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# On tau functions for orthogonal polynomials and matrix models

**Gordon Blower**

Department of Mathematics and Statistics

Lancaster University

Lancaster, LA1 4YF

England

g.blower@lancaster.ac.uk

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**Abstract.** Let $v$ be a real polynomial of even degree, representing an electrostatic field and $w$ the equilibrium density for charge on a long conducting wire. The system of orthogonal polynomials for $w$ gives rise to $2 \times 2$ rational matrix differential equations $Y' = A_nY$ which satisfy a recurrence relation. Here $w$ is algebraic with Riemann surface $\mathcal{E}$, and $\tau_n(t) = \det[\int_{-\infty}^t x^{j+k}w(x)dx]_{j,k=0}^{n-1}$ belongs to a Liouvillian tower over $\mathcal{E}$. The solutions of $Y' = A_nY$ give data for an inverse scattering problem that can be solved via the Gelfand–Levitan equation in terms of rational operator functions. Using linear systems, the paper shows that a multiple of $\sin x$ is the scattering function for Lamé's equation $-f'' + 2\wp f = \lambda f$ and realises elliptic potentials from periodic linear systems.

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## 1. Introduction

This paper is concerned with the scattering theory of linear differential equations

$$\frac{dY}{dx} = A_nY, \tag{1.1}$$

where $A_n(x)$ is a rational matrix function. Jimbo, Miwa and Uena [24, 25] introduced the tau function as a tool for describing the deformations of this equation that preserve monodromy, and found that tau functions have properties analogous to classical Abelian functions. In this paper, we consider the differential equations that arise from orthogonal polynomials associated with an algebraic weight. For applications to random matrices, see [41].

Such weights appear in electrostatics. We consider a unit of charge that is distributed along an infinite conducting wire in the presence of an electrical field. The field is represented by a real polynomial $v(x) = \sum_{j=0}^{2N} a_jx^j$ such that $a_{2N} > 0$, while the charge is represented by a Radon probability measure on the real line.

Boutet de Monvel *et al* [8, 36] prove the existence of the equilibrium density $w$ that minimises the electrostatic energy. They proved that there exists a constant $C_v$ such that

$$v(x) \geq 2 \int_S \log|x-y|\, w(y)dy + C_v \qquad (x \in \mathbf{R}), \tag{1.2}$$

and that equality holds if and only if $x$ belongs to a compact set $S$. Furthermore, there exists $g \geq 0$ and

$$-\infty < \delta_1 < \delta_2 \leq \delta_3 < \ldots < \delta_{2g+2} < \infty \tag{1.3}$$

such that

$$S = \cup_{j=1}^{g+1} [\delta_{2j-1}, \delta_{2j}] \tag{1.4}$$

has $g$ gaps. It is a tricky problem to find $S$ for a given $v$, and [15, Theorem 1.46 and p. 408] contains some significant results including the bound $g+1 \leq N+1$ on the number of intervals. Certainly, $v$ has less than or equal to $N$ local minima. When $v$ is convex, a relatively simple argument shows that $g = 0$, so $S$ is a single interval [13, 26].

In section 3, we introduce the system of orthogonal polynomials of the first and second kinds for $w$, and in section 4, derive the basic differential equation for this system. This has the form of (1.1), where $A_n$ is a $2 \times 2$ rational matrix function with simple poles at $\delta_j$. We also show that the $A_n$ satisfy a recurrence relation under $n \mapsto n+1$, which is an instance of a discrete Schlesinger transformation. As an illustration which is of importance in random matrix theory, we calculate the $A_n$ explicitly when $w$ is the semicircular law; see [33]. The recurrence relation involves the Hankel determinant of the weight $w$.

**Definition.** The $n^{th}$ order Hankel determinant for $w$ for $(-\infty, t)$ is

$$D_n(t) = \det\Big[\int_{S\cap(-\infty,t)} x^{j+k} w(x)dx\Big]_{j,k=0}^{n-1}, \tag{1.5}$$

and we let $D_n = D_n(\infty)$.

Schlesinger showed that when the positions of the poles $\delta_j$ are deformed, the solutions to (1.1) satisfy a system of partial differential equations. The Schlesinger equations are described in terms of a complex function $\tau(\delta_1, \ldots, \delta_{2g+2})$ such that $(\partial/\partial\delta_j)\log\tau$ give Hamiltonians in involution, as in [24, 25]. In this paper, we consider tau functions of one variable that are defined in terms of operators, as in [7].

**Definition.** Let $\mathbf{I}_{(t,\infty)}$ be the indicator function of $(t,\infty)$ and $P_{(t,\infty)} : L^2(w) \to L^2(w)$ be the orthogonal projection given by $f \mapsto \mathbf{I}_{(t,\infty)} f$, where the variable $t$ is often referred to as an edge. Given a self-adjoint and trace-class operator $K : L^2(w) \to L^2(w)$, the tau function of $K$ is $\tau(t) = \det(I - P_{[t,\infty)}K)$, and the potential is $q(t) = -2\frac{d^2}{dt^2}\log\tau(2t)$. (Usually one assumes that $0 \leq K \leq I$).

Let $E_n : L^2(w) \to \mathrm{span}\{x^j : j = 0, \dots, n-1\}$ the orthogonal projection. In Proposition 3.3 we note that the tau function of the edge is given by

$$\det(I - P_{[t,\infty)}E_{n+1}) = \frac{D_{n+1}(t)}{D_{n+1}}. \tag{1.6}$$

Chen and Lawrence [14] investigated a generalization of the Chebyshev polynomials to $[-1, \alpha] \cup [\beta, 1]$, and expressed their $D_n$ in terms of Jacobi's elliptic theta functions. Chen and Its [12] considered the $w$ that is analogous to the Chebyshev distribution on multiple intervals, and found their $D_n$ explicitly in terms of theta functions on a hyperelliptic Riemann surface.

In sections 5 we consider an algebraic weight $w(x) = c\sqrt{\prod_{k=1}^{4N-2}(x-\delta_j)}$ so that $w$ is a rational function on a Riemann surface $\mathcal{E}$. We show how the moments $\int^t x^j w(x)\,dx$ can be expressed in terms of Abelian integrals on $\mathcal{E}$, and hence we show that $D_n(t)$ belongs to a Liouvillian field extension of the rational functions on $\mathcal{E}$. The connection between the number of gaps $g$ and the genus of $\mathcal{E}$ is subtle. In section 6 we show that if $v$ is any quadratic, even quartic or even sextic, then $v$ has an equilibrium weight such that all of the $D_n(t)$ can be expressed in terms of elliptic and trigonometric integrals, or equivalently, in terms of rational functions on algebraic curves of genus zero or one. Generally, a quartic potential gives an $S$ which is the union of two intervals, and the Schlesinger equations reduce to Painlevé's equation VI, as in [35, 19, 28].

Jimbo *et al* [24, 25] observed that the formal solution of (1.1) plays the role of the Jost solution in inverse scattering. They introduced $\tau_n$ for (1.1) and showed that the logarithmic derivative $\tau_n'/\tau_n$ and the tau quotient $\tau_{n+1}/\tau_n$ both have rational expressions in terms of the coefficients of the formal solution; see [25, p. 409]. Following this approach, we consider the correspondence between tau functions and scattering functions. For $q(x)$ as above, we consider Schrödinger's equation $-\frac{d^2 f}{dx^2} + q(x)f(x) = \lambda f(x)$. Suppose momentarily that $q$ is smooth and rapidly decreasing as $x \to \infty$; then the scattering function $\phi(x)$ is determined by the Jost solutions $f(x)$. Furthermore, we can recover $q$, and hence $\tau$, by solving the Gelfand–Levitan equation

$$\phi(x+y) + T(x,y) + \int_x^\infty T(x,z)\phi(z+y)\,dz = 0 \qquad (0 < x < y) \tag{1.7}$$

and noting that $q(x) = -2\frac{d}{dx}T(x,x)$. The form of (1.7) suggests that we introduce the Hankel integral operator $\Gamma_\phi : L^2(0,\infty) \to L^2(0,\infty)$ for suitable $\phi \in L^2(0,\infty)$ by

$$\Gamma_\phi f(x) = \int_0^\infty \phi(x+y)f(y)\,dy. \tag{1.8}$$

The main purpose of this paper is to adapt these methods to establish a correspondence $\tau \leftrightarrow \phi$ when $\tau$ is an elliptic or trigonometric function such as arise in previous sections. In section 7 we introduce the scattering function for an elliptic $\tau$ and modify the Gelfand–Levitan equation to establish a correspondence with trigonometric $\phi$. In particular, we show that a multiple of the sine function is the scattering function for Lamé's equation with potential $\wp$ as in [29]. The main idea is to introduce a linear system $(-A, B, C)$ so as to realise $\phi(x) = Ce^{-xA}B$, and then solve the Gelfand–Levitan equation by rational expressions in $A, B, C$ and related operators. Thus we obtain explicit expressions linking $\phi$ with $\tau$.

