Complexity results for the gap inequalities for the max-cut problem

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A R T I C L E   I N F O
Article history:
Received 19 July 2011
Accepted 31 October 2011
Available online 28 January 2012

Keywords:
Computational complexity
Max-cut problem
Cutting planes

A B S T R A C T
We prove several complexity results about the gap inequalities for the max-cut problem, including (i) the gap-1 inequalities do not imply the other gap inequalities, unless \( NP = Co-NP \); (ii) there must exist non-redundant gap inequalities with exponentially large coefficients, unless \( NP = Co-NP \); (iii) the associated separation problem can be solved in finite (doubly exponential) time.

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1. Introduction

Given an edge-weighted undirected graph, the max-cut problem calls for a partition of the vertex set into two subsets, such that the total weight of the edges having exactly one end-vertex in each subset is maximized. The max-cut problem is a fundamental and well-known combinatorial optimization problem, proven to be strongly \( NP \)-hard in [12]. It has a surprisingly large number of important practical applications, and has received a great deal of attention (see, e.g., the book [9] and the survey [17]).

It is usual in combinatorial optimization to formulate a problem as a zero-one linear program, and then derive strong linear inequalities that must be satisfied by all feasible solutions. Such inequalities can then be exploited algorithmically within a branch-and-cut framework (see, e.g., [6]). A wide array of such inequalities have been discovered for the max-cut problem (see again [9]). In particular, Laurent and Poljak [18] introduced an intriguing class of inequalities, known as gap inequalities, which includes several other known classes as special cases.

Unfortunately, computing the right-hand side of a gap inequality is itself an \( NP \)-hard problem [18]. Perhaps for this reason, the gap inequalities have received very little attention in the literature. The present paper is concerned with certain complexity aspects of gap inequalities.

We assume throughout the paper that the reader is familiar with the fundamental concepts of computational complexity; in particular, the definition of the classes \( NP \) and Co-\( NP \) of decision problems (see, e.g., [11]). We also use the term extreme in several places. An inequality in a given class is said to be extreme if it cannot be expressed as a non-negative linear combination of two or more other inequalities in that class.

The structure of the paper is as follows. In Section 2, the relevant literature is reviewed. In Section 3, several results are proved concerned with the complexity of the coefficients that an extreme gap inequality can have. Then, in Section 4, some results are proved concerned with the complexity of the separation problem for gap inequalities and some of their special cases. Some open problems are also mentioned.

2. Literature review

Let \( G = (V, E) \) be an undirected graph. For any \( S \subseteq V \), the set of edges having exactly one end-vertex in \( S \) is called an edge-cutset or cut, and denoted by \( \delta(S) \). A vector \( x \in \{0, 1\}^{\frac{n(n-1)}{2}} \) is the incidence vector of a cut in the complete graph \( K_n \) if and only if it satisfies the following triangle inequalities:

\[ x_{ij} + x_{ik} + x_{jk} \leq 2 \quad (1 \leq i < j < k \leq n) \]
\[ x_{ij} - x_{ik} - x_{jk} \leq 0 \quad (1 \leq i < j \leq n; k \neq i, j). \]

The cut polytope, which we will denote by \( \text{CUT}_n \), is the convex hull in \( \mathbb{R}^{\frac{n(n-1)}{2}} \) of such incidence vectors [4].

Many classes of strong valid inequalities have been discovered for \( \text{CUT}_n \); see again [9,17]. Here, we are interested in the gap inequalities of Laurent and Poljak [18], which take the following form:

\[ \sum_{1 \leq i < j \leq n} b_{ij}x_{ij} \leq (\sigma(b))^2 - \gamma(b) |b|^2 / 4 \quad (\forall b \in \mathbb{Z}^n). \]

Here, \( \sigma(b) \) denotes \( \sum_{i \in V} b_i \), and

\[ \gamma(b) := \min\{|z|^2 | z \in \{\pm 1\}^n\} \]

is the so-called gap of \( b \).
Theorem 3 establishes a one-to-one correspondence between rounded psd inequalities for CUTₙ and hypermetric inequalities for CUTₙ₊₁. One can check that this correspondence preserves the property of being extreme. The result then follows from Theorem 2.

