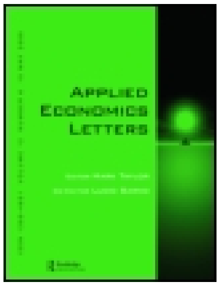

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On skewness of return and buying more than one ticket in a lottery

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The purpose in this article is to demonstrate that buying more than one ticket in a lottery is readily explicable in models of utility that permit gambling at actuarially unfair odds. However, contrary to popular view, we show this choice cannot be explained in terms of a variance–skew trade-off.

I. Introduction

Hirshleifer (1966) and Golec and Tamarkin (1998), amongst others, point out that the observed behaviour of some gamblers to engage in repetitive small-stake wagers, to buy more than one lottery ticket in a draw or to bet on more than one runner in a race is inconsistent with global risk-seeking behaviour. Global risk-seeking behaviour is the traditional explanation for explaining outcomes in gambling markets. (e.g. Ali (1977) and Quandt (1986)).

It has become quite common to hypothesize that betting at actuarially unfair odds might be explicable in terms of a preference for positive skewness of return, *ceteris paribus*. For instance Golec and Tamarkin (1998) state that if bettors are interested in moments other than the first two then betting more than one horse in a race can be explained as attempts by bettors to trade-off variance for skewness. They also state that bettors may appear to prefer variance when it is skewness they crave. This idea is borrowed from the standard expected utility model where it would appear that for utility functions that can be legitimately approximated by a Taylor expansion, agents exhibit a preference, *ceteris paribus*, for positive skewness.[1]

In fact the argument is in incorrect in general for the standard expected utility maximizer. Brockett and Garven (1998) show that one can always construct two distributions with a given moment ordering for which neither stochastically dominates the other at any degree of stochastic dominance.[2]

The purpose in this article is to demonstrate that betting more than one runner in a race or buying more than one ticket in a lottery is readily explicable in models of utility that permit gambling at actuarially unfair odds. However we show this choice cannot be sensibly explained in terms of a variance -skew trade-off or a preference for skewness. The models we employ are an expected utility model with an expo-cubic specification (see e.g. Friedman and Savage (1948) and the nonexpected model of utility of Markowitz (1952).[3] We construct examples in which the agent chooses between two lotteries,

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[1] This is because the third derivative of the utility function of a globally risk-averse agent is positive and therefore appears to imply a valid trade-off between mean, variance and skewness of return, that is, preferring a higher third moment for two random variables having equal means and variances.

[2] They prove and demonstrate with examples that expected utility preferences never universally translate into moment preferences. Cain and Peel (2004) also illustrate the potential fallacy in the context of simple gambles.

[3] We can obtain the same outcomes in Cumulative Prospect theory proposed by Kahneman and Tversky (1979) and Tversky and Kahneman (1992) or Rank-Dependent expected utility of Quiggin (1993). However probability distortion, a key element of their models, demand more space than is available here.

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A and B and prefers to buy more than one ticket in lottery A, even though it offers a lower expected return and less positive skewness than lottery B, which has the same variance.

The rest of the article is structured as follows. In the next section we set out our analysis and the final section of the article is a brief conclusion.

II. Some Analysis

The expo-cubic function

We model the Friedman–Savage expected utility function by the expo-power form. This function has the form envisaged by Friedman and Savage. It has the properties that utility is initially concave, then convex and finally concave over wealth. In addition the function is bounded from above, a sufficient condition for the resolution of the St Petersburg Paradox. The function, which nests the standard cubic specification. Utility, U, is given by

$$U = \frac{1}{a}\left[1 - e^{-a\left(w - bw^2 + cw^3\right)}\right]$$

where w is wealth and a, b and c are positive constants with $c > (4b/3)$ to ensure that marginal utility is everywhere positive. As $a \to 0$ by L'Hopitals rule we obtain the standard cubic function. Our specification obviates the theoretical problem of unbounded risk-seeking behaviour over large enough wealth levels in the standard specification. For ease of modelling, though the points made are generic, we assume that a lottery has a fixed number of tickets (T) each costing \$1. This is the form many lotteries take, e.g. those in airports. The set up is also applicable to betting on more than one number at roulette. Buying more than one ticket in a lottery is analogous to betting more than one runner in the race but is a convenient simplification of the analysis.