In section 8 we consider tau functions that are rational functions on hyperelliptic curves, such as arise from typical potentials $v$ of degree greater than or equal to six. A real periodic potential is algebro-geometric of the spectrum of $-f'' + qf = \lambda f$ has only finitely many gaps. We obtain results which are analogous to those of section 7, by introducing appropriate $\delta_j(x)$ and $\sqrt{\prod_{j=1}^{g}(\lambda - \delta_j(x))}$ to play a similar role to $w$ in earlier sections.

In section 9, we consider tau functions associated with kernels defined by the solutions of (1.1). We introduce the matrix

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{1.9}$$

and apply a simple gauge transformation to (1.1). Then for a sequence of real symmetric $2 \times 2$ matrices $J\beta_k(n)$, we consider solutions of the differential equation

$$J\frac{dZ}{dx} = \sum_{k=1}^{2g+2} \frac{J\beta_k(n)}{x - \delta_k} Z, \tag{1.10}$$

and, by analogy with the Bessel and Airy kernels from [40] and [41], form the kernel

$$K(x, y) = \frac{Z(y)^\dagger J Z(x)}{y - x}. \tag{1.11}$$

We show that the properties of $K$ depend crucially upon the sequence of signatures of the matrices $(\delta_j - \delta_k)J\beta_k(n)$. In Theorem 9.3, we introduce a symbol function $\phi$ from $Z$, a constant signature matrix $\sigma$ and a Hankel operator $\Gamma_\phi$ such that $K = \Gamma_\phi^\dagger \sigma \Gamma_\phi$.

In section 10, we derive an appropriate version of the Gelfand–Levitan equation for such a $\phi$, and solve by the method of linear systems from [5]. Thus we obtain the tau function as a uniform limit of finite determinants.

## 2. The equilibrium measure

In this section we recall some known results. Given the special form of the potential, the equilibrium measure and its support $S$ satisfy special properties. To describe these, we introduce the polynomial $u$ of degree $2N - 2$ by

$$u(z) = \int_S \frac{v'(z) - v'(x)}{z - x} w(x)dx \tag{2.1}$$

and the weight $w(x)$ such that $w(x)^2 = \pi^{-2}(4u(x) - v'(x)^2)$ and $w(x) \geq 0$ on $S$. Let $\mathcal{E}$ be the compact Riemann surface

$$\mathcal{E} = \{(x, w) \in \mathbf{C}^2 : w^2 = \pi^{-2}(4u(x) - v'(x)^2)\} \cup \{(\infty, \infty)\}. \tag{2.2}$$

**Proposition 2.1.** *The support of the equilibrium density is* $S = \{x \in \mathbf{R} : 4u(x) - v'(x)^2 \geq 0\}$, *which is the image of the real points on* $\mathcal{E}$ *under* $\pi : \mathcal{E} \to \mathbf{C} \cup \{\infty\} : (x, w) \mapsto x$. *The real endpoints* $\delta_1 < \ldots < \delta_{2g+2}$ *satisfy*

$$\sum_{j=1}^{g+1} \int_{\delta_{2j-1}}^{\delta_{2j}} \frac{x^k v'(x)\, dx}{\sqrt{-(x-\delta_1)(x-\delta_2)\ldots(x-\delta_{2g+2})}} = 0 \qquad (k = 0, \ldots, g), \tag{2.3}$$

$w(x) \to 0$ *as* $x$ *tends to any endpoint of* $S$, *and*

$$\int_S w(x) dx = 1. \tag{2.4}$$

**Proof.** By [11] and [34 , (91.12)], the endpoints of $S$ satisfy (2.3). See [36, 8, 37].

□

We have

$$w(x) = 2Na_{2N}\Big(-Q(x) \prod_{j=1}^{2g+2} (x - \delta_{2j-1})(x - \delta_{2j})\Big)^{1/2}, \tag{2.5}$$

where $Q(x)$ is a monic irreducible factor such that $w(x) = 4u(x) - v'(x)^2$ and $w(x) \geq 0$ on $S$. The polynomial $4u(x) - v'(x)^2$ has real zeros $\delta_1, \ldots, \delta_{2g+2}$ and may additionally have pairs of complex conjugate roots, which we list as $\delta_{2g+3}, \ldots, \delta_{4N-2}$ with regard to multiplicity.

The following result can be used to compute $w$ in significant special cases. See [39] for a discussion of hypothesis (i), and [43, p 18] for more about measure-preserving transformations.

**Proposition 2.2.** *Suppose that* $S$ *consists of* $m$ *disjoint closed intervals such that*

*(i) there exists a real polynomial* $\varphi$ *of degree* $m$ *such that* $S = \varphi^{-1}([a, b])$;

*(ii) there exists a polynomial* $U$ *such that* $v(x) = U(\varphi(x))$ *and such that the equilibrium measure of* $mU(x)$ *is* $\mu$, *with support* $[a, b]$.

*Then there exists* $\rho$, *a pull-back of* $\mu$ *onto* $S$ *via* $\varphi$, *so* $\rho(\varphi^{-1}(A)) = \mu(A)$ *for all Borel subsets* $A$ *of* $[a, b]$ *and* $\rho$ *is an equilibrium measure for* $v$.

**Proof.** Let $S = \cup_{j=1}^{m}[\delta_{2j-1}, \delta_{2j}]$. By the inverse function theorem, there exist locally analytic functions $\alpha_j$ such that $\alpha_j([a, b]) \subseteq [\delta_{2j-1}, \delta_{2j}]$ and

$$\varphi(x) - y = c \prod_{j=1}^{m} (x - \alpha_j(y)), \tag{2.6}$$

where $c$ is the leading coefficient of $\varphi$. Now we define $\rho$ to be the probability measure such that

$$\int_S f(y)\rho(dy) = \frac{1}{m}\sum_{j=1}^m \int_a^b f(\alpha_j(s))\mu(ds) \qquad (f \in C(S)) \tag{2.7}$$

and check that this has the required properties. From (2.6), we deduce that

$$\begin{aligned} \int_S \log|x-y|\,\rho(dy) &= \frac{1}{m}\sum_{j=1}^m \int_a^b \log|x-\alpha_j(s)|\,\mu(ds) \\ &= \frac{1}{m}\int_a^b \log|\varphi(x)-s|\mu(ds) - \frac{1}{m}\log c, \end{aligned} \tag{2.8}$$

and by the definition of the equilibrium measure $\mu$, we deduce that

$$\begin{aligned} 2\int_S \log|x-y|\,\rho(dy) &\leq U(\varphi(x)) - C_{mU} - \frac{2}{m}\log c \\ &= v(x) - C_{mU} - \frac{2}{m}\log c \end{aligned} \tag{2.9}$$

with equality for $x \in S$. Hence we have verified that $\rho$ is an equilibrium measure for $v$.

□

## 3. Orthogonal polynomials

First we introduce orthogonal polynomials for $w$, then the corresponding differential equations. Let $(p_j)_{j=0}^\infty$ be the sequence of monic orthogonal polynomials in $L^2(w)$, where $p_j$ has degree $j$ and let $h_j$ be the constants such that

$$\int_S p_j(x)p_k(x)w(x)dx = h_j\delta_{jk} \qquad (j,k = 0,1,\dots); \tag{3.1}$$

then let $(q_j)_{j=1}^\infty$ be the monic polynomials of the second kind, where

$$q_j(z) = \int_S \frac{p_j(z)-p_j(x)}{z-x}w(x)dx \qquad (j = 1,2,\dots) \tag{3.2}$$

has degree $j-1$. The orthogonal polynomials are semi classical in Magnus's sense [28], although the weight typically lives on several intervals.

**Lemma 3.1.** *Let $c_n = h_n/h_{n-1}$ and $b_n = h_n^{-1}\int_S xp_n(x)^2w(x)dx$. Then*

*(i) the polynomials $(p_n)_{n=0}^\infty$ satisfy the three-term recurrence relation*

$$xp_n(x) = p_{n+1}(x) + b_{n+1}p_n(x) + c_np_{n-1}(x); \tag{3.3}$$

*(ii) the polynomials $(q_j)_{j=1}^\infty$ likewise satisfy (3.3);*

*(iii) the Hankel determinant of (1.5) satisfies $D_n = h_0h_1\dots h_{n-1}$.*

**Proof.** This is standard in the theory of orthogonal polynomials; see [12; 41, section 6].

□

We introduce also

$$Y_n(z) = \begin{bmatrix} p_n(z) & \int_S \frac{p_n(t)w(t)dt}{z-t} \\ \frac{p_{n-1}(z)}{h_{n-1}} & \frac{1}{h_{n-1}} \int_S \frac{p_{n-1}(t)w(t)dt}{z-t} \end{bmatrix} \tag{3.4}$$

and

$$V_n(z) = \begin{bmatrix} z - b_{n+1} & -h_n \\ 1/h_n & 0 \end{bmatrix}. \tag{3.5}$$

The effect of passing from $n$ to $n+1$ is to add another row and column to the Hankel determinant, which by (iii) has the effect of multiplying by $h_n$, since $D_{n+1} = h_n D_n$. Our next result gives the corresponding recurrence relation for the $Y_n$.

