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Multiperiod Asset Pricing in the Presence of Transaction Costs and Taxes

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ABSTRACT

This paper models the effect of transaction costs and taxes on asset pricing in a multi-period setting. It extends the study by Dermody and Rockafellar (DR)(1991), where it was shown that term structure valuation is agent-specific owing to agents' different tax classes, and that a multiplicity of valuation operators exists owing to different costs associated with long and short trades. Unlike DR who focus solely on the riskless bond, this paper analyses both risky and riskless security pricing in a more general framework of taxation. Similar to DR, the tightest no arbitrage present value range for a claim is derived here without the knowledge of investor preferences. The Jouini and Kallal (1995) analysis of short sales in a tax free economy is a special case of our model. We also establish the existence of a set of pseudo risk neutral probability measures, under which the discounted long price is a supermartingale and the discounted short price is a submartingale, is the necessary and sufficient condition for no arbitrage.

JEL classification: G10, G12, C61.

1. INTRODUCTION

One of the most important and fundamental results in finance, assuming perfect markets and the absence of abitrage, is the existence of an equivalent martingale or risk neutral measure, whereby a valuation operator can be used to uniquely price all assets (see Ross 1976, Harrison and Kreps 1979). This result is the cornerstone of pricing by "no arbitrage".

The notion that every future cash flow has a net present value determinable from a single valuation operator breaks down when the assumption of a perfect market is removed and there are transaction costs and taxes (see Prisman 1986, Ross 1987, Dermody and Prisman 1988, 1993, Dermody and Rockafellar 1991, 1995). The fact that prices and transaction costs for long and short trades are not the same, and that taxation of long- and short-sides profits are not always symmetric further complicate the problem. Similar to the case of perfect markets, the minimal requirement necessary to achieve equilibrium in markets with transaction costs and taxes is the absence of after-tax arbitrage opportunities.

The earliest work in asset pricing focus on martingale measure in frictionless economy. Efforts have been made since to accommodate market frictions. Ross (1987) for example analyses complications that are due to taxation. He shows that different martingale measures exist when capital gains and ordinary income are taxed differently. Similarly, Dermody and Rockafellar (DR) (1991, 1995), who study the valuation of default free bond, show that different tax classes result in term structure that is agent specific. This agent specific term structure, together with the different transaction costs associated with long and short trades, leads to a multiplicity of valuation operators for pricing bond cash flows. In Dermody and Rockafellar's framework, every future cash flow can have a range of present values. Hence, there is a range of prices within which a bond may be traded and yet there is no arbitrage.

While a single valuation operator corresponds to the martingale measure, a multiplicity of valuation operators corresponds to the semimartingale measure. The semimartingale measure approach to asset pricing has become increasingly important recently. Using an abstract topology and with some constraints on the trading strategy, Frittelli (1997) shows that an adapted stochastic process is a semimartingale if and only if no global free lunch is allowed. In order to incorporate short sales constraints into the model, Jouini and Kallal (1995) restrict investors to hold nonnegative net positions. Under the assumption that the borrowing and lending rates are equal (or there is no bid-ask spread), they show that normalized (discounted) price processes for securities are either supermartingales or submartingales. Such restriction and assumption are not required in our model.

In this paper, the classic asset-pricing model is generalized to include a more realistic imperfect capital market with different interest rates for lending and borrowing, and different prices for long and short trades.[1] It

[1] A long position arises when an investor buys a security or unwinds a previously acquired short position. A short position arises when the investor sells or short borrows a security. Taking a "short-borrowed" position corresponds to a special transaction in which the investor assumes the obligation of paying out (for each unit of security i shorted) the expected pre-tax amount, in return for receiving a cash amount p_i in the present. Shorting is a form of borrowing, and shorting cost is the cost to be paid for such a borrowing in addition to other conventional transaction costs.

extends the study by Dermody and Rockafellar (DR) (1991, 1995) and Ross (1987) by studying the pricing of both risky and riskless securities in a more general framework of taxation. In markets with frictions, our model states that a security can have a range of prices. The models in Dermody and Rockafellar (DR) (1991, 1995), Jouini and Kallal (1995), and Ross (1987) can be viewed as special cases in our model. We demonstrate that the tightest bounds on the price of a claim, which we called the imputed long and short prices, can similarly be derived without the knowledge of investor preferences. Moreover, we show that the absence of arbitrage is a necessary but not a sufficient condition for the existence of a complete set of martingale (or risk-neutral) probabilities in both single- and multi-period frameworks. The necessary and sufficient condition for no arbitrage is the existence of a set of semimartingale probabilities under which the discounted long price is a supermartingale and the discounted short price is a submartingale.

Our finding has an important implication for pricing contingent claims for example. The Black Scholes option pricing model ignores transaction costs, and offers no guidance as to how transaction costs can be taken into account (see Jarrow and Turnbull, 1996). We provide an example to demonstrate how the model described here directly addresses this problem.

The remainder of the paper is organised as follows. Section 2 shows the equivalence between no arbitrage opportunities and the existence of a set of valuation operators in markets with friction. In this section, we follow Ross (1987) closely and recast Ross's theorems of no arbitrage into a more general framework that allow prices (and taxes) to differ between long and short trades. Section 3 derives the imputed long and short values for future after tax cash flows in an arbitrage free market, and establishes the relationship between no arbitrage and a pseudo semimartingale probability measure in a single period setting. Section 4 extends the pricing model to a multi-period setting. Finally, Section 5 concludes.

2. ARBITRAGE AND STATE PRICES IN A FRICTIONAL MARKET

In this section, we recast the analyses in Ross (1987) in the context where prices (and taxes) differ between long and short trades. Assume there are n primitive assets. Let A be a state space payoff tableaux– the $n \times s$ matrix whose entries are a_{ij}, for $i = 1, \ldots, n$ and $j = 1, \ldots, s$. An investor who holds one unit of security i receives amount a_{ij} if state j occurs. a_{ij} can be the coupon/principal of a fixed rate bond for example, or it may be the payoff/price of a risky asset. In this paper, the no arbitrage relationship is established by examining profitability generated by new trades. Investors' previously acquired positions are not considered. This assumption is one for simplicity. Even if investors can take advantage of their existing holdings by unwinding previously acquired positions and trade at favourable prices, such "free lunch" will be finite, and the analysis is omitted here for simplicity.

Next, let P_i and p_i be the long and short prices of security i, and let X_i and x_i be the numbers of units of security i longed and shorted respectively such that $X_i \geq 0$ and $x_i \geq 0$; $P = (P_1; \ldots; P_n)$; $p = (p_1; \ldots; p_n)$; $X = (X_1; \ldots; X_n)$; $x = (x_1; \ldots; x_n)$. The difference between P_i (the ask or long price) and p_i (the bid or short price) equal the bid-ask spread plus additional cost involved in short borrowing. We call this difference in price the long-short spread. For an investor with no previously acquired long position, the short price is significantly less than the long price for security i: $0 \leq p_i < P_i$ for $i = 1; \ldots; n$. Due to the additional shorting cost, the bid-ask spread may not be symmetrical. One immediately recognises that since a_{ij} represents the state price of a risky asset, it could be the bid or the ask price depending on the nature of the transaction. However, a_{ij} is involved in the unwinding trade only. It is reasonable to assume that, under normal circumstances, the bid-ask spread is roughly symmetrical for unwinding trades. Since the shorting cost has already been included in p_i; the distincion between long and short positions is omitted in the case of a_{ij} for simplicity.

Following Demody and Rockafellar (1991), investors are divided into separate classes indexed by $k = 1; \ldots; K$, according to the way in which their taxes are calculated. For investors in class k, the amounts of tax to be paid on a long portfolio X is denoted by $T^k(P;X)$; and the amount of tax to be rebated on a short portfolio x is denoted by $S^k(p;x)$. $T^k(P;X)$ and $S^k(p;x)$ are tax functions that map the current price P (or p) and position X (or x) of investors in class k, into a vector of state-dependent taxes. That is $T^k(\cdot;\cdot)$ and $S^k(\cdot;\cdot)$ map $(0;\infty) \times (0;\infty) \to (0;\infty)$, with $T^k(P;0) = S^k(p;0) = 0$. The after-tax payoff on this position is given by $Z(X;x;P;p) = A(X - x) - [T^k(P;X) - S^k(p;x)]$ where A is, as defined before, the state dependent payoff. The after-tax payoff is an s vector, whereby each of its elements is the after-tax payoff in the associated state of nature. Taxes depend in general on the prices P, p and positions X, x because these prices and positions set the basis for calculations of dividends, interest, and capital gains. In Ross (1987), the dependencies of taxes on state price, A, and current price, P (or p), are surpressed. Here, the impacts of P and p are specifically included. The impact of A is less critical if we observe the fact that although individual tax obligation and rebate may be netted off, all indivuduals are net tax payers, and the government is a net tax collector.