The moments of return for this type lottery are given by

$$\mathrm{ER} = pT\{(O - (T-1))\} + (1-pT)(-T) = T\mu$$

$$\sigma^2 = \frac{T(1+\mu)^2(1-pT)}{p}$$

$$\sigma_3 = \frac{T(1+\mu)^3(1-pT)(1-2pT)}{p^2}$$

$$\mu = pO - (1-p)$$

where ER is the expected return from purchasing T tickets in the lottery. p is the probability of one ticket winning. O are the odds against one ticket winning. μ is the expected return to a one unit bet (i.e. $T = 1$). σ^2 is the variance of return and σ_3 the skewness of return.

In order to gamble on the lottery we require that expected utility, EU, from betting

$$\mathrm{EU} = pT\left[1 - e^{-a(w+\lambda - b(w+\lambda)^2 + c(w+\lambda)^3}\right] + (1-pT)\left[1 - e^{-a(w-T-b(w-T))^2 + c(w-T)^3}\right]$$

is greater or equal to the utility of not betting.

We employ the following parameters in our example.

$$a = 0.0000001,\ b = 0.5,\ c = 0.43333.$$

The agent is risk-averse for wealth levels less than 0.385, risk seeking for wealth levels up 249.1 and risk-averse for wealth levels greater than this.

Consider the choice between the two lotteries, A and B.

$$\text{In Lottery A}:\ p = 0.01,\ O = 89,\quad \mathrm{ER} = -3.7,\ \mu = -0.1$$

$$\text{In lottery B}:\ p = 4.2882 \times 10^{-4},\quad O = 2097.8,\ \mathrm{ER} = -0.1, \mu = -0.1.$$

As we observe in Fig.1 in this example the agent would optimally choose to buy 37 tickets in lottery A. Purchasing one (or any number) ticket in lottery B has negative net utility. However the moments of lottery A and lottery B for purchasing 37 tickets and 1 ticket, respectively are given by

$$\text{A}:\ \mathrm{ER(A)} = -3.7,\quad \sigma^2(\mathrm{A}) = 1888.1,\ \sigma^3(\mathrm{A}) = 44182 \text{ and}$$

$$\text{B}:\ \mathrm{ER\,(B)} = -0.1,\ \sigma^2(\mathrm{B}) = 1888.1,\quad \sigma^3(\mathrm{B}) = 4.8881 \times 10^6.$$

In other words the variance of the two gambles is equal but the expected return of and positive skew of lottery A is lower than in lottery B. Nevertheless lottery A is preferred from a utility perspective.

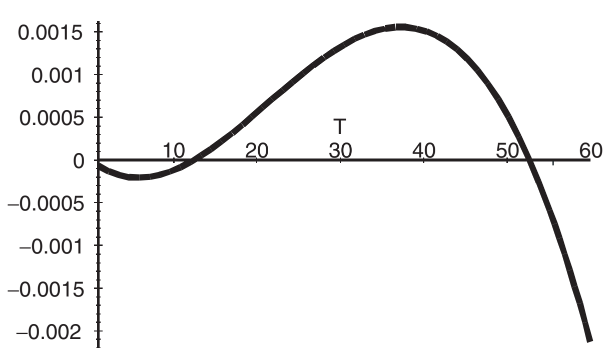

Fig. 1. Expected utility in excess of not gambling

The Markowitz model

In a seminal paper Markowitz (1952) assumed that from an agent's customary or normal level of wealth the agent was initially risk loving then risk-averse over gains whilst initially risk-adverse then risk seeking over losses. He also assumed that the typical agent was loss averse so that the curve falls faster to the left of the origin than it rises to the right. (i.e. $U(X) > |U(-X), X > 0)|''$

For our parametric specification of the Markowitz we employ the expo-power function (see Saha (1993)) and for the agent to gamble we require

$$\mathrm{EU} = pT\left(1 - e^{-r\alpha(O-(T-1))^{n}}\right) - (1 - pT)k\left(1 - e^{-\alpha T^{n}}\right) \geq 0$$

where r, α, k and n are positive constants with $n > 1$ $r \geq 1$, $k > r$.

With $n > 1$ the agent is risk–loving, (risk-averse), over gains as $(n-1)/r\alpha n$ is greater, (less), than $(O-(T-1))^{n}$ and risk-averse, risk loving over losses as $(n-1)/\alpha n$ is greater, less, than T^{n}.

The degree of loss aversion varies between r/k for symmetric small gambles and $1/k$ for large symmetric gambles.