**Proposition 3.2.** *(i) The matrices satisfy the recurrence relation*

$$Y_{n+1}(z) = V_n(z)Y_n(z). \tag{3.6}$$

*(ii) The matrix $Y_n(z)$ is invertible, and* $\det Y_n(z) = 1$.

**Proof.** (i) This follows from (i) and (ii) of the Lemma 3.1.

(ii) This follows by induction, where the induction step follows from the recurrence relation in (i).

□

**Definition.** We restrict $w$ to $(s,t) \cap S$ and let

$$\mu_j(s,t) = \int_{S\cap(s,t)} x^j w(x)dx \tag{3.7}$$

be the corresponding $j^{th}$ moment. Let $E_n : L^2(w) \to \mathrm{span}\{x^k : k = 0, \dots, n-1\}$ be the orthogonal projection.

**Proposition 3.3.** *(i) The Hankel determinant for $(s,t)$ satisfies*

$$\det(I - E_{n+1}P_{(s,t)}) = \frac{1}{D_{n+1}} \det\big[\mu_{j+k}(s,t)\big]_{j,k=0}^{n}. \tag{3.8}$$

*(ii) In particular, the tau function for $E_{n+1}$ is proportional to the Hankel determinant for $(-\infty, t)$ as in (1.5) and the corresponding potential satisfies*

$$q_n(t) = -2\frac{d^2}{dt^2}\log D_n(t). \tag{3.9}$$

**Proof.** (i) This is due to Borodin and Soshnikov [7, p. 599].

(ii) By definition,

$$q_n(t) = -2\frac{d^2}{dt^2}\log\det(I - E_{n+1}P_{(t,\infty)}), \tag{3.10}$$

and this simplifies by (i).

□

## 4. The basic differential equations and recurrence relations

In this section we derive a basic differential equation (4.14) which is a rational matrix differential equation of Fuchsian type, and a recurrence relation for the equations as $n$ changes to $n+1$. This section follows closely section 4 of [12], and essentially recovers some results from [4]. See also [25, p. 233].

Invoking Proposition 3.2(ii), we introduce the matrix function

$$A_n(z) = Y_n'(z)Y_n(z)^{-1} + Y_n(z)\begin{bmatrix} 0 & 0 \\ 0 & -w'(z)/w(z)\end{bmatrix} Y_n(z)^{-1}. \tag{4.1}$$

The basic properties of $A_n(z)$ are stated in the following Lemma.

**Lemma 4.1.** *Let $v'(z)^2 - 4u(z)$ have zeros at $\delta_j$ for $j = 1, \dots, 4N-2$. Then $A_n(z)$ is a proper rational function so that*

$$A_n(z) = \sum_{j=1}^{4N-2} \frac{\alpha_j(n)}{z-\delta_j}, \tag{4.2}$$

*where the residue matrices $\alpha_j(n)$ depend implicitly upon the $\delta_j$.*

**Proof.** The defining equation (4.1) for $A_n(z)$ may be written more explicitly as

$$\begin{bmatrix} p_n'(z) & -\int_S \frac{p_n(t)w(t)dt}{(z-t)^2} \\ \frac{p_{n-1}'(z)}{h_{n-1}} & -\frac{1}{h_{n-1}}\int_S \frac{p_{n-1}(t)w(t)dt}{(z-t)^2}\end{bmatrix}$$

$$= A_n(z)\begin{bmatrix} p_n(z) & \int_S \frac{p_n(t)w(t)dt}{z-t} \\ \frac{p_{n-1}(z)}{h_{n-1}} & \frac{1}{h_{n-1}}\int_S \frac{p_{n-1}(t)w(t)dt}{z-t}\end{bmatrix} + \begin{bmatrix} 0 & \frac{w'(z)}{w(z)}\int_S \frac{p_{n-1}(t)w(t)dt}{z-t} \\ 0 & \frac{w'(z)}{h_{n-1}w(z)}\int_S \frac{p_{n-1}(t)w(t)dt}{z-t}\end{bmatrix}. \tag{4.3}$$

By considering the entries, we see that $A_n(z)$ is a proper rational function with possible simple poles at the $\delta_j$, as in (4.2). Hence we have a Laurent expansion

$$A_n(z) = \frac{1}{z}\sum_{k=1}^{4N-2}\alpha_k(n) + \frac{1}{z^2}\sum_{k=1}^{4N-2}\delta_k\alpha_k(n) + O\Big(\frac{1}{z^3}\Big) \qquad (z\to\infty). \tag{4.4}$$

□

Let

$$\Phi_n(z) = \begin{bmatrix} \sqrt{2\pi i}\,p_n(z) & -\frac{i\pi w(z)p_n(z)+q_n(z)}{w(z)\sqrt{2\pi i}} \\ \frac{\sqrt{2\pi i}\,p_{n-1}(z)}{h_{n-1}} & -\frac{i\pi w(z)p_{n-1}(z)+q_{n-1}(z)}{w(z)h_{n-1}\sqrt{2\pi i}}\end{bmatrix}, \tag{4.5}$$

which is a matrix function with entries in $\mathbf{C}(z)[w]$; note that $\Phi_n$ also depends upon the $\delta_j$.

**Lemma 4.2.** *The matrix functions* $\Phi_n$ *satisfy*

*(i) the basic differential equation*

$$\frac{d\Phi_n(z)}{dz} = A_n(z)\Phi_n(z), \tag{4.6}$$

*(ii) and the recurrence relation* $\Phi_{n+1}(z) = V_n(z)\Phi_n(z)$;

*(iii) moreover,* $\Phi_n$ *is invertible since* $\det \Phi_n(z) = 1/w(z)$.

**Proof.** (i) We can write

$$\Phi_n(z) = Y_n(z)\begin{bmatrix} \sqrt{2\pi i} & 0 \\ 0 & 1/(w(z)\sqrt{2\pi i}) \end{bmatrix}, \tag{4.7}$$

and then the property (i) follows from (4.1).

(ii) The recurrence relation from Proposition 3.2(i).

(iii) Given (ii), this identity follows from Proposition 3.2(ii).

□

**Proposition 4.3.** *The discrete string equation holds*

$$A_{n+1}(z)V_n(z) - V_n(z)A_n(z) = \frac{d}{dz}V_n(z), \quad \frac{d}{dz}V_n(z) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.8}$$

**Proof.** The basic differential equation (4.6) and the recurrence relation in Lemma 4.2(iii) are consistent, and the Lax pair associated with these conditions gives

$$A_{n+1}(z)\Phi_{n+1}(z) = \frac{d}{dz}\Phi_{n+1}(z) = \frac{d}{dz}\Big(V_n(z)\Phi_n(z)\Big). \tag{4.9}$$

□

In the remainder of this section we consider the equilibrium measure for the Gaussian unitary ensemble, namely Wigner's semicircular law. For $a < b$, let

$$v(z) = \frac{8}{(b-a)^2}\Big(z - \frac{a+b}{2}\Big)^2, \tag{4.10}$$

so, by standard results used in random matrix theory [33], the equilibrium density

$$w(x) = \frac{8}{\pi(b-a)^2}\sqrt{(b-x)(x-a)}\,\mathbf{I}_{[a,b]}(x). \tag{4.11}$$

Let $U_n$ be the Chebyshev polynomial of the second kind of degree $n$, which satisfies

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \tag{4.12}$$

and let

$$p_n(x) = \frac{(b-a)^n}{2^{2n}} U_n\Big(\frac{x-(a+b)/2}{(b-a)/2}\Big) \tag{4.13}$$

which is monic and of degree $n$, and the $p_n$ are orthogonal with respect to the measure $w$. By elementary calculations involving trigonometric functions, one can find the terms in (4.5) and (4.6) explicitly, and hence show that

$$A_n(x) = \frac{1}{(x-b)(x-a)} \begin{bmatrix} n(x-(a+b)/2) & -(n+1)(b-a)^{2n}/2^{4n-1} \\ n2^{4n-3}/(b-a)^{2n-2} & -(n+1)(x-(a+b)/2) \end{bmatrix}, \tag{4.14}$$

which has poles at $a$ and $b$, as expected.