Next denote a change in portfolio position as (ξ, η) with $(\xi, \eta) \geq (0, 0)$. The increment in after-tax payoff due to the change from a position (X, x) to a new position $(X+\xi, x+\eta)$ is denoted by

$$
\begin{aligned}
F(X, x, \xi, \eta) &= Z(X+\xi, x+\eta, P, p) - Z(X, x, P, p) \\
&= A(\xi - \eta) - \{[T^k(P, X+\xi)] - [T^k(P, X)]\} \\
&\quad + \{[S^k(p, x+\eta)] - [S^k(p, x)]\}.
\end{aligned}
$$

The corresponding incremental investment cost for such a change is $P\xi - p\eta$.

Several basic definitions need to be established first.

Definition 2.1. Local weak no-arbitrage $LWNA^k$ is satisfied at (X, x) for investors in tax class k $(k = 1, \ldots, K)$ means that for every such investor with a previously acquired position vector (X, x), it is impossible to find a trade $(\xi, \eta) \geq (0, 0)$ such that the increment of after-tax payoff, $F(X, x, \xi, \eta) \geq 0$, and the increment of investment cost $P\xi - p\eta < 0$. If $LWNA^k$ holds for every $k = 1, \ldots, K$, we say local weak no-arbitrage, $LWNA$, is satisfied.

Definition 2.2. Local strong no-arbitrage for investor class k, $LSNA^k$, is satisfied at (X, x) for investors in class k when there is no trade $(\xi, \eta) \geq (0, 0)$, with $(\xi, \eta) \neq (0, 0)$ such that $F(X, x, \xi, \eta) \geq 0$ and $P\xi - p\eta \leq 0$ except perhaps ones satisfying $F(X, x, \xi, \eta) = 0$ and $P\xi - p\eta = 0$. If $LSNA^k$ holds for every $k = 1, \ldots, K$, we say local strong no-arbitrage, $LSNA$, is satisfied.

Obviously, when there is no initial investment, then the after-tax payoff is given by $A(\xi - \eta) - \{[T^k(P, \xi)] - [S^k(p, \eta)]\}$ and the investment cost is $P\xi - p\eta$. These are the definitions of (global) weak and strong form arbitrage at $(X;x) = (0;0)$ given by Dermody and Rockafellar (1991).[2] Note that the definition of arbitrage is agent-specific and depends on long and short prices and the respective previously acquired positions. The non-linearity of tax schedule forced the analyses to be based on the localised no arbitrage analysis. Extensions to global no arbitrage can be obtained along the lines of Ross (1987), Section IV (pages 376-377) using the concept of "extendable" arbitrage.

Assumption 2.1. *The tax functions $T^k(\cdot;\cdot)$ and $S^k(\cdot;\cdot)$ are, respectively, convex and concave in their second variable. All individuals are net tax payers and the after-tax payoff $Z(\cdot)$ is concave.*

Given Assumption 2.1 we have the following pair of primal-dual convex programming problems:

$$(PM): \min_{(\lambda;\gamma)\geq 0} P\lambda - p\gamma; \text{ subject to } F(X;x;\lambda;\gamma) \geq 0;$$

$$(DM): \max_{d\geq 0} \inf_{(\lambda;\gamma)\geq 0} \{P\lambda - p\gamma - dF(X;x;\lambda;\gamma)\}$$

where d is a valuation operator appearing in DM as a "shadow price". The function PM describes the process of identifying the portfolio that minimizes the current incremental investment cost while generate nonnegative increments of future after-tax payoff in every state of nature. Hence, the no-arbitrage condition can be defined in terms of PM. That is, there exists $LWNA^k$ at $(X;x)$ if and only if the optimal value of PM is zero. $LSNA^k$ holds at $(X;x)$ if and only if the optimal value of PM is zero, d is strictly positive, and the optimal solution satisfies the constraint as equality.

The theorem below verifies the existence of the valuation operator, d, in the presence of transaction costs and taxation, that can be used to price any change in after-tax income in an arbitrage-free market.

Theorem 2.2. *Assume Assumption 2.1 is satisfied. The following statements are equivalent:*

(1) there exists $LWNA^k$ at (X, x);

(2) there exists a semipositive valuation operator, d $(d \geq 0)$, such that, for any $(\lambda;\gamma) \geq (0;0)$,

$$P\lambda - p\gamma \geq dF(X;x;\lambda;\gamma);$$

[2] When $T^k = S^k$; $P = p$ this is the definition of arbitrage that appears in Ross (1987). If in addition $T^k = S^k = 0$, then this is the traditional definition of arbitrage.

(3) there exists a semipositive valuation operator d, such that, for any $(\xi,\zeta) \ge (0,0)$,

$$P\xi - d\{A\xi - [T^k(P,X+\xi) - T^k(P,X)]\} \ge 0,$$
$$p\zeta - d\{A\zeta - [S^k(p,x+\zeta) - S^k(p,x)]\} \le 0.$$

For $LSNA^k$ at (X,x) the constraints in statements (2) and (3) must be satisfied, and the valuation operator d must be strictly positive for any $(\xi,\zeta) \ge (0,0)$.

Proof: See Appendix.

Assumption 2.3. For a positive d, the point (X,x) is not a maximum of $d(X,x,P,p)$ with a zero derivative in some direction.

As stated in Ross (1987), the assumption that there is no zero derivative in any direction is a weak form of nonsatiation. If there is no satiation, then (X,x) is not a maximum of $Z(.)$ and assumption (2.3) is satisfied. Even if (X,x) is a maximum of $Z(.)$, if no directional derivative is zero, assumption (2.3) will still be satisfied.

Let $g(.)$ be a concave function and let $H_{X,x}$ be the set of directions on which $g(.)$ is nondecreasing:

$$\begin{aligned} H_{X,x} \equiv \{(H_1,H_2) \mid\ & H_1\xi - H_2\zeta \le 0 \\ & \Rightarrow g(X+\xi,x+\zeta,P,p) \le g(X,x,P,p)\} \end{aligned}$$

Denote $G_{X,x}$ as another set of directions on $g(.)$ where

$$\begin{aligned} G_{X,x} \equiv \{(G_1,G_2) \mid\ & g(X+\xi,x+\zeta,P,p) \le g(X,x,P,p) \\ & + G_1\xi - G_2\zeta\}. \end{aligned}$$

Lemma 2.1. (Ross's lemma 6) Any concave function g satisfies Assumption 2.3 if and only if $H_{X,x} = G_{X,x}$.

Proof. See Appendix.

Corollary 2.1. *Assume Assumptions 2.1 and 2.3 are satisfied. Lemma 2.1 implies conditions set out in Theorem 2.2 for* LSNA^k *are sufficient as well as necessary.*

Proof. See Appendix.

Denote the one-sided directional derivatives of $T^k(P;X)$ at X in the direction of ϕ be $\delta T^k(P;X;\phi)$; and that of $S^k(p;x)$ at x in the direction of φ be $\delta S^k(p;x;\varphi)$. Statement (3) in Theorem 2.2 implies that LWNA^k at $(X;x)$ holds if and only if there exists a semipositive valuation operator d, such that

$$P\phi \geq d\{A\phi - [\delta T^k(P;X;\phi)]\}; \qquad p\varphi \leq d\{A\varphi - [\delta S^k(p;x;\varphi)]\}. \qquad (1)$$

LSNA^k at $(X;x)$ holds if and only if equation (1) holds and d is strictly positive.

