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Abstract

In this paper we investigate the relationships between prudence, risk aversion, and cautiousness. These three measures explain investors' investment decisions in the money market, stock market, and option market respectively.[1] Thus to understand how investors' investment decisions in the three markets are related it is important to know the relationships between the three measures. We show that roughly speaking, if an investor's (absolute or relative) prudence has some feature then his (absolute or relative) risk aversion will have the same feature. We also show that roughly speaking, if an investor's cautiousness has some feature then his relative risk aversion will have the opposite feature.

[1]See Pratt (1964), Kimbal (1990), and Huang (2004). Also see Section 1 of this paper.

Introduction

Investors pursue financial activities in three main financial markets, namely the money market, stock market, and option market. It is, of course, of great interest to know how they make investment decisions in these three financial markets. Pratt (1964) and Arrow (1965) developed the theory of risk aversion to explain investors' behavior in the stock market. Pratt (1964) showed that the higher an investor's risk aversion the more risk premium he demands and the less investment he makes in the stock market.

According to Leland (1968), "the 'precautionary' demand for saving usually is described as the extra saving caused by future income being random rather than determinate." He concluded that the precautionary saving in response to risk is associated with a positive third derivative of a von Neumann-Morgenstern utility function. Kimball (1990) later developed the theory of prudence to measure the strength of an investor's motive to make precautionary savings. The theory is an anology to Pratt's (1964) theory of risk aversion. Kimball showed that the more prudent an investor, the more precautionary savings he will make and the more precautionary premium he demands.

The definition of cautiousness was first brought by Wilson (1968) without explanations. Cautiousness is equal to the ratio of prudence to risk aversion minus one. Leland (1980) suggested to use this measure to explain convexity of investors' optimal payoff functions which is related to the feature of options. Huang (2004) established that cautiousness measures of investors' tendencies to buy options. He showed that if investor i's lowest possible cautiousness is higher than j's highest possible cautiousness, then investor i has a stronger tendency to buy an option regardless of other conditions; and the reverse is also true.

Obviously to understand how investors' investment decisions in the three financial markets are related it is important to know the relationships between the three preference measures, namely prudnence, risk aversion, and cautiousness. Eeckhoudt and Schlesinger (1994) gave some good results on the relationship between prudence and risk aversion. In this paper we cast further light on this. We show that roughly speaking, if an investor's (absolute or relative) prudence has some feature then his (absolute or relative) risk aversion will have the same feature (sometimes with a minor additional condition).

We also give a thorough research on the relationship between cautiousness and risk aversion. We show that, roughly speaking, if an investor's cautiousness has some feature then his relative risk aversion will have the opposite feature (sometimes with a minor additional condition).

We also show some inter-personal chararacteristics of the relationships between the three measures. For example, we show that given two investors with different initial wealth, if one has higher relative prudence in all states than the other, he also has higher relative risk aversion in all states given that they have zero marginal utility of infinite wealth.

All these results have interesting implications for undstanding how investors' investment decisions in the three financial markets are related, which are explained in this paper.

The structure of this paper is as follows: In Section 1 we briefly introduce the three investment decision problems in the money market, stock market, and option market and show how the three preference measures explain investors' decisions regarding these three problems. In this section we also slightly extend the results on intra-personal comparative risk aversion and prudence in Pratt (1964) and Kimball (1990) to the inter-personal case. In Section 2 we discuss the relationship between absolute prudence and absolute risk aversion. Section 3 discusses the relationship between relative prudence and relative risk aversion. Section 4 shows the relationship between cautiousness and absolute risk aversion. Section 5 shows the relationship between cautiousness and relative risk aversion. Section 6 discusses extensions of the results on relative risk aversion and relative prudence to partial relative risk aversion and partial relative prudence. Section 7 concludes the paper. Lengthy proofs are put in appendices.

1 Prudence, Risk Aversion, and Cautiousness

Given a utility function $u(x)$, Arrow (1965) and Pratt (1964) defined its absolute risk aversion as the negative ratio of its second derivative to its first derivative.[2] If we use $R(x)$ to denote its absolute risk aversion, then we have $R(x) \equiv -u''(x)/u'(x)$. Kimball (1990) defined its absolute prudence as the negative ratio of its third derivative to its second derivative. If we use $P(x)$ to denote its absolute prudence, then we have $P(x) \equiv -u'''(x)/u''(x)$. The first derivative of risk tolerance is first called cautiousness by Wilson (1968). Equivalently it is equal to the ratio of prudence to risk aversion minus one. If we use $C(x)$ to denote cautiousness, then we have $C(x) \equiv P(x)/R(x) - 1$.[3]

1.1 Prudence and the Money Market

Consider a two-period economy. An investor with first-period utility function $v(x)$ and second-period utility function $u(x)$ has the following consumption-saving problem in the money market.

$$\text{(I)} \qquad \max_c v(c) + Eu((w_0 - c)r + \epsilon),$$

where w_0 is the investor's initial wealth, $c \equiv c(\epsilon)$ is his first period consumption, r is the total return of 1 unit of money saved in his bank account in the second period, and ϵ is the uncertainty in his wealth. Note $w_0 - c$ is the amount of money the investor will save in his bank account.

[2]In this paper we always assume that utility functions are strictly increasing, strictly concave, and three times differentiable unless stated otherwise.

[3]Throughout the paper we use P or P with subscripts to denote absolute prudence, R or R with subscripts to denote absolute risk aversion, and C or C with subscripts to denote cautiousness.

Leland (1968) and Kimball (1990) developed the theory of precautionary savings which explains investors' decisions with respect to problem (I). Kimball (1990) showed that prudence, which is defined as the negative ratio of the third derivative of $u(x)$ to its second derivative, measure the strength of the investor's motive to make precautionary savings responding to the uncertainty in his wealth.

Given two utility functions $u_1(x)$ and $u_2(x)$. Let P_i denote the absolute prudence of $u_i(x)$, $i = 1, 2$. Let $c = c_i(\epsilon)$ be the solution if $u(x) = u_i(x)$, $i = 1, 2$. Kimball (1990) showed that the following conditions are equivalent.

- For all $x \geq 0$, $P_1(x) \geq P_2(x)$.
- $c_1(\epsilon) \leq c_2(\epsilon)$ for all ϵ.

The same equivalences hold if attention is restricted throughout to an interval.

The result tells us that if an investor becomes uniformly more prudent then he will consume less and make more precautionary savings for all ϵ.

The above Kimball's (1990) original result is about intra-personal comparative prudence. This result can be extended to the interpersonal case where investors have different initial wealth. Before we proceed, we first introduce some notation. Given a utility function $u(x)$, its relative prudence is defined as its absolute prudence multiplying x. If we use $\beta(x)$ to denote its relative prudence then we have $\beta(x) \equiv xP(x)$, where $P(x)$ is its absolute prudence.[4]

In order to facilitate the comparison of investors' decisions with respect to the consumption-saving problem, we now assume that investors have the same utility functions in the first and second periods. Denote investor i's utility function by $u_i(x)$. Assume investor i initially has w_i units of wealth. Now the uncertainty $\hat{\epsilon}$ is proportional to his wealth. Let $\hat{c}_i \equiv \hat{c}_i(\hat{\epsilon})$ be his first period consumption as a proportion of his initial wealth. Then the consumption-saving problem becomes

$$(\mathrm{I}') \qquad \max_{\hat{c}_i} u_i(\hat{c}_i w_i) + \rho E u_i(w_i((1 - \hat{c}_i)r + \hat{\epsilon})).$$

where ρ is the time discount factor. We have the following result.

Proposition 1 *Let $\beta_i(x)$ be the relative prudence of $u_i, i = 1, 2$. Let $\hat{c}_i$ be the solution to investor i's problem in the money market.*

1. *Assume $\rho r = 1$. Then for all $x \geq 0$, $\beta_1(w_1 x) \geq \beta_2(w_2 x)$ if and only if for all ϵ $\hat{c}_1(w_1, \epsilon) \leq \hat{c}_2(w_2, \epsilon)$.*
2. *Assume $\rho r \leq 1$. Then for all $x \geq 0$, $\beta_1(w_1 x) \geq \beta_2(w_2 x)$ implies for all ϵ $\hat{c}_1(w_1, \epsilon) \leq \hat{c}_2(w_2, \epsilon)$.*

The same equivalences hold if attention is restricted throughout to an interval.