## 5. Algebraic integrability of tau functions

The algebraic properties of the tau functions $D_n(t)$ and their quotients $h_n = \tau_{n+1}/\tau_n$ are the subject of this section. Let $\mathbf{F}_n$ be the algebraic complex field that is generated by the elements of the matrix in (1.5) with determinant $D_n(t)$, so that $D_n(t) \in \mathbf{F}_n$. Also observe that $\mathbf{F}_n \subseteq \mathbf{F}_{n+1}$, and that $D_{n+1}(t)$ is an element of the field extension $\mathbf{F}_n(h_n)$, obtained by adjoining $h_n$ to $\mathbf{F}_n$. We consider such extensions systematically in this section.

Let $\mathbf{F}$ be a field (of complex functions) with differential $\partial$ that contains the subfield $\mathbf{C}$ of constants, and adjoin an element $h$ to form $\mathbf{F}(h)$, where either:

(i) $h = \int g$ for some $g \in \mathbf{F}$, so $\partial h = g$;

(ii) $h = \exp \int g$ for some $g \in \mathbf{F}$; or

(iii) $h$ is algebraic over $\mathbf{F}$.

**Definition.** Let $\mathbf{F}_j$ $(j = 1, \dots, n)$ be fields with differential $\partial$ that contain the subfield $\mathbf{C}$ of constants and suppose that

$$\mathbf{F}_1 \subseteq \mathbf{F}_2 \subseteq \ldots \subseteq \mathbf{F}_n, \tag{5.1}$$

where $\mathbf{F}_j$ arises from $\mathbf{F}_{j-1}$ by applying some operation (i), (ii) or (iii). Then the $\mathbf{F}_j$ are said to form a Liouvillian tower, and each $\mathbf{F}_j$ is a Liouvillian extension of $\mathbf{F}_1$. See [38].

In particular, let $\mathcal{E}$ be the algebraic curve defined by $W(x)^2 = \prod_{j=1}^m (x-\delta_j)$ where the $\delta_j \in \mathbf{C}$ are distinct, and let

$$\mathbf{F}_0 = \mathbf{C}(x), \mathbf{F}_1 = \mathbf{F}_0[W], \mathbf{F}_2 = \mathbf{F}_1\Big(\int \frac{dx}{W(x)}\Big), \dots, \mathbf{F}_{n+1} = \mathbf{F}_n\Big(\int \frac{x^{n-1}dx}{W(x)}\Big), \dots . \tag{5.2}$$

**Definition.** Let the Liouvillian field associated with $\mathcal{E}$ be $\mathbf{F}(\mathcal{E}) = \mathbf{F}_m$. (By Lemma 5.2, this is the largest field in (5.2), and can be strictly larger than the field of rational functions on $\mathcal{E}$.)

**Theorem 5.1.** *Suppose that the equilibrium density $w$ is supported on $g+1$ intervals, so that $w(x) = p(x)W(x)$ for some polynomial $p(x)$. Then $D_n(t)$ belongs to $\mathbf{F}(\mathcal{E})$ for all $n$.*

**Proof.** Recall from Proposition 3.3 that $D_n(t) = \det[\int_{S\cap(-\infty,t)} x^{j+k}p(x)W(x)\,dx]_{j,k=0}^{n-1}$ with $pW^2$ a polynomial of degree $2N+g$. Hence each element of the determinant is a linear combination of $\int_{\delta_1}^t x^{j+k+r}dx/W$, where $j,k = 0,\dots,n-1$ and $r = 0,\dots,2N+g$; hence by (i) each element belongs to $\mathbf{F}_{2n+2N+g}$.

To rest of the proof is contained in following lemma, which is no doubt known from the classical theory of Abelian functions.

**Lemma 5.2.** *The Liouvillian tower (5.2) stabilises, so that* $\mathbf{F}_n = \mathbf{F}_m$ *for all* $n \geq m$.

**Proof.** We introduce $P$ by $P'(u) = W(P(u))$, in local complex variables. Then we form $\mathbf{C}(P)$, and its algebraic extension $\mathbf{C}(P)[P']$, where $(P')^2 = \sum_{k=0}^m (-1)^{m-k}\sigma_{m-k}P^k$; then we adjoin $u = \int(1/P')dP$, $\zeta_1 = \int Pdu, \dots, \zeta_{m-2} = \int P(u)^{m-2}du$.

We deduce that $\int P(u)^k du = \int x^k dx/W(x)$ belongs to $\mathbf{C}(P)[u,\zeta_1,\dots,\zeta_{m-2}]$ for all $k = m-1, m, \dots$. Indeed, we observe that

$$2P'' = \sum_{k=1}^m \prod_{j=1;j\neq k}^m (P-\delta_j), \tag{5.3}$$

so we have coefficients $a_j$ such that

$$2\int P''P^\ell\,du = m\int P^{m+\ell-1}\,du + \sum_{j=0}^{m-2} a_j \int P^{j+\ell}\,du, \tag{5.4}$$

so for $\ell = 0$ we have

$$m\int P^{m-1}du = 2P' - \sum_{j=0}^{m-2} a_j \int P^j\,du \tag{5.5}$$

and for $\ell = 1,2,\dots$ we integrate (5.4) by parts, to obtain

$$2P'P^\ell - 2\ell\int (P')^2P^{\ell-1}\,du = m\int P^{m+\ell-1}\,du + \sum_{j=0}^{m-2} a_j \int P^{j+\ell}\,du, \tag{5.6}$$

and hence we obtain the recurrence relation

$$(2\ell+m)\int P^{m+\ell-1}\,du = 2P'P^\ell - 2\ell\sum_{k=0}^{m-1}(-1)^{m-k}\sigma_{m-k}\int P^{\ell+k-1}\,du - \sum_{j=0}^{m-2} a_j\int P^{j+\ell}\,du. \tag{5.7}$$

□

To take advantage of Theorem 5.1, we need to identify $\mathbf{F}(\mathcal{E})$ for $v$ of low degree and seek to deal with integrals over algebraic curves that have as small a genus as possible. We deal with $g = 0$ and $g = 1$ in this section, and in section 6 we refine the result to deal with even polynomials. In later sections, we will also need the exponential operation to integrate $\tau'/\tau$.

**Lemma 5.3.** *All integrals of the form*

$$\int_a^t \frac{x^j}{\sqrt{(b-x)(x-a)}}dx \qquad (j=0,1,\dots) \tag{5.8}$$

*belong to*

$$\mathbf{C}\Big[t, \sqrt{(b-t)(t-a)}, \cos^{-1}\Big(\frac{t-(a+b)/2}{(b-a)/2}\Big)\Big]. \tag{5.9}$$

**Proof.** These results follows from elementary integration.

□

The following result concerns the natural Liouvillian tower associated with elliptic functions, such as appear in subsequent sections. Let $\wp$ be Weierstrass's elliptic function with $e_3 < e_2 < e_1$ as in [32]; also, let $\zeta$ be Weierstrass's zeta function, which is meromorphic and singly periodic, but not elliptic. Let $\theta_1$ be Jacobi's elliptic theta function, which is entire.

**Lemma 5.4.** *Any integral of the form*

$$\int_{e_3}^t \frac{x^j\,dx}{\sqrt{(x-e_3)(x-e_2)(x-e_1)}} \qquad (j=0,1,2,\dots) \tag{5.10}$$

*may be reduced via the substitution* $x=\wp(u)$ *to an element of* $\mathbf{C}[u, \wp'(u), \wp(u), \zeta(u)]$.

**Proof.** This follows from Theorem 5.1 and its proof. The element $\wp'$ satisfies $(\wp')^2 = 4(\wp - e_3)(\wp - e_2)(\wp - e_1)$, so the elliptic function field $\mathbf{C}(\wp)[\wp']$ is an algebraic extension of $\mathbf{C}(\wp)$. Then we adjoin $u = \int d\wp/\wp'$ and $\zeta = \int (\wp/\wp')d\wp$ to the elliptic function field, and the resulting field $\mathbf{C}(\wp)[\wp'](u, \zeta)$ contains all $\int \wp^n du$.

□

## 6. The tau function for even quartic and sextic potentials

In this section we show that $D_n(t)$ for even quartics and sextics can be expressed in terms of trigonometric and elliptic integrals. If we wish to find $D_n(t)$ for a typical even polynomials $v$ of degree 8, then we need to consider hyperelliptic curves as in section 8.

