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Cumulative prospect theory and gambling

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Cumulative Prospect Theory and Gambling

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Abstract

Whilst Cumulative Prospect theory (CPT) provides an explanation of gambling on longshots at actuarially unfair odds, it cannot explain why people might bet on more favoured outcomes. This paper shows that this is explicable if the degree of loss aversion experienced by the agent is reduced for small-stake gambles (as a proportion of wealth), and probability distortions are greater over losses than gains. If the utility or value function is assumed to be bounded, the degree of loss aversion assumed by Kahneman and Tversky leads to absurd predictions, reminiscent of those pointed out by Rabin (2000), of refusal to accept infinite gain bets at low probabilities.
Boundedness of the value function in CPT implies that the indifference curve between expected-return and win-probability will typically exhibit both an asymptote (implying rejection of an infinite gain bet) and a minimum at low probabilities, as the shape of the value function dominates the probability weighting function. Also the high probability section of the indifference curve will exhibit a maximum. These implications are consistent with outcomes observed in gambling markets.

Keywords: Cumulative Prospect Theory; Exponential Value Function; Gambling

JEL classification: C72; C92; D80; D84

Cumulative Prospect Theory and Gambling

There is, however, one common observation which tells against the prevalence of risk aversion, namely, that people gamble ...I will not dwell on this point extensively, emulating rather the preacher, who, expounding a subtle theological point to his congregation, frankly stated: "Brethren, here there is a great difficulty; let us face it firmly and pass on': Kenneth Arrow (1965)

1. Introduction

The non-expected utility model proposed by Kahneman and Tversky (1979) and Tversky and Kahneman (1992), which they called Cumulative Prospect theory (CPT), has three key features. The first is that from a given reference point agents are risk-averse over potential gains but risk-loving over potential losses. Second, the utility or value function exhibits loss aversion so that the slope changes abruptly at the reference point. In particular, the function is postulated to fall roughly twice as fast over losses as it rises over gains, exhibiting diminishing sensitivity as the marginal impact of losses or gains diminishes with distance from the reference point [see e.g. Tversky and Kahneman (1992)]. Third, the probabilities of events are subjectively distorted by agents, via an inverted s-shaped probability weighting function so that small probabilities are exaggerated, and large probabilities are understated. The CPT model is able to resolve the Allais paradox [see e.g. Allais and Hagen (1979)] and also explains a variety of experimental evidence which is inconsistent with standard expected-utility theory [see e.g. Starmer (2000), Rabin (2000), Rabin and Thaler (2001), and Thaler (1985)]. [i]

Of particular importance is the probability weighting function, which can generate what Tversky and Kahneman (1992) call the most distinctive implication of CPT, namely the four-fold pattern of risk attitudes. This may arise because the normal risk-averse and risk-seeking preferences for gains and losses respectively may be reversed by the overweighting of small probabilities.

Prelec (2000) notes that for the four-fold pattern to emerge in general, probability weighting must over-ride the curvature of the value function; sometimes it works in favour and sometimes against the pattern[ii].

He suggests that the purchase of lottery tickets, for instance, indicates that probability over-weighting is strong enough to compensate for three factors which militate against such purchases, namely the concavity of the value function (which diminishes the value of the prize relative to the ticket price), loss aversion, and the fact that lottery tickets sell at an actuarially unfair price.

It is interesting that Prelec refers to outcomes in gambling markets as supporting CPT. This is also true of Kahneman and Tversky (1979), who note that CPT predicts insurance and gambling for small probabilities but state that "the present analysis falls far short of a fully adequate account of these complex phenomena". In fact there has been little discussion of whether CPT can provide a coherent explanation of gambling at actuarially unfair odds. Given that the great majority of people in developed countries participate in gambling, at least occasionally, [iii] and that gambles often involve large stakes,[iv] many would argue that an ability to explain outcomes observed in gambling markets [see Sauer (1998) and Vaughan

Williams (1999) for comprehensive surveys] is at least as important a test of a theoretical model as consistency with experimental evidence on the risk attitudes of small samples of students.

Of course, it is still the case that some economists explain gambling by invoking non-pecuniary returns such as excitement, buying a dream or entertainment [see e.g. Clotfelter and Cook (1989)].[v] However, there are convincing *a priori* and empirical reasons for giving little weight to this rationalisation in general. Friedman and Savage (1948) provide one convincing *a priori* critique of the entertainment rationale[vi]. Subsequently a number of surveys of gamblers have been conducted in which respondents are asked to cite the main reasons why they gamble. The predominant response, usually by 42%-70%, is for financial reasons - "to make money" [see e.g. Cornis (1978), and The Wager (2000 b)].[vii]

Given this background, the purpose in this paper is to consider the implications of CPT for gambling over mixed prospects. With the standard assumptions, gambling on longshots at actuarially unfair odds can optimally occur, but betting on 50/50 and odds-on chances cannot. We show the conditions in which the curvature of the value function can modify these results. In particular, (a) if stakes are not too large the assumption of ultimate boundedness of the value function will imply a minimum in the indifference curve in expected return-win probability space, (b) the indifference curve will typically exhibit an asymptote at very small probabilities, indicating that the agent would turn down a bet involving the possibility of an infinite gain; (c) depending on the degree of risk aversion assumed over gains, the asymptote can occur at any probability in the 0 -1 range; (d), in the absence of probability distortion agents will, paradoxically, ultimately accept very large bets on odds-on chances at actuarially unfair odds .

Finally, we illustrate how modification of the CPT model, such that agents are less loss averse over small- stake gambles than over large ones, and that probability distortions over gains are less than over losses, can explain both gambling on favoured outcomes, and also the favourite-longshot bias observed in most gambling markets.

The rest of the paper is structured as follows. In Section two we consider the implications of the CPT model for the shape of the indifference curve between expected-return and win-probability for mixed prospects. Section three develops further implications by assuming a particular parametric form of the Kahneman-Tversky function, and Section four contains a brief conclusion.

2. The Indifference Curve between Expected-return and Win-probability

Defining reference point utility as zero, for a gamble to occur in CPT we require expected utility or value to be non-negative:[viii]

$$EU = w^{+}(p)U^{r}(so) - \mathrm{w}^{-}(1\text{-}p)U^{l}(s) \;\geq\; 0 \qquad (1)$$

where the win-probability is given by p, and the functions $w^{+}(p)$ and $w^{-}(1-p)$ are non-linear s-shaped probability weighting functions. $U^{r}(so)$ is the value derived from a winning gamble, where o are the odds and s the stake. $U^{l}(s)$ is the disutility derived from a losing gamble.

From (1) the optimal stake is such that $\frac{\partial EU}{\partial s} = 0$ (and $\frac{\partial^2 EU}{\partial s^2} < 0$) so that

$$\frac{w^{+}(p)}{p}(\mu-p)U^{r'}(\frac{s(\mu-p)}{p}) \; = \; w^{-}(1-p)U^{l'}(s) \tag{2}$$

where the expected return from a unit gamble , μ , is defined as

$$\mu = p(1+o) \tag{3}$$

A bet is said to be actuarially fair when $\mu = 1$.