Lemma 1. If a rounded psd inequality is extreme, then the encoding length of the corresponding b-vector is bounded by a polynomial in n.

Proof. Theorem 3 establishes a one-to-one correspondence between rounded psd inequalities for CUTₙ and hypermetric inequalities for CUTₙ₊₁. One can check that this correspondence preserves the property of being extreme. The result then follows from Theorem 2.

Lemma 2. The following decision problem is \( \mathcal{N} \mathcal{P} \)-complete: given positive integers n and k and a vector \( b \in \mathbb{Z}^n \), is \( \gamma(b) < k \).

Proof. To show that \( \gamma(b) < k \), it suffices to exhibit a set \( S \subset V \) such that \( \sum_{i \in S} b_i - \sum_{i \in V \setminus S} b_i < k \). Therefore the problem lies in \( \mathcal{N} \mathcal{P} \).

Corollary 1. Suppose that every gap inequality is a non-negative linear combination of one or more rounded psd inequalities. Then \( \mathcal{N} \mathcal{P} = \mathcal{Co} \mathcal{N} \mathcal{P} \).

Proof. The inequality (5) is either a gap inequality or implied by a gap inequality. So, if the statement were true, the inequality (5) would be implied by rounded psd inequalities. In particular, by Carathéodory's theorem, there would exist a set of at most \( \binom{n}{2} \) extreme rounded psd inequalities that collectively implied the inequality (5). Now, Lemma 1 implies that, for each of those rounded psd inequalities, the corresponding b-vector would be a short certificate of validity. Thus, we would have a short certificate for a Co \( \mathcal{N} \mathcal{P} \)-complete problem, and \( \mathcal{N} \mathcal{P} \) would equal Co \( \mathcal{N} \mathcal{P} \).

Since the gap-1 inequalities are a special case of the rounded psd inequalities, Theorem 4 has the following corollary.

Corollary 1. Suppose that every gap inequality is a non-negative linear combination of one or more gap-1 inequalities. Then \( \mathcal{N} \mathcal{P} = \mathcal{Co} \mathcal{N} \mathcal{P} \).

Before presenting our second main result, we need the following lemma.

Lemma 4. Let \( ||b||_1 \) denote \( \sum_{i \in V} |b_i| \). One can compute \( \gamma(b) \) in \( O(n||b||_1) \) time.
Proof. To compute $\gamma(b)$, it suffices to solve the subset-sum problem
$$\text{SSP} = \max \left\{ \sum_{i \in V} |b_i|y_i : \sum_{i \in V} |b_i|y_i \leq \|b\|_1/2, y \in \{0, 1\}^n \right\},$$
and then set $\gamma(b) = \|b\|_1 - 2 \text{SSP}$. This subset-sum problem can be solved in $O(n\|b\|_1)$ time with dynamic programming [5]. □

Armed with this lemma, we can prove our second main result, which essentially states that there should exist extreme gap inequalities with ‘large’ coefficients.

Theorem 5. Suppose there exists a polynomial $p(n)$ such that every extreme gap inequality satisfies $\|b\|_1 \leq p(n)$. Then $\mathcal{P} = \mathcal{NP}$. □

Proof. The proof is similar to that of Theorem 4. The difference is that the inequality (5) would be implied by a set of $O(n^2)$ extreme gap inequalities, each with $\|b\|_1 \leq p(n)$. In light of Lemma 4, the $b$-vectors associated with these gap inequalities would provide a short certificate of validity. □

4. On the complexity of separation

The separation problem, for a given class of valid inequalities, is the problem of detecting when an inequality in that class is violated by some given input vector $x^*$. [14] The separation problem for triangle inequalities can be solved in $O(n^3)$ time by mere enumeration, and polynomial-time separation algorithms are known for psd inequalities [14,19] and negative-type inequalities [9]. To our knowledge, the complexity of separation for the remaining inequalities in Fig. 1 is unknown (even if Theorem 3 implies that rounded psd separation can be reduced to hypermetric separation).

The following two lemmas are relevant to the complexity of the separation problem for gap-1 inequalities.

Lemma 5. If a gap-1 inequality is extreme, then the encoding length of the corresponding $b$-vector is bounded by a polynomial in $n$.

