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Economic hysteresis effects and hitting time densities for CIR diffusions

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Economic Hysteresis Effects and Hitting Time Densities for CIR Diffusions

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Abstract

Using the so-called mean-reverting square-root process of Cox et al. (1985b) we generalize the work of Dias and Shackleton (2005) by introducing the mean reversion feature into the economic hysteresis analysis under stochastic interest rates and show that such issue highlights a tendency for a widening effect on the range of inaction, though both thresholds have risen when compared with the no mean-reverting case. In addition, using the work of Linetsky (2004) we compute the hitting time densities in order to have an idea of how long does it take for a current interest rate to revert and hit the investment thresholds that would induce idle firms to invest.

Keywords: real options, interest rate uncertainty, perpetuities, investment hysteresis, mean reversion, hitting time densities.
JEL Classification: G31; D92; D81; C61.

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1 Introduction

Decisions made in an uncertain economic environment where it is costly to reverse investment decisions will lead to an intermediate range of the state variable, known as the hysteretic band, where inaction is the optimal policy. Several models of entry and exit decisions have shown that the range of inaction can be remarkably large for many economic applications [see, for example, Brennan and Schwartz (1985), Dixit (1989a,b) and Abel and Eberly (1996)]. In such literature it is usually assumed that discount rates remain constant. However, since interest rates are also an important determinant of investment and disinvestment decisions it is important to analyze the economic hysteresis effect under stochastic interest rates.

In a very recent paper, Dias and Shackleton (2005) analyze the economic hysteresis problem under stochastic interest rates in a CIR economy, but using the so-called single-factor pure diffusion process thus ignoring the mean reversion effect in order to gain simplicity. More specifically, they generalize the work of Ingersoll and Ross (1992) in two ways. Firstly, they include real options on perpetuities, in addition to zero coupon cash flows. Secondly, they incorporate abandonment as well as investment options and thus model the interest rate hysteresis problem. However, the empirical evidence on interest rate behaviour seems to indicate that interest rates are pulled back to some long-run mean value over time, a phenomenon that is usually called mean reversion[1]. Thus, if the true generating rate process is mean-reverting the use of a single-factor pure diffusion process may produce gross errors of analysis. This issue is highly relevant for investment and disinvestment decisions of firms where the interest rate uncertainty is a key factor for the decision problem, since the upper interest rate threshold (trigger point that will induce the firm to disinvest) and the lower interest rate threshold (trigger point that will induce the firm to invest) will be surely influenced by this fact.

Using the so-called mean-reverting square-root process of Cox et al. (1985b) we generalize the work of Dias and Shackleton (2005) by introducing the mean reversion feature

[1]For a more detailed description of the empirical evidence on this issue, see, for example, Fama and Bliss (1987), Wu and Zhang (1996), Andersen and Lund (1997), Campbell et al. (1997, chap. 11) and the references contained therein.

into the economic hysteresis analysis under stochastic interest rates and show that such issue highlights a tendency for a widening effect on the range of inaction, though both thresholds have risen when compared with the no mean-reverting case. Other mean-reverting interest rate models have been used to examine the investment decision problem under stochastic interest rates. Lee (1997) uses the Ornstein-Uhlenbeck process to describe the interest rate dynamics (this model has been previously introduced in finance by Vasicek (1977) to price discount bonds). However, the criticism that is applied to Vasicek's arbitrage model does not apply to the CIR intertemporal general equilibrium term structure model, because the latter does not allow negative interest rates which is a desirable and more realistic feature for the term structure dynamics of interest rates [see Rogers (1995)]. Alvarez and Koskela (2006) study the impact of interest rate uncertainty on irreversible investment decisions using the mean-reverting model of Merton (1975) as the underlying stochastic interest rate dynamics. Again there is no evidence that this process has better features than the CIR model. But most importantly, neither Lee (1997) or Alvarez and Koskela (2006) considered the combined entry and exit strategy and thus they do not model the hysteresis problem under stochastic interest rates.

Using the work of Linetsky (2004) we also compute the hitting time densities in order to have an idea of how long does it take for a current interest rate to revert and hit the investment thresholds that would induce idle firms to invest. This information is relevant since it indicates how much time can the firm delay its decision to invest until the interest rate achieve a level where the firm's optimal decision is to invest immediately.

An outline of this paper is as follows: Section 2 presents the solutions for the perpetuity function in a CIR economy and provides some numerical computations. Section 3 discusses the economic hysteresis problem under stochastic interest rates and solves it numerically using the mean-reverting square-root model of Cox et al. (1985b). Section 4 computes CIR first hitting time down densities for our investment trigger points using the Linetsky (2004) framework. Section 5 concludes.

2 Valuation of Perpetuities under Stochastic Interest Rates within the CIR Framework

It is well known that under a CIR diffusion the price at time $t = t_0$ of a zero coupon bond maturing at time T is equal to:

$$P(r, t_0, T) = \mathbb{E}_{t_0}^{\mathcal{Q}}\left[e^{-\int_{t_0}^{T} r(s)\, ds}\right] = A(t_0, T)\; e^{-B(t_0,T)\, r(t_0)} \tag{1}$$

where

$$A(t_0, T) = \left[\frac{2\omega e^{[(\kappa+\lambda+\omega)(T-t_0)]/2}}{(\omega+\kappa+\lambda)(e^{\omega(T-t_0)}-1)+2\omega}\right]^{2\kappa\theta/\sigma^2} \tag{2a}$$

$$B(t_0, T) = \frac{2(e^{\omega(T-t_0)}-1)}{(\omega+\kappa+\lambda)(e^{\omega(T-t_0)}-1)+2\omega} \tag{2b}$$

$$\omega = \left[(\kappa+\lambda)^2 + 2\sigma^2\right]^{1/2} \tag{2c}$$

where $\mathbb{E}_{t_0}^{\mathcal{Q}}$ denotes the expectation under the risk-neutral probability $\mathcal{Q}$ (or martingale measure $\mathcal{Q}$), at time $t = t_0$, with respect to the risk-adjusted process for the instantaneous interest rate that can be written as the following stochastic differential equation:

$$dr_t = [\kappa\theta - (\lambda+\kappa)\, r_t]\, dt + \sigma\sqrt{r_t}\, dW_t \tag{3}$$

where κ is the parameter that determines the speed of adjustment (reversion rate), i.e., it measures the intensity with which the interest rate is drawn back towards its long-run mean, θ is the long-run mean of the instantaneous interest rate (asymptotic interest rate), σ is the volatility of the process, λ is the "market" risk parameter (positive premiums will exist if $\lambda < 0$)[2], r_t is the instantaneous interest rate and dW_t is a standard Brownian motion under $\mathcal{Q}$.

[2]It should be emphasized that although risk premiums for interest rates may be introduced, they cannot be observed or measured separately [see the detailed discussion in Dias and Shackleton (2005, section 2) as well as the references contained therein].

Under this framework, we can set the value of a perpetuity, that we will denote as $F(r)$, and the corresponding first derivative as follows[3]:

$$F(r) = \mathbb{E}_{t_0}^{\mathcal{Q}}\left[\int_{t_0}^{\infty} e^{-\int_{t_0}^{t} r(s)\, ds}\, dt \right] = \int_{t_0}^{\infty} P(r, t_0, t)\, dt \tag{4}$$

$$F'(r) = \frac{d}{dr} \int_{t_0}^{\infty} P(r, t_0, t)\, dt = \int_{t_0}^{\infty} \frac{\partial P(r, t_0, t)}{\partial r}\, dt = -\int_{t_0}^{\infty} A(t_0, t) B(t_0, t)\, e^{-B(t_0,t)\, r(t_0)}\, dt \tag{5}$$

In order to compute the value of a perpetuity and its derivative we will use the parameter values taken from the empirical work of Chan et al. (1992). Their values will be considered as our base case parameter values. Additionally, we are also interested in the special case where the term $\kappa\theta = 0$[4]. As it was pointed out by Ingersoll and Ross (1992) there is no consensus about the appropriate λ value to use. As a result, throughout this work we will consider no term premia ($\lambda = 0$). The parameter values for both cases are presented in Table 1[5].

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[3]The valuation of perpetuities using the methodology of Bessel processes under stochastic interest rates within the CIR's framework can be found in Delbaen (1993), Geman and Yor (1993) and Yor (1993).

[4]It is well known that one of the key issues of the square-root diffusion is the role played by the term $\kappa\theta$, which have important implications for the boundary conditions of the problem [see, for example, Feller (1951); for a complete description of the boundary classification for one-dimensional diffusions see Karlin and Taylor (1981, chap. 15) and Borodin and Salminen (2002, chap. II)]. Three important properties are of particular interest: (i) if $2\kappa\theta \geq \sigma^2$, $r = 0$ is an entrance, but not exit, boundary point for the process. This means that 0 acts both as absorbing and reflecting barrier such that no homogeneous boundary conditions can be imposed there. Thus, the origin is inaccessible and the CIR process stays strictly positive; (ii) if $0 < 2\kappa\theta < \sigma^2$, $r = 0$ is a reflecting boundary (exit and entrance), i.e., 0 is chosen to be an instantaneously reflecting regular boundary; (iii) if $\kappa\theta = 0$, $r = 0$ is a trap or an absorbing point and no boundary condition can be imposed there. Thus, when the CIR diffusion process hits 0 it is extinct, i.e., it remains at 0 forever (absorbing or exit boundary).

[5]It should be noted that we are making a small change on the parameters provided by Chan et al. (1992). It is well known that two of the parameters obtained from the model are $\kappa\theta$ and $\lambda + \kappa$, which corresponds, respectively, to α and $-\beta$ in their results. In our case, we are assuming that $\kappa = -\beta$ and then we are able to retrieve the value of θ for our numerical work. For the special case we are considering that both κ and θ are zero, but it is a sufficient condition that only κ be zero to generate such case, since

[Insert Table 1 Here]

Now it is interesting to present some numerical computations in order to understand the behaviour of the functions and the impact of considering, or not, a $\kappa\theta$ term equal to zero and different volatility levels. Table 2 shows the values of both functions for the base case parameter values, considering different maturities[6] and volatilities and with $r(0) = 0$[7].