Assumption 2.4. *$T^k(P;X)$ and $S^k(p;x)$ are differentiable at X and x respectively.*

If $T^k(P;X)$ and $S^k(p;x)$ are differentiable at X and x respectively, then the directional derivatives of $(T^k(P;X), S^k(p;x))$ at $(X;x)$ in all directions $(\phi;\varphi)$ are finite and,

$$\delta T^k(P;X;\phi) = [\nabla T^k(P;X)]\phi; \qquad \delta S^k(p;x;\varphi) = [\nabla S^k(p;x)]\varphi; \qquad (2)$$

where $\nabla T^k(P;X)$ and $\nabla S^k(p;x)$ are matrices of gradients of $T^k(P;X)$ and $S^k(p;x)$ evaluated at X and x respectively. From the above analysis, one can easily establish the following theorem.[3]

Theorem 2.5. *Assume Assumptions 2.1 and 2.4 are satisfied.* LWNA^k *at $(X;x)$ holds if and only if there exists a semipositive valuation operator d, such that,*

$$P \geq [A - \nabla T^k(P;X)]d; \quad p \leq [A - \nabla S^k(p;x)]d. \qquad (3)$$

[3] The main result in Ross (1987) is the case of $T^k = S^k$; $P = p$ in this theorem.

LSNA^k at $(X; x)$ holds if and only if equation (3) holds and d is strictly positive. ■

As we know, d is a vector that belongs to a convex set; it does not have to be a single vector. The jth component of the vector d is interpreted as the implicit price of an after-tax payoff of \$1 in state j. The vector d is a convex valuation operator that maps after-tax state-contingent payoffs into arbitrage-free prices. Theorem 2.5 verifies the existence of implicit state prices in the presence of taxation and transaction costs that can be used to price any change in after-tax income.

For small changes ξ and ζ, $[T^k(P; \xi)] = [T^k(P)]\xi$ and $[S^k(p; \zeta)] = [S^k(p)]\zeta$. PM and DM can then be rewritten as the following pair of primal-dual linear programming problems.

$$(PL): \inf_{(\xi;\zeta)\geq(0;0)} P\xi - p\zeta \quad s.t. [A - T^k(P)]\xi - [A - S^k(p)]\zeta \geq 0;$$

$$(DL): \sup_{d\geq 0} 0 d; d \in \mathbb{R}^S \quad s.t. [A - T^k(P)]d \leq P; [A - S^k(p)]d \geq p.$$

The duality theory of linear programming ensures that the optimal value of PL is zero if and only if the optimal value of its dual DL is zero. If a feasible solution to problem DL exists, obviously its optimal value is zero. We thus obtain the following result.

Corollary 2.2. LWNA^k at (0,0) holds if and only if there exists a semipositive valuation operator d, such that $[A - T^k(P)]d \leq P$, and $[A - S^k(p)]d \geq p$. LSNA^k at (0,0) holds if $[A - T^k(P)]d \leq P$, $[A - S^k(p)]d \geq p$ and d is strictly positive. ■

In Dermody and Rockafellar (1991), D^k is used to denote a term structure packet for investor of tax class k. Corollary 2.2 is consistent with Theorem 3.2 and Theorem 4.3 of Dermody and Rockafellar (1991), if we denote

$$D^k \equiv \{d \mid d \geq 0; [A - T^k(P)]d \leq P; [A - S^k(p)]d \geq p\}$$

and regard maturities for riskless securities as equivalent to states for risky securities in the single period.

Using the properties of convexity and subgradients or supergradients, Theorem 2.5 means that $Ad - P$ is a subgradient of $T^k(P; X + \xi)$ at X and $Ad - p$ is a supergradient of $S^k(p; x + \zeta)$ at x, since $[T^k(P; X + \xi) -$

$T^k(P;X)]d \geq \xi(Ad - P)$ and $[S^k(p;x;\theta) - S^k(p;x)]d \leq \theta(Ad - p)$. Therefore, we have $\Omega_1^k d = Ad - P$ and $\Omega_2^k d = Ad - p$, that is $(A - \Omega_1^k)d = P$, and $(A - \Omega_2^k)d = p$, where Ω_1^k and Ω_2^k are matrices of subgradients of $T^k(P;X+\xi)$ at X and supergradients of $S^k(p;x+\theta)$ at x respectively. If the ranks of $A - \Omega_1^k$ and $A - \Omega_2^k$ are not full, then there is an infinite, convex, and closed set of semimartingale probability measures, D^k, for investors in tax class k. In another words, for unique pricing, we need $A - \Omega_1^k$ and $A - \Omega_2^k$ to be full rank.

3. IMPUTED VALUES OF ASSETS AND PSEUDO SEMIMARTINGALE MEASURES

A multiplicity of valuation operators has important implications for asset pricing. A multiplicity of valuation operators was shown to posses two distinguishing characteristics. First, the absence of arbitrage allows every future after-tax cash flow to have a range of present values within which a security may be traded. Second, the properties of probability measures induced by a multiplicity of valuation operators differ from those of established risk-neutral probability measures. Instead, we get a set of semimartingale measures that can either be super-martingales or submartingales.

3.1. Imputed long and short values and no arbitrage bound

In this subsection we define no arbitrage bound for single-period long and short prices of both risky and riskless payoffs in a manner similar to the way in which a multi-period term structure was defined in Dermody and Rockafellar (1991). Following Dermody and Rockafellar (1991), the imputed long and short values of a traded claim, whose payoff can be hedged by an admissible trading strategy, can be defined as follows:

Definition 3.1. (Dermody and Rockafellar's equations (3.2) and (3.3)) The imputed long value, or long price for the after-tax cash flow w relative to tax class k and position $(X;x)$ is the amount,

$$V^k(w) = \min(P\xi - p\theta); \qquad (4)$$

subject to $[A - \nabla T^k(P;X)]\xi - [A - \nabla S^k(p;x)]\theta \geq w$, with $\xi \geq 0$ and $\theta \geq 0$. The imputed short value, or short price for the after-tax cash flow w relative to tax class k and position $(X;x)$ is the amount,

$$v^k(w) = \max(-P\xi + p\theta); \qquad (5)$$

subject to $-[A - rT^k(P;X)]\theta + [A - rS^k(p;x)]\vartheta \le w$, with $\theta \ge 0$ and $\vartheta \ge 0$.

The imputed long value is the lowest current cost necessary for an investor in tax class k whose initial position is $(X;x)$, through some trade $(\theta;\vartheta)$ to get a future after-tax cash flow at least as good as w. Likewise, the imputed short value is the largest amount of cash that any investor in class k with initial position $(X;x)$ can obtain immediately by taking on the obligation of paying a future cash flow for which the after-tax burdens are no worse than w. These imputed values define bounds for the long and short prices within which a security w may be traded. No investor in class k with previously acquired position $(X;x)$ should be willing to pay more than the amount $V^k(w)$, the upper bound, since there would be a cheaper way of getting at least the same payoff. Similarly, no investor in class k with previously acquired position $(X;x)$ would sell (or short sell) the same payoff for less than the lower bound $v^k(w)$ since a better alternative is to borrow a larger amount and use w to repay all future obligation.

Therefore, the interval $[-V^k(-w); V^k(w)]$ defines arbitrage bounds on the price of the claim w. Without any further knowledge about the preferences of an investor in tax class k who has a previously acquired position $(X;x)$, $-V^k(w)$ and $V^k(w)$ are the tightest bounds that can be derived for the price of the claim w. It is easy to see that the imputed long and short values, $V^k(w)$ and $v^k(w)$, satisfy $v^k(w) = -V^k(-w)$ if $LWNA^k$ is satisfied.