Proof: See Appendix Appendix A.

[4]Throughout this paper given a utility function $u(x)$ we will always use $\beta(x)$ to denote its relative prudence. When the utility function $u(x)$ has a hat or a subscript, the corresponding $\beta(x)$ will also have a hat or the subscript.

1.2 Risk Aversion and the Equity Market

An investor may also have the following investment problem in the equity market.

$$\text{(II)} \qquad \max_x Eu(w_0 r + x(\epsilon - r)),$$

where $\varepsilon = S/S_0$, S_0 and S are the values of the equity at time 0 and 1 respectively, and x is his position in the equity market.

Arrow (1965) and Pratt (1964) developed the theory of risk aversion which explains investors' decisions with respect to problem (II). They showed that the higher an investor's measure of risk aversion, the more money he invests in the equity market. As in the first problem, assume two investors have the same initial wealth. Let R_i denote the absolute risk aversion of u_i, $i = 1, 2$. Let $x = x_i(\varepsilon)$ be the solution if $u(x) = u_i(x)$, $i = 1, 2$. They showed that the following conditions are equivalent.

- For all $x \geq 0$, $R_1(x) \geq R_2(x)$.
- $x_1(\varepsilon) \leq x_2(\varepsilon)$ for all ε .
- $g(t) = u_1(u_2^{-1}(t))$ is a concave function of t.

The same equivalences hold if attention is restricted throughout to an interval.

The result tells us that if an investor becomes uniformly more risk averse then he will invest less in the equity market whatever the return on the equity is.

The above Pratt's (1964) original result is about intra-personal comparative risk aversion. This result can be easily extended to the interpersonal case where investors have different initial wealth. Before we proceed, we first explain the notation. Given a utility function $u(x)$, its relative risk aversion is defined as its absolute risk aversion multiplying x. If we use $\gamma(x)$ to denote its relative risk aversion then we have $\gamma(x) \equiv xR(x)$, where $R(x)$ is its absolute risk aversion.[5]

Assume investor i invest $\hat{x}_i$ proportion of his wealth in the equity market. Then investor i's investment problem in the equity market becomes:

$$\text{(II}'\text{)} \qquad \max_{x_i} E[u_i(w_i r + x_i w_i(\varepsilon - r))].$$

We have the following result.

Proposition 2 *Let $\gamma_i(x)$ and x_i be the relative risk aversion and optimally invested proportion of initial wealth corresponding to investor i, $i = 1, 2$. Then the following conditions are equivalent.*

1. *For any $x \geq 0$, $\gamma_1(w_1 x) \geq \gamma_2(w_2 x)$.*
2. *$x_1(w_1, \varepsilon) \leq x_2(w_2, \varepsilon)$ for all ε [and $<$ if $0 < x_1(w_1, \varepsilon) < 1$].*

[5]Throughout this paper given a utility function $u(x)$ we will always use $\gamma(x)$ to denote its relative risk aversion. When the utility function $u(x)$ has a hat or a subscript, the corresponding $\gamma(x)$ will also have a hat or the subscript.

3. $g(t) = u_1(\frac{w_1}{w_2} u_2^{-1}(t))$ *is a concave function of* t.

The same equivalences hold if attention is restricted throughout to an interval.

Proof: Given investor i's utility function $u_i(x)$, define

$$\hat{u}_i(x) \equiv u_i(w_i x). \tag{1}$$

We call $\hat{u}_i(x)$ investor i's transformed utility function. Although investors have different initial wealth, that is in terms of their original utility functions. In terms of the transformed utility function all investors have the same initial wealth; thus we can apply Pratt's (1964) results on the transformed utility functions.

Let $\hat{R}_i(x)$ and $\hat{P}_i(x)$ be the absolute risk aversion and absolute prudence of $\hat{u}_i(x)$. Let $\gamma_i(x)$ and $\beta_i(x)$ be the relative risk aversion and relative prudence of $u_i(x)$. From (1), we have

$$\hat{R}_i(x) = \gamma_i(w_i x)/x, \tag{2}$$

and

$$\hat{P}_i(x) = \beta_i(w_i x)/x. \tag{3}$$

Applying Pratt's (1964) Theorem 1 and Theorem 7, noting that

$$\hat{R}_i(x) = w_i R_i(w_i x) = \gamma_i(w_i x)/x,$$

we immediately obtain the proposition. This completes the proof.

1.3 Cautiousness and the Option Market

Now assume there is an option written on the stock. The third investment problem, which involves the option, is

$$\text{(III)} \qquad \max_{x,y} Eu(w_0 r + x(\frac{S}{S_0} - r) + y(\frac{c(S)}{c_0} - r)),$$

where c_0 and $c(S)$ are the values of the equity at time 0 and 1 respectively and y is the investor's position in the option.

Huang (2004) studied the third problem and established that cautiousness, which is equal to the ratio of prudence to risk aversion minus one, measures an investor's tendency to buy options. He showed that the following two conditions are equivalent:

- There exists a constant $C \geq 0$ such that for all x $C_i(x) \geq C \geq C_j(x)$.
- Investor j buys the derivative only if investor i does, and investor i sells the derivative only if investor j does, regardless of their initial wealth, the interest rate, the stock price, the derivative price, and the distribution of the future stock price.

In Sections 2 of this paper we will show the relation between an investor's decisions in Problems (I) and (II) by investigating the relationship between prudence and risk aversion. In Sections 3 we will show the relation between an investor's decisions in Problems (I′) and (II′) by investigating the relationship between relative prudence and relative risk aversion. In Sections 5 we will show the relation between an investor's decisions in Problems (I) and (III) by investigating the relationship between cautiousness and absolute risk aversion. In Sections 5 we will show the relation between an investor's decisions in Problems (I′) and (III) by investigating the relationship between cautiousness and relative risk aversion.

2 Absolute Prudence and Absolute Risk Aversion

An investor's decisions on precautionary saving and investment in equity must be closely related, and prudence must be closely associated with risk aversion. In the early days, the difference between investor's motivation of precautionary saving and risk aversion was sometimes even ignored. As pointed out by Leland (1968), "when rigor is absent, economists have tended to equate the precautionary demand for saving with the concept of risk avoidance." Although it is inappropriate to equate the concept of prudence to that of risk aversion, there is no doubt that the two are closely related to each other. In this section we show the relationship between absolute prudence and absolute risk aversion, which will reveal how an investor's decision in problem (I) is related to his decision in problem (II).

Lemma 1 *Given two utility functions $u_1(x)$ and $u_2(x)$, assume for all $x \in (a, A)$, $P_1(x) > P_2(x)$, where A can be $+\infty$. Then the following statements are true.*

1. *For some $x \in (a, A)$, $R_1(x) > R_2(x)$, if and only if there exists $x^\circ \in (x, A) \cup \{A\}$, such that $u_1'(x^\circ)/u_1'(x) \leq u_2'(x^\circ)/u_2'(x)$.*[6]
2. *For some $x \in (a, A)$, $R_1(x) < R_2(x)$, if and only if there exists $x^\circ \in [a, x)$, such that $u_1'(x^\circ)/u_2'(x^\circ) < u_1'(x)/u_2'(x)$.*[7]

Both statements hold if all $>$'s and $<$'s are replaced by $\geq$'s and $\leq$'s respectively.

Proof: See Appendix B.

The first statement generalizes a result obtained by Eeckhoudt and Schlesinger (1994). Their result gives a sufficient condition for the above relationship between the two preference measures, which states that if utility function $u_1(x)$

[6] Throughout this paper, given a utility function $u(x)$, which is defined for $x \in (a, A)$, $u_i'(a)$ is defined as $\lim_{x\to a} u_i'(x)$ and $u_i'(A)$ is defined as $\lim_{x\to A} u_i'(x)$, $i = 1, 2$.