[Insert Table 2 Here]

The results from Table 2 seems to indicate that the use of a fixed number T in the upper limit of the integral, instead of using infinity, will not generate any problem for the base case. We also have tried other interest rate values and we reach the same conclusions. Thus, considering $T = 500$ or $T = 1000$ seems quite reasonable for the analysis and it will simplify the numerical computations if we use this approach. Even the use of $T = 100$ will not produce too much differences. But instead of using equations (4) and (5) to compute the value of a perpetuity and its derivative, we can use, as an alternative, the analytic functions proposed by Delbaen (1993) and Geman and Yor (1993) [see appendix A for a short description of these alternative formulations]. Using the formulae presented in the appendix we also achieve the same values that we present in the table when $T = \infty$.

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it is this parameter that plays a key role on the distribution of the future interest rates. As Cox et al. (1985b) have shown as $\kappa \to 0$ the conditional mean goes to the current interest rate and the conditional variance of $r(s)$ given $r(t)$ (where $s > t$) goes to $\sigma^2\, r(t) \times (s-t)$. Therefore, this leads to the single-factor pure diffusion process of Ingersoll and Ross (1992) that we will use as our special case. But θ also plays a significant role even if the κ parameter is not zero. In this case, if we consider that $\theta = 0$ it still would not be possible to impose a boundary condition at $r = 0$. However, we would continue to have a mean-reverting process but now with an asymptotic interest rate equal to zero.

[6]It should be noted that with this approach the resulting formulae does still hold even if the upper limit of the integral in $F(r)$ is a fixed number T. For example, since the stochastic nature of interest rates is particularly relevant for actuarial purposes, considering a fixed number T in the upper limit of the integral may be a better description of the finite nature of human life for such applications.

[7]We choose this interest rate since the values of the functions at this point are of particular interest for the one-factor model that we are using, but similar computations can be done for any positive interest rate wanted. For the special case, however, the value of the perpetuity at $r = 0$ diverges for infinity since the term $\kappa\theta = 0$. For this reason it is not necessary to reproduce the results here.

Therefore, the choice between one of the two ways to compute perpetuities under the CIR framework is at the decision of the user. Using a finite maturity may be useful in some insurance applications in which human life nature may be better described. However, for other applications such as real options, where it is usually considered that the problems under study are time-independent, any of the approaches presented in the appendix are quite suitable. Yet, the use of $T = 500$ or $T = 1000$ in the first approach would generate similar results.

It is interesting to compare both cases with the most basic perpetuity that we can use in finance, the case where we have a zero volatility level. In this case the perpetuity function is just $F(r) = 1/r$. Figures 1 and 2 present the value of a perpetuity as a function of the interest rate using the base case values and special case values, respectively (Figure 2 clearly shows why it is not possible to impose a boundary condition at $r = 0$ when $\kappa\theta = 0$). The particular case with zero volatility is just the case where $F(r) = 1/r$. Thus, it is easy to see that volatility plays a key role for both cases. Moreover, if the true generating rate process is mean-reverting the use of a single-factor pure diffusion process may produce gross errors of analysis. This issue is highly relevant for investment and disinvestment decisions of firms where the interest rate uncertainty is a key factor for the decision problem, since the upper interest rate threshold (trigger point that will induce the firm to disinvest) and the lower interest rate threshold (trigger point that will induce the firm to invest) will be surely influenced by this fact.

[Insert Figures 1 & 2 Here]

3 Economic Hysteresis in a Mean-Reverting CIR Diffision

3.1 Extension of the Problem for Mean Reversion

Dias and Shackleton (2005) analyze the economic hysteresis effect under stochastic interest rates using the single-factor pure diffusion process of Ingersoll and Ross (1992). This allow them to concentrate exclusively on the effects of interest rate uncertainty on investment and disinvestment decisions. However, empirical evidence on interest rate behaviour seems to indicate the existence of mean reversion. Therefore, it is interesting to analyze the behaviour of the interest rate thresholds when the mean-reverting effect is considered. To do so, we will use the square-root mean-reverting process of Cox et al. (1985b). Obviously, this complicates the problem since we have to use some functions that are usually applied in the mathematics of physics. As a result, it is extremely complicated to get analytical comparative static expressions, but the procedure would be similar to the one used for the no mean-reverting case. Since the qualitative properties would also be true with the more general stochastic process with mean reversion, we will essentially resort some numerical analysis in order to understand the effect of considering the drift term of the process on the investment and disinvestment thresholds, though as it is shown below it is still possible to model the interest rate hysteresis effect in a very straightforward way.

If we consider a very long time to maturity options the interest rate contingent claim has to satisfy the following ordinary differential equation [see Cox et al. (1985a,b)]:

$$\frac{1}{2}\sigma^2 r \frac{\partial^2 F(r)}{\partial r^2} + \kappa(\theta - r)\frac{\partial F(r)}{\partial r} - \lambda r \frac{\partial F(r)}{\partial r} - rF(r) + 1 = 0 \tag{6}$$

The general solution to equation (6) is the sum of a complementary solution and a particular solution. Similarly as for the no mean-reverting case, the value of an idle firm, $F_0(r)$, is obtained by the solution of the respective complementary function (i.e., without the term $C(r,t) = 1$ included in the ODE stated above) and the value of an active firm, $F_1(r)$, is the solution of the complete differential equation (6). Now, finding a solution for the differential equations is a little more complicated than it was for the no mean-reverting case. However, the solution of the complementary function is of a series solution form that can be easily obtained in *Mathematica*. Thus, it turns out that the general solution for an idle firm and an active firm can be given, respectively, by:

$$F_0(r) = C_1\, e^{\zeta\, r + \eta\, ln[r]}\, U(a,b,z) + C_2\, e^{\zeta\, r + \eta\, ln[r]}\, L_n^{\beta}(x) \tag{7}$$

and

$$F_1(r) = C_3\, e^{\zeta\, r+\eta\, ln[r]}\, U(a,b,z) + C_4\, e^{\zeta\, r+\eta\, ln[r]}\, L_n^{\beta}(x) + Y(r) \tag{8}$$

where C_1, C_2, C_3 and C_4 are constants to be determined from boundary conditions, $Y(r)$ is the solution of the particular integral (i.e., is the perpetuity function computed as $\int_0^\infty P(r,0,t)\; dt$, where we are setting $t_0 = 0$), $U(a,b,z)$ and $L_n^{\beta}(x)$ are, respectively, the Tricomi confluent hypergeometric function (see appendix B for a short description of this special function) and the generalized Laguerre polynomial (see appendix C for a short description of this special function) and where:

$$\nu = \left[(\kappa+\lambda)^2 + 2\sigma^2\right]^{1/2} \tag{9a}$$

$$\zeta = \frac{\kappa+\lambda-\nu}{\sigma^2} \tag{9b}$$

$$\eta = \frac{-2\kappa\theta+\sigma^2}{\sigma^2} \tag{9c}$$

$$a = -\frac{\kappa\theta\,(\kappa+\lambda+\nu) - \sigma^2\,\nu}{\sigma^2\,\nu} \tag{10a}$$

$$b = 1+\eta \tag{10b}$$

$$z = \frac{2\,\nu\,r}{\sigma^2} \tag{10c}$$

$$n = \frac{\kappa\theta\,(\kappa+\lambda+\nu) - \sigma^2\,\nu}{\sigma^2\,\nu} \tag{11a}$$

$$\beta = \eta \tag{11b}$$

$$x = \frac{2\,\nu\,r}{\sigma^2} \tag{11c}$$

We know that the option of activating an idle firm should be nearly worthless for high interest rate levels. Therefore, we must set $C_2 = 0$. Similarly, the option of shutting an operating project should be nearly worthless for low interest rate levels. Thus, we must set $C_3 = 0$. As a result, the expect net present value of making an investment in the idle state is:

$$F_0(r) = C_1\, e^{\zeta\, r+\eta\, ln[r]}\, U(a,b,z) \tag{12}$$

Similarly, the value of an active firm is:

$$F_1(r) = C_4\, e^{\zeta\, r+\eta\, ln[r]}\, L_n^{\beta}(x) + F(r) \tag{13}$$

where $F(r)$ is the value of a perpetuity making a continuous payment of one unit over time. It turns out that we will need to use the first derivative of the functions (12) and (13). We already know the derivative of the perpetuity function, so we just need to get the derivatives of the investment and disinvestment opportunities. Noting that z and x are functions of r, the corresponding derivatives can be computed as follows (see appendixes B and C for additional details):

$$\frac{\partial F_0(r)}{\partial r} = C_1\,(\zeta + \eta/r)\, e^{\zeta\, r+\eta\, ln[r]}\, U(a,b,z) + C_1\, e^{\zeta\, r+\eta\, ln[r]}\,(-2\,a\,\nu/\sigma^2)\, U(1+a, 1+b, z) \tag{14}$$

and

$$\frac{\partial F_1(r)}{\partial r} = C_4\,(\zeta + \eta/r)\, e^{\zeta\, r+\eta\, ln[r]}\, L_n^{\beta}(x) + C_4\, e^{\zeta\, r+\eta\, ln[r]}\,(-2\,\nu/\sigma^2)\, L_{n-1}^{\beta+1}(x) + F'(r) \tag{15}$$