From Definition (3.1), the long and short values, $V^k(w)$ and $v^k(w)$; can be calculated by solving certain optimization problems in $(\theta;\vartheta)$. Specifically, the absence of arbitrage opportunities for an investor in class k with previously acquired position $(X;x)$ implies that there exists at least a trading strategy $(\theta;\vartheta)$ and a nonnegative valuation operator $d \in D^k$, where

$$D^k \equiv \{d | P \ge [A - rT^k(P;X)]d;\; p \le [A - rS^k(p;x)]d\}$$

provided that T^k and S^k are differentiable for any $(\theta;\vartheta) \ge (0;0)$. This is similar to Dermody and Rockafellar's (1991) equation (3.8). Under the equivalent conditions of $P\theta - p\vartheta \ge dw$ and Dermody and Rockafellar's Theorem 3.2, the long and short values, $V^k(w)$ and $v^k(w)$; for the after-tax cash flow w can be computed as

$$V^k(w) = \min_{(\theta;\vartheta) \ge (0;0)} P\theta - p\vartheta = \max_{d \in D^k} dw;$$

and $v^k(w) = -V^k(-w) = \min_{d \in D^k} dw$. On the other hand, D^k is completely determinable by knowledge of V^k and v^k:

$$D^k = \{d | dw \le V^k(w) \text{ for all } w\} = \{d | dw \ge v^k(w) \text{ for all } w\}.$$

3.2. Example: Valuation of an European Call Option

Here is an example showing how the valuation European call option can be derived when prices are different in long and short trades. (We suppressed the issue of differential taxation in this example for simplicity.) It is well known that in a perfect market we can price and hedge a derivative security via synthetic replication. That is, a portfolio consists of the underlying stock and a riskless bond is constructed to mimic the value of the option. This synthetic option must, by the absence of arbitrage, equal the price of the traded option. Suppose we want to price a European call option with maturity at the end of one period. Let the strike price of the option be K, and let the option be written on a stock whose current price is S_0^0. It is known that at the end of the period the stock price will be either S^u in the up state, or S^d in the down state ($S^d < K < S^u$). If the stock price is S^u, the traded option is worth $S^u - K$. If the stock price is S^d, the option is worthless. Assuming that this portfolio consists of y shares of stock and x units of the riskless bond. By design, the future cash flow of this replicating portfolio equals the cash flow of the traded option no matter which state occurs. That is,

$$x(1+r) + yS^u = S^u - K; \qquad x(1+r) + yS^d = 0;$$

where r is the riskless rate. The cost of the replicating portfolio is,

$$C = x + yS_0^0 = \frac{-S^d + S_0^0(1+r)}{(S^u - S^d)(1+r)}.$$

To avoid arbitrage, the traded option must have this value.

In the presence of transaction costs, we suppose that the current long and short prices of the stock are (S_0, s_0), and the lending and borrowing rates are r_L and r_B respectively. The no arbitrage term structure in a tax free environment will be,

$$D = \left\{ (d_1, d_2) \,\middle|\, \begin{pmatrix} (1+r_B)^{-1} \\ s_0 \end{pmatrix} \le \begin{pmatrix} 1 & 1 \\ S^u & S^d \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \le \begin{pmatrix} (1+r_L)^{-1} \\ S_0 \end{pmatrix} \right\},$$

that is,

$$\frac{1}{1+r_B} - d_1 \le d_2 \le \frac{1}{1+r_L} - d_1.$$

Next, substitute the relationship between d_1 and d_2 into $s_0 \le S^u d_1 + S^d d_2 \le S_0$, we get

$$\frac{s_0(1+r_L) - S^d}{(1+r_L)(S^u - S^d)} \le d_1 \le \frac{S_0(1+r_B) - S^d}{(1+r_B)(S^u - S^d)}.$$

Therefore, the long value of the option should be

$$V = \max_{d \in D} d_1(S^u - K) + d_2 0 = \frac{S_0(1+r_B) - S^d}{(1+r_B)(S^u - S^d)}(S^u - K);$$

and the corresponding short value is,

$$v = \frac{s_0(1+r_L) - S^d}{(1+r_L)(S^u - S^d)}(S^u - K).$$

It is easy to see that $v \le C \le V$, since $s_0 \le S_0^0 \le S_0$ and $r_L \le r \le r_B$.

3.3. Risk Neutral Probabilities in Frictional markets

A multiplicity of valuation operators not only results in a range of imputed values for a claim, but it also means that no-arbitrage is a necessary, but not a sufficient, condition for the existence of risk-neutral probabilities in markets with transaction costs and taxes. Indeed, as shown below, the risk-neutral probability measure in a perfect market is induced by a very conservative strategy.

Given a valuation operator d and $d = (d_1, \cdots, d_s)$. Let $d_0 = \sum_{j=1}^{s} d_j$. $\hat{d}_j \equiv d_j/d_0$ is the martingale or risk-neutral probability for state j ($j = 1, \ldots, s$) in markets without transaction costs and taxes. In frictional markets, it is more precise to call it a pseudo-risk-neutral or semimartingale probability for reasons we will elaborate in the next subsection. Theorems 2.2 and 2.5 guarantee that $d_0 > 0$ if LSNA^k holds at (X, x).

Suppose there exists a trading strategy (ξ^0, θ^0) such that $P\xi^0 - p\theta^0 \neq 0$, and

$$\frac{F(X, x; \xi^0, \theta^0)}{P\xi^0 - p\theta^0} = R1.$$

Since the return in every state is the same, R is the riskless return, and we refer to this strategy as riskless trading. $r = R - 1$ is interpreted as the implicit after-tax riskless rate. The discount on this riskless trading is,

$$d_0 = \frac{dF(X;x;\xi';\eta')}{(P\xi' - p\eta')R}:$$

Obviously, $d_0 \leq 1/R$ if LSNA holds at $(X;x)$.

Suppose $(A;P;p;T;S)$ admits no local strong arbitrage. For any portfolio change $(\xi;\eta)$ with $P\xi - p\eta \neq 0$, the return on $(\xi;\eta)$ is the vector $R^{\xi;\eta}$ in $\mathbb{R}^S$ defined by

$$R_s^{\xi;\eta} = \frac{F(X;x;\xi';\eta')_s}{(P\xi - p\eta)}$$

for state s. Theorem 2.5 implies that LSNA holds at $(X;x)$ if and only if for any $(\xi;\eta)$, we have $dR^{\xi;\eta} \leq 1$; that is $\sum_{j=1}^S \hat{d}_j R_j^{\xi;\eta} \leq 1/d_0$. By convention $\hat{d}_j$ $(j = 1;\dots;s)$ are risk-neutral probabilities if we can choose a state-price vector d from the multiplicity such that $\sum_{j=1}^S \hat{d}_j R_j^{\xi;\eta} = R$. In other words, the risk-neutral probability measure is induced by a very risk-averse strategy. In markets with transaction costs and taxes, it rules out all the other cases where $\sum_{j=1}^S \hat{d}_j R_j^{\xi;\eta} \neq R$; but $\sum_{j=1}^S \hat{d}_j R_j^{\xi;\eta} \leq 1/d_0$: In fact, that LSNA holds is equivalent to the existence of a set of pseudo-risk-neutral probabilities. We will discuss this further in the multiperiod setting in Section 4.

3.4. Adjusted Risk Neutral Probabilities and Frictional Markets

An equivalent interpretation of Theorem 2.2 and Theorem 2.5 in terms of $(\hat{d}_1;\dots;\hat{d}_S)$ and tax-adjusted pseudo-risk-neutral or semimartingales probabilities is possible. Theorem 2.5 indicates that if Assumption 2.4 is satisfied, then the necessary and sufficient condition for LSNA^k at $(X;x)$ is the existence of a pseudo-risk-neutral probability measure such that the expected after-tax return cannot exceed the original value under this measure and the cost of investment is nonpositive, that is

$$P\xi - p\eta \leq 0 \implies \hat{E}[R(X+\xi;x+\eta;P;p)] \leq \hat{E}[R(X;x;P;p)]:$$

In addition, Theorem 2.5 formally implies corollary (3.1) below. Let the amount of tax obligation on each unit of security i held long be $t_{ij}^k(P_i)$, and the amount of tax subsidy for each unit of security i held short be $s_{ij}^k(p_i)$. $t_{ij}^k(P_i)$ are entries in $T^k(P)$ and $s_{ij}^k(p_i)$ are the entries in $S^k(p)$.