[7] The ratio $u_1'(a)/u_2'(a)$ is defined as $\lim_{x\to a} u_1'(x)/u_2'(x)$.

always has higher absolute prudence than utility function $u_2(x)$, then $u_1(x)$ has higher absolute risk aversion than $u_2(x)$ at some x, if the ratio of the latter's marginal utility and the former's marginal utility at x is no larger than the limit of the ratio when x approaches infinity.

Proposition 3 *Given two utility functions $u_1(x)$ and $u_2(x)$, assume for all $x \in (a, A)$, $P_1(x) > P_2(x)$, where A can be $+\infty$. Then the following statements are true.*

1. *For all $x \in (a, A)$, $R_1(x) > R_2(x)$, if and only if for all $x \in (a, A)$, $u_1'(A)/u_1'(x) \leq u_2'(A)/u_2'(x)$.*
2. *For all $x \in (a, A)$, $R_1(x) < R_2(x)$, if and only if for all $x \in (a, A)$, $u_1'(a)/u_2'(a) < u_1'(x)/u_2'(x)$.*

Both statements hold if all $>$'s and $<$'s are replaced by $\geq$'s and $\leq$'s respectively.

Proof: We first prove the first statement. The sufficiency part is immediately implied by the first statement of Lemma 1. Thus we need only show the necessity part. If for every $x \in (a, A)$, $R_1(x) > R_2(x)$, then for any $x \in (a, A)$, we have $(\ln(u_1'(x)/u_2'(x)))' < 0$, which implies that $ln(u_1'(A)/u_2'(A)) < ln(u_1'(x)/u_2'(x))$. Hence the necessity is proved.

The proof of the second statement is similar. Q.E.D.

Roughly speaking, the two statements of Proposition 3 give necessary and sufficient conditions for a utility function to associate uniformly high absolute prudence with uniformly high/low absolute risk aversion when compared with another utility function.

Assume two inevstor A and B with the same initial wealth have utility functions $u_1(x)$ and $u_2(x)$ respectively.

If the condition in the first statement is satisfied then $u_1(x)$ is uniformly more prudent than $u_2(x)$ implies $u_1(x)$ is uniformly more risk averse than $u_2(x)$. According to the results from Kimball (1990) and Pratt (1964) (shown in Subsections 1.1 and 1.2 of this paper) this implies that if investor A decides to make more precautionary savings than B in the money market w.r.t. problem (I) for any uncertainty ϵ in their wealth, then A will decide to invest less than B in the equity w.r.t. problem (II) for any equity return ε.[8] This seems reasonable.

However, if the condition in the second statement is satisfied then $u_1(x)$ is uniformly more prudent than $u_2(x)$ implies $u_1(x)$ is uniformly less risk averse than $u_2(x)$. According to the results from Kimball (1990) and Pratt (1964) (shown in Subsections 1.1 and 1.2 of this paper) this implies that if investor A decides to make more precautionary savings in the money market w.r.t. problem (I) than B for any uncertainty ϵ in their wealth, then A will decide to invest

[8]Consider all random variables ϵ and ε satisfy the conditions that for $c, x \in (0, w)$, $a < (w-c)R + \epsilon < A$ and $a < (w-x)R + x\varepsilon < A$.

more in the equity w.r.t. problem (II) than B for any equity return ε. This looks less reasonable.

From the above proposition we can see clearly that if for all $x \in (a, +\infty)$, $P_1(x) > (\geq) P_2(x)$, and $u_1'(+\infty) = 0$, then for all $x \in (a, +\infty)$, $R_1(x) > (\geq) R_2(x)$. That is, assuming the marginal utility of infinite wealth is zero, if a utility function always has higher absolute prudence than another utility function, then the former will always have higher absolute risk aversion than the latter. This shows that if we use utility functions that have zero marginal utility of infinite wealth to model investors' preferences then the more prudent an investor is globally in the money market then the more risk averse globally the investor in the equity market.

It is not difficult to see that most common utility functions satisfy the condition that marginal utility of infinite wealth is zero. For example, all risk-averse HARA class utility functions satisfy this condition. However, there are utility functions that do not have zero marginal utility of infinite wealth. These utility functions will not associate globally high prudence with globally high risk aversion. We give the following example.

Let $u_1(x) = \ln x$ and $u_2(x) = 10x^{0.1}$. Note that $u_1'(+\infty) = u_2'(+\infty) = 0$. Let $\hat{u}_1(x)$ be a utility function such that

$$\hat{u}_1'(x) = u_1'(x) + \frac{1}{9} = \frac{1}{x} + \frac{1}{9}.$$

Apparently, $\hat{u}_1(+\infty) = \frac{1}{9} > 0$. It is obvious that the prudence of $\hat{u}_1(x)$ is equal to the prudence of $u_1(x)$. Since the prudence of $u_1(x)$ is higher than that of $u_2(x)$, we conclude that the prudence of $\hat{u}_1(x)$ is higher than that of $u_2(x)$. On the other hand we have

$$\hat{u}_1'(1)/\hat{u}_1'(x) = (1 + \frac{1}{9})/(\frac{1}{x} + \frac{1}{9}) \quad \text{and} \quad u_2'(1)/u_2'(x) = x^{0.9}.$$

Since for $x \neq 1$

$$x^{-0.1} + \frac{1}{9}x^{0.9} = \frac{10}{9}\frac{9x^{-0.1} + x^{0.9}}{10} > \frac{10}{9},$$

the preceding equation implies that for $x \neq 1$, $\hat{u}_1'(1)/\hat{u}_1'(x) < u_2'(1)/u_2'(x)$. Thus from the second statement of Proposition 3 we conclude that for $x > 1$, $\hat{R}_1(x) < R_2(x)$. For $x > 1$, applying Corollary 1, we also have $\hat{C}_1(x) > C_2(x)$, where $\hat{C}_2(x)$ and $C_2(x)$ are the cautiousness of $\hat{u}_1(x)$ and $u_2(x)$. The above conclusions can be verified by calculating the risk aversion and cautiousness $\hat{u}_1(x)$ and $u_2(x)$.

Note that if $u_2'(a) = +\infty$ and $u_1'(a) < +\infty$, then we always have $u_1'(a)/u_1'(x) \leq u_2'(a)/u_2'(x)$. Thus it appears that we have a similar result implied by the second statement of Proposition 3. That is, assume for all $x \in (a, +\infty)$, $P_1(x) > P_2(x)$, $u_1'(a) < +\infty$, and $u_2'(a) = +\infty$ then for all $x \in (a, +\infty)$, $R_1(x) < R_2(x)$. It can be proved that the above statement is true. However, the conditions in the statement can never be satisfied. According to Pratt's (1964) Theorem 8, if $P_1(x) \geq P_2(x)$ then $u_2'(a) = +\infty$ implies $u_1'(a) = +\infty$.

Since cautiousness is equal to the ratio of prudence to risk aversion minus one, a relationship between prudence and risk aversion may imply a relationship between cautiousness and prudence. From the second statement of Proposition 3, we have the following result.

Corollary 1 *Given two utility functions $u_1(x)$ and $u_2(x)$, assume for all $x \in (a, A)$, $P_1(x) > (\geq) P_2(x)$, where A can be $+\infty$. If for all $x \in (a, A)$, $u_1'(a)/u_2'(a) \leq u_1'(x)/u_2'(x)$, then for all $x \in (a, A)$, $C_1(x) > (\geq) C_2(x)$.*

Proof: Given utility function $u(x)$, we have

$$(\ln[1/R(x)])' = (\ln u'(x))' - (\ln[-u''(x)])' = P(x) - R(x),$$

where $P(x)$ and $R(x)$ are its absolute prudence and absolute risk aversion respectively. This implies

$$[1/R(x)]' = P(x)/R(x) - 1.$$

Thus we have

$$C(x) = P(x)/R(x) - 1. \tag{4}$$

On the other hand, under the condition given in this corollary, Proposition 3 implies that for all $x \in (a, A)$ $R_1(x) < R_2(x)$. Since for all $x \in (a, A)$ $P_1(x) > P_2(x)$, this and (4) imply that for all $x \in (a, A)$ $C_1(x) > C_2(x)$. This completes the proof.