The intuition behind the optimal strategies is similar as for the no mean-reverting case, yet the resulting systems of non-linear equations will be much more complex now. The investment strategy can be stated as follows:

$$\overline{I} + IO(r) \rightarrow F(r)$$

The optimal investment policy is determined using one value matching condition and one smooth pasting condition that yields a system of two non-linear equations in two variables (C_1 and $\underline{r}$):

$$\overline{I} + C_1\, e^{\zeta\, \underline{r}+\eta\, ln[\underline{r}]}\, U(a, b, 2\,\nu\,\underline{r}/\sigma^2) = \int_0^\infty A(0,t) e^{-B(0,t)\underline{r}}\, dt \tag{16a}$$

$$\begin{aligned} &C_1\,(\zeta + \eta/\underline{r})\, e^{\zeta\, \underline{r}+\eta\, ln[\underline{r}]}\, U(a, b, 2\,\nu\,\underline{r}/\sigma^2) + \\ +C_1\, e^{\zeta\, \underline{r}+\eta\, ln[\underline{r}]}\,(-2\,a\,\nu/\sigma^2)\, U(1+a, 1+b, 2\,\nu\,\underline{r}/\sigma^2) &= \int_0^\infty -B(0,t) A(0,t) e^{-B(0,t)\underline{r}}\, dt \end{aligned} \tag{16b}$$

The disinvestment strategy can be stated as follows:

$$\underline{I} \leftarrow F(r) + DO(r)$$

where $\underline{I}$ takes a positive value since when the firm close its operations will not incur any cost to disinvest, because we want to focus our analysis on the possibility that some fraction of the lump-sum cost $\overline{I}$ can be recouped if firms decide to abandon its operations. Therefore, we will define a new variable α that will measure the degree of reversibility, i.e., $\alpha = \underline{I}/\overline{I}$. $\alpha = 0$ corresponds to an option in which the decision taken is irreversible and can be exercised only once. The case $0 < \alpha < 1$ corresponds to partial reversibility. We will consider three cases: $\alpha = 0.25$, $\alpha = 0.50$ and $\alpha = 0.75$. The limiting case of perfect reversibility where $\alpha = 1$ will also be considered. The optimal policy to disinvest is determined by the solution of the system of two non-linear equations with two variables (C_4 and $\overline{r}$):

$$\int_0^{\infty} A(0,t)e^{-B(0,t)\overline{r}}\, dt + C_4\, e^{\zeta\,\overline{r}+\eta\, ln[\overline{r}]}\, L_n^{\beta}(2\,\nu\,\overline{r}/\sigma^2) = \underline{I} \tag{17a}$$

$$\begin{aligned}\int_0^{\infty} -B(0,t)A(0,t)e^{-B(0,t)\overline{r}}\, dt + C_4\,(\zeta + \eta/\overline{r})\, e^{\zeta\,\overline{r}+\eta\, ln[\overline{r}]}\, L_n^{\beta}(2\,\nu\,\overline{r}/\sigma^2) + \\ + C_4\, e^{\zeta\,\overline{r}+\eta\, ln[\overline{r}]}\,(-2\,\nu/\sigma^2)\, L_{n-1}^{\beta+1}(2\,\nu\,\overline{r}/\sigma^2) = 0\end{aligned} \tag{17b}$$

Finally, the strategy for the entry and exit case can be stated as follows:

$$\overline{I} + IO(r) \rightarrow F(r) + DO(r)$$
$$\underline{I} + IO(r) \leftarrow F(r) + DO(r)$$

The solution for the combined entry and exit strategy under stochastic interest rates and mean reversion is obtained by the system of four non-linear equations in four variables (C_1, C_4, $\underline{r}$ and $\overline{r}$):

$$\overline{I} + C_1\, e^{\zeta\,\underline{r}+\eta\, ln[\underline{r}]}\, U(a, b, 2\,\nu\,\underline{r}/\sigma^2) = \int_0^{\infty} A(0,t)e^{-B(0,t)\underline{r}}\, dt + C_4\, e^{\zeta\,\underline{r}+\eta\, ln[\underline{r}]}\, L_n^{\beta}(2\,\nu\,\underline{r}/\sigma^2) \tag{18a}$$

$$C_1\,(\zeta+\eta/\underline{r})\,e^{\zeta\,\underline{r}+\eta\,ln[\underline{r}]}\,U(a,b,2\,\nu\,\underline{r}/\sigma^2)+$$
$$+C_1\,e^{\zeta\,\underline{r}+\eta\,ln[\underline{r}]}\,(-2\,a\,\nu/\sigma^2)\,U(1+a,1+b,2\,\nu\,\underline{r}/\sigma^2)=\int_0^\infty -B(0,t)A(0,t)e^{-B(0,t)\underline{r}}\,dt+$$
$$+C_4\,(\zeta+\eta/\underline{r})\,e^{\zeta\,\underline{r}+\eta\,ln[\underline{r}]}\,L_n^{\beta}(2\,\nu\,\underline{r}/\sigma^2)+C_4\,e^{\zeta\,\underline{r}+\eta\,ln[\underline{r}]}\,(-2\,\nu/\sigma^2)\,L_{n-1}^{\beta+1}(2\,\nu\,\underline{r}/\sigma^2) \tag{18b}$$

$$\int_0^\infty A(0,t)e^{-B(0,t)\overline{r}}\,dt+C_4\,e^{\zeta\,\overline{r}+\eta\,ln[\overline{r}]}\,L_n^{\beta}(2\,\nu\,\overline{r}/\sigma^2)=\underline{I}+C_1\,e^{\zeta\,\overline{r}+\eta\,ln[\overline{r}]}\,U(a,b,2\,\nu\,\overline{r}/\sigma^2) \tag{18c}$$

$$\int_0^\infty -B(0,t)A(0,t)e^{-B(0,t)\overline{r}}\,dt+C_4\,(\zeta+\eta/\overline{r})\,e^{\zeta\,\overline{r}+\eta\,ln[\overline{r}]}\,L_n^{\beta}(2\,\nu\,\overline{r}/\sigma^2)+$$
$$+C_4\,e^{\zeta\,\overline{r}+\eta\,ln[\overline{r}]}\,(-2\,\nu/\sigma^2)\,L_{n-1}^{\beta+1}(2\,\nu\,\overline{r}/\sigma^2)=C_1\,(\zeta+\eta/\overline{r})\,e^{\zeta\,\overline{r}+\eta\,ln[\overline{r}]}\,U(a,b,2\,\nu\,\overline{r}/\sigma^2)+$$
$$+C_1\,e^{\zeta\,\overline{r}+\eta\,ln[\overline{r}]}\,(-2\,a\,\nu/\sigma^2)\,U(1+a,1+b,2\,\nu\,\overline{r}/\sigma^2) \tag{18d}$$

3.2 Economic Hysteresis Effect

Dias and Shackleton (2005) have examined the economic hysteresis effect provoked by interest rate uncertainty assuming that changes in the instantaneous interest rate follow the single-factor pure diffusion process of Ingersoll and Ross (1992). They have concluded that when uncertainty comes from the stochastic nature of the interest rate term structure the range of inaction can be remarkably large. Now we are assuming that interest rates follow the square-root mean-reverting stochastic process of Cox et al. (1985b). This turns the problem much more complex, but it is still possible to model the interest rate hysteresis effect in a very straightforward way as it is explained below.

Let us define the following function:

$$V(r)=F_1(r)-F_0(r) \tag{19}$$

Using the solutions of equations (12) and (13) we have:

$$V(r)=C_4\,e^{\zeta\,r+\eta\,ln[r]}\,L_n^{\beta}(x)-C_1\,e^{\zeta\,r+\eta\,ln[r]}\,U(a,b,z)+F(r) \tag{20}$$

where $F(r)$ represents the perpetuity value. The two value matching and the two smooth pasting conditions can be defined in terms of V as:

$$V(\underline{r}) = \overline{I}\,,\;\; V'(\underline{r}) = 0\,,\;\; V(\overline{r}) = \underline{I}\,,\;\; V'(\overline{r}) = 0 \tag{21}$$

Once again, it is possible to conclude that there exists an optimal policy similar to the no mean reversion case, which indicates that there always exist an unique optimal policy for combined entry and exit decisions. Thus, an idle firm will only invest when interest rates fall to $\underline{r}$ and an operating firm will disinvest once the interest rates rise to $\overline{r}$. The range $(\underline{r}, \overline{r})$ is the hysteretic band of the problem since idle firms do not invest and operating firms do not abandon at this intermediate level of interest rates. As we will show later, one important conclusion is that this range of inaction has a tendency to be even higher when the mean reversion feature is introduced.

3.3 Numerical Analysis

The three systems of equations are represented by highly non-linear equations and no closed-form solution is possible. In addition, the equations are dependent of the Tricomi confluent hypergeometric function and the generalized Laguerre polynomial, which turns the numerical problem much more complex. Fortunately, both functions are available as built-in-functions in many scientific computing software such as *Mathematica* or *Maple*. In our case, we use *Mathematica* to compute the value functions as well as the numerical routines for solving simultaneous non-linear equations[8]. In order to find the numerical

[8] A technical detail now arises. In addition to the definitions presented in appendixes B and C for both functions other representations are possible such as integral representations, asymptotic expansions, etc. [see, for example, Slater (1960), Buchholz (1969) and Lebedev (1972)]. Depending on the index values of the functions, some representations may be computationally more efficient than others. To our knowledge, these software packages may use several different representations in order to compute the functions as most efficient as possible for each parameter set. Even so, for some parameter values the functions may lose accuracy and computation time may increase. For example, for low volatility levels ($\sigma = 0.03$ in our case) it turns out that the value functions become extremely large in absolute terms (for instance, assuming that $r = 0.10$ and using the base case parameters the values of the functions are $U(a, b, z) = 1.77486 \times 10^{70}$ and $L_n^{\beta}(x) = -8.51909 \times 10^{21}$). Thus, although we can compute each value

solutions of the three systems described above we need to use functions for both the perpetuity and its derivative. Although the analytical solutions that we present in appendix A are suitable to compute the isolated corresponding values when there is mean reversion, the use of such formulas within the numerical routines creates problems of convergence. Therefore, we will use equations (4) and (5) to compute the perpetuity and the derivative of the perpetuity functions, respectively, and impose an upper limit T for the integrals. However, as we have shown before (see Table 2) the use of $T = 500$ and $T = 1000$ produce the same results for a perpetuity and its derivative as if we have used $T = \infty$. Thus, we will use such time values in the numerical routines. It should be noted that the use of these time values does not produce any change on the thresholds of the three different strategies as we will confirm below.