Corollary 3.1. Assume Assumptions 2.1 and 2.4 are satisfied. The necessary and sufficient condition for LSNA^k at $(X;x)$ is that there exists pseudo-risk-neutral probabilities and an expectation operator for security i such that

$$\frac{P_i}{d_0} \geq \sum_{j=1}^{S} \hat{\pi}_j [a_{ij} - \nabla t_{ij}^k (P;X)] \equiv \hat{E}[a_i - \nabla t_i^k (P;X)]$$

$$\frac{p_i}{d_0} \leq \sum_{j=1}^{S} \hat{\pi}_j [a_{ij} - \nabla s_{ij}^k (p;x)] \equiv \hat{E}[a_i - \nabla s_i^k (p;x)]$$

where $\nabla T^k(P;X) \equiv (\nabla t_{ij}^k (P;X))_{n \times s}$, $\nabla S^k(p;x) \equiv (\nabla s_{ij}^k (p;x))_{n \times s}$ viewing the expected after-tax payoff of the security under these specially chosen pseudo-risk-neutral probabilities as the number between its normalized long and short prices. ¥

For any security $i; P_i \geq d_0 \hat{E}(a_i - \nabla t_i^k)$ and $p_i \leq d_0 \hat{E}(a_i - \nabla s_i^k)$ imply that any security's discounted expected after-tax payoff with respect to the artificially chosen probabilities is a number between its long and short prices. Specifically, the following corollary is true.

Corollary 3.2. Assume Assumptions 2.1 and 2.4 are satisfied. LSNA ^{k} at $(X;x)$ holds if and only if there exists a martingale process which is between the normalized long and short prices under this probability measure. ¥

4. VALUATION WITH FRICTIONS IN A MULTIPERIOD SETTING

In a multiperiod setting, the valuation process can be derived in a manner similar to that derived in a single period case previously. The equivalence between no arbitrage opportunities and the existence of a multiplicity of state-price deflators corresponds to a pseudo semimartingale probability measure can similarly be established for multi-period and in markets with transaction costs and taxes. The tightest bounds for the price of a traded claim can again be obtained without considering investor preferences. In the following analyses, we stop making the difference between local and global arbitrage, and the difference of different tax classes. This is to keep the focus of the analysis on the multi-period context of the valuation model.

Suppose the set of dates is $\{0, 1, \ldots, T\}$. As usual, a discrete-time multiperiod financial model is built on a finite filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$, where Ω is a finite set of states, $\mathcal{F}$ is the tribe of subsets of Ω that are events, P is a probability measure assigning to any event B in $\mathcal{F}$ its probability $P(B)$, $\mathcal{F}_t$ is a σ-algebra of events up to time t, which can be seen as the set of events corresponding to the information available at time t. We will assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$. For any random variable X, we let $E_t(X) = E(X|\mathcal{F}_t)$ denote the conditional expectation of X given $\mathcal{F}_t$. An adapted process is a sequence $Y = \{Y_0, \ldots, Y_T\}$ such that Y_t is a random variable with respect to $(\Omega, \mathcal{F}_t)$ at t. The market consists of n securities defined by an R^n-valued adapted process $P_t + \delta_t$ or $p_t + \delta_t$, the cum-dividend security price at time t, where $P_t = (P_t^1, \ldots, P_t^n)$, $p_t = (p_t^1, \ldots, p_t^n)$ and $\delta_t = (\delta_t^1, \ldots, \delta_t^n)$ are these securities' adapted long price processes, short price processes and dividend processes. P_t^j, p_t^j and δ_t^j denote the ex-dividend long price, short price, and the dividend paid by security j at time t respectively, with P_t, p_t, and δ_t measurable with respect to $\mathcal{F}_t$.

A trading strategy is an adapted process $\theta_t = (X_t, x_t)$ for $0 \le t \le T$ that represents the position held by that portfolio after trading at time t, with $\theta_{-1} \equiv 0$, which means zero initial investment. We assume that θ is predictable, that is, (X_t^i, x_t^i) is $\mathcal{F}_{t-1}$-measurable for $t \ge 1$ and for any asset i.

The pre-tax payoff is $X_{t-1}(P_t + \delta_t) - x_{t-1}(p_t + \delta_t)$, and the tax amount, which is a function of stochastic variables in general, is $T_t^k(\delta_t, P_t, X_{t-1})$ and $S_t^k(\delta_t, p_t, x_{t-1})$ for long and short positions respectively at time t. Once again, we ignore an investor's chance to trade at advantageous prices by unwinding a previously acquired position when discussing arbitrage. For the sake of simplicity, we omit the superscript k for tax class.

The after-tax cash flow process $M^{X,x}$ generated at time t by a trading strategy (X, x) is defined by,

$$M_t^{X,x} = X_{t-1}(P_t + \delta_t - T_t) - x_{t-1}(p_t + \delta_t - S_t) - X_t P_t + x_t p_t. \qquad (6)$$

For a self-financing trading strategy, $M^{X,x}$ is often set equal to zero. Following Duffie (1992), the definitions of an arbitrage opportunity and a state-price deflator for a multiperiod setup can be introduced based on $M^{X,x}$.

Definition 4.1. A trading strategy $(X; x)$ is an arbitrage opportunity if $M_t^{X;x} \geq 0$ for every $t = 1; \dots; T$, and $M_t^{X;x} > 0$ for at least one t; and $X_T P_T - x_T p_T \geq 0$.

Definition 4.2. We call any strictly positive adapted process a deflator. π is a state-price deflator if, for all $t = 0; 1; \dots; T-1$, we have,

$$E_t\{(P_{t+1} + \delta_{t+1} - T_{t+1})\pi_{t+1}\} \leq P_t \pi_t; \tag{7}$$

$$E_t\{(p_{t+1} + \delta_{t+1} - S_{t+1})\pi_{t+1}\} \geq p_t \pi_t; \tag{8}$$

for a "buy and hold" strategy. Here, E_t represents the expectation operator conditional on all available information at time t.

Using the property of iterated expectations, it is easy to show that π is a state-price deflator if and only if,

$$P_t \geq \frac{1}{\pi_t} E_t\Big[\sum_{j=t+1} \pi_j (\delta_j - T_j)\Big]; \quad p_t \leq \frac{1}{\pi_t} E_t\Big[\sum_{j=t+1} \pi_j (\delta_j - S_j)\Big] \tag{9}$$

for a dividend-paying security assuming that $P_T = p_T = 0$; and

$$P_t \geq \frac{1}{\pi_t} E_t[\pi_T (P_T - T_T)]; \quad p_t \leq \frac{1}{\pi_t} E_t[\pi_T (p_T - S_T)]: \tag{10}$$

for a non dividend-paying security.[4] This means that the long (short) price is never less (greater) than the expected state-price discounted future after-tax cash flows. In addition, this shows the different notions of a state-price deflator between the single period and multiperiod frameworks. The following theorem provides the basis for calculating the tightest bound on the price of a claim in a frictional environment.

Theorem 4.1. π is a state-price deflator if and only if, for $t = 0; \cdots; T - 1$,

$$P_t X_t - p_t x_t \geq \frac{1}{\pi_t} \Big\{E_t\Big[\sum_{j=t+1} \pi_j M_j^{X;x}\Big] + E_t[\pi_T (P_T X_T - p_T x_T)]\Big\}: \tag{11}$$

[4] In equations (9) and (10), maturity T may be changed to any exercise date τ ($> t$).

Proof: See Appendix.

If we let $T = t+1$, then we have from Theorem 4.1

$$P_t X_t - p_t x_t \ge \frac{1}{\pi_t} E_t \{\pi_{t+1}[X_t(P_{t+1} + \delta_{t+1} - T_{t+1}) - x_t(p_{t+1} + \delta_{t+1} - S_{t+1})]\} .$$

This link to Theorem 2.2 described previously for the single period setting. Next, define the after-tax gain process G for a dividend-price package $(\delta; P, p; T; S)$ by $G_{lt} = P_t + \sum_{j=1}^{t} (\delta_j - T_j)$ for a long position and $G_{st} = p_t + \sum_{j=1}^{t} (\delta_j - S_j)$ for a short position and define the deflated after-tax gain processes G^{γ}, by $G^{\gamma}_{lt} = \gamma_t P_t + \sum_{j=1}^{t} \gamma_j (\delta_j - T_j)$ for a long position and $G^{\gamma}_{st} = \gamma_t p_t + \sum_{j=1}^{t} \gamma_j (\delta_j - S_j)$ for a short position, given a deflator γ. We then have the following theorem.