From Proposition 3 we can derive an interesting result.

Proposition 4 *The following two statements are true.*

- (i) *If there exists a constant $\delta > 0$ such that for all $x \in (a, +\infty)$, $P(x) < (\leq)\delta$, then for all $x \in (a, +\infty)$, $R(x) < (\leq)\delta$.*
- (ii) *If there exists a constant $\delta > 0$ such that for all $x \in (a, +\infty)$, $P(x) > (\geq)\delta$, then for all $x \in (a, +\infty)$, $R(x) > (\geq)\delta$ if and only if $u'(+\infty) = 0$.*

Proof: We only prove Statement (ii). Statement (i) can be similarly proved. Let $u_0(x) \equiv -e^{-\delta x}$. It is straightforward that its prudence and risk aversion are $P_0(x) = \delta$ and $R_0(x) = \delta$. If $u'(+\infty) = 0$, since $P(x) > P_0(x)$, applying Proposition 3, we have $R(x) > R_0(x) = \delta$. This proves the sufficiency part. The necessity part follows from the fact that $R(x) > \delta > 0$ implies

$$\ln u'(+\infty) = \ln u'(x) - \int_x^{+\infty} R(x)dx = -\infty.$$

This completes the proof.

Pratt (1964) and Arrow (1965) defined a special type of preference with which an investor will invest more in equity when he becomes wealthier. Such investors are said to have decreasing absolute risk aversion (hereafter DARA). Investors who have the opposite feature are said to have increasing absolute risk aversion (hereafter IARA). They also defined a special type of preference with which an investor will invest more proportion of his wealth in equity when he becomes wealthier. Such investors are said to have decreasing relative risk aversion. Investors who have the opposite feature are said to have increasing relative risk aversion.

Kimball (1993) defined a special type of preference with which an investor will make less precautionary savings when he becomes wealthier. Such investors are said to have decreasing absolute prudence (hereafter DAP). Investors who have the opposite feature are said to have increasing absolute prudence (hereafter IAP). Kimball (1993) also defined a special type of preference with which an investor will make less precautionary savings proportional to his wealth when he becomes wealthier. Such investors are said to have decreasing relative prudence. Investors who have the opposite feature are said to have increasing relative prudence.

Kimball (1993) showed that any utility function that has DAP with wealth $x \in (a, +\infty)$ also has DARA. In this paper, we show that any utility function that has decreasing relative prudence with wealth $x \in (a, +\infty)$ also has decreasing relative risk aversion. Moreover we show that any utility function that has increasing absolute (relative) prudence with wealth $x \in (a, +\infty)$ also has increasing absolute risk aversion if and only if the marginal utility of infinite wealth is zero. We first state Kimball's result as follows and give a different proof.

> [Kimball (1993)] If $P(x)$ is decreasing with wealth $x \in (a, +\infty)$, then $R(x)$ is decreasing with wealth $x \in (a, +\infty)$.

Proof: For any $\varepsilon > 0$, let $u_1(x) = u(x)$ and $u_2(x) = u(x + \varepsilon)$. Since $P(x)$ is decreasing with $x \in (a, +\infty)$, we have that $P_1(x) > P_2(x)$. Since $u'(x)$ is decreasing and positive, $u'(+\infty)$ exists. It follows that $u_1'(\infty) = u_2'(\infty) \geq 0$. But since $u'(x)$ is decreasing, we obtain $u_1'(\infty)/u_1'(x) \leq u_2'(\infty)/u_2'(x)$. Applying Proposition 3, we obtain $R_1(x) > R_2(x)$. Thus we conclude that for any $\varepsilon > 0$, $R(x) > R(x + \varepsilon)$, which proves the first half. The second half can be proved similarly. This completes the proof.

We now show the following result.

Proposition 5 *Given a utility function $u(x)$, assume its absolute prudence is (strictly) increasing with wealth $x \in (a, +\infty)$. Then its absolute risk aversion is also (strictly) increasing with wealth $x \in (a, +\infty)$ if and only if the utility function has zero marginal utility of infinite wealth.*

Proof: The proof is similar to the one of Kimball's (1993) result we give in the context preceding this proposition, thus it is omitted.

The proposition tells us that when utility functions have zero marginal utility of infinite wealth globally increasing prudence implies globally increasing risk aversion; however, when utility functions do not have zero marginal utility of infinite wealth, globally increasing prudence will not imply globally increasing risk aversion.

3 Relative Prudence and Relative Risk Aversion

In the last section we have shown the relationship between absolute prudence and absolute risk aversion. In this section we show the relationship between relative prudence and relative risk aversion. By doing this we help to understand how investors' decisions w.r.t problem (I$'$) are related to their decisions w.r.t problem (II$'$). More specifically we will answer the following question. If an investor decides to make more proportional precautionary savings responding to a proportional risk in his wealth in problem (I$'$) than another investor under what necessary and sufficient conditions will the first investor decide to invest more or less proportion of wealth in the equity market in problem (II$'$) than the second investor? Note that different investors not only have different utility functions but have different wealth as well, thus the problem is more complicated than the problem in the last section.

We now present the following result.

Lemma 2 *Given two utility functions $u_1(x)$ and $u_2(x)$, assume for all $x \in (a, A)$, $\beta_1(w_1 x) > \beta_2(w_2 x)$, where $a \geq 0$, A can be $+\infty$, and $w_i > 0$, $i = 1, 2$. Then the following two statements are true.*

1. *For some $x \in (a, A)$, $\gamma_1(w_1 x) > \gamma_2(w_2 x)$, if and only if there exists $x^\circ \in (x, A) \cup \{A\}$, such that $u_1'(w_1 x^\circ)/u_1'(w_1 x) \leq u_2'(w_2 x^\circ)/u_2'(w_2 x)$.*

2. *For some $x \in (a, A)$, $\gamma_1(w_1 x) < \gamma_2(w_2 x)$, if and only if there exists $x^\circ \in [a, x)$, such that $u_1'(w_1 x^\circ)/u_2'(w_2 x^\circ) < u_1'(w_1 x)/u_2'(w_2 x)$.*

Both statements hold if all $>$'s and $<$'s are replaced by $\geq$'s and $\leq$'s respectively.

Proof: Let $\hat{u}_1(x) = u_1(w_1 x)$ and $\hat{u}_2(x) = u_2(w_2 x)$. Let $\hat{\gamma}_i(x)$ and $\hat{\beta}_i(x)$ be the relative risk aversion and relative prudence of $\hat{u}_i(x)$, $i = 1, 2$. Applying Lemma 3 on $\hat{u}_i(x)$, $i = 1, 2$ and noting that $\hat{\gamma}_i(x) = \gamma_i(w_i x)$ and $\hat{\beta}_i(x) = \beta_i(w_i x)$, $i = 1, 2$, we immediately obtain Lemma 2. This completes the proof.

Proposition 6 *Given two utility functions $u_1(x)$ and $u_2(x)$, assume for all $x \in (a, A)$, $\beta_1(w_1 x) > \beta_2(w_2 x)$, where $a \geq 0$, A can be $+\infty$, and $w_i > 0$, $i = 1, 2$. Then the following two statements are true.*

1. *For all $x \in (a, A)$, $\gamma_1(w_1 x) > \gamma_2(w_2 x)$, if and only if for all $x \in (a, A)$, $u_1'(w_1 A)/u_1'(w_1 x) \leq u_2'(w_2 A)/u_2'(w_2 x)$.*

2. *For all $x \in (a, A)$, $\gamma_1(w_1 x) < \gamma_2(w_2 x)$, if and only if for all $x \in (a, A)$, $u_1'(w_1 a)/u_2'(w_2 a) < u_1'(w_1 x)/u_2'(w_2 x)$.*

Both statements hold if all $>$'s and $<$'s are replaced by $\geq$'s and $\leq$'s respectively.