From (2) we have that $s = s(\mu, p)$ if $EU \geq 0$. Substituting $s = s(\mu, p)$ into (1) gives expected utility or value, EU, as a function of μ and p, and hence an indifference map in (μ, p) space may be obtained by differentiating (1) with respect to p and equating to zero, in order to find the combinations of expected return, μ, and probability, p, between which the bettor is indifferent. This produces:

$$\frac{dEU}{dp} = \{\frac{\partial w^{+}(p)}{\partial p}U^{r}(so) - \frac{\partial w^{-}(1-p)}{\partial p}U^{l}(s) - \frac{w^{+}(p)s\mu}{p^{2}}U^{r'}(so) + s\frac{w^{+}(p)}{p}U^{r'}(so)\}\frac{d\mu}{dp}$$
$$+[\frac{w^{+}(p)}{p}(\mu-p)U^{r'}(\frac{s(\mu-p)}{p}) - w^{-}(1-p)U^{l'}(s)]\frac{ds}{dp} \; = 0 \tag{4}$$

and hence, in view of (2), (4) reduces to

$$\frac{d\mu}{dp} = \{1 + o - \frac{\varepsilon^{gp}o}{\varepsilon^{u}} + \frac{\varepsilon^{lp}w^{-}(1-p)U^{l}(s)o}{\varepsilon^{u}w^{+}(p)U^{r}(so)}\} \tag{5}$$

where $\varepsilon^{u} = \dfrac{soU^{r'}(so)}{U^{r}(so)}$, $\varepsilon^{gp} = \dfrac{\partial w^{+}(p)}{\partial p}\dfrac{p}{w^{+}(p)}$, $\varepsilon^{lp} = \dfrac{\partial w^{-}(1-p)}{\partial p}\dfrac{p}{w^{-}(1-p)}$

where ε^{u} is the elasticity of $U(.)$, ε^{gp} is the elasticity of the probability weighting function over gains (strictly positive), and ε^{lp} is the elasticity of the probability weighting function over losses (strictly negative). Equation (5) also holds for any arbitrary fixed level of stake.

We observe from inspection of (5) that the slope of the indifference curve can exhibit turning points, and can be positive or negative depending on particular parameter values. For risk-loving behaviour it is necessary that the slope of the indifference curve be positive over some region of its domain. From inspection of (5) we observe that this possibility is enhanced when the elasticities of the probability weighting functions are small compared to the elasticity of the value function over gains. The size of stake will also influence the slope of the indifference curve by changing the elasticity of the value function over gains, and by affecting the ratio of the utility loss to the utility gain from the gamble.

It turns out, as shown below, that the slope of the indifference curve, $\frac{d\mu}{dp}$, can be negative at actuarially unfair odds, $\mu < 1$. This is not possible in the standard expected-utility model, where $\frac{d\mu}{dp}$<1 and $\mu > 1$ everywhere.

3. A Parametric Example of the Kahneman - Tversky Model

In order to generate further predictions from the analytical framework set out above, we need to specify a parametric form for the Kahneman -Tversky model. Because of serious limitations of the power value function (assumed by Kahneman–Tversky) in this framework we employ the exponential value function,[ix] where EU is given by

$$EU = w^{+}(p)(1-e^{-r\delta so}) - \mathrm{w}^{-}(1\text{-}p)\lambda(1\text{-}e^{-\delta s}) \geq 0 \qquad (6)$$

where r, δ and λ are positive constants.

The value function in (6) has upper and lower bounds as is commonly assumed, e.g. Markowitz (1952) and Machina (1982). The resolution of the St. Petersburg Paradox requires this assumption [see, e.g. Menger (1967) and Bassett (1997)]. The degree of loss aversion, (LA), for this value function is defined by the ratio of the utility gain to the utility loss from a symmetric gamble, given by

$$LA = \frac{(1 - e^{-r\delta s})}{\lambda(1 - e^{-\delta s})} \tag{7}$$

From (7), as stake size approaches zero, loss aversion would require that $\frac{r}{\lambda} < 1$, and as it becomes large, loss aversion requires that $\frac{1}{\lambda} < 1$. In order to ensure that $\frac{\partial LA}{\partial s} \leq 0$, so that the degree of loss aversion increases with stake size, we also require that $r \geq 1$.

From (6) the optimal stake, where $\frac{\partial EU}{\partial s} = 0$, is given by

$$s = \frac{\ln\{\frac{w^{+}(p)ro}{w^{-}(1-p)\lambda}\}}{\delta(ro - 1)} \tag{8}$$

The second-order condition for a maximum requires that $ro - 1 > 0$.

Note from (8) that both the numerator and denominator of (8) are positive at an optimum. The second-order condition implies that rapidly diminishing returns to increases in wealth (a large r) are a necessary condition for optimally betting on more favoured outcomes.

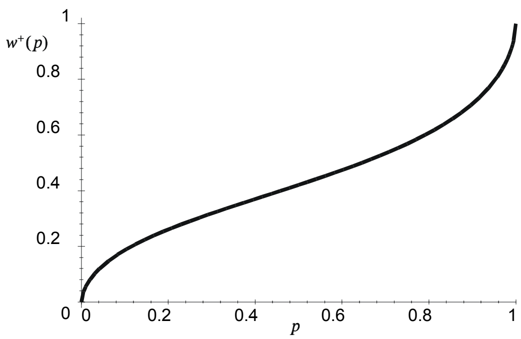
The numerator of (8) shows how the probability weighting function interacts with the degree of loss aversion (measured by $\frac{\mathrm{r}}{\lambda}$). Obviously, loss aversion itself militates against gambling; for the degree postulated by Kahneman and Tversky, $\frac{r}{k} < \frac{1}{2}$. For a typical functional form and parameter values of the probability weighting function [Tversky and Kahneman (1992)],[x] we plot in Figures 1(a) to 1(f) some probability weighting functions and their elasticities, $(\varepsilon^{gp},\ \varepsilon^{lp})$, over gains and losses to illustrate their numeric values over the probability range.

The ratio of the weighting functions, $\frac{w^{+}(p)}{w^{-}(1-p)}$, appears in the optimal stake equation (8) above, and the magnitude of its impact on the decision to gamble, relative to the distortion-free case is given by $\frac{w^{+}(p)}{w^{-}(1-p)} - \frac{p}{1-p}$. This is plotted in Figures 2(a) and 2(b). We observe that the elasticity of the probability weighting function becomes infinitely large as p approaches 1, and is less than one when p=0.

