Lancaster University Management School
Working Paper
2002/013

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The Role of Options in an Economy with Background Risk: A Note

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March 24, 2002

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Abstract

This note presents three results closely related to Franke, Stapleton and Subrahmanyam's work [11] on the role of options in an economy with non-hedgeable background risk. It first shows two necessary conditions for the existence of equilibrium when negative terminal wealth is not allowed. It then shows the impact on investors' cautiousness of background risk and gives a simple proof of one main result in their work. Thirdly, it shows how investors construct their optimal sharing rules using call options. Keywords: Options, role of options, optimal sharing rule, portfolio insurance, background risk, cautiousness. *Journal of Economic Literature* Classification Numbers: D52, D81, G11, G13.

1 Introduction

Since the setup of the problem was given in details by Franke, Stapleton and Subrahmanyam (see [11]), we shall introduce the notation without much comment. The reader is encouraged to consult [11] for further details.

We assume a two-date, pure exchange economy. There are N investors indexed by $i = 1, 2, ..., N$. An investor, say i, may have non-hedgeable background risk denoted by e_i, which is bounded from below: $e_i \geq \underline{e_i}$. We assume that $\underline{e_i}$ is the largest lower bound of e_i, i.e., for any $z > \underline{e_i}$, $Prob(e_i > z) < 1$. The investor is allowed to buy contingent claims on the market portfolio to construct his optimal sharing rule. X is the ending-time payoff on the market portfolio and is assumed to be continuous on $R^+ = (0, +\infty)$. We use $x_i(X)$ to denote investor i's sharing rule. Thus his ending-time income may be written as $y_i = x_i(X) + e_i$. Investor i's utility function is assumed to be of the hyperbolic absolute risk aversion (HARA) form

$$v_i(x) = \frac{1}{1-\gamma}(A_i + x)^{1-\gamma} \tag{1}$$

where $\gamma > 0$ is a constant across investors and A_i is a threshold parameter. We call $v_i(x)$ the direct utility function to differentiate it from

$$u_i(x) = E_{e_i}(v_i(x + e_i)),$$

which is called the indirect utility function. The investor solves the following maximization problem

$$\max_{x_i} E[E_{e_i}(v_i(x_i + e_i))]$$

$$\text{s.t.} \quad E[(x_i - x_{i0}(X))\phi(X)] = 0,$$

where $\phi(X)$ is the pricing kernel whose functional form is determined in equilibrium and $x_{i0}(X)$ is investor i's initial endowment. From [11] we have the first order condition to investor i's utility maximization problem:

$$u_i'(x_i) = \lambda_i \phi(X), \tag{2}$$

where λ_i is the Lagrangian multiplier.

2 On the Existence of Equilibrium

In [11] conditions for the existence of equilibrium were not discussed. Although there is a rich literature on the existence of equilibrium, the economies discussed are different. To show that the problem is not trivial we here discuss some necessary conditions for the existence of equilibrium.

We now give the following result.

Lemma 1 *Assume investors are not homogeneous. If terminal wealth is not allowed to be negative, then the following two equations are necessary conditions for the existence of an equilibrium with interior solutions in the economy.*

$$\lim_{x \to 0} u_i'(x) = +\infty. \tag{3}$$

$$A_i + \underline{e_i} = 0. \tag{4}$$

Proof: Assume that an equilibrium with interior solutions exists. Thus all first order conditions are equalities. Suppose for some i, $\lim_{x \to 0} u_i'(x) < +\infty$ and for some j, $\lim_{x \to 0} u_j'(x) = +\infty$. When $X \to 0$, we naturally have $x_i(X) \to 0$

and $x_j(X) \to 0$. But from (2) we can see that $x_i \to 0$ and $x_j \to 0$ cannot hold simultaneously. This implies that an equilibrium with interior solutions cannot exist.