Proof: The proof is almost the same as the proof of Proposition 3.

Proposition 6 shows necessary and sufficient conditions for a utility function to associate uniformly high relative prudence with uniformly high/low relative risk aversion when compared with another utility function given that the investors whose preferences are represented by these two utility functions have different initial wealth.

From the second statement of Proposition 6 we have the following corollary.

Corollary 2 *Given two utility functions $u_1(x)$ and $u_2(x)$, assume for all $x \in (a, A)$, $\beta_1(w_1x) > (\geq)\beta_2(w_2x)$, where $a \geq 0$, A can be $+\infty$, and $w_i > 0$, $i = 1, 2$. If for all $x \in (a, A)$, $u_1'(w_1a)/u_2'(w_2a) < (\leq)u_1'(w_1x)/u_2'(w_2x)$, then for all $x \in (a, A)$, $C_1(w_1x) > (\geq)C_2(w_2x)$, where $C_i(x)$ is the cautiousness of $u_i(x)$, $i = 1, 2$.*

Proof: The proof is similar to the proof of Corollary 1.

From Proposition 6 we have the following result.

Proposition 7 *Given two utility functions $u_1(x)$ and $u_2(x)$, assume $u_1'(+\infty) = 0$, and for all $x \in (a, +\infty)$, $\beta_1(w_1x) > (\geq)\beta_2(w_2x)$, where $a \geq 0$ and $w_i > 0$, $i = 1, 2$. Then for all $x \in (a, +\infty)$, $\gamma_1(w_1x) > (\geq)\gamma_2(w_2x)$.*

Proof: Since $u_2'(x)$ is positive and decreasing thus $u_2'(\infty)$ must exist. Since $u_1'(\infty) = 0$, for all $x \in (a, +\infty)$, $u_1'(w_1(+\infty))/u_1'(w_1x) \leq u_2'(w_2(+\infty))/u_2'(w_2x)$. Applying Proposition 6, we immediately obtain the result.

We have shown in Subsection 1.1 that if for all x, $\beta_1(w_1x) \geq \beta_2(w_2x)$ then $\hat{c}_1 \leq \hat{c}_2$, where $\hat{c}_i$ is the solution to investor i's investment problem (I$'$), $i = 1, 2$. We have also shown in Subsection 1.2 that if for all x, $\gamma_1(w_1x) \geq \gamma_2(w_2x)$ then $x_1 \leq x_2$, where x_i is the solution to investor i's investment problem (II$'$), $i = 1, 2$.

Thus assuming utility functions have zero marginal utility of infinite wealth, if $u_1(w_1x)$ has uniformly higher prudence than $u_2(w_2x)$ implies $u_1(w_1x)$ has uniformly higher risk aversion than $u_2(x)$. According to the results in Subsections 1.1 and 1.2 of this paper, this implies that if investor A decides to make more proportional precautionary savings than B in the money market w.r.t. problem (I) for any uncertainty ϵ in their wealth, then A will decide to invest less proportion of his wealth than B in the equity w.r.t. problem (II) for any equity return ε.

From the above proposition we can derive another interesting result.

Proposition 8 *The following two statements are true.*

1. *If there exists a constant $\delta > 1$ such that for all $x \in (a, +\infty)$, $\beta(x) < (\leq)\delta$, then for all $x \in (a, +\infty)$, $\gamma(x) < (\leq)\delta - 1$.*

2. *If there exists a constant $\delta > 1$ such that for all $x \in (a, +\infty)$, $\beta(x) > (\geq)\delta$, then for all $x \in (a, +\infty)$, $\gamma(x) > (\geq)\delta - 1$ if and only if $u'(+\infty) = 0$.*

Proof: We only prove Statement (ii). Statement (i) can be similarly proved. Let $u_0(x) \equiv x^{2-\delta}/(2-\delta)$. It is straightforward that its relative prudence and relative risk aversion are $\beta_0(x) = \delta$ and $\gamma_0(x) = \delta - 1$. If $u'(+\infty) = 0$, since $\beta(x) > \beta_0(x)$, applying Proposition 7 we have $\gamma(x) > \gamma_0(x) = \delta - 1$. This proves the sufficiency part. The necessity part follows from the fact that $\gamma(x) > \delta - 1 > 0$ implies

$$\ln u'(+\infty) = \ln u'(x) - \int_x^{+\infty} \gamma(x) d \ln x = -\infty.$$

This completes the proof.

Pratt (1964) and Arrow (1965) defined a special type of preference with which an investor will invest more proportion of his wealth in equity when he becomes wealthier. Such investors are said to have decreasing relative risk aversion. Investors who have the opposite feature are said to have increasing relative risk aversion. Kimball (1993) defined a special type of preference with which an investor will make less precautionary savings proportional to his wealth when he becomes wealthier. Such investors are said to have decreasing relative prudence. Investors who have the opposite feature are said to have increasing relative prudence.

We now show the following result.

Proposition 9 *The following statements are true.*

1. *If its relative prudence is (strictly) decreasing with wealth $x \in (a, +\infty)$, then its relative risk aversion is also (strictly) decreasing with wealth $x \in (a, +\infty)$.*

2. *Assume its relative prudence is (strictly) increasing with wealth $x \in (a, +\infty)$. Then its relative risk aversion is also (strictly) increasing with wealth $x \in (a, +\infty)$ if and only if the utility function has zero marginal utility of infinite wealth.*

Proof: We only prove the first statement. The second statement can be similarly proved.

Let $\beta(x)$ and $\gamma(x)$ denote the relative prudence and relative risk aversion of the utility function respectively. For any $x > a \geq 0$ and $\varepsilon > 0$, let $w_1 = x$ and $w_2 = x + \varepsilon$. Given $y > a/x$, let $u_1(w_1 y) = u(xy)$ and $u_2(w_2 y) = u((x + \varepsilon)y)$. Let $\beta_i(x)$ and $\gamma_i(x)$ denote the relative prudence and relative risk aversion of $u_i(x)$, $i = 1, 2$. It is straightforward that $\beta_1(w_1 y) = \beta(xy)$ and $\beta_2(w_2 y) = \beta((x + \varepsilon)y)$. This and the condition that $\beta(x)$ is decreasing with $x \in (a, +\infty)$ imply that for any $y > a/x$, $\beta_1(w_1 y) > \beta_2(w_2 y)$. On the other hand, we have $u_1'(\infty) = u_2'(\infty) \geq 0$. But since $u'(x)$ is decreasing, we obtain $u_1'(\infty)/u_1'(xy) = u'(\infty)/u'(xy) \leq u_2'(\infty)/u_2'(xy)$. Applying Lemma 6, we obtain

$\gamma_1(w_1 y) > \gamma_2(w_2 y)$. But we have $\gamma_1(w_1 y) = \gamma(xy)$ and $\gamma_2(w_2 y) = \gamma((x+\varepsilon)y)$. Hence we obtain for any $y > a/x$, $\gamma(xy) > \gamma((x+\varepsilon)y)$. Letting $y = 1$ in the last equation, we obtain $\gamma(x) > \gamma(x+\varepsilon)$. This completes the proof.

When utility functions do not have zero marginal utility of infinite wealth, globally increasing relative prudence will not imply globally increasing relative risk aversion. We give the following example.

Let $u(x) \equiv \ln(x+1) + x$. It is straightforward that $u'(x) = \frac{1}{x+1} + 1$. Apparently, $u'(+\infty) = 1 > 0$. It is also straightforward that the relative prudence of $u(x)$ is $\beta(x) = 2x/(x+1)$, which is increasing with x. However, its relative risk aversion is

$$\gamma(x) = \frac{x}{(x+1)^2} / (\frac{1}{x+1} + 1) = \frac{x}{(x+1)(x+2)}.$$

We have

$$\gamma'(x) = \frac{3 - x^2}{(x+1)^2(x+2)^2},$$

which is negative for $x > \sqrt{3}$. This example shows increasing relative prudence does not imply increasing relative risk aversion when the marginal utility of infinite wealth is not zero.