Taking the estimates reported by Tversky and Kahneman (1992), we observe from Figure 2(a) that the probability weighting function enhances the attraction of longshot gambles *per se*, but diminishes the attraction of more favoured outcomes, with the cross-over occurring at probabilities of around 0.45. When probability distortion over gains exceeds that over losses, the cross-over can

Figures 1 (a) -1(d)
Shape and Impact of the Probability Weighting Function (PWF)

(a) Probability Weighting Function over Gains



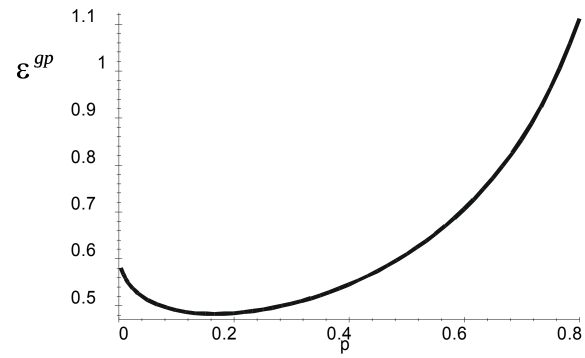
$$w^{+}(p)=\frac{p^{\sigma}}{(p^{\sigma}+(1-p)^{\sigma})^{\frac{1}{\sigma}}},$$

$\sigma = 0.61$ (Tversky and Kahneman (1992))

$p = 0.9$, $w^{+}(p) = 0.712$, $p = 0.5$, $w^{+}(p) = 0.421$

$p = 0.1$, $w^{+}(p) = 0.186$, $p = 0.001$, $w^{+}(p) = 0.014$

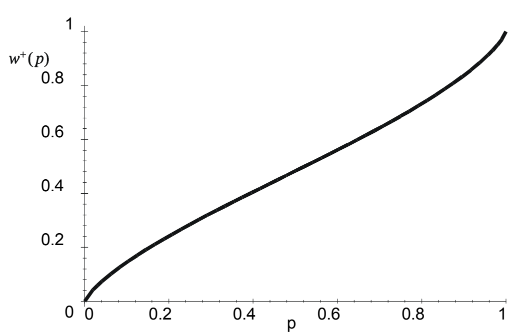
(b) Elasticity of the PWF over Gains



$$w^{+}(p)=\frac{p^{\sigma}}{(p^{\sigma}+(1-p)^{\sigma})^{\frac{1}{\sigma}}}, \quad \varepsilon^{gp}=\frac{\partial w^{+}(p)}{\partial p}\frac{p}{w^{+}(p)}$$

$\sigma = 0.61$ (Tversky and Kahneman (1992))

(c) PWF



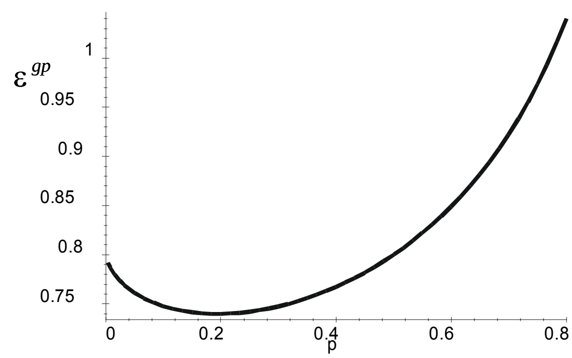
$$w^{+}(p)=\frac{p^{\sigma}}{(p^{\sigma}+(1-p)^{\sigma})^{\frac{1}{\sigma}}},$$

$\sigma = 0.8$

$p = 0.9$, $w^{+}(p) = 0.837$, $p = 0.5$, $w^{+}(p) = 0.482$

$p = 0.1$, $w^{+}(p) = 0.144$, $p = 0.001$, $w^{+}(p) = 0.0039$

(d) Elasticity of the PWF

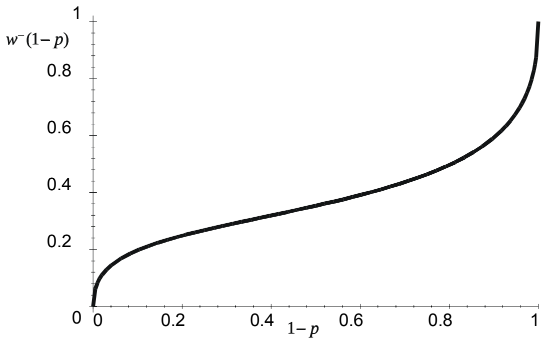


$$w^{+}(p)=\frac{p^{\sigma}}{(p^{\sigma}+(1-p)^{\sigma})^{\frac{1}{\sigma}}}, \quad \varepsilon^{gp}=\frac{\partial w^{+}(p)}{\partial p}\frac{p}{w^{+}(p)}$$

$\sigma = 0.8$

Figures 1 (e) -1(f)
Shape and Impact of the Probability Weighting Function (PWF)

(e) Probability Weighting Function



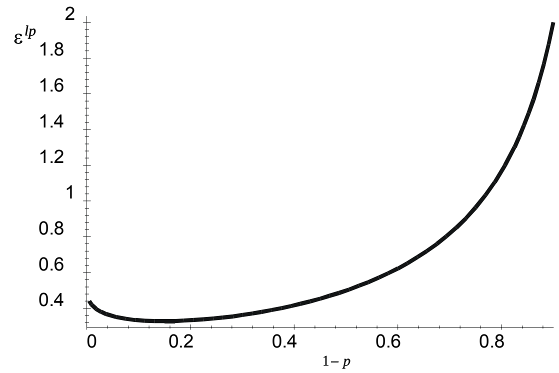
$$w^{-}(1-p) = \frac{(1-p)^{\rho}}{(p^{\rho}+(1-p)^{\rho})^{\frac{1}{\rho}}} \ ,$$

$\rho = 0.5$

$1-p = 0.9$, $w^{-}(1-p) = 0.592$, $1-p = 0.5, w^{-}(1-p) = 0.354$

$1-p = 0.1, w^{-}(1-p) = 0.198$, $1-p = 0.001, w^{-}(1-p) = 0.0297$

(f) Elasticity of the PWF over Losses

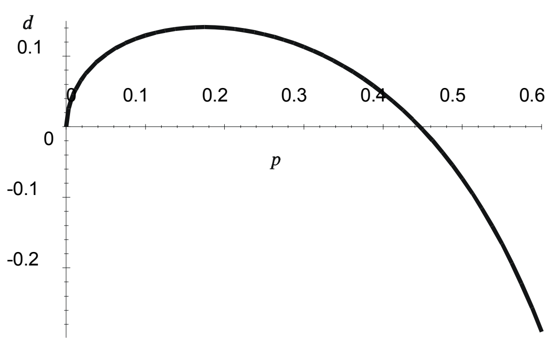


$$w^{-}(1-p) = \frac{(1-p)^{\rho}}{(p^{\rho}+(1-p)^{\rho})^{\frac{1}{\rho}}} \ , \qquad \varepsilon^{lp} = \frac{\partial w^{-}(1-p)}{\partial(1-p)}\frac{(1-p)}{w^{-}(1-p)}$$