Table 3 presents the lower trigger points for the single investment strategy, under mean reversion, considering different investment cost levels and different interest rate volatilities. It should be noted that these trigger points indicate at which interest rate level an idle firm is induced to invest and continuing its operations forever. Two immediate comparisons with the no mean-reverting case are possible. Firstly, we can see that all trigger points are higher when the mean reversion effect is considered. For example, considering $\sigma = 0.0854$ and $\overline{I} = 10$ it would be necessary that the interest rate falls to 1.94% to induce an idle firm to invest in the no mean-reverting case. Now, it is only necessary that the interest rate falls to 7.23%. Therefore, with mean reversion it seems that firms are induced to invest sooner. This also suggests that not taking into account mean reversion, when the true generating rate process is mean-reverting, may lead to incorrect decisions such as projects being delayed when they should be undertaken immediately. Secondly, when the high volatility level is considered, $\sigma = 0.3$, the lower trigger points are all positive now. Thus, even for high volatility levels there is an interest rate threshold that induce idle firms to invest. Two other issues are worthwhile to mention, both of which are in consensus

functions it is much more difficult to solve the systems of equations when $\sigma = 0.03$ due to the unfavorable values of the index functions, especially a, b, n and β. Thus, we will not present any result for the low volatility case. For the other volatility values, however, we will determine the numerical solutions of the corresponding systems which it is sufficient to compare these results with those obtained for the no mean-reverting case.

with the no mean-reverting case: (i) for a given volatility level there is a tendency for a fall in the lower trigger point as the investment cost rises; and (ii) for a given investment cost level there is a tendency for a fall in the lower threshold as the volatility level rises.

[Insert Table 3 Here]

Table 4 presents the upper threshold levels for the single disinvestment strategy, in the mean-reverting case, considering different interest rate volatilities, different investment levels and different fractions of disinvestment proceeds to investment costs. From the table we can see immediately that in this case the choice of T is not sensible for the disinvestment single strategy since we achieve the same upper trigger points when we use $T = 500$ and $T = 1000$. In addition, we can reach some of the conclusions that we have observed in the no mean-reverting case: (i) for a given volatility level and investment cost there is a tendency for the upper trigger point to fall as α rises; (ii) for a given volatility level and α parameter the upper threshold falls as the investment cost rises (this issue originates a rise on the disinvestment proceeds); and (iii) for a given investment cost and α parameter the upper threshold rises as volatility rises. The main difference between the mean-reverting and the no mean-reverting cases arises when we compare the respective thresholds for different volatility levels. Thus, for high volatility levels, $\sigma = 0.3$, the optimal disinvestment points of the mean-reverting case are all smaller than the ones when the mean reversion effect is not considered. For the volatility level of $\sigma = 0.0854$, however, this is not the case. Thus, for high α parameters the upper trigger points under mean reversion effects may be smaller than the corresponding ones of the no mean-reverting case, but for lower levels of α there is a tendency for a rise in the upper trigger point.

[Insert Table 4 Here]

Tables 5 and 6 present both the optimal entry and exit points of the combined strategy considering different investment cost levels, $\overline{I} = 10$ for Table 5 and $\overline{I} = 7.5$ for Table 6, and different interest rate volatilities and α parameters. It should be noted that for one of the cases where the investment cost level is $\overline{I} = 7.5$ we were not able to get real solutions.

As a result, we present NA in the corresponding table to highlight this issue. Let us now compare both cases. Thus, considering a volatility level of $\sigma = 0.0854$, an investment cost $\overline{I} = 10$ and a fraction $\alpha = 0.5$ we have concluded that the range of inaction is 0.2442 under the no mean-reverting case [see Dias and Shackleton (2005)]. When the mean reversion effect is considered we observe that the range of inaction is now 0.3246 ($\overline{r} - \underline{r} = 0.3969 - 0.0723$). Figure 3 depicts this feature. It is clearly demonstrated that both thresholds rise in the mean-reverting case and the hysteretic range is wider. In this case, the lower trigger point is now approximately 28 percent below the Marshallian investment threshold and the upper trigger point is approximately 98 percent above the corresponding Marshallian exit point. Once again, uncertainty effects are responsible for these differences.

[Insert Table 5 Here]

[Insert Table 6 Here]

[Insert Figure 3 Here]

We may conclude that considering a tendency toward some predictable long-run interest rate level there is a widening effect on the range of inaction, at least for moderate volatility levels, although both thresholds have risen. Figure 4 depicts this[9]. There is an economic intuition for this behaviour. When interest rates are high the economy tends to slow down and borrowers will require less capital. As a result, interest rates have a tendency to decline (mean reversion tends to imply a negative drift). Thus, firms are more reluctant to abandon its operations and the upper threshold has tendency to rise when compared with the one of the no mean-reverting case. When interest rates are at low levels borrowers tend to require more funds and the interest rates tend to rise (mean reversion tends to imply a positive drift). Since interest rates are at low interest rate levels it is not expected that they fall even more. Thus, firms are induced to invest before the likely rise in the interest rate and as a result the lower trigger point with mean reversion

[9] It should be noted that for greater visual appeal we show the pictures as continuous curves and not as step functions.

tends to be higher than the one where there is no mean reversion. For $\sigma = 0.3$, once again, both thresholds show a tendency to rise when compared with the ones of the no mean-reverting case. However, there is a tendency to have a narrower hysteretic band, due to the fact that entry trigger points are positive now.

[Insert Figure 4 Here]

In the mean reversion case it is possible to see that the lower trigger points for the single investment strategy and the combined entry and exit strategy are equal for the volatility level of $\sigma = 0.0854$ and all α parameters (the exception is obviously $\alpha = 1.00$). This imply that the firm's option to shut down later if interest rates start rising for very high levels is almost worthless for moderate volatility levels. For higher levels of volatility, however, this option has some value but not to much pronounced. When comparing the upper trigger points between the single disinvestment strategy and the combined strategy we see that the differences are much more significant. Once again, the differences comes from the reentry option values.

As for the case with no mean reversion effects we can take the following conclusions: (i) for a given level of volatility and investment cost the lower threshold rises (for moderate volatility levels the rise is insignificant) and the upper threshold falls as the α parameter rises. This imply a narrower range of inaction for higher α parameters; (ii) for a given volatility level and α parameter both the lower and the upper thresholds falls as the investment cost rises; and (iii) for a given investment cost level and α parameter the lower trigger point falls and the upper threshold rises as volatility rises, which originates a wider range of inaction.

4 Hitting Time Densities for CIR Diffusions

The computation of hitting time densities for CIR or OU diffusions have many applications in finance, for example to the analysis of mean-reverting CIR or OU models for interest rates, credit spreads, stochastic volatility, commodity convenience yields and other mean-reverting financial variables [see Linetsky (2004)]. The interest rate application motivate us to extend further the results of the preceding section. Therein, we analyzed the economic hysteresis effects under stochastic interest rates using the mean-reverting CIR diffusion and we achieved a set of lower and upper thresholds that would induce a firm to invest or disinvest. Thus, we know the interest rate levels that would correspond to optimal actions of a firm. However, it would be also very interesting to know how long does it take for a firm to decide for such actions. Using previous results we want to answer the following question: How long does it take for an interest rate following a mean-reverting CIR diffusion type to hit the thresholds? Using the work of Linetsky (2004) we will try to shed some light on such question, in particular for the lower trigger points to invest.

4.1 Theoretical Background

Hitting time densities for CIR processes have been previously recovered using numerical Laplace transform inversion procedures, since the Laplace transform of the first hitting time for the CIR diffusion is well known in the literature [see, for example, Giorno et al. (1986), Leblanc and Scaillet (1998) and Göing-Jaeschke and Yor (2003)]. The numerical inversion of the Laplace transform is, however, a difficult task since it poses several problems for the numerical implementation [see, for example, the discussion of Leblanc and Scaillet (1998) and the references contained therein]. Linetsky (2004) presents an alternative approach by providing explicit analytical characterizations for first hitting time densities for CIR diffusions (as well as for OU diffusions) in terms of relevant Sturm-Liouville eigenfunction expansions. The explicit analytical forms of the hitting time distributions for CIR diffusions are given in terms of confluent hypergeometric functions, but large-n asymptotics in terms of elementary functions are also given. From the point of view of practical applications in finance, the series expansion is generally preferable and easier to implement since no numerical integration is required.