On the other hand, suppose for every i, $\lim_{x\to 0} u_i'(x) < +\infty$. Let $x_i \to 0$ for every i. From (2), we obtain $\lambda_i = u_i'(0)/\nu$, where $\nu = \phi(0)$ is a constant, for $i = 1, 2, ..., N$. From (2) we obtain

$$x_i = u_i^{-1}((u_i'(0)/\nu)\phi). \tag{5}$$

Since $\sum_{i=1}^{N} x_i = X$ we obtain $\phi = \phi(X;\nu)$. Substituting this into the above equation we obtain $x_i = x_i(X;\nu)$. So far the derivation of ϕ and x_i has nothing to do with the N budget constraints $E((x_i - x_{i0})\phi) = 0$, $i = 1, 2, ..., N$ and they do not depend on x_{i0}, $i = 1, 2, ..., N$. Now substituting $x_i = x_i(X;\nu)$ and $\phi = \phi(X;\nu)$ into the N budget constraints, we obtain N equations with just one unknown variable ν. The fact that ϕ and x_i do not depend on x_{i0}, $i = 1, 2, ..., N$ implies that we can freely choose x_{i0}, $i = 1, 2, ..., N$ and the N equations always hold. This is obviously contradictory unless the investors are homogeneous. Thus there does not exist such kind of equilibrium.

We now prove the second half. Given background risk e_i distributed in $(\underline{e_i}, +\infty)$, we must have $A_i + e_i \geq 0$ otherwise the (direct) utility function $v_i(x_i + e_i)$ will not be defined for $0 < x_i < -(A_i + \underline{z_i})$. Moreover, if $A_i + \underline{e_i} > 0$, we will have $u_i'(0) = E_{e_i}(v_i'(e_i)) < (A_i + \underline{e_i})^{-\gamma} < +\infty$. According to the first half of the lemma, there will be no equilibrium with interior solutions. Hence the equation is a necessary condition. Q.E.D.

3 Impact on Investors' Cautiousness of Background Risk

As is well known, the convexity of an investor's optimal sharing rule is related to his cautiousness which is defined as the first derivative of his risk tolerance.[1] In this section we first give a result showing the impact on an investor's cautiousness of his non-hedgeable background risk. Later we use it to give a simple proof of one main result in [11]. To avoid confusion, in the rest of the paper we will always use $R_0(x)$ and $C_0(x)$ to denote the absolute risk aversion and cautiousness of the direct utility function and use $R(x)$ and $C(x)$ to denote those of the indirect utility function.

Proposition 1 *When an investor with HARA class utility is exposed to background risk, his cautiousness will become strictly higher.*

Proof: Given (direct) utility function $v(x)$, by definition, cautiousness $C_0(x) = (1/R_0(x))'$, where $R_0(x) = -v''(x)/v'(x)$ is absolute risk aversion. It can be written as

$$C_0(x) = P_0(x)/R_0(x) - 1,$$

where $P_0(x) = -v'''(x)/v''(x)$ is absolute prudence. For HARA class utility, $C_0(x)$ is a positive constant. Thus $P_0(x)/R_0(x)$ is also a positive constant. Write $P_0(x)/R_0(x) = \lambda$ for any x. Let $A = P_0(x+a)/R_0(x+b)$ and $B =$

[1]The definition of cautiousness can be found in Wilson (1968). Leland [15] stated in his Proposition I that everything being equal the optimal sharing rule of an investor with higher cautiousness is more likely to be convex.

$P_0(x+b)/R_0(x+a)$. Since $(A+B)^2 \geq 4AB$, we have, for any a and b,

$$P_0(x+a)/R_0(x+b) + P_0(x+b)/R_0(x+a) \geq 2\lambda \tag{6}$$

Rearranging the terms in (6), we have, for any a and b,

$$v'''(x+a)v'(x+b) + v'''(x+b)v'(x+a) \geq 2\lambda v''(x+a)v''(x+b) \tag{7}$$

Assuming a and b are independent and have identical distribution as e and taking expectations of both sides of (7), we obtain

$$2E(v'''(x+e))E(v'(x+e)) \geq 2\lambda(E(v''(x+e)))^2 \tag{8}$$

Rearranging the terms in (8), we immediately obtain

$$\frac{E(v'''(x+e))E(v'(x+e))}{E(v''(x+e)))^2} \geq \lambda,$$

which implies that $C(x) > C_0(x)$, where $C(x)$ is the cautiousness of the indirect utility function. Q.E.D.

Proposition 1 states that when an investor with HARA utility exposed to background risk, he will have higher cautiousness than before. We now use this result to give a simple proof of Theorem 3 in [11].