$\rho = 0.5$

Figures 2(a) -2(b)
Impact of the Probability Weighting Function on Gambling

(a) Impact of the PWF on the gambling decision



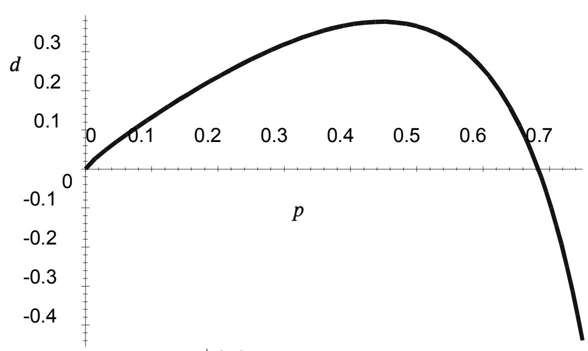
$$d = \frac{w^{+}(p)}{w^{-}(1-p)} - \frac{p}{1-p}$$

$$w^{+} = \frac{p^{\sigma}}{(p^{\sigma}+(1-p)^{\sigma})^{\frac{1}{\sigma}}}$$

$$w^{-}(1-p) = \frac{(1-p)^{\rho}}{(p^{\rho}+(1-p)^{\rho})^{\frac{1}{\rho}}}$$

$\sigma = 0.61, \rho = 0.60$ (Tversky and Kahneman (1992))

(b) Impact of the PWF on the gambling decision



$$d = \frac{w^{+}(p)}{w^{-}(1-p)} - \frac{p}{1-p}$$

$$w^{+} = \frac{p^{\sigma}}{(p^{\sigma}+(1-p)^{\sigma})^{\frac{1}{\sigma}}}$$

$$w^{-}(1-p) = \frac{(1-p)^{\rho}}{(p^{\rho}+(1-p)^{\rho})^{\frac{1}{\rho}}}$$

$\sigma = 0.8, \rho = 0.50$

occur at much higher probabilities, illustrated in Figure 2(b).[xi] However, from inspection of (8), even in this case the probability distortion is insufficient to overcome the degree of loss aversion assumed by Kahneman and Tversky (at least 2), so that betting on even-money or odds-on chances is ruled out.

Experiments reported in the literature are often based on asking agents what they would need to win in order to induce them to bet a particular fixed amount at specific win-probabilities (usually 0.5). Intuitively, there is no reason to expect required winnings to increase linearly either with stake size or with win-probability. It is instructive, therefore, to examine the indifference curve between expected -return and win-probability for a given fixed stake.

Using the exponential value function described earlier, the slope of the indifference curve is given by

$$\frac{d\mu}{dp} = \{1 + o - e^{gp}\frac{(e^{rso}-1)}{rs} + \frac{e^{lp}}{rs}\lambda e^{s(ro-1)}(e^{s}-1)\} \tag{9}$$

It is extremely unlikely that any plausible mix of parameter values would make this equation equal to zero everywhere (implying a flat linear indifference curve between expected return and win-probability), and it is clearly possible for the indifference curve to exhibit a maximum or a minimum, depending on the particular parameter values assumed.

Note from (9) that as $o \to \infty$, then $\frac{d\mu}{dp}$ will ultimately become negative so that the agent will act as an expected-utility maximiser, provided that $\mu \geq 1$. In this case, contrary to Prelec's (2000, p.90) conjecture, the boundedness of the value function dominates the probability weighting function, so that there is a range of behaviour not obtainable in his analysis.

Also, note that increasing stake size will ultimately lead to $\frac{d\mu}{dp} < 0$, and this may occur over the whole of the probability range so that the agent will again appear to behave as a expected utility maximiser, so long as μ>1 which will be the case for large enough stakes.

There are two further important implications of boundedness. First, from (1) the agent will gamble if

$$\frac{w^{+}(p)}{w^{-}(1-p)} > \frac{U^{l}(s)}{U^{r}(so)+U^{l}(s)} \tag{10}$$

When utility is bounded from above, $U^{r}(so) \rightarrow$ a limit, say 1, as $\mathrm{so} \rightarrow \infty$. The agent would then turn down an infinite gain bet if

$$\frac{w^{+}(p)}{w^{-}(1-p)} < \frac{U^{l}(s)}{1+U^{l}(s)} \tag{11}$$

Consequently, there is a win-probability threshold beyond which infinite gain bets will be turned down, even with small stakes. This is an implication of utility bounded from above; the precise threshold will depend on particular parameter values, as illustrated below.

Second, if bet size becomes very large, then from (10) the agent would gamble if

$$\frac{w^{+}(p)}{w^{-}(1-p)} > \frac{\lambda}{1+\lambda} \tag{12}$$

since $U^{l}(s) \to \lambda$ and $U^{r}(so) \to 1$ as $s \to \infty$. As a consequence, the agent would then be prepared to gamble at actuarially unfair odds at some large enough stake. Essentially, risk-loving behaviour over losses (which are bounded) implies that the agent would accept a large bet at actuarially unfair odds, since the size of the losses ceases to matter.

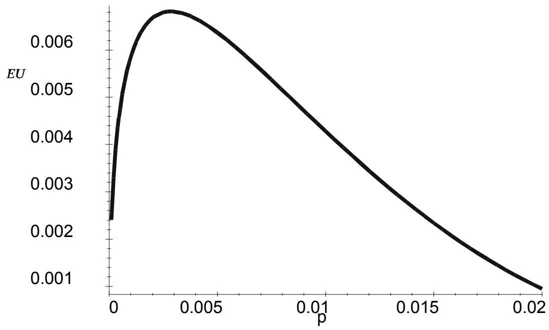
In order to rule out such gambles arising from curvature of the value function *per se*, the degree of loss aversion has to become very large over large-stake gambles (a large value of λ). With λ=90, for example, so that the gain from a symmetric gamble that could lead to bankruptcy is ninety times less than the pain of loss, gambles would be rejected unless they offered win-probabilities of more than 90/91. It seems relevant to note that such gambles are not observed in practice.

Some of the above possibilities are illustrated in Figures 3(a) – 4. In Figure 3(a) expected utility is plotted against win-probability when the stake is set optimally, the degree of loss aversion is as postulated by Kahneman and Tversky, and the probability weighting function has the parameter estimates suggested in the experimental literature. The agent is observed optimally gambling on a longshot where the expected loss per unit staked is 0.45, so $\mu = 0.55$. The distortion to probabilities caused by the probability weighting function overcomes the disinclination to gamble caused by the degree of loss aversion, so that the agent bets on longshots.