**Theorem 6.1.** *Let $v$ be an even polynomial of degree $2N$ that has positive leading coefficient and has equilibrium density $w$ supported on $N$ disjoint intervals. Then there exist algebraic curves $\mathcal{E}_1$ and $\mathcal{E}_2$, both of genus less than or equal to $\lfloor N/2 \rfloor$, such that $D_n(t)$ belongs to the Liouvillian function field generated by $\mathbf{F}(\mathcal{E}_1)$ and $\mathbf{F}(\mathcal{E}_2)$.*

**Proof.** We introduce the polynomial $U$ of degree $N$ by $U(x^2) = v(x)$, and consider the variational problem

$$\inf_\mu \Big\{ 2\int_0^\infty U(x)\,\mu(dx) + \int_0^\infty\int_0^\infty \log\frac{1}{|x-y|}\mu(dx)\mu(dy)\Big\}, \tag{6.1}$$

where the infimum is taken over all Radon probability measures on $(0,\infty)$ that have no atoms. By results of [37], there exists a unique $\mu$ with compact support $S_+$, and a constant $C$, such that

$$U(x) \geq \int_{S_+} \log|x-y|\,\mu(dy) + C \tag{6.2}$$

with equality on $S_+$. Furthermore, $S_+$ is a finite union of intervals and $\mu$ is absolutely continuous, so we can recover $\mu$ by solving the singular integral equation

$$U'(x) = \mathrm{PV}\int_{S_+} \frac{1}{x-y}\frac{d\mu}{dy}dy \qquad (y \in S_+) \tag{6.3}$$

by the method of [15, Theorem 1.38]. The density $w$ with support $S$ is symmetric about zero, and as in Proposition 2.2, the map $x \mapsto x^2$ pushes $w(x)dx$ forwards to $\mu$ with support $S_+$. The most difficult case to deal with is when $N$ is odd, and $\mu$ tends to accumulate mass near to the hard edge at zero. We consider

$$S_+ = [0, a^2] \cup \bigcup_{j=1}^{(N-1)/2} [c_j^2, b_j^2] \tag{6.4}$$

and introduce, as in [34, section 84],

$$R(z) = \frac{z-a^2}{z} \prod_{j=1}^{(N-1)/2} (z-b_j^2)(z-c_j^2) \tag{6.5}$$

which has a pole at zero and zeros at the other endpoints of $S_+$ and such that $\sqrt{R(z)}$ is analytic on $\mathbf{C} \setminus S_+$ and $z^{(1-N)/2}\sqrt{R(z)} \to 1$ as $|z| \to \infty$. We have

$$\sqrt{(x+i0-b_j^2)(x+i0-c_j^2)} = i\sqrt{(b_j^2-x)(x-c_j^2)} \qquad (c_j^2 < x < b_j^2).$$

Let $C_r$ be the circle of centre $0$ and radius $r > |b_{(N-1)/2}|^2 + |z|$, taken once in the positive sense, and observe that

$$p(z) = \frac{1}{2\pi}\int_{C_r} \frac{U'(\zeta)d\zeta}{\sqrt{R(\zeta)}(\zeta - z)} \tag{6.6}$$

does not depend upon $r$; indeed, the Taylor expansion reduces to

$$p(z) = \sum_{\ell=0}^{(N-1)/2} \frac{z^\ell}{2\pi}\int_{C_r} \frac{U'(\zeta)d\zeta}{\sqrt{R(\zeta)}\zeta^{\ell+1}}. \tag{6.7}$$

Now we consider the function $F(z) = U'(z) + ip(z)\sqrt{R(z)}$ which is holomorphic on $\mathbf{C} \setminus S_+$ and has jump $F(x+i0) - F(x-i0) = 2i\sqrt{R(x+i0)}p(x)$ across $S_+$ and $F(x+i0)+F(x-i0) = 2U'(x)$ on $S_+$. Hence by Plemelj's formula

$$U'(x) = \mathrm{PV}\frac{1}{\pi}\int_{S_+} \frac{p(y)\sqrt{R(y+i0)}\,dy}{y-x} \qquad (x \in S_+). \tag{6.8}$$

The integrand is unbounded at $y = 0$. From Proposition 2.2, we deduce that $w(x)dx$ has support

$$S = [-a, a] \cup \bigcup_{j=1}^{(N-1)/2} [-b_j, -c_j] \cup [c_j, b_j], \tag{6.9}$$

and is given by the weight

$$w(x) = -|x|\sqrt{R(x^2 + i0)}p(x^2)/\pi. \tag{6.10}$$

We introduce the algebraic curves

$$\mathcal{E}_1 : \quad W_1^2 = (x - a^2) \prod_{j=1}^{(N-1)/2} (x - b_j^2)(x - c_j^2) \tag{6.11}$$

of genus $(N - 1)/2$ and

$$\mathcal{E}_2 : \quad W_2^2 = x(x - a^2) \prod_{j=1}^{(N-1)/2} (x - b_j^2)(x - c_j^2) \tag{6.12}$$

of genus $(N - 1)/2$ and observe that

$$\int_0^t x^k w(x)\, dx = \frac{-1}{2\pi} \int_0^{t^2} u^{k/2}(u - a^2)p(u) \prod_{j=1}^{(N-1)/2} (u - b_j^2)(u - c_j^2)\frac{du}{W_2(u)}, \tag{6.13}$$

which belongs to $\mathbf{F}(\mathcal{E}_1)$ for $k$ odd and $\mathbf{F}(\mathcal{E}_2)$ for $k$ even. Whereas the curves $\mathcal{E}_1$ and $\mathcal{E}_2$ are birationally isomorphic when $N$ is odd, we need elements of both Liouvillian towers to express $D_n(t)$. When $N$ is even, the required curves can have distinct genus.

The field that is generated by $\mathbf{F}(\mathcal{E}_1)$ and $\mathbf{F}(\mathcal{E}_2)$ is a Liouvillian extension of $\mathbf{C}(x)$. Indeed, given Liouvillian towers

$$\mathbf{C}(x) \subseteq \mathbf{F}_1 \subseteq \mathbf{F}_2 \subseteq \dots \quad \text{and} \quad \mathbf{C}(x) \subseteq \mathbf{F}'_1 \subseteq \mathbf{F}'_2 \subseteq \dots,$$

there exists a Liouvillian tower $\mathbf{C}(x) \subseteq \mathbf{F}''_1 \subseteq \mathbf{F}''_2 \subseteq \dots$ such that $\mathbf{F}_j, \mathbf{F}'_j \subseteq \mathbf{F}''_{2j}$ and that is constructed by alternately applying the operations that produced the original towers.

□

**Corollary 6.2.** *Let $v$ be an even quartic with positive leading coefficient and equilibrium density $w$.*

*(i) If the density $w$ is supported on the pair of intervals $[-b, -a] \cup [a, b]$, then $D_n(t)$ may be expressed as trigonometric and elliptic integrals as in Lemmas 5.3 and 5.4.*

*(ii) If the density $w$ is supported on a single interval $[-b, b]$, then $D_n(t)$ may be expressed as a trigonometric integral.*

**Proof.** (i) By scaling, one can reduce to the case of $v(x) = (1/4)x^4 - (1/2)m^2x^2 + a_0$, where $m > 2^{1/2}$. Then we let $U(x) = (1/4)x^2 - m^4/4$ and $\varphi(x) = x^2 - m^2$ in Proposition 2.2, and obtain the equilibrium measure for $v$ from the semicircular law on $[-2, 2]$ for $2U$. With $a^2 = m^2 - 2$ and $b^2 = 2 + m^2$, we obtain a constant $c$ such that

$$w(x) = c|x|\sqrt{(b^2 - x^2)(x^2 - a^2)}. \tag{6.14}$$

Now for any polynomial $f(x)$, there exist a constant $c_0$ and polynomials $f_1(x)$ and $f_2(x)$ such that $f(x) = c_0 + xf_1(x^2) + x^2f_2(x^2)$. Hence we can express the indefinite integral $\int f(x)dx/w(x)$ as a sum of

$$\int \frac{c_0dx}{x\sqrt{(b^2 - x^2)(x^2 - a^2)}} + \int \frac{f_1(x^2)dx}{\sqrt{(b^2 - x^2)(x^2 - a^2)}} + \int \frac{xf_2(x^2)dx}{\sqrt{(b^2 - x^2)(x^2 - a^2)}} \tag{6.15}$$

which reduce by $x^2 = u$ or $x^2 = 1/u$ to

$$-\int \frac{c_0du}{2\sqrt{(b^2u - 1)(1 - a^2u)}} + \int \frac{f_1(u)du}{\sqrt{u(b^2 - u)(u - a^2)}} + \int \frac{f_2(u)du}{\sqrt{(b^2 - u)(u - a^2)}}. \tag{6.16}$$

By Lemma 5.3, the first and last of these integrals are trigonometric, whereas by Lemma 5.4 middle integral is elliptic.

(ii) In this case, the integral equation shows that $w(x) = (c_2x^2 + c_0)\sqrt{b^2 - x^2}$ for some constants $c_0$ and $c_2$. Hence we can obtain the result from Lemma 5.3.