4 Cautiousness and Absolute Risk Aversion

In the last two sections we have shown the relationship between prudence and risk aversion, which helps to understand how investors' investment decisions in the money market and equity market are related. In this section and next section we show the relationship between cautiousness and risk aversion, which will help us to understand how investors' investment decisions in the equity market and option market are related. In this section we first show the relationship between cautiousness and absolute risk aversion. This will help to understand the relation between problem (II) and problem (III). Moreover, we know that cautiousness is the ratio of prudence to risk aversion minus one. On the other hand, according to Kimball's (1990) precautionary premium and Pratt's (1964) risk premium for a small risk are approximately equal to half the prudence and risk aversion multiplied by the variance of the risk respectively. Thus the relationship between cautiousness and risk aversion will also reveal how Kimball's precautionary premium is related to pratt's risk premium.

As in the previous context, given a utility function $u(x)$, we always use $C(x)$ to denote its cautiousness. When the utility function $u(x)$ has a hat or a subscript, the corresponding $C(x)$ will also have a hat or the subscript.

Lemma 3 *Given two utility functions $u_1(x)$ and $u_2(x)$, assume for all $x \in (0, A)$, $C_1(x) < (\leq) C_2(x)$. Then for all $x \in (0, A)$, $R_1(x) > (\geq) R_2(x)$, if and only if $1/R_1(0) \leq 1/R_2(0)$.*

Proof: Since for all $x \in (0, A)$, $C_1(x) \leq C_2(x)$, we have

$$(1/R_1(x) - 1/R_2(x))' = C_1(x) - C_2(x) \leq 0.$$

Thus if $1/R_1(0) - 1/R_2(0) \leq 0$, then for all $x \in (0, A)$, $1/R_1(x) - 1/R_2(x) \leq 0$. This proves the sufficiency part. The necessity part follows from the fact that if $1/R_1(0) - 1/R_2(0) > 0$, then for sufficiently small $x > 0$, $1/R_1(x) - 1/R_2(x) > 0$. This completes the proof.

From the above lemma we have the following result.

Proposition 10 *Given two utility functions $u_1(x)$ and $u_2(x)$, assume for all $x \in (0, A)$, $0 < C_1(x) < (\leq) C_2(x)$. If $u_1'(0) = +\infty$, then for all $x \in (0, A)$, $R_1(x) > (\geq) R_2(x)$.*

Proof: Noting that the conditions $u_1'(0) = +\infty$ and for all $x \in (0, A)$, $C_1(x) > 0$ imply that $R_1(0) = +\infty$. Hence we always have $1/R_1(0) \leq 1/R_2(0)$. Applying Lemma 3, we obtain the corollary. This completes the proof.

This result shows that if we use utility functions that have infinite marginal utilities of zero wealth to model investors' preferences, then with the same initial wealth if an investor's cautiousness become uniformly higher then the investor will be uniformly less risk averse.

5 Cautiousness and Relative Risk Aversion

In this section we show the relationship between cautiousness and relative risk aversion. This will help to understand the relation between problem (II′) and problem (III). Similar to the last section, this section will also helps to understand the relation between precautionary premium and risk premium. For example, after this section we will be able to answer the following question: if an investor's ratio of proportional precautionary premium to proportional risk premium for a small proportional risk is larger than that of another investor, what will be the relation between their proportional risk premiums?

Note that γ or γ with subscripts denote relative risk aversion.

Lemma 4 *Given two utility functions $u_1(x)$ and $u_2(x)$, assume for $w_1 > 0$, $w_2 > 0$, and all $x \in (0, A)$, $C_1(w_1 x) < (\leq) C_2(w_2 x)$. Then for all $x \in (0, A)$, $\gamma_1(w_1 x) > (\geq) \gamma_2(w_2 x)$, if and only if $1/(w_1 R_1(0)) \leq 1/(w_2 R_2(0))$.*

Proof: Let $\hat{u}_1(x) = u_1(w_1 x)$ and $\hat{u}_2(x) = u_2(w_2 x)$. Let $\hat{R}_i(x)$ and $\hat{C}_i(x)$ be the absolute risk aversion and cautiousness of $\hat{u}_i(x)$, $i = 1, 2$. Applying Lemma 3 on $\hat{u}_i(x)$, $i = 1, 2$ and noting that $\hat{R}_i(x) = w_i R_i(w_i x)$ and $\hat{C}_i(x) = C_i(w_i x)$, $i = 1, 2$, we immediately obtain Lemma 6. This completes the proof.

From the above lemma we have the following result.

Proposition 11 *Given two utility functions $u_1(x)$ and $u_2(x)$, assume for all $x \in (0, A)$, $0 < C_1(w_1 x) < (\leq) C_2(w_2 x)$. If $u_1'(0) = +\infty$, then for all $x \in (0, A)$, $\gamma_1(w_1 x) > (\geq) \gamma_2(w_2 x)$.*

Proof: Noting that the conditions $u_1'(0) = +\infty$ and for all $x \in (0, A)$, $C_1(w_1 x) > 0$ imply that $R_1(0) = +\infty$. Hence we always have $1/(w_1 R_1(0)) \leq 1/(w_2 R_2(0))$. Applying Lemma 4, we obtain the corollary. This completes the proof.

This shows that utility functions that have infinite marginal utility of zero wealth associate high cautiousness with low relative risk aversion when compared with each other. Moreover, we have the following result.

Proposition 12 *The following two statements are true.*

1. *If for all $x \in (0, A)$ $\;C(x) \geq (>) C_0 > 0$, then for all $x \in (0, A)$ $\;\gamma(x) \leq (<) 1/C_0$.*
2. *If for all $x \in (0, A)$ $\;0 < C(x) \leq (<) C_0$, then for all $x \in (0, A)$ $\;\gamma(x) \geq 1/C_0$ if and only if $u'(0) = +\infty$.*

Proof: Let $u_1(x) \equiv u(x)$ and $u_2(x) \equiv x^{1-1/C_0}/(1 - 1/C_0)$. Since $C_2(x) = C_0$ and $\gamma_2(x) = 1/C_0$, applying Proposition 11, we prove the first statement and the sufficiency part of the second statement. The necessity part of the second statement follows from the fact that for all $x \in (0, A)$ $\;\gamma(x) \geq 1/C_0$ implies $u'(0) = +\infty$. This completes the proof.

The proposition tells us that given that an investor's marginal utility of zero wealth is infinity, if an investor's cautiousness is bounded above (below) by some positive constant then his relative risk aversion is bounded below (above) by the inverse of the constant.

For a power utility function $u(x) = x^{1-\nu}/(1-\nu)$, it is straightforward that the relative risk aversion $\gamma(x) = \nu$ and the cautiousness $C(x) = 1/\nu$. Thus in this special case we have $C(x) = 1/\gamma(x)$. Only power utility functions and logarithmic utility functions have this property. This can be shown by some simple calculations.

$C(x) = 1/\gamma(x)$ is equivalent to $(1/R(x))' = 1/[xR(x)]$. It can be written as $d \ln[1/R(x)]/d \ln x = 1$, which is equivalent to $1/R(x) = \gamma x$, where $\gamma > 0$ is a constant. We rewrite it as $xR(x) = \gamma$, which implies that $u(x)$ is a power utility function or a logarithmic utility function.