Theorem 4.2. *The dividend-price-tax package $(\delta; P; p; T; S)$ admits no arbitrage iff there is a state-price deflator, π; and that the after-tax gain process G^{π}_{lt} is a supermartingale and G^{π}_{st} is a submartingale.*

Proof: See Appendix.

Therefore, no arbitrage implies a multiplicity of valuation operators π in a multiperiod setting, just as in the single period case.

4.1. The concept of pseudo-semimartingale measure

Next, we show the equivalence between the absence of arbitrage and the existence of a pseudo-semimartingale probability measure Q with the property that the discounted after-tax payoff is bounded between the long and short prices under Q.

Suppose there are short-term riskless borrowing and lending ($r_{Lt} \le r_{Bt}$) at strictly positive discount factors, d_{Bt} and d_{Lt}, for each $t < T$.[5] We

[5] Borrowing (lending) at a borrowing rate r_{Bt} (lending rate r_{Lt}) at a given time $t < T$ is possible if there is a trading strategy $(X; x)$ with,

(i) $M_s^{X;x} = 0;\ s < t;$

(ii) $M_s^{X;x} = M_t^{X;x};\ s = t$, which gives the discount factor d_{Bt} (d_{Lt}) at period t

(iii) $M_s^{X;x} = 1;\ s = t+1;$

(iv) $M_s^{X;x} = 0;\ s > t+1.$

define,

$$R^B_{t,\tau} = (d_{Bt}\, d_{Bt+1} \cdots d_{B\tau-1})^{-1}; \quad R^L_{t,\tau} = (d_{Lt}\, d_{Lt+1} \cdots d_{L\tau-1})^{-1};$$

for any times t and $\tau > t$. $R^B_{t,\tau}$ can be interpreted as the payback at time τ to one unit of account borrowed risklessly at time t and rolled over in short-term borrowing contracts repeatedly until date τ. Letting $d_{Bt} \le d_t \le d_{Lt}$,

$$R_{t,\tau} = (d_t d_{t+1} \cdots d_{\tau-1})^{-1}.$$

Note that $R_{t,t} \equiv 1$; $R_{t,s} = R_{t,\tau} R_{\tau,s}$ for $t < \tau < s$.

We can adjust the original probability measure P to the equivalent probability measure Q which means P and Q assign zero probabilities to the same events, such that any security's long and short prices are related to the expected discounted after-tax cash flows of the security for pseudo-risk-neutral investors. An equivalent probability measure Q is an equivalent pseudo-semimartingale measure if, for $t < T$,

$$P_t \ge E^Q_t\left[\frac{P_{t+1} + \delta_{t+1} - T_{t+1}}{R_{t,t+1}}\right] \quad \text{and}$$

$$p_t \le E^Q_t\left[\frac{p_{t+1} + \delta_{t+1} - S_{t+1}}{R_{t,t+1}}\right].$$

Let Q^0 be the probability measure defined by the Radon-Nikodym derivative. The density process ξ for Q^0 is defined by,

$$\xi_t = \frac{\pi_t R_{0,t}}{\pi_0}.$$

Since ξ_t is strictly positive, Q^0 and P are equivalent probability measures. We show that Q^0 is an equivalent pseudo-semimartingale measure.

In fact, since π is a state-price deflator, we have,

$$E_{j-1}\left[\frac{(P_j + \delta_j - T_j)\pi_0 \xi_j}{R_{0,j}}\right] \le \frac{P_{j-1}\pi_0 \xi_{j-1}}{R_{0,j-1}};$$

$$E_{j-1}\left[\frac{(p_j + \delta_j - S_j)\pi_0 \xi_j}{R_{0,j}}\right] \ge \frac{p_{j-1}\pi_0 \xi_{j-1}}{R_{0,j-1}}.$$

That is,

$$E_{j-1}\Big[\frac{(P_j+\delta_j-T_j)\xi_j}{R_{t,j}}\Big] \le \frac{P_{j-1}\xi_{j-1}}{R_{t,j-1}};$$
$$E_{j-1}\Big[\frac{(p_j+\delta_j-S_j)\xi_j}{R_{t,j}}\Big] \ge \frac{p_{j-1}\xi_{j-1}}{R_{t,j-1}}.$$

Note that $R_{0,j}=R_{0,t}R_{t,j}$ for $0<t<j$. The above inequalities imply,

$$E^{Q'}_{j-1}\Big[\frac{P_j+\delta_j-T_j}{R_{t,j}}\Big] \le \frac{P_{j-1}}{R_{t,j-1}}; \qquad E^{Q'}_{j-1}\Big[\frac{p_j+\delta_j-S_j}{R_{t,j}}\Big] \ge \frac{p_{j-1}}{R_{t,j-1}}.$$

Therefore, for predictable decision variables X and x, we have

$$\begin{aligned}
E_t^{Q'}\sum_{j=t+1}\Big[\frac{M_j^{X,x}}{R_{t,j}}\Big] &= E_t^{Q'}\Big\{\sum_{j=t+1}\Big[X_{j-1}E^{Q'}_{j-1}\frac{P_j+\delta_j-T_j}{R_{t,j}}\,\Big|\,\mathcal{F}_{j-1}\Big] \\
&\quad-\sum_{j=t+1}\Big[x_{j-1}E^{Q'}_{j-1}\frac{p_j+\delta_j-S_j}{R_{t,j}}\,\Big|\,\mathcal{F}_{j-1}\Big] \\
&\quad-E_t^{Q'}\sum_{j=t+1}\Big[X_j\frac{P_j}{R_{t,j}}\Big]+E_t^{Q'}\sum_{j=t+1}\Big[x_j\frac{p_j}{R_{t,j}}\Big]\Big\} \\
&\le E_t^{Q'}\sum_{j=t+1}\Big[X_{j-1}\frac{P_{j-1}}{R_{t,j-1}}\Big]-E_t^{Q'}\sum_{j=t+1}\Big[x_{j-1}\frac{p_{j-1}}{R_{t,j-1}}\Big] \\
&\quad-E_t^{Q'}\sum_{j=t+1}\Big[X_j\frac{P_j}{R_{t,j}}\Big]+E_t^{Q'}\sum_{j=t+1}\Big[x_j\frac{p_j}{R_{t,j}}\Big] \\
&= X_tE_t^{Q'}\Big[\frac{P_t}{R_{t,t}}\Big]-x_tE_t^{Q'}\Big[\frac{p_t}{R_{t,t}}\Big]-X_TE_t^{Q'}\Big[\frac{P_T}{R_{t,T}}\Big]+x_TE_t^{Q'}\Big[\frac{p_T}{R_{t,T}}\Big] \\
&= P_tX_t-p_tx_t-E_t^{Q'}\Big[\frac{X_TP_T-x_Tp_T}{R_{t,T}}\Big].
\end{aligned}$$

Hence, we obtain the following result.

Theorem 4.3. *The absence of arbitrage is equivalent to the existence of an equivalent pseudo-semimartingale measure. Moreover, if π is a state-price deflator, then the equivalent pseudo-semimartingale measure Q has the density process ξ defined by $\xi_t=\pi_tR_{0,t}/\pi_0$.* ∎

In other words, there are no arbitrage opportunities if and only if there exists an equivalent probability measure that transforms the long (short) price process into a pseudo-supermartingale (submartingale) or some process that is between the long and the short price processes into a pseudo-martingale (after a normalization). In the absence of transaction costs and taxes, this pseudo-martingale measure changes to a martingale (super- or sub-martingale) measure. Once again, we have the same conclusion as in the single period setting. That is, the absence of arbitrage opportunities is a necessary but not a sufficient condition for the existence of martingale or risk-neutral probabilities in a frictional market.