In Figures 3(b), 3(c) and 3(d) we plot the indifference curves between expected return and win-probability for a small constant $s\delta$.[xii] Figure 3(b) illustrates that

Figures 3(a) -3(d)
Expected Utility, Expected Return and Probability for CPT

(a) Expected Utility-probability indifference curve



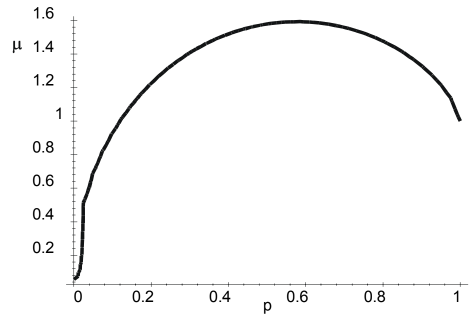
$$EU = w^{+}(p)(1-e^{-r\delta so}) - w^{-}(1-p)\lambda(1-e^{-\delta s})$$

$$s = \frac{\ln[\frac{w^{+}(p)or}{w^{-}(1-p)\lambda}]}{\delta(ro-1)}, \mu = 0.55, r = 45, \lambda = 90.$$

$$w^{+}(p) = \frac{p^{\sigma}}{(p^{\sigma}+(1-p)^{\sigma})^{\frac{1}{\sigma}}}, \sigma = 0.61,$$

$$w^{-}(1-p) = \frac{(1-p)^{\rho}}{(p^{\rho}+(1-p)^{\rho})^{\frac{1}{\rho}}}, \rho = 0.69.$$

(b) (μ, p) indifference curve $0.000002 \le p \le 1$

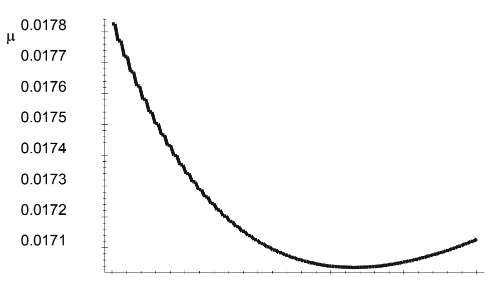


$$EU = w^{+}(p)(1-e^{-r\alpha so}) - w^{-}(1-p)\lambda(1-e^{-\alpha s}) = 0$$

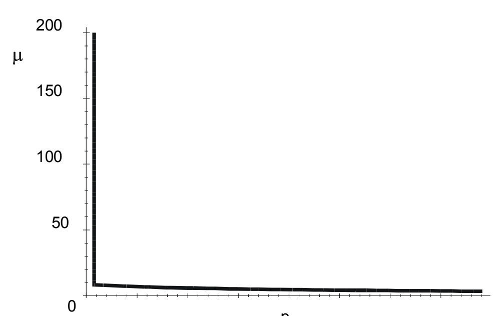
$s = 1, \alpha = 0.000001,\ r = 45,\ \lambda = 90.$

$w^{+}(p), w^{-}(1-p)$ as in 3(a).

(c) (μ, p) induifference curve : $0.000001 \le p < 0.000002$



(d)) (μ, p) indifference curve: $0 \le p \le 0.000001$



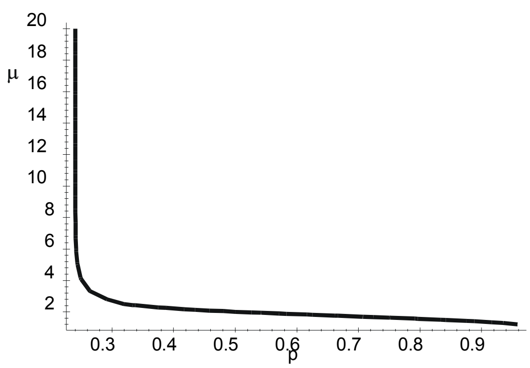
$\frac{d\mu}{dp}$ can be negative when expected returns are less than unity, a feature that cannot occur in the standard expected-utility model. Also note from Figure 3(b) that the indifference curve has a maximum in the favourite end of the spectrum at better than actuarially fair odds. In Figure 3(c) we observe that the indifference curve exhibits a minimum, and in Figure 3(d) an asymptote, so that the agent turns down a gamble with infinite expected return, at an extremely small probability. In this case, the boundedness of the value function ultimately "overpowers" the probability weighting function contrary to previous models in the literature.

In Figure 4 we plot the indifference curves between expected return and win-probability for a large $s\delta$, noting that with power value functions the magnitude of the stake has little influence on the gambling decision. The key features are that the asymptote now occurs at a higher win-probability, and the indifference curve is negatively sloped throughout its range. The interaction of high stakes and the curvature of the value function dominate the influence of the probability weighting function.

By choice of $s\delta$ and other parameter values we can position the asymptote at any win-probability. In this context, we note that the high degree of loss aversion assumed by Kahneman and Tversky for the symmetric ten dollar bet apparently implies "absurd" behaviour for non-symmetric ten dollar gambles involving lower win-probabilities. For example, using the exponential value function, with parameters $r = 45,\ \lambda = 90,\ \delta = 0.0001$, and the probability weighting functions of Kahneman and Tversky (1992), with parameters of 0.61 for gains and 0.69 for losses, we calculate that in order to bet \$10, with win-probability of 0.5, the agent would need to win at least \$18.13 (plausibly less than that of the students in the Kahneman-Tversky experiments). In addition, this agent would accept a bet to win infinity or lose \$100 at win probability of 0.5 ($EU = 0.093$), unlike the expected-utility maximiser, who would (absurdly) reject this gamble, as demonstrated by Rabin (2000). Indeed, our agent would accept this \$100 gamble if the potential gain were more than \$336.1, which is much more plausible than Rabin's example. However, this Kahneman-Tversky agent would reject a bet to win infinity or lose \$10 at a win-probability of 0.02 or less. This rejection seems just as absurd as that of the expected-utility maximiser. A similar result is obtained with the more flexible expo-power value function. [xiii]

Figure 4
Expected Return and Probability for CPT

(a) (μ, p) indifference curve in range $0 \le p \le 1$



$$EU = w^{+}(p)(1-e^{-r\alpha s o}) - w^{-}(1-p)\lambda(1-e^{-\alpha s}) = 0$$
$s = 1, \alpha = 0.000001,\ r = 45,\ \lambda = 90.$
$w^{+}(p), w^{-}(1-p)$ as in 3(a).

This type of calibration raises the question as to whether the degree of loss aversion assumed by Kahneman and Tversky, based on student responses, is too large to be widely applicable to other agents.[xiv]

Certainly, the under-weighting of high probabilities, in conjunction with the degree of loss aversion assumed by Kahneman and Tversky , makes an explanation of observed gambling on even-money or odds-on chances impossible in Cumulative Prospect theory; for example, gambling at actuarially unfair odds on the NFL, evens chances at roulette, and odds-on favourites in horse-racing. This seems to be a major failure of the theory. Leroy (2003) makes a related point about the assumed degree of loss aversion in the context of more traditional asset markets. He questions who would actually turn down a bet to win $11 or lose $10 at a win-probability of 0.5 (as the Kahneman-Tversky students do), noting that such gambles have risk−return characteristics superior to those of the daily returns on common stocks, which individuals generally find acceptable.