For more general utility functions, however, we have the following result:

Proposition 13 *The following two statements are true.*

1. *If $0 < C(\infty) < +\infty$, then $\gamma(\infty) = 1/C(\infty)$.*
2. *If $0 < C(0) < +\infty$, then $\gamma(0) = 1/C(0)$, if and only if $u'(0) = +\infty$.*

Proof:The first statement is proved in Appendix C. Thus we only prove the second statement here. Since $0 < C(0) < +\infty$, given any $\delta \in (0, C_0)$ for sufficiently small $\nu > 0$, for all $x \in (0, \nu)$, $C(0) - \delta < C(x) < C(0) + \delta$. Applying Proposition 12, we conclude that for all $x \in (0, \nu)$, $1/(C(0) + \delta) < \gamma(x) < 1/(C(0) - \delta)$ if and only if $u'(0) = +\infty$. This completes the proof.

This result tells us that when wealth approaches zero or infinity, if the limit of the cautiousness exists, then the limit of the relative risk aversion also exists and in limit they have the relationship, which holds for power utility functions. This is not surprising since in limit the cautiousness is a positive constant and it is natural for the utility function to behave like power utility functions, which have positive constant cautiousness.

Proposition 14 *The following two statements are true.*

1. *If its cautiousness is (strictly) increasing with wealth $x \in (0, A)$, then its relative risk aversion is (strictly) decreasing with wealth $x \in (0, A)$ if and only if $u'(0) = +\infty$.*

2. *If its cautiousness is (strictly) decreasing with wealth $x \in (0, A)$, then its relative risk aversion is (strictly) increasing with wealth $x \in (0, A)$.*

Proof: For any $x \in (0, A)$, arbitrarily given $\Delta x > 0$, let $w_1 = x$, $w_2 = x + \Delta x$, $u_1(w_1 y) \equiv u(w_1 y)$, and $u_2(w_2 y) \equiv u(w_2 y)$. Since the cautiousness of $u(x)$ is increasing with x, we have for all $y \in (0, A/w_2)$ $C_1(w_1 y) \leq C_2(w_2 y)$, where $C_i(x)$ denotes the cautiousness of $u_i(x)$, $i = 1, 2$. Applying Lemma 3, we conclude that $\gamma_1(w_1 y) \geq \gamma_2(w_2 y)$ if and only if $1/(w_1 R_1(0)) \leq 1/(w_2 R_2(0))$, which is equivalent to $1/(xR(0)) \leq 1/(x + \Delta x)R(0))$. The last condition holds if and only if $R(0) = +\infty$. Noting that $\gamma_1(w_1 y) = \gamma(xy)$ and $\gamma_2(w_2 y) = \gamma((x + \Delta)y)$, from the above result we conclude that the relative risk aversion is decreasing with $x \in (0, A)$ if and only if $R(0) = +\infty$. The last condition is equivalent to $u'(0) = +\infty$, given that $C(x)$ is increasing with $x \in (0, A)$. This proves the second statement.

The second statement can be similarly proved. This completes the proof.

6 Extensions

All the results in previous sections related to relative risk aversion or relative prudence can be generalized. Given a utility function $u(x)$, let $P(x)$ and $R(x)$ be its absolute prudence and absolute risk aversion. Define $\beta_b(x) \equiv (x - br)P(x)$ and $\gamma_b(x) \equiv (x - br)R(x)$, where b is a constant and r is the total return on a unit money account. $\beta_b(x)$ and $\gamma_b(x)$ are called partial relative prudence and partial relative risk aversion respectively. All the results in Sections 2 and 3 related to relative risk aversion or relative prudence will still hold if we replace $\beta(x)$ and $\gamma(x)$ with $\beta_b(x)$ and $\gamma_b(x)$ respectively and replace $\beta_i(w_i x)$, $\gamma_i(w_i x)$, $C(w_i x)$, and $u_i'(w_i x)$ with $\beta_{b_i}(b_i+(w_i-b_i)x)$, $\gamma_{b_i}(b_i+(w_i-b_i)x)$, $C(b_i+(w_i-b_i)x)$, and $u_i'(b_i+(w_i-b_i)x)$ respectively. All the results in Sections 4 related to relative risk aversion will still hold if we replace $\gamma(x)$ with $\gamma_b(x)$ and replace $\gamma_i(w_i x)$, $C(w_i x)$, $u_i'(0)$, and $w_i R_i(0)$ with $\beta_{b_i}(b_i+(w_i-b_i)x)$, $\gamma_{b_i}(b_i+(w_i-b_i)x)$, $C(b_i+(w_i-b_i)x)$, $u_i'(b_i r)$, and $(w_i - b_i)R_i(b_i r)$ respectively. The proofs are virtually the same.

Note that if for some reasons an investor must hold a certain amount of cash, say b_i, then given his wealth x, he will consider to invest at most $x - b_i$ units of wealth in risky assets. An example of such a case is an investor with a utility function that has infinite marginal utility of b_i units of wealth. Such an investor cannot live with an amount of wealth below b_i. Thus he always holds (at least) b_i units of cash to avoid this situation. In this case the investor's sonsumption-saving problem becomes:

$$(\mathrm{I}'') \qquad \max_{\hat{c}} u_i(\hat{c}(w_i - b_i)) + \rho E u_i(b_i r + (w_i - b_i)((1-\hat{c})r + \epsilon)).$$

where w_i is the investor's initial wealth, $\hat{c}$ is his first period consumption as a proportion of his initial wealth minus b_i, r is the total return on a unit money account in the second period, and ϵ is the uncertainty in his wealth. Note $1-\hat{c}$ is the proportion of his initial wealth minus b_i the investor saves in his bank account.

The investor's investment problem in the equity market becomes:

$$(\mathrm{II}'') \qquad \max_{\hat{x}} E[u_i(w_i r + \hat{x}(w_i - b_i)(\varepsilon - r))]$$

where ε is the total return of 1 unit of money invested in the equity market in the second period and $\hat{x}$ is the proportion of his initial wealth minus b_i units of cash he invests in the equity market.

The results on the relationship between partial relative prudence and partial relative risk aversion and the relationship between cautiousness and partial relative risk aversion can be used in the exactly same way to explain the relationship between the relationship of investors' decisions in the above two problems as the results on the relationship between relative prudence and relative risk aversion and the relationship between cautiousness and relative risk aversion.

7 Conclusions

In this paper we have thoroughly investigated the relationships between prudence, risk aversion, and cautiousness. This helps to understand how investors' investment decisions in the money market, stock market, and option market are related. We have shown that roughly speaking, if an investor's (absolute or relative) prudence has some feature then his (absolute or relative) risk aversion will have similar feature (sometimes with a minor additional condition). We have also shown that roughly speaking, if an investor's cautiousness has some feature then his relative risk aversion will have the opposite feature (sometimes with a minor additional condition).

We have shown the implications of prudence and cautiousness for risk aversion; however, it is more difficult to show the implications of risk aversion for prudence and cautiousness. This is due to the mathematical feature of the three preference measures under the utility representation framework. Note that prudence and cautiousness involve higher order derivatives of a utility function than the risk aversion. Thus it is not surprising that both prudence and cautiousness have implications for risk aversion. Mathematically, the higher order derivatives have implications for the lower order derivatives given certain boundary conditions. However, for the same reason it is difficult to find implications of risk aversion for prudence and cautiousness[9].

[9] Eeckhoudt and Schlesinger (1994) managed to reveal some implications of risk aversion for prudence under certain conditions.

Appendix A Proof of Proposition 1

We first prove the first statement. Let $t = u_2'(w_2((1-c)r+\epsilon))$. Hence $(1-\hat{c})r + \epsilon = u_2'^{-1}(t)/w_2$. It follows that

$$Eu_1'(w_1((1-\hat{c})r+\epsilon)) = Eu_1'(\frac{w_1}{w_2}u_2'^{-1}(t))$$

Using the analogy between risk aversion and prudence established by Kimball (1990) and the result on interpersonal comparative risk aversion, we conclude that the condition that for all $x \geq 0$, $\beta_1(w_1x) \geq \beta_2(w_2x)$ is equivalent to $u_1'(\frac{w_1}{w_2}u_2'^{-1}(t))$ is concave in t, which is equivalent to

$$Eu_1'(\frac{w_1}{w_2}u_2'^{-1}(t)) \geq [>] u_1'(\frac{w_1}{w_2}u_2'^{-1}(Et)) \tag{5}$$

for all t.