$\square$

**Corollary 6.3.** *Suppose that $v$ is an even sextic such that the equilibrium measure is supported on the three intervals $[-b, -c] \cup [-a, a] \cup [c, b]$. Then $D_n(t)$ belongs to the Liouvillian fields $\mathbf{F}(\mathcal{E}_1)$ and $\mathbf{F}(\mathcal{E}_2)$ over elliptic curves $\mathcal{E}_1$ and $\mathcal{E}_2$ as in Lemma 5.4.*

**Proof.** This follows directly from Theorem 6.1 and Lemma 5.4.

$\square$

In the remainder of this section, we describe the situation for typical quartic $v$, and how Schlesinger's equation reduces to Painlevé's equation $P_{VI}$ in the present situation. Suppose that $S_+ = [\delta_1, \delta_2] \cup [\delta_3, \delta_4]$. There exists a Möbius transformation $\varphi$ such that $\varphi(\delta_1) = 0$, $\varphi(\delta_2) = 1$ and $\varphi(\delta_4) = \infty$; then we let $t = \varphi(\delta_3)$. Having fixed three of the endpoints, we can introduce the differential equations from section 4, and then describe the effect of varying the endpoint $t$. Let

$$A(x, t) = \frac{\alpha_0}{x} + \frac{\alpha_1}{x - 1} + \frac{\alpha_t}{x - t}; \tag{6.17}$$

then by Lemma 4.2,

$$\frac{d}{dx}\Phi(x) = A(x, t)\Phi. \tag{6.18}$$

We now deform the differential equation by varying the position of the branch point at $t$, while keeping fixed the singularities at $0, 1$ and $\infty$. By analysis of the formal solution as in [24], one can show that this deformation gives the differential equation

$$\frac{\partial \Phi}{\partial t} = \frac{-\alpha_t}{x-t}\Phi \tag{6.19}$$

which is consistent with (6.18). By forming the Lax pair of (6.18) and (6.19), one can show that the matrix coefficients satisfy

$$\frac{\partial \alpha_0}{\partial t} = \frac{[\alpha_0, \alpha_t]}{-t}, \qquad \frac{\partial \alpha_1}{\partial t} = \frac{[\alpha_1, \alpha_t]}{1-t},$$

$$\frac{\partial \alpha_t}{\partial t} = \frac{[\alpha_t, \alpha_0]}{-t} + \frac{[\alpha_t, \alpha_1]}{1-t}, \tag{6.20}$$

which is a particular case of Schlesinger's equation; see [19, 4.0.3]. Considering the top right corner of the matrices, we introduce $x(t) = t((\alpha_0)_{1,2})/(t((\alpha_0)_{1,2}+(\alpha_1)_{1,2})-(\alpha_1)_{1,2})$ such that $A(x,t)_{1,2} = 0$; then by [25, C.57], the corresponding Schlesinger equations give a version of $P_{VI}$ in terms of $x$, namely

$$\begin{aligned}\frac{d^2x}{dt^2} &+ \Big(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{x-t}\Big)\frac{dx}{dt} - \frac{1}{2}\Big(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t}\Big)\Big(\frac{dx}{dt}\Big)^2 \\ &= \frac{1}{2}\frac{x(x-1)(x-t)}{t^2(t-1)^2}\Big(k_\infty - \frac{k_0 t}{x^2} + \frac{k_1(t-1)}{(x-1)^2} - \frac{(k_t-1)t(t-1)}{(x-t)^2}\Big).\end{aligned} \tag{6.21}$$

**Proposition 6.4** *Let* $\mathbf{F}$ *be a differential field that contains* $\mathbf{C}(t)$ *and a solution* $x(t)$ *of (6.21). Then* $\tau$*, defined by*

$$\frac{d}{dt}\log\tau(t) = \mathrm{trace}\Big(\frac{\alpha_0\alpha_t}{t} + \frac{\alpha_1\alpha_t}{t-1}\Big), \tag{6.22}$$

*belongs to a Liouvillian extension field of* $\mathbf{F}$*.*

**Proof.** The matrices $\alpha_0, \alpha_1$ and $\alpha_t$ can be expressed as rational functions of $t$, $x(t), x'(t)$ and $\int x(t)dt$ as in [23, p. 11975] and [25, (C.57)]. Starting from one solution of (6.21), one can construct a solution of Schlesinger's system (6.19) and (6.20) by using the formulas from [25], hence once can obtain all of the solutions of (6.21). It follows that the right-hand side of (6.22) belongs to a Liouvillian extension of $\mathbf{F}$, and we can solve for $\tau(t)$ in a further Liouvillian extension of this field. See also [27, p. 12043].

□

Fuchs [21] used the elliptic change of variable

$$u(x) = \int_0^x \frac{ds}{\sqrt{s(s-1)(s-t)}} \tag{6.23}$$

to obtain solutions of (6.21) when $k_0 = k_1 = k_\infty = 0$ and $k_t = 1$. Magnus [28, p. 228] obtained $P_{VI}$ for the orthogonal polynomials with a generalized Jacobi weight with three factors. See also [20].

## 7. Scattering functions for elliptic potentials

In previous section we have obtained elliptic and trigonometric $\tau$ functions from algebraic weights. In this section, we show how to realise elliptic $\tau$ from linear systems and establish a correspondence between $\tau$ and scattering functions $\phi$. We take the scattering function $\phi$ as the starting point and, as in inverse scattering, seek to reconstruct $q$.

**Definition.** A linear system $(-A, B, C)$ consists of complex Hilbert spaces called the state space $H$ and the input and output space $H_0$ and (bounded) linear operators $A : H \to H$, $B : H_0 \to H$ and $C : H \to H_0$. The associated linear differential equation is

$$\begin{aligned} \frac{dX}{dt} &= -AX + BU \\ Y &= CX \end{aligned} \tag{7.1}$$

for $U, Y : (0, \infty) \to H_0$ and $X : (0, \infty) \to H$. The scattering function is $\phi(x) = Ce^{-xA}B$ and the transfer function is its Laplace transform $\hat{\phi}(s) = C(sI + A)^{-1}B$.

**Proposition 7.1.** *Let $t = 2^{-1}(b + a) + 2^{-1}(b - a)\cos x$, so a typical element of (5.9) is*

$$\phi(x) = \sum_{j=1}^{m} \bigl(a_j x^{n_j} \cos m_j x + b_j x^{k_j} \sin m_j x\bigr) \tag{7.2}$$

*where $m_j, n_j, k_j \in \{0, 1, \dots\}$ and $a_j, b_j \in \mathbf{C}$. Then $\phi(x)$ is the scattering function $Ce^{-xA}B$ of a linear system with $H_0 = \mathbf{C}$ and $H = \mathbf{C}^N$ for some $N < \infty$. If $n_j = k_j = 0$ for all $j$, then $A$ may be chosen to be real and skew symmetric.*

**Proof.** By induction on degree, one proves using the trigonometric addition rules that (7.2) gives a typical element of (5.9). For this $\phi$, the transfer function is a proper rational function and hence may be realised from a linear system with finite-dimensional state space. One can produce $A, B$ and $C$ by considering the Jordan canonical forms of various matrices. See [10, p. 55].

□

We formulate a version of the Gelfand–Levitan equation that is appropriate when $\phi(x) = Ce^{-xA}B$ is periodic. Demontis and van der Mee used a variant of this idea in [16] to solve the matrix nonlinear Schrödinger equation explicitly by linear algebra. McKean and Ercolani also considered scattering for periodic functions in [17].

**Definition** (Periodic linear system $(-A, B, C; E)$). Let $A, B, C$ and $E$ be finite square matrices of equal size; let $\varepsilon = \pm 1$, and suppose that $BC = \varepsilon(AE + EA)$, $BE = EB$, $EA = AE$ and $\exp 2\pi A = I$. Define $\phi(x) = Ce^{-xA}B$ to be the scattering function for $(-A, B, C)$ and then introduce

$$W(x,y) = Ce^{-xA}\big(I - e^{-xA}Ee^{-xA}\big)^{-1}e^{-yA}B. \tag{7.3}$$

We define the tau function to be

$$\tau(x) = \exp\Big(\int_0^x \operatorname{trace} W(y,y)\, dy\Big) \tag{7.4}$$

and let $q(x) = -2\frac{d^2}{dx^2}\log\tau(x)$ be the potential function. See also [10, p. 114].

These definitions are justified by the following, which is analogous to [5, p. 324].

**Lemma 7.2.** *(i) The matrices satisfy the Gelfand–Levitan equation*

$$-\phi(x+y) + W(x,y) - \varepsilon\int_x^{2\pi} W(x,z)\phi(z+y)\, dz = W(x,y)E \qquad (0 < x < y < 2\pi), \tag{7.5}$$

*and*

$$\frac{d}{dx}\log\det(I - e^{-xA}Ee^{-xA}) = \varepsilon \operatorname{trace} W(x,x). \tag{7.6}$$

*(ii) Let* $\mathbf{F}$ *be a differential field that contains all the entries of* $e^{-xA}$*. Then* $\tau(x)$ *belongs to a Liouvillian extension of* $\mathbf{F}$*, and* $\tau(x + 2\pi) = \kappa\tau(x)$ where $\kappa = \exp\int_0^{2\pi} \operatorname{trace} W(y,y)\, dy$.