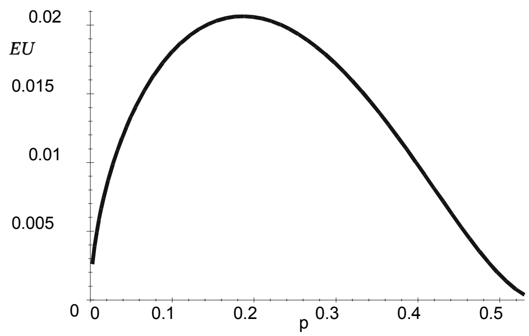
Kahneman-Tversky agents with less loss aversion

With this point in mind, we relax the degree of loss aversion over small stakes.[xv] In addition, we allow the probability distortion over losses to be slightly greater than over gains, as suggested by the empirical work of Jullien and Salanie (2000).[xvi] In Figures 5(a) and 5(b) the agent exhibits loss aversion over all wealth ranges, but initially *less* than assumed by Kahneman and Tversky. Observe in Figure 5(b) that over the win-probability range typically observed in horse-racing

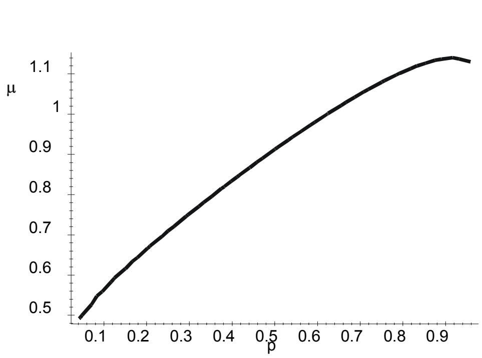
(0.01 - 0.7), the indifference curve has the shape of the typical favourite–longshot bias reported in the literature. In addition, the probability weighting function

Figures 5(a) - 5(c)
Expected Utility, Expected Return and Probability for a Less Loss Averse Gambler under CPT

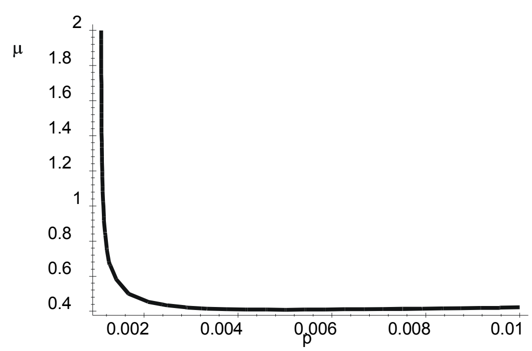
(a) Expected Utility



(b) ((μ, p) indifference curve: $0.01 \leq p \leq 1$



(c) (μ, p) indifference curve: $0 \leq p < 0.01$



$$EU = w^{+}(p)(1-e^{-r\delta o}) - w^{-}(1-p)\lambda(1-e^{-\alpha s}) = 0$$
$$s = 1, \delta = 0.00005,\ r = 80,\ \lambda = 90$$
$$w^{+}(p) = \frac{p^{\sigma}}{(p^{\sigma} + (1-p)^{\sigma})^{\frac{1}{\sigma}}}, \sigma = 0.8$$
$$w^{-}(1-p) = \frac{(1-p)^{\rho}}{(p^{\rho} + (1-p)^{\rho})^{\frac{1}{\rho}}}, \rho = 0.5$$

induces a maximum in the indifference curve for extreme favourites, requiring positive rates of return.

Clearly, if a Kahneman-Tversky agent is assumed not to be loss averse for small stakes, but is gain-loving instead, so that $\frac{r}{k} > 1,$ the indifference curve is qualitatively similar in shape to the previous case, except that the range of win-probabilities at which the agent would accept actuarially unfair bets is extended to include very strong favourites in excess of 0.7 win probability.

4. Conclusion

Surveys show that a high proportion of adults regularly gamble at actuarially unfair odds in most developed countries; many bets are sizeable and most of the money bet is on favourites; most people gamble primarily for financial gain. It therefore appears from survey evidence, from consideration of the pattern of money bet and from *a priori* reasoning that entertainment *per se* cannot explain outcomes in gambling markets. Whilst Cumulative Prospect theory provides an explanation of gambling on longshots (low probability bets) at actuarially unfair odds, gambling on more favoured outcomes is inexplicable.

This paper shows that gambling on more favoured outcomes, at actuarially unfair odds, can be explained if the degree of loss aversion experienced by the agent is reduced over small-stake gambles (as a proportion of wealth), and probability distortions are assumed to be greater over losses than gains. It is also suggested that the degree of loss aversion assumed by Kahneman and Tversky leads to absurd predictions that infinite gain bets would be rejected at low win-probabilities if the value function is assumed to be bounded.

Boundedness of the value function in Cumulative Prospect theory implies that the indifference curve between expected-return and win-probability will exhibit both an asymptote (implying rejection of an infinite-gain bet) and a minimum at low win-probabilities, because the shape of the value function dominates the probability weighting function, contrary to Prelec's (2000) conjecture. Also, a maximum will occur at high win-probabilities. These implications, which seem to be new, are consistent with gambling market outcomes, and may explain why lotteries typically offer relatively high expected returns compared to betting on longshots in horse-racing, and why there may sometimes be a reverse bias in horse-racing.

It is also demonstrated that boundedness of the value function paradoxically creates an incentive for an agent to engage in large-stake gambles involving high probabilities at actuarially unfair odds. Increasing the degree of loss aversion exhibited over large-stake gambles ensures that this property has no practical relevance.

REFERENCES

M. Allais and O. Hagen (1979), editors, *Expected Utility Hypotheses and the Allais Paradox.* Dordrecht: D. Reidel.

Arrow, K.J. (1965), The Theory of Risk Aversion, in K.J. Arrow, *Aspects of the Theory of Risk-Bearing* (Helsinki: Yrjo Jahnssonin saatio, lecture 2).

Bassett, G.W. (1987), “The St. Petersburg Paradox and Bounded Utility”, *History of Political Economy*, 1987, v. 19, n. 4, reprinted in *Expected Utility, Fair Gambles and Rational Choice*, O.F. Hamouda and J.C.R. Rowley editors, Elger, 1997.

Bell, D.E. (1985), “Disappointment in Decision Making Under Uncertainty”, *Operations Research*, 33, 1-27.

Boyd, W.H. (1976), “Excitement: The Gambler’s Drug”, in *Gambling and Society*, W.R. Eadington, ed. Springfield, 11.Thomas.