Assume $\hat{c}$ is the solution to

$$u_2'(\hat{c}w_2) = \rho r E u_2'(w_2((1-\hat{c})r+\epsilon))$$

Since $\rho r = 1$, we have $\hat{c} = \frac{1}{w_2}u_2'^{-1}(Et)$, where $t = u_2'(w_2((1-\hat{c})r+\epsilon))$. Hence we have

$$u_1'(\hat{c}w_1) - \rho r E u_1'(w_1((1-\hat{c})r+\epsilon)) = u_1'(\frac{w_1}{w_2}u_2'^{-1}(Et)) - Eu_1'(\frac{w_1}{w_2}u_2'^{-1}(t))$$

This and (5) imply that Statement 1 is equivalent to $u_1'(\hat{c}w_1) - \rho r E u_1'(w_1((1-\hat{c})r+\epsilon)) \leq [<] 0$. It follows that the statement is equivalent to $\hat{c}_1 \leq \hat{c}$ [and $<$ if $0 < \hat{c}_1(w_1, \epsilon) < 1$]. This proves the first statement.

Now we prove the second statement. Let $\hat{\rho} \equiv \rho r$. We have

$$\frac{d}{dt}[u_1'(\frac{w_1}{w_2}u_2'^{-1}(\hat{\rho}t)) - \hat{\rho}u_1'(\frac{w_1}{w_2}u_2'^{-1}(t))] = \hat{\rho}\frac{w_1}{w_2}[\frac{u_1''(\frac{w_1}{w_2}u_2'^{-1}(\hat{\rho}t))}{u_2''(u_2'^{-1}(\hat{\rho}t))} - \frac{u_1''(\frac{w_1}{w_2}u_2'^{-1}(t))}{u_2''(u_2'^{-1}(t))}]$$

Since

$$\frac{d}{dt}\ln\frac{u_1''(\frac{w_1}{w_2}u_2'^{-1}(t))}{u_2''(u_2'^{-1}(t))} = -[\beta_1(\frac{w_1}{w_2}u_2'^{-1}(t)) - \beta_2(u_2'^{-1}(t))]u_2''(u_2'^{-1}(t))/u_2'^{-1}(t) > 0$$

for $\hat{\rho} \leq 1$, we have

$$\frac{u_1''(\frac{w_1}{w_2}u_2'^{-1}(\hat{\rho}t))}{u_2''(u_2'^{-1}(\hat{\rho}t))} < \frac{u_1''(\frac{w_1}{w_2}u_2'^{-1}(t))}{u_2''(u_2'^{-1}(t))}$$

Hence we obtain

$$\frac{d}{dt}[u_1'(\frac{w_1}{w_2}u_2'^{-1}(\hat{\rho}t)) - \hat{\rho}u_1'(\frac{w_1}{w_2}u_2'^{-1}(t))] < 0$$

Since $u_1'(+\infty) = u_2'(+\infty) = 0$, it follows that

$$u_1'(\frac{w_1}{w_2}u_2'^{-1}(\hat{\rho}t)) - \hat{\rho}u_1'(\frac{w_1}{w_2}u_2'^{-1}(t)) < 0 \tag{6}$$

Assume c is the solution to

$$u_2'(cw_2) = \hat{\rho} E u_2'(w_2((1-c)r+\varepsilon))$$

we have $c = \frac{1}{w_2} u_2'^{-1}(\hat{\rho} E t)$, where $t = u_2'(w_2((1-c)r+\varepsilon))$. Hence we have

$$u_1'(cw_1) - \hat{\rho} E u_1'(w_1((1-c)r+\varepsilon)) = u_1'(\frac{w_1}{w_2} u_2'^{-1}(\hat{\rho} E t)) - \hat{\rho} E u_1'(\frac{w_1}{w_2} u_2'^{-1}(t))$$

This, (5), and (6) imply that $u_1'(cw_1) - \hat{\rho} E u_1'(w_1((1-c)r+\varepsilon)) \leq [<] 0$. It follows that $c_1 \leq c$ [and $<$ if $0 < c_1(x_1, \varepsilon) < 1$]. This completes the proof.

Appendix B Proof of Lemma 3

We first prove the first statement. The necessity is rather obvious. Suppose for all $y \in (x, A) \cup \{A\}$, we have $u_1'(y)/u_1'(x) > u_2'(y)/u_2'(x)$, then we have that

$$\frac{\ln u_1'(y) - \ln u_1'(x)}{y - x} > \frac{\ln u_2'(y) - \ln u_2'(x)}{y - x}.$$

Let $y \to x$, we immediately conclude that $R_1(x) \leq R_2(x)$.

Now we prove the sufficiency: Let $u_1'(x) = t(u_2'(x))$. The existence of the function $t(x) = u_1'(u_2'^{-1}(x))$ is rather obvious given that $u_2'(x)$ is strictly decreasing in x. We have

$$P_1(x) = -\frac{u_1'''(x)}{u_1''(x)} = -\frac{t''(u_2')}{t'(u_2')} u_2'' + P_2(x).$$

Since $P_1(x) > P_2(x)$, we obtain $t''(x) > 0$ ($t'(x) > 0$). We also have

$$R_1(x) = u_2'(x) \frac{t'(u_2'(x))}{t(u_2'(x))} R_2(x). \tag{7}$$

Since there exists $x^\circ \in (x, A) \cup \{A\}$, such that $u_1'(x^\circ)/u_1'(x) \leq u_2'(x^\circ)/u_2'(x)$, if $u_2'(x^\circ) \neq 0$, we have

$$\frac{t(u_2'(x^\circ))}{u_2'(x^\circ)} \leq \frac{t(u_2'(x))}{u_2'(x)} = k.$$

It follows that

$$\frac{t(u_2'(x)) - t(u_2'(x^\circ))}{u_2'(x) - u_2'(x^\circ)} \geq \frac{k u_2'(x) - k u_2'(x^\circ)}{u_2'(x) - u_2'(x^\circ)} = k = \frac{t(u_2'(x))}{u_2'(x)} \tag{8}$$

But if $u_2'(x^\circ) = 0$, we must have $u_1'(x^\circ) = u_2'(x^\circ) = 0$, thus (8) naturally holds. Hence we always have (8).

On the other hand, since $t''(x) > 0$ ($t'(x) > 0$) and the utility function $u_2(x)$ is concave, we conclude that

$$t'(u_2'(x)) > \frac{t(u_2'(x)) - t(u_2'(x^\circ))}{u_2'(x) - u_2'(x^\circ)}. \tag{9}$$

From (8) and (9) we obtain

$$t'(u_2'(x)) > \frac{t(u_2'(x))}{u_2'(x)}. \tag{10}$$

From (7) and (10) we immediately conclude that $R_1(x) > R_2(x)$.

Hence the first statement is proved. The proof of the second statement is similar. Q.E.D.

Appendix C Proof of Proposition 13

We have

$$C(x) = (\frac{1}{R(x)})'$$

Since $C(+\infty)$ exists, $\lim_{x\to+\infty}(1/R(x))'$ exists. Given sufficiently small $\varepsilon > 0$ for sufficiently large x, we have

$$C(\infty) - \varepsilon < (\frac{1}{R(x)})' < C(\infty) + \varepsilon.$$

Thus for sufficiently large a and $x > a$,

$$(C(\infty) - \varepsilon)(x - a) < \frac{1}{R(x)} - \frac{1}{R(a)} < (C(\infty) + \varepsilon)(x - a).$$

It can be written as

$$(C(\infty) - \varepsilon)\frac{x - a}{x} < \frac{1}{xR(x)} - \frac{1}{xR(a)} < (C(\infty) + \varepsilon)\frac{x - a}{x}.$$

Let $x \to +\infty$ and $\varepsilon \to 0$, we conclude that $\lim_{x\to+\infty} xR(x)$ exists and

$$\gamma(\infty) = \frac{1}{C(\infty)}.$$

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