For the analysis of the first hitting time densities for the mean-reverting CIR diffusion we will need the following results from Linetsky (2004, prop. 2), that we present here for the sake of completeness. Consider that we fix two interest rate levels x, $y \in I$ where I represents the state space of the problem. Then, the first hitting time densities $f_{\mathcal{T}_{x\to y}}(t)$ are computed as:

$$f_{\mathcal{T}_{x\to y}}(t) = \sum_{n=1}^{\infty} c_n \, \lambda_n \, e^{-\lambda_n t}, \quad t > 0 \tag{22}$$

where the eigenvalues λ_n (with $0 < \lambda_1 < \lambda_2 < \ldots \lambda_n \to \infty$ as $n \to \infty$) and the coefficients c_n are presented below. For all $t_0 > 0$, the series representation for the density (22) converges uniformly on $[t_0, \infty)$. Now suppose that $0 < x < y < \infty$ and let us introduce the following notation that is in use for both the CIR first hitting time up and CIR first hitting time down:

$$b = \frac{2\,\kappa\,\theta}{\sigma^2} \tag{23a}$$

$$\overline{x} = \frac{2\,\kappa\,x}{\sigma^2} \tag{23b}$$

$$\overline{y} = \frac{2\,\kappa\,y}{\sigma^2} \tag{23c}$$

$$a_n = -\frac{\lambda_n}{\kappa} \tag{23d}$$

Then, it turns out that in the case of the CIR first hitting time up the values of a_n (with $0 > a_1 > a_2 > \ldots a_n \to -\infty$ as $n \to \infty$) are the negative roots of the equation:

$$M(a, b, \overline{y}) = 0 \tag{24}$$

where $M(a, b, z)$ is the Kummer confluent hypergeometric function (see appendix B for a short description of this special function) and

$$c_n = -\frac{M(a_n, b, \overline{x})}{a_n \frac{d}{da}\{M(a, b, \overline{y})\}|_{a=a_n}} \tag{25}$$

Instead of computing hitting time densities in terms of special functions it is possible to compute it in terms of elementary functions. In this case, the eigenvalues λ_n and the coefficients c_n have the following large-n asymptotics:

$$\lambda_n \sim \frac{\kappa\,\pi^2}{4\,\overline{y}} \left(n + \frac{b}{2} - \frac{3}{4}\right)^2 - \frac{\kappa\,b}{2} \tag{26}$$

$$
\begin{aligned}
c_n \sim \frac{(-1)^{n+1}\, 2\,\pi\,(n+b/2-3/4)}{\pi^2\,(n+b/2-3/4)^2 - 2\,b\,\overline{y}} \times \\
\times\, e^{\frac{1}{2}\,(\overline{x}-\overline{y})} \left(\frac{\overline{x}}{\overline{y}}\right)^{\frac{1}{4}-\frac{b}{2}} cos\left(\pi\left(n+\frac{b}{2}-\frac{3}{4}\right)\sqrt{\frac{\overline{x}}{\overline{y}}} - \frac{\pi\,b}{2} + \frac{\pi}{4}\right)
\end{aligned}
\tag{27}
$$

Similarly, in the case of the CIR first hitting time down we suppose that $0 < y < x < \infty$ and then the values of a_n are the negative roots of the equation:

$$
U(a, b, \overline{y}) = 0 \tag{28}
$$

where $U(a, b, z)$ is the Tricomi confluent hypergeometric function (see appendix B for a short description of this special function) and

$$
c_n = -\frac{U(a_n, b, \overline{x})}{a_n \frac{d}{da}\{U(a, b, \overline{y})\}|_{a=a_n}} \tag{29}
$$

In this case, the eigenvalues λ_n and the coefficients c_n have the following large-n asymptotics:

$$
\lambda_n = \kappa\left(k_n - \frac{b}{2}\right), \quad \text{where } k_n \sim n - \frac{1}{4} + \frac{2\,\overline{y}}{\pi^2} + \frac{2}{\pi}\sqrt{\left(n-\frac{1}{4}\right)\overline{y} + \frac{\overline{y}^2}{\pi^2}} \tag{30}
$$

$$
c_n \sim \frac{(-1)^{n+1}\sqrt{k_n}}{\left(k_n - b/2\right)\left(\pi\sqrt{k_n} - \sqrt{\overline{y}}\right)}\, e^{\frac{1}{2}\,(\overline{x}-\overline{y})} \left(\frac{\overline{x}}{\overline{y}}\right)^{\frac{1}{4}-\frac{b}{2}} cos\left(2\sqrt{k_n\,\overline{x}} - \pi\,k_n + \frac{\pi}{4}\right) \tag{31}
$$

The large-n estimates of the eigenvalues λ_n and the coefficients c_n, for both the CIR first hitting time up and the CIR first hitting time down, are very important for the numerical analysis of the series representation for the density (22). For all $t_0 > 0$, the series representation for the density (22) is uniformly convergent on $[t_0, \infty)$. The rate of numerical convergence of this series is determined by the choice of t_0. Truncating the series after $N-1$ terms implies that the absolute value of the first omitted term $|c_N \lambda_N e^{-\lambda_N t}|$ is maximized at $t = t_0$. For a fixed t_0 and using equations (26), (27) and (23a) - (23d) for the CIR first hitting time up we have a large-N estimate as:

$$|c_N \lambda_N e^{-\lambda_N t_0}| \sim A\, N e^{-B\, N^2\, t_0} \tag{32}$$

with the two constants represented by:

$$A = \frac{\sigma^2\, \pi}{4\, y}\, e^{\frac{1}{2}\,(\overline{x} - \overline{y})} \left(\frac{x}{y}\right)^{\frac{1}{4} - \frac{b}{2}} \tag{33a}$$

$$B = \frac{\sigma^2\, \pi^2}{8\, y} \tag{33b}$$

Similarly, using equations (30), (31) and (23a) - (23d) for the CIR first hitting time down we have a large-N estimate as:

$$|c_N \lambda_N e^{-\lambda_N t_0}| \sim A e^{-B\, N\, t_0} \tag{34}$$

with the two constants represented by[10]:

$$A = \frac{2\, \kappa}{\pi}\, e^{\frac{1}{2}\,(\overline{x} - \overline{y})} \left(\frac{x}{y}\right)^{\frac{1}{4} - \frac{b}{2}} \tag{35a}$$

$$B = \kappa \tag{35b}$$

Using the estimates for both the CIR first hitting time up and CIR first hitting time down will help the determination of how many terms are needed in the series to achieve some desired error tolerance. It turns out that in the case of the CIR first hitting time down more terms will be needed since the rate of convergence will be slower. Once we have the density for the first hitting time, we can compute its mean (e.g., its expected first hitting time) by simply integrating time against this density:

$$\text{mean} = E[t] = \int_{t_0}^{\infty} t\, \times\, \sum_{n=1}^{\infty} c_n\, \lambda_n\, e^{-\lambda_n\, t}\, dt\,, \quad t_0 > 0 \tag{36}$$

After presenting the major insights of the explicit analytical characterizations for first hitting time densities for CIR diffusions proposed by Linetsky (2004) we will present an application for our previous results concerning the investment trigger points. For the upper trigger points the steps would be similar.

[10]Equation (35a) corrects a typo in Linetsky (2004, eq. 38).

4.2 Applications for Our Investment Thresholds

Computing hitting time densities for CIR diffusions will allow us to have an idea of how long does it take for a current interest rate level to revert and hit a specific investment threshold that would induce a firm to invest. The application for the disinvestment trigger points would also be possible but is less interesting to analyze. Therefore, we will focus our analysis on the investment thresholds using as an example the value of $\underline{r} = 0.0723$ taken from Table 3. Thus, we are interested in computing the mean first hitting time that a current interest rate level x will take to hit an investment threshold level $y = \underline{r} = 0.0723$, with $x > y$. In addition, we want to determine the specific relationship of the difference $x - y$ and the respective mean hitting values through time, by using different values of x.

We assume that the instantaneous interest rate follows a CIR diffusion with a mean reversion rate $\kappa = 0.2339$, a long-run mean level $\theta = 0.0808$, a volatility parameter $\sigma = 0.0854$ and a price of interest rate risk $\lambda = 0$. Let us also assume an investment threshold level $y = \underline{r} = 0.0723$ and a current interest rate level $x = 0.1023$ (i.e., the initial interest rate level is 300 basis points above the investment trigger point). In this case, we are interested in computing the first hitting time density of the investment threshold level $y = \underline{r} = 0.0723$, starting from $x > y$. A detailed analysis is provided for the case where $x = 0.1023$. For the other values of x, with $x = 0.0973$, 0.0923, 0.0873, 0.0823, 0.0773 and 0.0723, it will be provided the mean first hitting time values and the number of terms N included in the series.

Fixing $t_0 > 0$, the first hitting time density is given by the series represented by equation (22) which is uniformly convergent on $[t_0, \infty)$. The constants for the large-N estimate (34) are obtained using equations (35a) and (35b), which gives $A = 0.1729$ and $B = 0.2339$. The rate of numerical convergence of the series (22) is determined by the choice of t_0. After some calculations, we choose to fix $t_0 = 0.07$ and an error tolerance $\epsilon = 10^{-6}$, but the choice of other values would generate similar results, being the only difference the computation time of the numerical analysis. Using equation (34) we estimate that we need 736 terms to achieve this accuracy. Since in the case of the CIR first hitting time down the series converges more slowly it is necessary to use more terms. The maximum absolute value of the first omitted term with $N = 737$ is $|c_{737}\lambda_{737}e^{-\lambda_{737}t_0}| =$

1.9×10^{-7}, approximately.

To compute the exact eigenvalues λ_n we use the root-finding procedure FindRoot available in *Mathematica* to determine the roots of equation (28). As a starting point value for the FindRoot function we use the estimated value obtained by equation (30) for each $n \geq 1$. The exact values of the expansion coefficients c_n and their estimates are computed using equations (29) and (31), respectively. Table 7 presents the exact values and their estimates of the eigenvalues λ_n and the expansion coefficients c_n for $n = 1, 2, ..., 10$.

Figure 5 plots the first hitting time density on the interval $[0.07, 5]$ with $N = 736$ terms included in the series (22). The solid line represents the exact density, resulting in a mean hitting time of 3.607 years. The dashed line with short dashes plots the estimated density using the exact eigenvalues λ_n and coefficients c_n for $n \leq 10$ and estimates (30) and (31) for $n > 10$, which gives a mean hitting time of 3.598 years. The dashed line with long dashes is the estimated density using estimates (30) and (31) instead of exact λ_n and c_n for all n, resulting in a mean hitting time of 7.521 years. The first two densities are quite close since the exact density and the estimated density can hardly be distinguished. Thus, by using just a few exact values of λ_n and c_n based on special functions calculations and then use the estimates expressed in terms of elementary functions for the others, it will lead to a sharply decrease on computation time and it will give a density close to the exact density. Since the accuracy of the estimates increases with n, we can determine a few first exact eigenvalues λ_n and expansion coefficients c_n and then use estimates for the rest in order to achieve a very accurate calculation. Using only estimates (30) and (31) instead of exact λ_n and c_n for all n will produce very high errors.