Our parameter values are $r = 50$, $k = 100$, $\alpha = 1 \times 10^{-9}$ and $n = 1.5$.

$$\text{In Lottery A}: \ p = 1 \times 10^{-6}, \ O = 899999, \quad \mathrm{ER} = -5, \ \mu = -0.1$$

$$\text{In lottery B}: p = 2.00009996 \times 10^{-8}, \quad O = 4.499775 \times 10^{7}, \quad \mathrm{ER} = -0.1, \mu = -0.1.$$

From a utility perspective, lottery A with a purchase of 50 tickets is preferred to purchasing 1 ticket in lottery B even though the variance of return in lottery B is identical to that in A and has a higher expected return and higher positive skew.[4]

III. Conclusion

Depending on the parameters of the utility function and the lottery structure an agent can optimally choose to purchase zero, one or more tickets in a lottery. We have constructed examples where we model utility by either the Friedman–Savage or Markowitz form and the agent optimally purchases more than one ticket. These lotteries have the same variances but lower expected returns and skewness of returns than those generated by the purchase of one ticket in alternative lotteries. Nevertheless they are preferred from a utility perspective. The examples therefore illustrate that arguments to explain gambling based on higher moments, and in particular a variance–skewness trade-off can be invalid.

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[4] σ^2 (A) $= 4.049 \times 10^{7}$ $\sigma_3(\mathrm{A}) = 3.644 \times 10^{13}$. $\sigma^2(\mathrm{B}) = 4.049 \times 10^{7}$ $\sigma_3(\mathrm{B}) = 1.82232 \times 10^{15}$. $\mathrm{EU(A)} = 1.465 \times 10^{-5}$, $\mathrm{EU(B)} = -7.99 \times 10^{-8}$.

Appendix

Markowitz Utility Function

If we define the current level of wealth as W, and the level of utility associated with W as $\overline{U}$

$$U = \overline{U} + U(W + x) \qquad (a)$$

defines utility for increases in wealth above W, where $W + x$ is wealth measured W to ∞. We require that marginal utility, $\partial U/\partial x$ is positive and the second derivative, $\partial^2 U/\partial x^2$, is initially positive then negative for an increase in wealth. For a decrease in wealth below W, we define the utility function as

$$U = \overline{U} - U(W - x) \qquad (b)$$

where $W - x$ is wealth measured 0 to W. We require that the marginal utility, $\partial U/\partial x$, be positive and the second derivative, $\partial^2 U/\partial x^2$, be initially negative then positive for a decrease in wealth as postulated by Markowitz. These features are exhibited in our calibrated examples below.

Expected Utility of a gamble over gains (G) and losses (L) in the Markowitz model is given by

$$\mathrm{EU} = pU(G) - (1 - p)U(L) \qquad (c)$$

For our expo-power function and the definition of gains and loses this gives

$$\mathrm{EU} = pT\Big(1 - e^{-r\alpha(O-(T-1))^n}\big) - (1 - pT)k\big(1 - e^{-\alpha T^n}\big) \qquad (d)$$

For the utility function in (c), the degree of loss aversion, (LA), is defined by the ratio of the utility gain to the utility loss from a symmetric gamble, given by

$$\mathrm{LA} = \frac{\left(1 - e^{-r\alpha G^n}\right)}{k(1 - e^{-\alpha L^n})} \quad G = L \qquad (e)$$

As stake size approaches zero, the assumption of loss aversion requires that $r/k < 1$, (by L'Hopital's Rule) and as it becomes large that $1/k < 1$.

From the definition of the moments for a T ticket purchase in Lottery we obtain

$$\mathrm{ER} = pT\{(O - (T - 1))\} + (1 - pT)(-T) = T\mu$$

$$\sigma^2 = \frac{T(1 + \mu)^2(1 - pT)}{p}$$

$$\sigma_3 = \frac{T(1 + \mu)^3(1 - pT)(1 - 2pT)}{p^2}$$

$$\mu = pO - (1 - p)$$

Lottery, B, a one-ticket lottery, has moments

$$\mathrm{ER} = q\delta + (1 - q)(-1) = \mu$$

$$\sigma^2 = \frac{(1 + \mu)^2(1 - q)}{q}$$

$$\sigma_3 = \frac{(1 + \mu)^3(1 - q)(1 - 2q)}{q^2}$$

where q is the probability of winning in Lottery B and δ are the odds. We set the expected returns for a one-stake gamble to be equal in both lotteries. In order to ensure that the variances are identical q is set equal to the solution of

$$\frac{(1 + \mu)^2(1 - q)}{q} = \frac{T(1 + \mu)^2(1 - pT)}{p}$$

This enables us to calculate δ, given the rates of return to a 1-unit gamble are the same.

All calculations were done in Scientific WorkPlace.