> Theorem 3 in [11]: *Suppose that there is an investor who has no background risk in an economy where some other investors face background risk. The sharing rule of this investor is strictly concave.*

Proof: Differentiating both sides of (2) we have

$$x_i'(X) = R_e(X)/R_i(x_i), \tag{9}$$

where $R_e(X)$ is the representative investor's absolute risk aversion and $R_i(x_i)$ is investor i's indirect absolute risk aversion respectively. Differentiating both sides of the above equation, we obtain

$$x_i''(X) = R_e^2(X)R_i^{-1}(x_i)[C_i(x_i) - C_e(X)], \tag{10}$$

where $C_e(X)$ is the representative investor's cautiousness and $C_i(x_i)$ is investor i's indirect cautiousness respectively. Noting that $\sum_i x_i = X$, from the above equation we have

$$C_e(X) = \sum_i s_i C_i(x_i), \tag{11}$$

where $s_i = R_e(X)/R_i(x_i)$. According to Proposition 1, in a HARA economy, the cautiousness of the indirect utility function is strictly larger than that of the direct utility function. From Eq. (11), we easily verify that the representative investor's cautiousness will become strictly larger when some investors exposed to background risk, thus it will become strictly larger than the cautiousness of those without background risk. It follows (10) that the optimal sharing rules of those without background risk will be strictly concave. Q.E.D.

This proof is transparent and intuitive. It shows clearly that the reason that investors without background have concave optimal sharing rules is that their coefficients of cautiousness become relatively lower compared with those of the investors with background risk.

4 The Role of Options

In their Theorem 2, [11] derived investors' optimal sharing rules in an economy where investors have power utility functions with an identical power coefficient.[2] However, the form of the optimal sharing rules are complicated and it is still not clear what contingent claims the investors buy. In this section we show precisely how investors construct their optimal sharing rules using call options. For convenience, we use $\gamma_i(x)$ to denote investor i's coefficient of indirect relative risk aversion, i.e.,

$$\gamma_i(x) \equiv -xE_{e_i}[v_i''(x+e_i)]/E_{e_i}[v_i'(x+e_i)],$$

[2]Theorem 2 in [11]: Suppose that investors in the economy have power utility functions with an identical power coefficient $\gamma > 0$. Then investor i's optimal sharing rule is

$$x_i = A_i^* + a_i X + a_i[\psi_i^*(x_) - \psi(X)],$$

where

a) $A_i^* = \alpha_i A - A_i$ is the investor's risk free income at time 1, where $A = \sum_i A_i$ and

$$a_i = \lambda_i^{-\frac{1}{\gamma}} / \sum_i \lambda_i^{-\frac{1}{\gamma}},$$

b) $a_i X$ is the investor's linear share of the market portfolio payoff,

c) $a_i[\psi_i^*(x_i) - \psi(X)]$ is the investor's payoff from contingent claims, where $\psi_i^* = \psi_i/a_i$ and

$$\psi(X) = \sum_i \psi_i(x_i).$$

where $v_i(x)$ is his direct utility function given in (1). We assume that negative terminal wealth is not allowed.[3] Thus (4) must hold, which implies that

$$\text{for every investor } i \text{ with background risk,} \quad \gamma_i(0) < \gamma. \tag{12}$$

For convenience, given function $h(x)$ we will always use $h(0)$ to denote $\lim_{x\to 0} h(x)$. We first give the following lemma.

Lemma 2 *Assume for every i, $0 < \gamma_i(0) < +\infty$. Then for every investor i without background risk, $\lim_{X\to 0} x_i(X)/X > 0$ and $x_i'(0) > 0$. For any investor i with background risk, $\lim_{X\to 0} x_i(X)/X = 0$ and $x_i'(0) = 0$.*

Proof: See the Appendix.

We are now ready to present the following result.[4]

Proposition 2 *Assume that investors have power utility functions with an identical power coefficient $\gamma > 0$ and for every i, $0 < \gamma_i(0) < +\infty$. Then investor i's optimal sharing rule can be constructed as follows.*

$$x_i = x_i'(0)X + \int_0^{+\infty} x_i''(K)[c(X;K)]dK, \tag{13}$$

[3]This will be guaranteed by the condition that for every i, $0 < \gamma_i(0) < +\infty$. It can be verified that this condition implies (3) and (4).