[Insert Table 7 Here]

[Insert Figure 5 Here]

Table 8 presents the mean first hitting time values, using the three forms of computing the series density (22), for different initial interest rate levels x (for $x = 0.1023$, 0.0973, 0.0923, 0.0873, 0.0823, 0.0773 and 0.0723), as well as the number of terms N included in each of the series. For example, considering the initial level $x = 0.1023$ it would imply that it would take 3.607 years for the interest rate to hit the threshold and induce an idle firm to invest. This information is relevant since it indicates how much time can the firm delay its decision to invest until the interest rate achieve a level where the firm's optimal decision is to invest immediately. Figure 6 plots the mean hitting time for the different initial interest rate levels computed using the exact density. The dashed line presents the interpolating spline and the solid line presents a linear fitting, that can be represented by a simple equation $E[t] = 119.5643\,x - 8.4674$ with $R^2 = 0.9906$. Thus, there is an approximate linear relationship between the initial interest rate levels and the corresponding mean hitting time values.

[Insert Table 8 Here]

[Insert Figure 6 Here]

5 Conclusions

Using the so-called mean-reverting square-root process of Cox et al. (1985b) we generalize the work of Dias and Shackleton (2005) by introducing the mean reversion feature into the economic hysteresis analysis under stochastic interest rates and show that such issue highlights a tendency for a widening effect on the range of inaction, though both thresholds have risen when compared with the no mean-reverting case. Therefore, with mean reversion it seems that firms are induced to invest sooner. This also suggests that not taking into account mean reversion, when the true generating rate process is mean-reverting, may lead to incorrect decisions such as projects being delayed when they should be undertaken immediately.

Using the work of Linetsky (2004) we compute the hitting time densities in order to have an idea of how long does it take for a current interest rate to revert and hit the investment thresholds that would induce idle firms to invest. This information is relevant since it indicates how much time can the firm delay its decision to invest until the interest rate achieve a level where the firm's optimal decision is to invest immediately.

Appendix A: Analytic Functions of Delbaen (1993) and Geman and Yor (1993)

The purpose of this appendix is to present a short description of the analytic functions proposed by Delbaen (1993) and Geman and Yor (1993) to value perpetuities under the CIR framework. In addition, we will present two functions to compute the value of the first derivative of the perpetuity function, both of which were not mentioned by the authors. We will also present explicitly the parameter λ in our formulation in order to allow a better and easy visualization on the use of the risk premium of the single factor that drives the economy.

Let us start with the formulation of Delbaen (1993, pg. 127). Herein, we change his γ by θ to be consistent with our notation and avoid any sort of confusion. Therefore, the value of a perpetuity making a continuous payment of one unit over time has the form:

$$
\begin{aligned}
F(r) = \int_0^1 \frac{2}{\kappa+\lambda+\omega} e^{-z(2r/(\kappa+\lambda+\omega))}(1+\beta z)^{[(\omega+\kappa+\lambda)/2\omega](2\kappa\theta/\sigma^2)-1} \times \\
\times (1-z)^{[(\omega-\kappa-\lambda)/2\omega](2\kappa\theta/\sigma^2)-1} dz
\end{aligned}
\tag{A.1}
$$

where

$$
\omega = \left[(\kappa+\lambda)^2 + 2\sigma^2\right]^{1/2} \tag{A.2a}
$$

$$
\beta = \frac{\omega-\kappa-\lambda}{\kappa+\lambda+\omega} \tag{A.2b}
$$

$F(r)$ can be extended to an analytic function defined on the complex plane. Since differentiation under the integral sign is allowed, it turns out that $F'(r)$ is given by:

$$
\begin{aligned}
F'(r) = -\int_0^1 z\left(\frac{2}{\kappa+\lambda+\omega}\right)^2 e^{-z(2r/(\kappa+\lambda+\omega))}(1+\beta z)^{[(\omega+\kappa+\lambda)/2\omega](2\kappa\theta/\sigma^2)-1} \times \\
\times (1-z)^{[(\omega-\kappa-\lambda)/2\omega](2\kappa\theta/\sigma^2)-1} dz
\end{aligned}
\tag{A.3}
$$

As it was shown by Delbaen (1993, pg. 129) we can use an equivalent formulation:

$$
F(r) = \frac{\omega}{\kappa\,\theta}\,\Phi_1\big(a\,,b\,,c\,,x\,,y\big) \tag{A.4}
$$

where Φ_1 is the degenerate hypergeometric function defined as (it is one of the confluent series of the Horn function)[11]:

$$\Phi_1\big(a,b,c,x,y\big) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{m!}\,\frac{1}{n!}\,\frac{(a)_{m+n}}{(c)_{m+n}}\,(b)_m\,x^m\,y^n \tag{A.5}$$

where $(\alpha)_j$ is the Pochhammer symbol defined as $(\alpha)_0 = 1$ and $(\alpha)_j = \alpha\,(\alpha+1)\ldots(\alpha+j-1) = \Gamma(\alpha+j)/\Gamma(\alpha)$, where $\Gamma(.)$ is the Euler gamma function [see Abramowitz and Stegun (1972, pg. 255)], and where

$$a = 1 \tag{A.6a}$$

$$b = -\frac{\omega+\kappa+\lambda}{2\,\omega}\,\frac{2\,\kappa\,\theta}{\sigma^2} + 1 \tag{A.6b}$$

$$c = \frac{\omega-\kappa-\lambda}{2\,\omega}\,\frac{2\,\kappa\,\theta}{\sigma^2} + 1 \tag{A.6c}$$

$$x = -\frac{\omega-\kappa-\lambda}{\omega+\kappa+\lambda} \tag{A.6d}$$

$$y = -\frac{2\,r}{\kappa+\lambda+\omega} \tag{A.6e}$$

The first derivative of equation (A.4) is not presented by Delbaen (1993), but it follows that $F'(r)$ is defined as:

$$F'(r) = \frac{\omega}{\kappa\,\theta}\,\frac{d}{dr}\Phi_1\big(a\,,b\,,c\,,x\,,y\big) \tag{A.7}$$

where

$$\frac{d}{dr}\Phi_1\big(a\,,b\,,c\,,x\,,y\big) = -\frac{2}{\kappa+\lambda+\omega}\,\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} n\,\frac{1}{m!}\,\frac{1}{n!}\,\frac{(a)_{m+n}}{(c)_{m+n}}\,(b)_m\,x^m\,y^{n-1} \tag{A.8}$$

Let us turn now for the formulation proposed by Geman and Yor (1993, pg. 370). Once again, we change some of the notation used by them, with the intention of not generating any confusion with the notation that we are using throughout this paper. Therefore, their

[11] It should be noted that for the case where $r = 0$ it implies that $y = 0$, so that the degenerate hypergeometric function turns out to be the hypergeometric function ${}_2F_1(a,b,c,x)$ available as a built-in function in *Mathematica* [Hypergeometric2F1(a,b,c,x)].

θ, λ and $\underline{\theta}$ corresponds, respectively, to ϕ, ψ and φ in our paper. Under this framework, the value of a perpetuity making a continuous payment of one unit over time has the form:

$$F(r) = \frac{\psi}{\omega} e^{\phi\psi r/2} \int_0^1 \frac{(1+u)^p\,(1-u)^q\,e^{-\frac{r\omega\,(u+\varphi)}{2\,(1+\varphi u)}}}{(1+\varphi u)^{\delta\psi/2}}\,du \tag{A.9}$$

where

$$p = \frac{\phi\delta\psi^2}{4\omega} + \frac{\delta\psi}{4} - 1 \tag{A.10a}$$

$$q = \frac{\delta\psi}{4} - \frac{\phi\delta\psi^2}{4\omega} - 1 \tag{A.10b}$$

$$\omega = (2\psi + \phi^2\psi^2)^{1/2} \tag{A.10c}$$

$$\delta = \kappa\theta \tag{A.10d}$$

$$\phi = \frac{\kappa + \lambda}{2} \tag{A.10e}$$

$$\psi = \frac{4}{\sigma^2} \tag{A.10f}$$

$$\varphi = \frac{\phi\psi}{\omega} \tag{A.10g}$$

Although the first derivative of equation (A.9) is not presented by Geman and Yor (1993), it follows that $F(r)$ can be extended to an analytic function defined on the complex plane. Since differentiation under the integral sign is allowed, it turns out that $F'(r)$ is given by:

$$F'(r) = \frac{\psi}{\omega} e^{\phi\psi r/2} \int_0^1 \frac{(1+u)^p\,(1-u)^q\,e^{-\frac{r\omega\,(u+\varphi)}{2\,(1+\varphi u)}}\left(\frac{\phi\,\psi}{2} - \frac{\omega\,(u+\varphi)}{2\,(1+\varphi u)}\right)}{(1+\varphi u)^{\delta\psi/2}}\,du \tag{A.11}$$

Appendix B: Confluent Hypergeometric Functions

The purpose of this appendix is to present a short description of the confluent hypergeometric functions $M(a, b, z)$ and $U(a, b, z)$. A detailed description of these functions can be found in Slater (1960) and Buchholz (1969). Herein, we will only provide a very short description of them. Let us start with the confluent hypergeometric function $M(a, b, z)$, usually known as the Kummer confluent hypergeometric function. The function has also the alternative notations $\Phi(a,b,z)$ or ${}_1F_1(a,b,z)$ and is available as a built-in function in *Mathematica* [Hypergeometric1F1(a,b,z)]. The function is defined as [see Slater (1960, pg. 2)]:

$$M\big(a,b,z\big) = \sum_{n=0}^{\infty} \frac{(a)_n\, z^n}{(b)_n\, n!} \tag{B.1}$$

where $(\alpha)_n$ is the Pochhammer symbol defined as $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha\,(\alpha+1)\ldots(\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha)$, where $\Gamma(.)$ is the Euler gamma function [see Abramowitz and Stegun (1972, pg. 255)]. We will need to use the derivative of the Kummer function with respect to its first index and this is given by:

$$\frac{d}{da} M(a,b,z) = \sum_{n=0}^{\infty} \frac{(a)_n\, \Psi(a+n)}{(b)_n\, n!}\, z^n \,-\, \Psi(a)\, M(a,b,z) \tag{B.2}$$

where

$$\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \sum_{n=0}^{\infty} \left(\frac{1}{n} - \frac{1}{n+z-1} \right) - \gamma \tag{B.3}$$

is the digamma function [see Abramowitz and Stegun (1972, pg. 258)]. The derivative of the Kummer confluent hypergeometric function with respect to its first index is also available in *Mathematica* as a built-in function [Hypergeometric1F1$^{(1,0,0)}$(a,b,z)].