4.2. Imputed long and short prices in a multi-period context

Similarly, the imputed long and short prices for a traded claim, whose payoff can be hedged by an admissible trading strategy, can be defined. The imputed long price at time t for the after-tax cash flow w_j (w_j can be positive, zero, or negative) at time j $(j = t+1, \ldots, T)$ relative to tax class k is the amount

$$V_t^k(w) = \min P_t X_t - p_t x_t;$$
$$\text{subject to } \Delta_j^{X;x} \geq w_j \ (j = T+1, \ldots, T) \text{ for } P_T = p_T = 0; \text{ or}$$
$$P_T X_T - p_T x_T \geq w_T \text{ for } w_j = 0 \ (j = t+1, \ldots, T-1)$$

and the self-financing strategy $\delta_j = 0$ $(j = t+1, \ldots, T)$ with $X \geq 0; x \geq 0$. On the other hand, the imputed short price at time t for the after-tax cash flow w_j relative to tax class k is the amount

$$v_t^k(w) = \max -P_t X_t + p_t x_t;$$
$$\text{subject to } -\Delta_j^{X;x} \leq w_j \ (j = t+1, \ldots, T) \text{ for } P_T = p_T = 0; \text{ or}$$
$$-P_T X_T + p_T x_T \leq w_T \text{ for } w_j = 0 \ (j = t+1, \ldots, T-1)$$

and the self-financing strategy $\delta_j = 0$ $(j = t+1, \ldots, T)$ with $X \geq 0; x \geq 0$.

The imputed long price is the lowest cost necessary at time t for an investor in tax class k to get at least as good as the future after-tax cash flow associated with w_j $(j = t+1, \ldots, T)$ through trading strategy $(X; x)$. Likewise, the imputed short price is the highest amount of present cash that every investor in class k can obtain at time t by taking on the obligation of paying a future cash flow for which the after-tax burdens are no worse than w_j $(j = t+1, \ldots, T)$. These prices define a possible range for the long and the short prices at which a new security could be traded if it were to be introduced to the market. No investor in class k should be willing to pay more than the upper bound $V_t^k(w)$ or would (short) sell for less than the lower bound $v_t^k(w)$.

The imputed long and short prices $V^k(w)$ and $v^k(w)$ satisfy $v_t^k(w) = -V_t^k(-w)$ if the no arbitrage condition is satisfied. Prices in the interval $[-V^k(-w), V^k(w)]$ define arbitrage bounds for the price of the claim w_t $(j = t+1, \ldots, T)$. Without any further knowledge about the preferences of an investor in tax class k, $-V_t^k(-w)$ and $V_t^k(w)$ are the tightest bounds that can be placed on the price of the claim w_j $(j = t+1, \ldots, T)$. The absence of an arbitrage opportunity for investor in class k implies that there exists a trading strategy (X, x) and a nonnegative valuation operator $d \in D_t^k$, where

$$D_t^k \equiv \{d | E_t[\sum_{j=t+1}^{T} (\delta_j - T_j) d_j] \le P_t ; E_t[\sum_{j=t+1}^{T} (\delta_j - S_j) d_j] \ge p_t\}$$

for a security with intermediate cash flows, and

$$D_t^k \equiv \{d | E_t[(P_T - T_T) d_T] \le P_t ; E_t[(p_T - S_T) d_T] \ge p_t\}$$

for a security without intermediate cash flows. Under the equivalent conditions,

$$P_t X_t - p_t x_t \ge E_t[\sum_{j=t+1} d_j \delta_j^{X,x}] + E_t[d_T (P_T X_T - p_T x_T)];$$

the long and short prices $V_t^k(w)$ and $v_t^k(w)$ for after-tax cash flow w of some claim can be calculated by

$$V_t^k(w) = \min_{(X,x) \ge (0,0)} P_t X_t - p_t x_t = \max_{d \in D_t^k} E_t[\sum_{j=t+1}^{T} d_j w_j]; \quad \text{and}$$

$$v^k(w) = -V^k(-w).$$

Therefore, the long and short prices reflect the multiplicity of no-arbitrage term structures that may exist when transaction costs and taxes are present in a multiperiod setting.

5. SUMMARY AND CONCLUSION

When long-short spreads and taxes are present, a multiplicity of valuation operators replaces the single valuation operator in an arbitrage-free financial market. Previous research shows that a valuation operator is agent-specific when agents have different tax classes. We demonstrate that a multiplicity of operators exists because of long-short spreads. Taxes and long-short spreads together lead to the condition of no arbitrage whereby every future cash flow may be represented by range of present values.

In our framework, the tightest bound for the price of a claim can still be derived without any knowledge of investor preferences. In fact, the lower bound represents the lowest current cost of which an investor in a tax class may get at least as good an amount of after-tax cash flow in the future. The upper bound of the price is the highest amount of present cash that an investor in some tax class can obtain by taking on the obligation of paying an amount of future cash flow for which the after-tax burdens are no worse than this amount. Furthermore, we show in this paper that the existence of a set of pseudo-risk-neutral or pseudo-semimartingale probabilities is a necessary and sufficient condition for the absence of arbitrage opportunities. This implies that the pseudo-semimartingale or pseudo-risk-neutral probability measure, as opposed to the martingale measure, can be widely used to price securities by arbitrage in a frictional market even though no arbitrage is not sufficient to guarantee the existence of a set of risk-neutral or martingale probabilities.

The ideas described here can be adapted to price assets in a continuous-time setting. By making use of this approach, the classic asset pricing model can be generalized to a more realistic, imperfect capital market setting with different interest rates for lending and borrowing and different prices for long- and short-positions.

APPENDIX

Proof of Theorem 2.2. Consider,

$$\inf_{(\xi;\eta)\geq 0} \{P\xi - p\eta - d\{A(\xi-\eta) - [T^k(P;X+\xi) - T^k(P;X) \\ - S^k(p;x+\eta) + S^k(p;x)]\}\};$$

under the condition $d \geq 0$. For a fixed initial investment position $(X;x)$, it reaches its minimum of 0 at $(\xi;\eta) = (0;0)$ if, and only if,

$$P\xi - p\eta - d\{A(\xi-\eta) - [T^k(P;X+\xi) - T^k(P;X) \\ - S^k(p;x+\eta) + S^k(p;x)]\} \geq 0, \text{ for any } (\xi;\eta) \geq (0;0);$$

this holds, in turn, if and only if

$$P\xi - d\{A\xi - \{[T^k(P;X+\xi)] - [T^k(P;X)]\}\} \geq 0, \text{ for any } \xi \geq 0;$$

and

$$p\eta - d\{A\eta - \{[S^k(p;x+\eta)] - [S^k(p;x)]\}\} \leq 0 \text{ for any } \eta \geq 0:$$

Since PM is stable, by the Strong Duality Theorem of convex programming (see Avriel 1976), the first part of our theorem has been proved. Similarly, we can show the second part of the theorem for the local strong no-arbitrage condition. ■

Proof of Lemma 2.1. Sufficiency. Suppose $(X;x)$ is a maximum with a zero derivative in some direction $(\varphi_X;\varphi_x)$. Since $(X;x)$ is a maximum, $H_{X;x} = \mathbb{R}^n \times \mathbb{R}^n$. Let

$$(H_1;H_2) = -\lambda(\varphi_X;-\varphi_x) \in H_{X;x} (\lambda > 0):$$

If $(H_1;H_2) \in G_{X;x}$ for some $\lambda > 0$, and all $(\xi;\eta) \geq (0;0)$, we have

$$g(X+\xi;x+\eta;P;p) \leq g(X;x;P;p) + H_1\xi - H_2\eta:$$

Letting $(\xi;\eta) = t(\varphi_X;\varphi_x)$, then

$$g(X+\xi;x+\eta;P;p) \leq g(X;x;P;p) - \lambda t(\varphi_X^2 + \varphi_x^2);$$

that is,

$$\frac{1}{t}[g(X+t\varphi_X, x+t\varphi_x; P, p) - g(X, x; P, p)] \le -\lambda(\varphi_X^2+\varphi_x^2) < 0.$$

This contradicts the assumption that the derivative is zero in the direction of (φ_X, φ_x).

