. Since $j_n\rho_n x \in \operatorname{span}\{e_0, e_1, \ldots, e_n\}$, there is a subset A of $[0, n+1] \cap \mathbb{N}_0$ such that $\|j_n\rho_n x\|_{J_p} = \nu_p(j_n\rho_n x, A)$. Similarly, as $R_nUy \in \overline{\operatorname{span}}\,\{e_{n+2}, e_{n+3}, \ldots\}$, we can find a finite subset B of $[n+1,\infty) \cap \mathbb{N}$ such that $\|R_nUy\|^p_{J_p} \leqslant \nu_p(R_nUy, B)^p + \varepsilon$. Combining these identities with (3.5), we conclude that

$$\|x\|^p_{J_p^{(\infty)}} - \varepsilon \leqslant \nu_p(j_n\rho_n x, A)^p + \nu_p(R_nUy, B)^p \leqslant \nu_p\bigl(j_n\rho_n x + R_nUy, A \cup B\bigr)^p \leqslant \|S_n x\|^p_{J_p},$$

and letting ε tend to 0, we see that S_n is bounded below by 1, as stated.

Thus Remark 3.3 implies that $d_{\mathrm{BM}}\big(F_n, S_n(F_n)\big) \leqslant \|S_n\| \leqslant 1 + 2^{1/p}\|U\|$, and therefore $\mathrm{uc}\big(S_n(F_n)\big) \leqslant C\big(1 + 2^{1/p}\|U\|\big)$ by (3.1). Moreover, for each $k \in \{0, 1, \ldots, n\}$, we have $\iota_n e_k \in F_n$, so that $S_n(F_n) \ni S_n(\iota_n e_k) = e_k$ because $j_n \rho_n \iota_n e_k = e_k$ and $(I_{J_p^{(\infty)}} - \iota_n \rho_n)\iota_n = 0$. Hence the sequence $\big(S_n(F_n)\big)_{n\in\mathbb{N}_0}$ satisfies both parts of (3.2), so Lemma 3.6 implies that J_p has l.u.st., contradicting the above-mentioned theorem of Johnson and Tzafriri that this is impossible for a quasi-reflexive Banach space. $\square$

Corollary 3.8. *The Banach spaces G_p and $J_p^{(\infty)}$ are not isomorphic for any $p \in (1, \infty)$.*

Proof. This is clear because, as remarked above, G_p has an unconditional basis and thus l.u.st., whereas $J_p^{(\infty)}$ does not by Proposition 3.7. $\square$

The *proof of Theorem* 1.2 is now easy. Recall that $\mathscr{W}(J_p) = \overline{\mathscr{G}}_{J_p^{(\infty)}}(J_p)$. The inclusions $\overline{\mathscr{G}}_{\ell_p}(J_p) \subsetneq \overline{\mathscr{G}}_{G_p}(J_p)$ and $\overline{\mathscr{G}}_{G_p}(J_p) \subsetneq \overline{\mathscr{G}}_{J_p^{(\infty)}}(J_p)$ both follow from Lemma 3.1, which applies by Lemma 3.2 and the facts that $\ell_p \ncong G_p$ and $G_p \ncong J_p^{(\infty)}$. The second of these facts was proved in Corollary 3.8, while the first can be justified in various ways; for instance, ℓ_p is uniformly convex with type $\min\{2, p\}$ and cotype $\max\{2, p\}$, whereas G_p is not uniformly convexifiable, has type 1 and fails to have finite cotype. $\square$

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