Let us turn now to the confluent hypergeometric function $U(a,b,z)$, sometimes also known as the Tricomi confluent hypergeometric function or Kummer confluent hypergeometric function of the second kind. The function has also the alternative notation $\Psi(a,b,z)$ and is available as a built-in function in *Mathematica* [HypergeometricU(a,b,z)]. The function is defined as [see Slater (1960, pg. 5)]:

$$U(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)}\, M(a,b,z) + \frac{\Gamma(b-1)}{\Gamma(a)}\, z^{1-b}\, M(1+a-b, 2-b, z) \tag{B.4}$$

In this case we will need to use the derivative of the Tricomi function with respect to its first index as well as its third index. The first derivative is given by:

$$
\begin{aligned}
\frac{d}{da}U(a,b,z) &= \frac{\Gamma(b-1)\, z^{1-b}}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Psi(a-b+n+1)\,(a-b+1)_n\, z^n}{n!\,(2-b)_n} + \\
&+ \frac{\Gamma(1-b)}{\Gamma(a-b+1)} \sum_{n=0}^{\infty} \frac{\Psi(a+n)\,(a)_n\, z^n}{n!\,(b)_n} - \Big[\Psi(a) + \Psi(a-b+1)\Big]\, U(a,b,z)
\end{aligned}
\tag{B.5}
$$

and is available as a built-in function in *Mathematica* [HypergeometricU$^{(1,0,0)}$(a,b,z)]. The derivative with respect to its third index can be computed by:

$$
\frac{d}{dz}U(a,b,z) = -a\,U(a+1,b+1,z) \tag{B.6}
$$

It should be noted that in our case z is a function of the interest rate ($z = 2\,\nu\,r/\sigma^2$). Therefore, we will have:

$$
\frac{d}{dr}U(a,b,2\,\nu\,r/\sigma^2) = -2\,a\,\nu/\sigma^2\,U(a+1,b+1,2\,\nu\,r/\sigma^2) \tag{B.7}
$$

In addition to the definitions presented for both functions other representations are possible such as integral representations, asymptotic expansions, etc. [see Slater (1960) and Buchholz (1969)]. Depending on the index values of the functions, some representations may be computationally more efficient than others. To our knowledge, software packages such as *Mathematica* or *Maple* use several different representations in order to compute the functions as most efficient as possible for each parameter set.

Appendix C: Generalized Laguerre Polynomial

The generalized Laguerre polynomial $L_n^{\beta}(x)$ is one of the orthogonal polynomials classes. A detailed description of this function can be found in Lebedev (1972). It should be noted that the generalized Laguerre polynomial is related with both the Kummer and Tricomi confluent hypergeometric functions in some special cases [see Slater (1960, pg. 95)]. The generalized Laguerre polynomial is also available as a built-in function in *Mathematica* [LaguerreL(n,β,x)]. Once again, we will only provide a short description of it. The function is defined as [see Lebedev (1972, pg. 76)]:

$$L_n^\beta(x) = e^x \, \frac{x^{-\beta}}{n!} \, \frac{d^n}{dx^n} \, (e^{-x} \, x^{n+\beta}) \tag{C.1}$$

The first derivative with respect to its third index is obtained by:

$$\frac{d}{dx} L_n^\beta(x) = -L_{n-1}^{\beta+1}(x) \tag{C.2}$$

Since in our case we have x as a function of the interest rate ($x = 2\,\nu\,r/\sigma^2$), it turns out that the derivative will be:

$$\frac{d}{dr} L_n^\beta(2\,\nu\,r/\sigma^2) = -2\,\nu/\sigma^2 \, L_{n-1}^{\beta+1}(2\,\nu\,r/\sigma^2) \tag{C.3}$$

Table 1: Parameter values for the base and special cases.

Parameter	Base Case Value	Special Case Value
κ	0.2339	0
θ	0.0808	0
σ	0.0854	0.0854
λ	0	0

Table 2: Values of the perpetuity function and the first derivative of the perpetuity function using the base case parameter values for different levels of volatility. CIR parameters: $\kappa = 0.2339$, $\theta = 0.0808$ and $\lambda = 0$.

		$r(0) = 0.00$		
Function	T	$\sigma = 0.03$	$\sigma = 0.0854$	$\sigma = 0.3$
$F(r)$	100	16.136	16.595	21.163
$F'(r)$	100	-52.888	-52.877	-52.601
$F(r)$	500	16.141	16.604	21.275
$F'(r)$	500	-52.913	-52.913	-52.913
$F(r)$	1000	16.141	16.604	21.275
$F'(r)$	1000	-52.913	-52.913	-52.913
$F(r)$	∞	16.141	16.604	21.275
$F'(r)$	∞	-52.913	-52.913	-52.913

Table 3: Lower thresholds for the investment option of the base case under different levels of investment costs and interest rate volatility. CIR parameters: $\kappa = 0.2339$, $\theta = 0.0808$ and $\lambda = 0$.

	$\overline{I} = 10$		$\overline{I} = 7.5$	
T	$\sigma = 0.0854$	$\sigma = 0.3$	$\sigma = 0.0854$	$\sigma = 0.3$
500	0.0723	0.0238	0.1101	0.0490
1000	0.0723	0.0238	0.1101	0.0490

Table 4: Upper thresholds for the disinvestment option of the base case under different levels of disinvestment proceeds and interest rate volatility and different ratios of the disinvestment proceeds to the investment costs. CIR parameters: $\kappa = 0.2339$, $\theta = 0.0808$ and $\lambda = 0$.

		$\overline{I} = 10$		$\overline{I} = 7.5$	
T	α	$\sigma = 0.0854$	$\sigma = 0.3$	$\sigma = 0.0854$	$\sigma = 0.3$
500	0.25	0.7303	1.0819	0.8835	1.2473
1000		0.7303	1.0819	0.8835	1.2473
500	0.50	0.4304	0.7351	0.5460	0.8722
1000		0.4304	0.7351	0.5460	0.8722
500	0.75	0.2828	0.5539	0.3858	0.6812
1000		0.2828	0.5539	0.3858	0.6812
500	1.00	0.1900	0.4320	0.2828	0.5539
1000		0.1900	0.4320	0.2828	0.5539

Table 5: Upper and lower thresholds for the switching option of the base case under an investment cost of 10 for different ratios of the disinvestment proceeds to the investment costs and different interest rate volatilities. CIR parameters: $\kappa = 0.2339$, $\theta = 0.0808$ and $\lambda = 0$.

		$\overline{I} = 10$			
		$\sigma = 0.0854$		$\sigma = 0.3$	
T	α	$\underline{r}$	$\overline{r}$	$\underline{r}$	$\overline{r}$
500	0.25	0.0723	0.7098	0.0238	0.9328
1000		0.0723	0.7098	0.0238	0.9328
500	0.50	0.0723	0.3969	0.0244	0.5505
1000		0.0723	0.3969	0.0244	0.5505
500	0.75	0.0723	0.2375	0.0288	0.3416
1000		0.0723	0.2375	0.0288	0.3416
500	1.00	0.1000	0.1000	0.1000	0.1000
1000		0.1000	0.1000	0.1000	0.1000

Table 6: Upper and lower thresholds for the switching option of the base case under an investment cost of 7.5 for different ratios of the disinvestment proceeds to the investment costs and different interest rate volatilities. CIR parameters: $\kappa = 0.2339$, $\theta = 0.0808$ and $\lambda = 0$.

		$\overline{I} = 7.5$			
		$\sigma = 0.0854$		$\sigma = 0.3$	
T	α	$\underline{r}$	$\overline{r}$	$\underline{r}$	$\overline{r}$
500	0.25	0.1101	0.8510	0.0490	1.0647
1000		0.1101	0.8510	0.0490	1.0647
500	0.50	0.1101	0.4871	0.0495	0.6327
1000		0.1101	0.4871	0.0495	0.6327
500	0.75	NA	NA	0.0541	0.4004
1000		NA	NA	0.0541	0.4004
500	1.00	0.1333	0.1333	0.1333	0.1333
1000		0.1333	0.1333	0.1333	0.1333

Table 7: Eigenvalues λ_n and coefficients c_n for the CIR first hitting time of the investment trigger point. CIR parameters: $y = 0.0723$, $x = 0.1023$, $\kappa = 0.2339$, $\theta = 0.0808$ and $\sigma = 0.0854$.

n	Exact λ_n	Estimated λ_n	Exact c_n	Estimated c_n
1	0.19834	0.14328	0.57571	0.97199
2	0.54707	0.50078	0.22555	0.24680
3	0.87302	0.83232	0.13603	0.12906
4	1.18631	1.14954	0.09139	0.07891
5	1.49115	1.45734	0.06388	0.05070
6	1.78981	1.75834	0.04507	0.03258
7	2.08370	2.05414	0.03142	0.02003
8	2.37374	2.34577	0.02114	0.01094
9	2.66060	2.63399	0.01321	0.00416
10	2.94477	2.91934	0.00699	-0.00098