Bruce, A.C. and Johnson, J.E.V. (1992), “Towards an Explanation of Betting as a Leisure Pursuit”, *Leisure Studies*, 11, 201-218.

Clotfelter, C.T. and Cook, P.J. (1989), *Selling Hope: State Lotteries in America*, Cambridge, MA: Harvard University Press.

Conlisk, J. (1993), “The Utility of Gambling”, *Journal of Risk and Uncertainty*, 6, 255-275.

Cornish, D.B. (1978), *Gambling: A Review of the Literature*, HMSO, London.

Friedman, M. and Savage, L.J. (1948), “The Utility Analysis of Choices Involving Risk,” *Journal of Political Economy*, LV1, 279-304.

Golec, J. and Tamarkin, M. (1998), “Bettors Love Skewness, Not Risk, at the Horse Track”, *Journal of Political Economy*, 106, 205-225.

Hacking, I. (1980),”Strange Expectations”, *Philosophy of Science,* 47, 562-567.

Jeffrey, R. C. (1983), *The Logic of Decision*, Second Edition, University of Chicago Press.

.

Jullien, B. and Salanie, B. (2000), “Estimating Preferences under Risk: The Case of Racetrack Bettors”, *Journal of Political Economy*, 108, 503-530.

Kahneman, D. and Tversky, A. (1979), “Prospect Theory: An Analysis of Decision under Risk,” *Econometrica,* 2, 263-91.

Ladouceur, R.D., Dube, I.Giroux, Legendre, N. (1995), “Cognitive Biases in Gambling: American Roulette and 6/49 Lottery”, *Journal of Social Behavior and Personality*” 10, 473-479.

Le Roy, S. (2003), "Expected Utility: a Defense", *Economics Bulletin*, Vol. 7, No. 7, 1−3.

Lidner, R.M. (1950), “The Psychodynamics of Gambling“, *Annals of the American Academy of Political and Social Science*, 269, 93-107.

Machina, M.J. (1982), “Expected Utility” Analysis without the Independence Axiom*”, Econometrica*, 50, 2, 277-324.

Markowitz, H. (1952), “The Utility of Wealth”, *Journal of Political Economy*, 56, 151-154.

Menger, K.(1967), “The Role of Uncertainty in Economics”, in *Essays in Mathematical Economics in Honor of Oskar Morgenstern* (ed. Martin Shubik), Princeton University Press.

Prelec, D. (1998), “The Probability Weighting Function,” *Econometrica*, 66, 497-527.

Prelec, D. (2000), “Compound Invariant Weighting Functions in Prospect Theory”, pp. 67-92 in *Choices, Values, and Frames,* (eds.) Daniel Kahneman and Amos Tversky , Cambridge University Press.

(1999) Pathological Gambling: A Critical Review. Committee on the Social and Economic Impact of Pathological Gambling, National Research Council.

Rabin, M (2000), "Risk Aversion and Expected-Utility Theory: A Calibration Theorem", Econometrica 68, 5, 1281-1292.

Rabin, M. and Thaler, R. (2001), "Risk Aversion", *Journal of Economic Perspectives,* 15(1), Winter 2001, 219-232.

Samuelson, P.A. (1963), "Risk and Uncertainty: A Fallacy of Large Numbers", *Scientia,* 153-158.

Sauer, R.D. (1998), “The Economics of Wagering Markets”, *Journal of Economic Literature,* XXXV1, 2021-2064.

Starmer, C. (2000), “Developments in Non-Expected Utility Theory: The Hunt for a Descriptive Theory of Choice under Risk,” *Journal of Economic Literature*, Vol. XXXV111, 332-382.

Strumpf, K.S. (2002), “Illegal Sports Bookmakers” mimeo. University of North Carolina at Chapel Hill.

Thaler, R.H. (1985), “Mental Accounting and Consumer Choice”, *Marketing Science*, 4, 199-214.

Thaler, R.H. and Johnson, E.J. (1990), “Gambling with the House Money and Trying to Break Even: The Effects of Prior Outcomes on Risky Choice”, *Management Science*, 36, 643-660.

Tversky, A.and Kahneman, D. (1992), “Advances in Prospect Theory: Cumulative Representation of Uncertainty,” *Journal of Risk and Uncertainty*, 5:4, 297-323.

Vaughan Williams, L. (1999), “Information Efficiency in Betting Markets: a Survey”', *Bulletin of Economic Research*, 53, 1-30.

The Wager (2000a), “The Prevalence of Problem Gambling: The British Experience”, Gamble” 5, 32. Massachusetts Council on Compulsive Gambling.

The Wager (2000b), “Stress, Anxiety, and Why Gamblers Gamble”, 5, 27. Massachusetts Council on Compulsive Gambling.

Walther, H. (2002), “Economics of Gambling”, mimeo., University of Economics, Vienna.

Wu, G. and Gonzalez, R. (1996), “Curvature of the Probability Weighting Function*”, Management Science*, 42, 1676-1688.

Endnotes:

[i] For instance the apparent preference of some agents for segregated gains reported by Thaler (1985, p. 203) whose survey evidence indicated that most people believe that a person would be happier to win $50 plus $25 in separate lotteries rather than $75 in a single lottery. An excellent discussion of this experimental evidence can be found in Starmer (2000). Rabin (2000) provides further indirect support for CPT, in demonstrating that the assumption of global risk-aversion has implications for agents' preferences with respect to small and large gambles that appear untenable *a priori*. In particular, he shows that if an agent turns down a gamble to win $11 or lose $10, each with probability 0.5, at all prevailing wealth levels, then she will also turn down a bet to win infinity or lose $100 gamble, each with probability 0.5. In addition, Rabin notes that the assumption of global risk-aversion implies that agents who turn down a gamble to lose $100 or win $200 with win-probability 0.5, would turn down a sequence of N such bets, say, N=100, as shown by Samuelson (1963). Again, this appears absurd *a priori*. As a consequence of these implications, Rabin suggests that economists should reject standard expected-utility theory in favour of some version of the non-expected utility model,such as that proposed by Kahneman and Tversky.

[ii] He suggests that probability non-linearity will eventually be recognised as a more important determinant of risk attitudes than money non-linearity, at least in situations in which one is comparing only amongst gain (or loss) prospects. He notes, however, that for mixed prospects involving losses and gains, the assumption of loss aversion will become a critical additional factor.

[iii] The proportion of people reported as gambling varies little between countries and is uniformly high. For instance, in 1998 68% of respondents in the United States reported gambling at least once in the previous year. Legal gambling losses in America totalled over $50 billion, and illegal gambling has been estimated at over $100 billion - greater than the estimated expenditure on illegal drugs [see e. g. Strumpf (2003), Pathological Gambling (1999), and The Wager (2000a)].