[4]The result is closely related to Theorem 1 in Carr and Madan (2001). In fact, the proposition can be seen as an application of their result. They stated that if $\lim_{X\to 0} x_i'(X)$ exists then every twice differentiable function $x_i(X)$ can be written as (13). But their result does not tell you when $\lim_{X\to 0} x_i'(X)$ exists so that every investor's optimal sharing rule can be constructed as (13).

where $x_i'(0) = 0$ *for any investor* i *with background risk,* $x_i'(0) > 0$ *for any investor* i *without background risk,* $x_i''(K) < 0$ *for any investor* i *without background risk and* $c(X;K)$ *is the payoff of the call option on the market portfolio with strike price* K*.*

Proof: Applying Lemma 2, we have for any investor i with background risk, $x_i'(0) = 0$; and for any investor i without background risk, $x_i'(0) > 0$. From Theorem 3 in [11] shown in the last section we have for any investor i without background risk $x_i''(K) < 0$.

Now the right side of (13) can be written as

$$x_i'(0)X + \int_0^X x_i''(K)[X - K]dK,$$

which can be written as

$$x_i'(0)X + X\int_0^X x_i''(K)dK - \int_0^X Kx_i''(K)dK.$$

This is equivalent to $x_i(X) - x_i(0)$. But $x_i(0) = 0$. Hence (13) is proved. Q.E.D.

The proposition tells us that any investor without background risk holds a fraction of the market portfolio plus a portfolio of written call options on the market portfolio while any investor with background risk just holds a portfolio of long or short positions in call options on the market portfolio.

Unlike Theorem 2 in [11], the above proposition clearly tells us how to construct investors' optimal sharing rules using call options. An interesting point implied by the result is that additional to the market portfolio, call options on the market portfolio with strike prices from zero to infinity can sufficiently satisfy the needs of the investors and obtain Pareto efficiency in the economy.

5 Conclusions

We show that when exposed to background risk, the cautiousness of investors with HARA class utility becomes strictly higher. Because of this investors without background risk in the economy have concave optimal sharing rules, which implies that they sell options. This is consistent with the result obtained by Leland in [15], which states that (other things being equal) investors with lower cautiousness are more likely to sell options. We also show how investors construct their optimal sharing rules using call options. Interestingly, the result implies that additional to the market portfolio, a set of call options with all strike prices are sufficient to obtain Pareto efficiency in the economy.

A Proof of Lemma 2

Since $\sum_i x_i'(X) = 1$, from (9) we have

$$R_e(X) = 1/\sum_i R_i^{-1}(x_i) \tag{14}$$

or

$$\gamma_e(X) = \sum_i x_i'(X)\gamma_i(x_i). \tag{15}$$

Let I_0 be the set of investors without background risk. We now assert that

$$\text{for every } i \notin I_0, \quad \lim_{X\to 0} w_i(X) = 0, \tag{16}$$

where

$$w_i(X) \equiv x_i(X)/X.$$

Otherwise, we have for some investor $i \notin I_0$,

$$\limsup_{X\to 0} w_i(X) > 0,$$

which implies that

$$\limsup_{X\to 0} \sum_{i\in I_0} w_i(X) < 1. \tag{17}$$

This together with Equations (15) and (12) implies that $\limsup_{X\to 0} \gamma_e(x_i) < \gamma$. Hence from (9) for any investor $i \in I_0$

$$\limsup_{X\to 0} \frac{d\ln w_i(X)}{d\ln X} = \limsup_{X\to 0} \frac{\gamma_e(X) - \gamma}{\gamma} < 0$$

This implies that $\lim_{X\to 0} w_i(X) = +\infty$ which is impossible. Hence the assertion is proved.

Moreover, from (9) and (14) we have

$$x_i'(X) = w_i(X)\frac{\gamma_i^{-1}(x_i)}{\sum_i w_i(X)\gamma_i^{-1}(x_i)}. \tag{18}$$

Since for every i, $0 < \gamma_i(0) < +\infty$ it follows that

$$\lim_{X\to 0} w_i(x) = 0 \text{ if and only if } \lim_{X\to 0} x_i'(x) = 0$$

Hence from (16) we have for any investor $i \notin I_0$,

$$\lim_{X\to 0} x_i'(X) = 0.$$

Now noting that for all investors without background risk they have the same power utility function. From (2) we conclude that for any $i, j \in I_0$, x_i/x_j is a constant. But from (16) we have

$$\lim_{X\to 0}\sum_{i\in I_0} w_i(X) = 1.$$

It follows that for every $i \in I_0$, $\lim_{X\to 0} w_i(X) > 0$

Similarly we have for every investor $i \in I_0$, $\lim_{X\to 0} x_i'(X) > 0$. Q.E.D.

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