Table 8: Mean hitting time. Computed using: (i) the exact density; (ii) the approximate density (estimated density using the exact eigenvalues λ_n and coefficients c_n for $n \le 10$ and estimates (30) and (31) for $n > 10$); (iii) the estimated density (using estimates (30) and (31) instead of exact λ_n and c_n for all n). CIR parameters: $y = 0.0723$, $\kappa = 0.2339$, $\theta = 0.0808$ and $\sigma = 0.0854$.

x	Mean using the exact density	Mean using the approximate density	Mean using the estimated density	Terms included in the series
0.1023	3.607	3.598	7.521	736
0.0973	3.155	3.145	6.986	733
0.0923	2.658	2.651	6.133	731
0.0873	2.106	2.103	4.981	729
0.0823	1.486	1.487	3.551	728
0.0773	0.782	0.784	1.870	727
0.0723	0.000	0.000	0.000	727

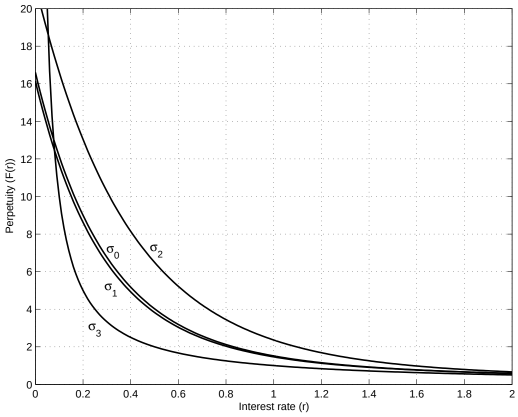

Figure 1: Value of a perpetuity as a function of the interest rate using the base case parameter values for different levels of volatility. CIR parameters: $\kappa = 0.2339$, $\theta = 0.0808$, $\lambda = 0$, $\sigma_0 = 0.0854$, $\sigma_1 = 0.03$, $\sigma_2 = 0.3$ and $\sigma_3 = 0$.

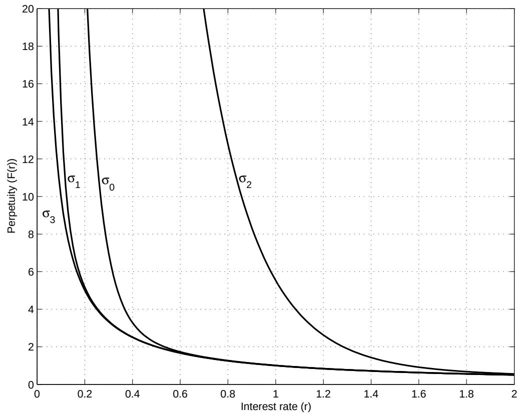

Figure 2: Value of a perpetuity as a function of the interest rate using the special case parameter values for different levels of volatility. CIR parameters: $\kappa = \theta = \lambda = 0$, $\sigma_0 = 0.0854$, $\sigma_1 = 0.03$, $\sigma_2 = 0.3$ and $\sigma_3 = 0$.

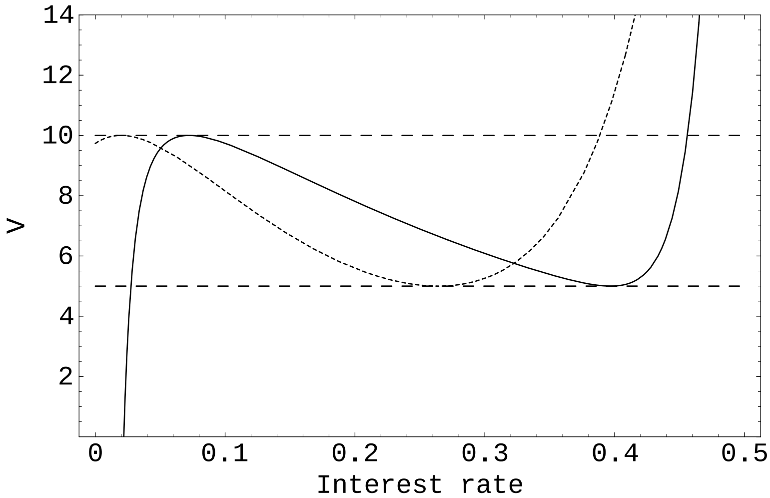

Figure 3: Determination of the numerical upper and lower thresholds. CIR parameters for the base case (mean reversion): $\kappa = 0.2339$, $\theta = 0.0808$, $\lambda = 0$ and $\sigma = 0.0854$. CIR parameters for the special case (no mean reversion): $\kappa = \theta = \lambda = 0$ and $\sigma = 0.0854$. $\overline{I} = 10$ and $\alpha = 0.5$. Solid line: mean-reverting case. Dashed line with short dashes: no mean-reverting case. Dashed line with long dashes: value matching condition at $\underline{r}$ (i.e., $V(\underline{r}) = \overline{I} = 10$) and value matching condition at $\overline{r}$ (i.e., $V(\overline{r}) = \underline{I} = \alpha\,\overline{I} = 5$).

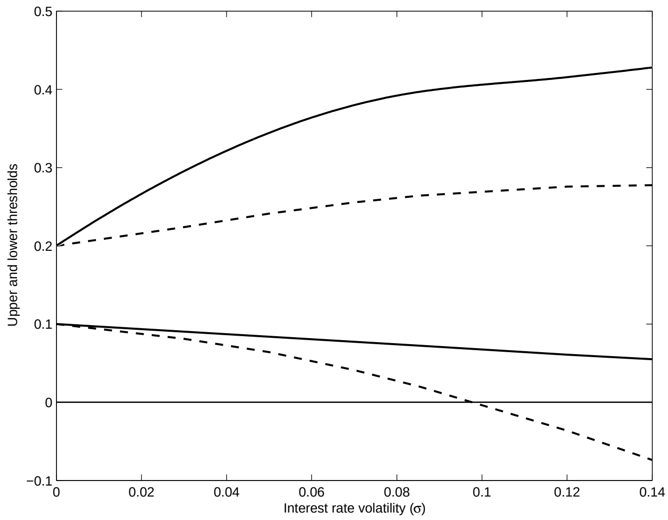

Figure 4: Entry and exit thresholds as functions of interest rate volatility. CIR parameters for the base case (mean reversion): $\kappa = 0.2339$, $\theta = 0.0808$ and $\lambda = 0$. CIR parameters for the special case (no mean reversion): $\kappa = \theta = \lambda = 0$. $\overline{I} = 10$ and $\alpha = 0.5$. Solid line: mean-reverting case. Dashed line: no mean-reverting case.

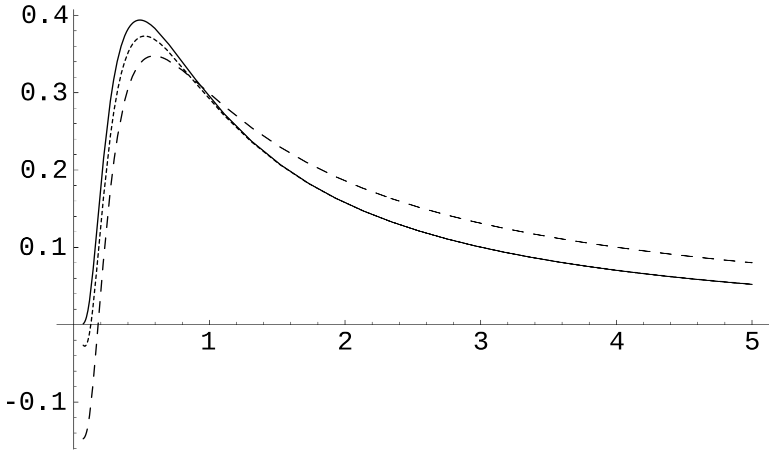

Figure 5: CIR first hitting time down density of the investment trigger point on the interval $[0.07, 5]$. CIR parameters: $y = 0.0723$, $x = 0.1023$, $\kappa = 0.2339$, $\theta = 0.0808$ and $\sigma = 0.0854$. Mean first hitting time using the exact density: 3.607 years. Solid line: exact density. Dashed line with short dashes: estimated density using the exact eigenvalues λ_n and coefficients c_n for $n \leq 10$ and estimates (30) and (31) for $n > 10$. Dashed line with long dashes: estimated density using estimates (30) and (31) instead of exact λ_n and c_n for all n. In all three cases the series representation for the density (22) is truncated after 736 terms.

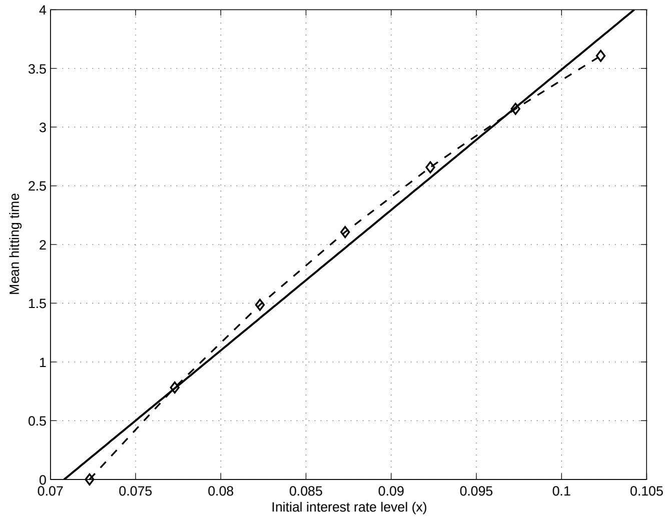

Figure 6: Mean hitting time for different initial interest rate levels computed using the exact density. Dashed line: interpolating spline. Solid line: linear fitting ($E[t] = 119.5643\,x - 8.4674$ and $R^2 = 0.9906$). CIR parameters: $y = 0.0723$, $\kappa = 0.2339$, $\theta = 0.0808$ and $\sigma = 0.0854$.

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