Necessity. It is easy to see $G_{X,x} \subseteq H_{X,x}$. We only need show that $H_{X,x} \subseteq G_{X,x}$ under the condition that the concave function g satisfies Assumption 2.3. Suppose $\exists (H_1, H_2) \in H_{X,x}$, but $(H_1, H_2) \notin G_{X,x}$. Since $G_{X,x}$ is a convex set, by the separation theorem there exists (φ_X, φ_x) such that,

$$\varphi_X H_1 - \varphi_x H_2 \le 0 \le \varphi_X G_1 - \varphi_x G_2 \quad \text{for all } (G_1, G_2) \in G_{X,x}.$$

Because $(H_1, H_2) \in H_{X,x}$, the first inequality implies that the directional derivative of $g(\cdot,\cdot)$ in the direction of (φ_X, φ_x) at (X, x), $D^+g \le 0$. The second inequality implies that the same directional derivative $D^+g \ge 0$. Hence we have $D^+g = 0$. If (X, x) is not a maximum of g with a zero directional derivative, then $0 \notin G_{X,x}$, the separation is strict, thus $D^+g > 0$, a contradiction. This verifies that $H_{X,x} \subseteq G_{X,x}$. So $H_{X,x} = G_{X,x}$. ∎

Proof of Corollary 2.1. Necessary condition. From Theorem 2.2, we know that there exists LSNA^k at (X, x) if, and only if, there exist positive valuation operators d, such that, for any $(\xi, \eta) \ge (0, 0)$,

$$\begin{aligned} P\xi - d\{A\xi - [T^k(P, X+\xi) - T^k(P, X)]\} &\ge 0 \text{ and} \\ p\eta - d\{A\eta - [S^k(p, x+\eta) - S^k(p, x)]\} &\le 0. \end{aligned}$$

This yields

$$d[R(X+\xi, x+\eta; P, p) - R(X, x; P, p)] \le P\xi - p\eta.$$

We can easily get the first part of our corollary.

Sufficient condition. This follows from Theorem 2.2 and Lemma 2.1 above. ∎

Proof of Theorem 4.1. *Necessity.* Since,

$$
\begin{aligned}
E_t\Big[\sum_{j=t+1}^{T} \pi_j M_j^{X,x}\Big] = {} & E_t\Big\{\sum_{j=t+1}^{T} [X_{j-1}\pi_j(P_j+\delta_j-T_j) \\
& - x_{j-1}\pi_j(p_j+\delta_j-S_j) \\
& - \pi_j P_j X_j + \pi_j p_j x_j]\Big\} \\
= {} & E_t\Big\{\sum_{j=t+1}^{T} E_{j-1}[X_{j-1}\pi_j(P_j+\delta_j-T_j) \\
& - x_{j-1}\pi_j(p_j+\delta_j-S_j)|F_{j-1}]\,|F_t\Big\} \\
& + E_t\Big\{\sum_{j=t+1}^{T} [-\pi_j P_j X_j + \pi_j p_j x_j]\Big\} \\
= {} & E_t\Big\{\sum_{j=t+1}^{T} \{X_{j-1}E_{j-1}[\pi_j(P_j+\delta_j-T_j)|F_{j-1}] \\
& - x_{j-1}E_{j-1}[\pi_j(p_j+\delta_j-S_j)|F_{j-1}]\}\,|F_t\Big\} \\
& + E_t\Big\{\sum_{j=t+1}^{T} [-\pi_j P_j X_j + \pi_j p_j x_j]\Big\};
\end{aligned}
$$

and since π is a state-price deflator,

$$E_j\{P_{j+1}+\delta_{j+1}-T_{j+1}\} \le P_j\pi_j; \text{ and } E_j\{p_{j+1}+\delta_{j+1}-S_{j+1}\} \le p_j\pi_j;$$

for $j = 0,1,\ldots,T-1$. So for predictable decision processes X and x, we have,

$$
\begin{aligned}
E_t\Big[\sum_{j=t+1}^{T} \pi_j M_j^{X,x}\Big] \le {} & E_t\Big\{\sum_{j=t+1}^{T} [X_{j-1}\pi_{j-1}P_{j-1} - x_{j-1}\pi_{j-1}p_{j-1}]\Big\} \\
& + E_t\Big\{\sum_{j=t+1}^{T} [-\pi_j P_j X_j + \pi_j p_j x_j]\Big\} \\
= {} & \pi_t(P_tX_t - p_tx_t) - E_t[\pi_T(P_TX_T - p_Tx_T)].
\end{aligned}
$$

Therefore, Theorem 4.1 can be obtained by reorganizing the above inequality.

Sufficiency. If

$$P_tX_t - p_tx_t \ge E_t\Big[\sum_{j=t+1}^{T} \pi_j M_j^{X,x}\Big] + E_t[\pi_T(P_TX_T - p_Tx_T)]$$

holds, then consider, for an arbitrary security i, the trading strategy $(X^i; x^i)$ defined by,

(i) Let $x^i = 0$ and $X^i_t = 1; X^i_{t+j} = 0$, for arbitrary $t = 0, 1, \ldots, T$; $j = 1, \ldots, T - t$. We can easily get

$$E_t[(P_{t+1} + \delta_{t+1} - T_{t+1})\pi_{t+1}] \le P_t \pi_t$$

for all securities $i = 1, \ldots, n$.

(ii) Let $X^i = 0$ and $x^i_t = 1; x^i_{t+j} = 0$, for arbitrary $t = 0, 1, \ldots, T$; $j = 1, \ldots, T - t$, we have

$$E_t[(p_{t+1} + \delta_{t+1} - S_{t+1})\pi_{t+1}] \ge p_t \pi_t$$

for all securities $i = 1, \ldots, n$. Since t is arbitrary, π is a state-price deflator. □

Proof of Theorem 4.2. Necessity. We need to show that G^π_{lt} is a supermartingale and G^π_{st} is a submartingale. From the definition, an adapted process Y is a supermartingale (submartingale) if and only if $E(Y_\tau) \le Y_0$ ($\ge Y_0$) for any stopping time $\tau \le T$. Consider, for an arbitrary security i and an arbitrary stopping time $\tau \le T$, the trading strategy $(X^i; x^i)$ defined by $x^i \equiv 0$, and $X^i_j = 1$ for $j < \tau$, with $X^i_j = 0$ for $j \ge \tau$. Since there exists π, such that $E(\sum_{j=0}^{T} \pi_j \delta_j^{X,x}) \le 0$, we have

$$E\Big[-\pi_0 P^i_0 + \sum_{j=1}^{\tau-1} \pi_j (\delta^i_j - T^i_j) + \pi_\tau (P^i_\tau + \delta^i_\tau - T^i_\tau)\Big] \le 0.$$

Since i is arbitrary, so

$$E\Big[-\pi_0 P_0 + \sum_{j=1}^{\tau-1} \pi_j (\delta_j - T_j) + \pi_\tau (P_\tau + \delta_\tau - T_\tau)\Big] \le 0;$$

that is $E(G^\pi_{l\tau}) \le \pi_0 P_0 = E(G^\pi_{l0})$, since $X_{-1} = 0$.

Similarly, we can define a trading strategy $(X^i; x^i)$ by $X^i \equiv 0$, and $x^i_j = 1$ for $j < \tau$, with $x^i_j = 0$ for $j \ge \tau$, and get $E(G^\pi_{s\tau}) \ge \pi_0 p_0 = E(G^\pi_{s0})$, since $x_{-1} = 0$. Since τ is arbitrary, G^π_{lt} (G^π_{st}) is a supermartingale (submartingale).

Sufficiency. Since

$$E\left(\sum_{j=0} \pi_j \delta_j^{X,x}\right) = E\left(\sum_{j=0} \pi_j [X_{j-1}(P_j + \delta_j - T_j) - x_{j-1}(p_j + \delta_j - s_j)] - P_j X_j + p_j x_j\right)$$

and π is a state-price deflator,

$$E\{(P_{j+1} + \delta_{j+1} - T_{j+1})\pi_{j+1}\} \leq P_j \pi_j;$$
$$E\{(p_{j+1} + \delta_{j+1} - S_{j+1})\pi_{j+1}\} \leq p_j \pi_j$$

for $j = 0, 1, \ldots, T-1$.

Using the property of iterated expectations, we have,

$$E\left(\sum_{j=0} \pi_j M_j^{X,x}\right) \leq E\left(\sum_{j=0} X_{j-1} P_{j-1} \pi_{j-1} - x_{j-1} p_{j-1} \pi_{j-1} - \pi_j X_j P_j + \pi_j x_j p_j\right)$$
$$= -\pi_T X_T P_T + \pi_T x_T p_T$$

If $X_T P_T - x_T p_T \geq 0$, then $E\left(\sum_{j=0}^{T} \pi_j M_j^{X,x}\right) \leq 0$. From the definition of arbitrage, this proves sufficiency. ∎

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