[iv] Strumpf (2003), in his study of six illegal bookmakers in New York City over the period 1995-2000 (two of which had turnover in excess $100 million per annum), reports that average bet size was relatively large for these firms, averaging in excess of $1000. We also note that observation of high rollers on odd/even bets at roulette is folklore.

[v] Psychologists provide many explanations of gambling; for example, that it is a substitute for masturbation, or represents an erotisation of fear, or is a sublimation of oedipal aggression towards the father. All these and other psychological explanations are reviewed in Lidner (1950). More recent explanations of pathological gambling have hypothesised a genetic rationale (see, e.g. The Wager, Feb 20, 2002).

[vi] They note that (a) entertainment could be purchased separately, in principle, by paying admission to participate in a game using valueless chips, (b) that the gambler could buy the gamble by having an agent play the game for him according to detailed instructions and (c) gambles are often purchased in almost pure form: Friedman and Savage gave the example of the Irish sweepstake tickets at that time, where the purchaser is not a spectator to the drawing of the winner.

[vii] Also see Bruce and Johnson (1992), who examine 1200 bets on horse races randomly selected from a larger sample of bets in the UK in March and April 1987. The betting pattern before the off, and the much larger absolute average size of bet immediately before the off, leads them to conclude that "undoubtedly, the most striking conclusion relates to the unambiguous confirmation of the existence of a subset motivated by financial returns".

It is also important to note that a fixed entertainment value of gambling, with small-stake betting by near risk-neutral agents, predicts equality of expected returns across the expected return-win probability indifference curve. This poses a major problem, since it is inconsistent with the favourite-longshot bias, (in which bets on longshots (low-probability bets) have low mean returns relative to bets on favourites (high probability bets), one of the key empirical findings observed in numerous studies of horse-race betting and other gambling markets [see e.g. Sauer (1998), Vaughan Williams (1999)].

Modification to allow for differential excitement based on the outcome odds can explain small-stake gambles on longshots [see Conlisk (1994)], but cannot explain betting on more favoured horses or the fact that, by construction, the greatest volume of money is bet on such horses, often involving large stakes. For instance, in US racetrack betting Golec and Tamarkin (1998) point out that for a race with an even-money favourite (i.e. with odds of 1/1), about 42% of the money bet is on the favourite (with track take of 17%). For a race favourite at 2/1 it would be about 28%. Bruce and Johnson (1992) report that average stakes on first favourites in the UK were £22.63, and on second favourites £6.40.

[viii] If we define the current level of wealth as W, and the level of utility associated with W as $\overline{U}$ then the exponential utility function

$$U = \overline{U} + U(W + x) \qquad \text{(a)}$$

defines utility for increases in wealth above W, where $W + x$ is wealth measured from W to ∞. We require that the marginal utility and the second derivative for an increase in wealth, $\frac{\partial U}{\partial x} > 0, \frac{\partial^2 U}{\partial x^2} < 0$. For a decrease in wealth below W, we define the utility function as

$$U = \overline{U} - U(W - x) \qquad \text{(b)}$$

where $W - x$ is wealth measured from 0 to W. We require that the marginal utility and the second derivative for a decrease in wealth are both positive, as postulated by Kahneman and Tversky.

[ix] Tversky and Kahneman (1992) assumed that the value function was of the power form. However, this is not suitable for the analysis of optimal gambling over mixed prospects, since for small stakes the assumption of loss-aversion is violated and the agent becomes infinitely gain loving as the stake approaches zero. Let the value function be

$$v(x) = \begin{cases} x^{\alpha} & x \geq 0 \quad (0<\alpha<1 \\ -\lambda(-x)^{\beta} & x < 0 \quad (\lambda > 0, 0<\beta<1) \end{cases} \qquad \text{(c)}$$

with $\beta > \alpha$ to ensure that stake size is determinate.

The shape of the indifference curve in expected return-win probability space is given by

$$\frac{d\mu}{dp} = (1 + o - \frac{\varepsilon^{gp} o}{\alpha} + \frac{\varepsilon^{lp} w^{-}(1-p)\lambda s^{\beta-\alpha} o^{1-\alpha}}{\alpha w^{+}(p)}) \qquad \text{(d)}$$

For the power function, $EU \geq 0$ implies that $1 \geq \dfrac{\lambda w^{-}(1-p)s^{\beta-\alpha}}{w^{+}(p)o^{\alpha}}$

Consequently, since $\varepsilon^{gp} > 0$ and $\varepsilon^{lp} < 0$, we note from (d) that if $\alpha < \varepsilon^{gp}$ then $\dfrac{d\mu}{dp} < 0$ for large odds, so that the agent will not exhibit risk-loving behaviour over low probability gambles. However, it can be shown that the agent will bet at actuarially unfair odds in this case if stakes are low enough. With $\varepsilon^{gp} > \alpha$ the slope of the indifference curve in expected return- win probability space will be positive over some range, given that stake size is not too large.

[x] Employing alternative functional forms made no qualitative difference [e.g. Prelec (1998), Wu and Gonzalez (1996)].

[xi] This possibility receives some support from the empirical analysis of race-track betting by Jullien and Sallanie (2000).

[xii] There is an observational equivalence between increasing (decreasing) s and decreasing (increasing) δ in this model.

[xiii] In this case EU is given by

$$EU = w^{+}(p)(1-e^{-r\delta g^{n}}) - w^{-}(1-p)\lambda(1-e^{-\delta l^{n}}) \geq 0 \quad \text{(a)}$$

where g is the gain from the gamble and l the loss; n is a constant, $0 < n \leq 1$.
Other parameters are constants as defined in the text. As $\delta \to 0$, equation (a) simplifies to the power function. With n = 1 we have the exponential case.
We assume the following values: $r = 14$, $\lambda = 28$, $\delta = 0.0015$, $n = 0.7$.
For the probability weighting functions we assume the values found by Tversky and Kahneman, namely 0.61 for gains and 0.69 for losses. We find with these parameter values that (1) the agent requires to win at least $25.31 in order to accept a gamble incurring a loss of $10 at probability 0.5. These numbers are close to those reported by Kahneman and Tversky.

(2) The agent will accept the gamble to win infinity or lose $100 at probability 0.5, and will, in fact, accept the gamble at probability 0.5 for any gain exceeding $828.
Implications (1) and (2) might appear reasonable a priori. However, the same agent (3) will also turn down a gamble involving the loss of $10 with probability 0.924 and a win of infinity with probability p=0.076. This seems absurd.

[xiv] The postulated degree of loss aversion is based partly on experimental evidence in which students required a "substantial" win of approximately $30 in order to induce them to bet $10 on a 50/50 chance.

[xv] Note that in the exponential value function we can redefine s as the percentage of wealth (by deflating by total wealth and redefining the parameters).

[xvi] The literature on the psychology of gambling includes reference to the denial by pathological gamblers of the reality of their gambling situation, including the odds of winning or losing [Ladouceur et al. (1995)].