*(iii) Suppose moreover that* $\varepsilon = 1$ *and* $2\pi\|\phi\|_\infty < 1$*. Then*

$$\frac{\partial^2 W}{\partial x^2} - \frac{\partial^2 W}{\partial y^2} = -2\Big(\frac{d}{dx}W(x,x)\Big)W(x,y). \tag{7.7}$$

**Proof.** (i) One can check that

$$\int_x^{2\pi} e^{-zA}BCe^{-zA}\, dz = \varepsilon e^{-xA}Ee^{-xA} - \varepsilon E \tag{7.8}$$

and it is then a simple matter to verify the integral equation (7.5).

By rearranging terms, one checks that

$$\begin{aligned}\operatorname{trace} W(x,x) &= \operatorname{trace}\big((I - e^{-xA}Ee^{-xA})^{-1}e^{-xA}BCe^{-xA}\big)\\ &= \varepsilon\frac{d}{dx}\operatorname{trace}\log(I - e^{-xA}Ee^{-xA})\\ &= \varepsilon\frac{d}{dx}\log\det(I - e^{-xA}Ee^{-xA}).\end{aligned} \tag{7.9}$$

(ii) By (i), $\tau$ is given by exponential integrals of the entries of $e^{-xA}$. Note that $W(x,y)$ is periodic in both $x$ and $y$, so $W(x,x)$ is periodic and hence $\int_0^x \operatorname{trace} W(y,y)\, dy$ changes by the same amount as $x$ increases through any interval of length $2\pi$.

(iii) By repeatedly differentiating (7.5), and using periodicity, one derives the identity

$$\frac{\partial^2 W}{\partial x^2}-\frac{\partial^2 W}{\partial y^2}+2\Big(\frac{d}{dx}W(x,x)\Big)\phi(x+y)+W(x,0)\phi'(y)-\frac{\partial W}{\partial y}(x,0)\phi(y)$$

$$-\int_x^{2\pi}\Big(\frac{\partial^2 W}{\partial x^2}-\frac{\partial^2 W}{\partial y^2}\Big)\phi(z+y)\,dz=\frac{\partial^2 W}{\partial x^2}E-\frac{\partial^2 W}{\partial y^2}E. \tag{7.10}$$

Since $ABC-CBA=0$, we obtain

$$W(x,0)\phi'(y)-\frac{\partial W}{\partial y}(x,0)\phi(y)=0, \tag{7.11}$$

so (7.10) is a multiple of the original integral equation by $-2\frac{d}{dx}W(x,x)$. By the assumptions on $\|\phi\|_\infty$, the solutions are unique, hence the differential equation (7.7) is satisfied.

□

In view of Corollaries 6.2 and 6.3, we aim to realise elliptic tau functions in terms of linear systems. By introducing infinite block matrices, we obtain an analogue of Proposition 7.1. Clearly we can replace $\varepsilon$ in (7.3) by a diagonal matrix with blocks of $\pm 1$ entries on the diagonal. One can interpret the following result as saying that Lamé's operator $-\frac{d^2}{dx^2}+2\wp$ has the scattering function proportional to $\sin x$. Note that $\phi$ lies in the ring of trigonometric functions as in (5.9), and trigonometric functions may be viewed as rational functions on $\mathbf{C}\cup\{\infty\}$ via the substitution $t=\tan x/2$. However, the potential lies in the elliptic function field, which gives the rational functions on a curve of genus one.

Elliptic functions may also be viewed as doubly periodic meromorphic function on $\mathbf{C}$. Let $\omega_1$ and $\omega_2$ be the periods, so that $\omega=\omega_2/\omega_1$ has $\Im\omega>0$; then let $e_1=\wp(\omega_1/2)$, $e_2=\wp((\omega_1+\omega_2)/2)$ and $e_3=\wp(\omega_2/2)$; then let Jacobi's modulus be $\mathrm{m}^2=(e_2-e_3)/(e_1-e_3)$ and the elliptic nome be $\mathrm{q}=e^{i\omega\pi}$. To be specific, we choose $\omega_1=2\pi$ and $\omega_2=2\pi i$. Let $A,B$ and $C$ be the infinite block diagonal matrices with $2\times 2$ diagonal blocks

$$\begin{aligned} A&=\mathrm{diagonal}\big[J\big]_{n=-\infty}^{\infty}, & C&=A,\\ E&=\mathrm{diagonal}\big[\mathrm{q}^{2|n|}I_2\big]_{n=-\infty}^{\infty}, & B&=2E. \end{aligned} \tag{7.12}$$

**Proposition 7.3.** *(i) The functions $\phi(x)=Ce^{-xA}B$ and $W(x,y)$ of (7.3) satisfy the Gelfand–Levitan equation (7.5) and*

$$\mathrm{trace}\,\phi(x)=4\frac{1+\mathrm{q}^2}{1-\mathrm{q}^2}\sin x \qquad (x\in\mathbf{R}). \tag{7.13}$$

*(ii) The corresponding tau function is $\tau(x)=c(\mathrm{q})\theta_1(x)^2$ with $c(\mathrm{q})$ constant, so $\tau$ is entire, belongs to the Liouvillian extension $\mathbf{C}(\wp',\wp,\zeta,\theta_1)$ of the standard elliptic function field and satisfies*

$$2\wp(x)=-\frac{d^2}{dx^2}\log\tau(x) \qquad (x\in\mathbf{R}). \tag{7.14}$$

**Proof.** (i) The matrices satisfy $EB = BE$, $AE = EA$ and $BC = AE + EA$, so Lemma 7.2(i) applies. Note that the entries of $E$ are summable, so $E$ defines a trace-class operator, hence the trace exists and a simple calculation gives (7.13).

(ii) Observe also that

$$\det(I - \mathrm{q}^{2|n|}e^{-2xA}) = \det \begin{bmatrix} 1 - \mathrm{q}^{2|n|}\cos 2x & -\mathrm{q}^{2|n|}\sin 2x \\ \mathrm{q}^{2|n|}\sin 2x & 1 - \mathrm{q}^{2|n|}\cos 2x \end{bmatrix} = 1 - 2\mathrm{q}^{2|n|}\cos 2x + \mathrm{q}^{4|n|}, \tag{7.15}$$

so one has

$$\det(I - e^{-xA}Ee^{-xA}) = 4\sin^2 x \prod_{n=1}^{\infty}(1 - 2\mathrm{q}^{2n}\cos 2x + \mathrm{q}^{4n})^2; \tag{7.16}$$

for comparison, by [32, p 135] the Jacobi elliptic function satisfies

$$\theta_1(x) = 2\mathrm{q}^{1/4}\sin x \prod_{n=1}^{\infty}\big(1 - 2\mathrm{q}^{2n}\cos 2x + \mathrm{q}^{4n}\big)(1 - \mathrm{q}^{2n}) \tag{7.17}$$

where the infinite product is absolutely and uniformly convergent over compact subsets of $\mathbf{C}$. So we have an entire function

$$\tau(x) = \det(I - e^{-xA}Ee^{-xA}) = \frac{\theta_1(x)^2}{\mathrm{q}^{1/2}\prod_{n=1}^{\infty}(1 - \mathrm{q}^{2n})^2}. \tag{7.18}$$

Moreover, we have [32, p. 132]

$$\wp(x) = -\frac{d^2}{dx^2}\log\theta_1(x) + e_1 + \frac{d^2}{dx^2}\log\theta_1(x)\big|_{x=1/2}, \tag{7.19}$$

hence we obtain (7.14). The differential field $\mathbf{C}(\wp, \wp', \zeta, \theta_1)$ is a Liouvillian extension of the standard elliptic function field $\mathbf{C}(\wp)[\wp']$ and contains $\tau$ and $\wp$.

□

**Theorem 7.4.** *Let $\tau$ be an elliptic function.*

*(i) Then there exists a periodic linear system $(-A, B, C; E)$, where $A, B, C$ and $E$ are infinite block diagonal matrices with $2 \times 2$ blocks, such that*

$$\frac{\tau'(x)}{\tau(x)} = \operatorname{trace} W(x, x). \tag{7.20}$$

*(ii) There exists a sequence of periodic linear systems with finite matrices and tau functions $\tau_N$ such that $\tau_N(x) \to \tau(x)$ as $N \to \infty$, and with $e^{ix} = z$, each quotient $h_N(z) = \tau_N(x)/\tau_{N-1}(x)$ is a rational function such that $h_N(0) = h_N(\infty) = 1$.*

**Proof.** Any elliptic function is of rational character on $\mathbf{C}/(\omega_1\mathbf{Z}+\omega_2\mathbf{Z})$, and is the quotient of theta functions by [32, p 105], so

$$\tau(x)=c\prod_{j=1}^m \frac{\theta_1(x-a_j)}{\theta_1(x-b_j)} \tag{7.21}$$

where $a_j$, $b_j$ and $c$ are constants. Suppose for notational simplicity that $c=1$.