Proof. It is well known and easy to see that the gap-1 inequalities are nothing but the inequalities that can be obtained from hypermetric inequalities by the switching operation, mentioned in Theorem 1. Moreover, if a gap-1 inequality is extreme, it will be a switching of an extreme hypermetric inequality. The result then follows from Theorem 2. □

Lemma 6. The following problem is in $\mathcal{NP}$: ‘Given an integer $n \geq 2$ and a vector $x^* \in \{0, 1\}^{\binom{n}{2}}$, does $x^*$ violate a gap-1 inequality?’.

Proof. If $x^*$ violates a gap-1 inequality, then it violates an extreme gap-1 inequality. From Lemma 5, the encoding length of the associated $b$ vector is polynomially bounded. This $b$ vector, along with a set $S$ such that $\sum_{i \in S} b_i - \sum_{i \in V \setminus S} b_i = 1$, constitutes a short certificate of validity of the gap-1 inequality, and therefore also of violation. □

In fact, it is possible to formulate the separation problem for gap-1 inequalities as an Integer Quadratic Program (IQP) of ‘small’ size:

Theorem 6. The separation problem for gap-1 inequalities can be formulated as an IQP with $\mathcal{O}(n)$ variables and $\mathcal{O}(n)$ constraints.

Proof. Let $x^* \in \{0, 1\}^{\binom{n}{2}}$ be the point to be separated, and let $U$ be an upper bound on the value of $\|b\|_1$ implied by Lemma 5. From the form of the gap inequality (3), a violated gap-1 inequality exists if and only if the solution to the following optimization problem has a cost of less than 1:

$$\min \left\{ \sum_{i=1}^{n} b_i^2 + \sum_{1 \leq i < j \leq n} (2 - 4x_{ij})b_ib_j : \gamma(b) = 1, b \in [-U, U]^n \cap \mathbb{Z}^n \right\}.$$
solution. Now, observe that the condition \( \gamma(b) \geq 1 \) is equivalent to an exponential number of disjunctions:

\[
\left( \sum_{i \in S} b_i - \sum_{i \notin V \setminus S} b_i \geq 1 \right) \lor \left( \sum_{i \in S} b_i - \sum_{i \notin V \setminus S} b_i \leq -1 \right) \quad (\forall S \subseteq V).
\]

Accordingly, we let \( \mathcal{F} = 2^V \) denote the family of all possible sets \( S \subseteq V \). To solve the separation problem, it suffices to solve the following Convex Quadratic Program (CQP) for all subsets \( \mathcal{F}' \subseteq \mathcal{F} \):

\[
\begin{align*}
\min & \quad \sum_{i=1}^{n} b_i^2 + \sum_{1 \leq i < j \leq n} (2 - 4x_i^* b_i) b_j \\
\text{s.t.} & \quad \sum_{i \in S} b_i - \sum_{i \in V \setminus S} b_i \geq 1 \quad (\forall S \in \mathcal{F}') \\
& \quad \sum_{i \in S} b_i - \sum_{i \in V \setminus S} b_i \leq -1 \quad (\forall S \in \mathcal{F} \setminus \mathcal{F}') \\
& \quad b \in \mathbb{Q}^n.
\end{align*}
\]

Each of these CQP instances can be solved in finite time using, e.g., the simplex method of Wolfe [22].

Observe that the running time of this algorithm is doubly exponential. We leave it as an open question whether an algorithm can be devised whose running time is singly exponential. In any case, it is clear that fast heuristics for separation would be essential if one wished to use gap inequalities as cutting planes in an exact algorithm for the max-cut problem and related problems. We hope to address this issue in a future paper.

To close, we mention two other open questions. The first is whether the gap inequalities define a polyhedron. (It is known that the hypermetric and rounded psd inequalities define polyhedra [8, 20], whereas the negative type and psd inequalities do not [9, 19].) The second is whether there exists a gap inequality with \( \gamma(b) > 1 \) that induces a facet of \( \text{CUT}_n \). (As mentioned in Section 2, it is conjectured in [18] that no such inequality exists.)

Acknowledgment

The second and third authors were supported by the Engineering and Physical Sciences Research Council (EPSRC), under grants EP/F033613/1 and EP/D072662/1, respectively.

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