First we construct a periodic linear system with $\theta_1$ as its tau function. For $n=0$, let $A_0=J/2$, $E_0=-iJ$, $B_0=iI$ and $C_0=I$, then $(-A_0,B_0,C_0;E_0)$ is a periodic linear system such that $\det(I-e^{-xA_0}E_0e^{-xA_0})=2i\sin x$.

For $n=1,2,\dots$ and q as in (7.12), let $A_n=C_n=J$, $E_n=\mathrm{q}^{2n}I$ and $B_n=2E_n$; then $(-A_n,B_n,C_n;E_n)$ is a periodic linear system such that $\det(I-e^{-xA_n}E_ne^{-xA_n})=1-2\mathrm{q}^{2n}\cos 2x+\mathrm{q}^{4n}$. Hence we can introduce block diagonal matrices $A=\mathrm{diagonal}[A_0,A_1,\dots]$ and $E=\mathrm{diagonal}[E_0,E_1,\dots]$, and so on to give a periodic linear system $(-A,B,C;E)$ such that

$$\begin{aligned}\det(I-e^{-xA}Ee^{-xA})&=2i\sin x\prod_{n=1}^\infty(1-2\mathrm{q}^{2n}\cos 2x+\mathrm{q}^{4n})\\&=\frac{i\theta(x)}{\mathrm{q}^{1/4}\prod_{n=1}^\infty(1-\mathrm{q}^{2n})}.\end{aligned} \tag{7.22}$$

Next we replace $(-A,B,C;E)$ by the terms $(-A,e^{a_jA}B,Ce^{a_jA};e^{a_jA}Ee^{a_jA})$ which give $W_j$ by (7.3); likewise we introduce $(-A,e^{b_jA}B,-Ce^{b_jA};e^{b_jA}Ee^{b_jA})$ which give $\hat W_j$ by (7.3). We then form the block diagonal matrix

$$\oplus_{j=1}^m\Big((-A)\oplus(-A),e^{a_jA}B\oplus e^{b_jA}B,Ce^{a_jA}\oplus(-Ce^{b_jA});e^{a_jA}Ee^{a_jA}\oplus e^{b_jA}Ee^{b_jA}\Big) \tag{7.23}$$

which gives the required $W(x,y)=\oplus_{j=1}^m W_j(x,y)\oplus\hat W_j(x,y)$ by (7.3), and we verify

$$\begin{aligned}\mathrm{trace}\,W(x,x)&=\sum_{j=1}^m\Big(\mathrm{trace}\,W_j(x,x)+\mathrm{trace}\,\hat W_j(x,x)\Big)\\&=\frac{d}{dx}\sum_{j=1}^m\Big(\log\theta_1(x-a_j)-\log\theta_1(x-b_j)\Big)\\&=\frac{d}{dx}\log\tau(x).\end{aligned} \tag{7.24}$$

One can check that $W$ satisfies (7.5) with $\varepsilon$ replaced by a diagonal matrix with diagonal entries $\pm1$.

(ii) We introduce the $2(N+1)\times 2(N+1)$ block diagonal matrices

$$A_{(N)}=\mathrm{diagonal}\,[A_0,A_1,\dots,A_N]\quad\text{and}\quad E_{(N)}=\mathrm{diagonal}\,[E_0,E_1,\dots,E_N], \tag{7.25}$$

with $2 \times 2$ blocks $A_j$ and $E_j$ as above, such that

$$\det(I - e^{-xA_{(N)}} E_{(N)} e^{-xA_{(N)}}) = 2i \sin x \prod_{j=1}^{N} (1 - 2\mathrm{q}^{2n} \cos 2x + \mathrm{q}^{4n}). \tag{7.26}$$

Then by (7.21) and (7.22), we have

$$\tau_N(x) = \prod_{j=1}^{m} \frac{\det(I - e^{-xA_{(N)}} e^{a_j A_{(N)}} E_{(N)} e^{a_j A_{(N)}} e^{-xA_{(N)}})}{\det(I - e^{-xA_{(N)}} e^{b_j A_{(N)}} E_{(N)} e^{b_j A_{(N)}} e^{-xA_{(N)}})} \to \prod_{j=1}^{m} \frac{\theta(x - a_j)}{\theta(x - b_j)} \qquad (N \to \infty). \tag{7.27}$$

Hence we can adjust the construction to produce the sequence of finite linear systems as above. By passing from $N-1$ to $N$, we introduce an extra factor

$$\begin{aligned} \frac{\tau_N(x)}{\tau_{N-1}(x)} &= \prod_{j=1}^{m} \frac{1 - 2\mathrm{q}^{2n} \cos(x - a_j) + \mathrm{q}^{4n}}{1 - 2\mathrm{q}^{2n} \cos(x - b_j) + \mathrm{q}^{4n}} \\ &= \prod_{j=1}^{m} \frac{-\mathrm{q}^{2n} e^{-2ia_j} z^4 + (1 + \mathrm{q}^{2n}) z^2 - e^{2ia_j} \mathrm{q}^{2n}}{-\mathrm{q}^{2n} e^{-2ib_j} z^4 + (1 + \mathrm{q}^{2n}) z^2 - e^{2ib_j} \mathrm{q}^{2n}}, \end{aligned} \tag{7.28}$$

which corresponds to the blocks on the diagonal indexed by $N$ in the block diagonal form of the Gelfand–Levitan equation. Evidently this is a rational expression, and at $z = 0$,

$$\frac{\tau_N(i\infty)}{\tau_{N-1}(i\infty)} = \exp\Big(2i \sum_{j=1}^{m} (a_j - b_j)\Big) = 1, \tag{7.29}$$

where the final identity follows from Abel's theorem [32, p. 105]; likewise, the quotient $\tau_N/\tau_{N-1}$ converges to one as $z \to \infty$.

$\square$

Let $q(t) = -2\frac{d^2}{dt^2} \log \tau(2t)$ be an elliptic potential as in Theorem 7.3. Then $-d^2f/dt^2 + q(t)f = \lambda f$ is Hill's equation on the torus, as considered in [22]. In the next section, we consider potentials on hyperelliptic curves.

## 8. Linear systems for potentials on hyperelliptic curves

In this section, we extend results from section 8 from elliptic to hyperelliptic curves. Suppose that $q : \mathbf{R} \to \mathbf{R}$ is $C^2$ and periodic with period one; introduce Hill's operator $-\frac{d^2}{dx^2} + q(x)$ in $L^2(\mathbf{R})$. Let $\Phi$ be the $2 \times 2$ fundamental solution matrix that satisfies

$$\frac{d}{dx}\Phi(x) = \begin{bmatrix} 0 & 1 \\ -\lambda + q(x) & 0 \end{bmatrix} \Phi(x), \qquad \Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{8.1}$$

and let $\Delta(\lambda) = \operatorname{trace} \Phi(1)$ be the discriminant of Hill's equation. We can characterize $S = \{\lambda \in \mathbf{R} : \Delta(\lambda)^2 \leq 4\}$, and its connected components are known as the intervals of stability. Suppose further that $q$ is algebro-geometric, so that the Bloch spectrum

$$S = [\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3] \cup \ldots \cup [\lambda_{2g}, \infty) \tag{8.2}$$

has $g$ gaps. The $\lambda_j$ are the points of the simple periodic spectrum, such that $-f''+qf=\lambda_j f$ has a unique solution, up to scalar multiples, that is periodic with period one or two. Let $\lambda_j'$ be the zeros of $\Delta'(\lambda)$ that are not zeros of $\sqrt{4-\Delta(\lambda)^2}$; then Hochstadt [31, p 219] proved that

$$\frac{\Delta'(\lambda)}{\sqrt{4-\Delta(\lambda)^2}} = c\frac{\prod_{j=1}^{g}(\lambda-\lambda_j')}{\sqrt{\prod_{j=0}^{2g}(\lambda-\lambda_j)}} \tag{8.3}$$

for some constant $c$. Moreover, Hochstadt [31] proved that $g=1$ if and only if $q(x)=c_1+2\wp(x+c_2)$ where $c_1$ and $c_2$ are constants, as in Proposition 7.3.

Now supposing that $g>1$, we form the hyperelliptic curve

$$\mathcal{C}: \quad w^2 = -\prod_{j=0}^{2g}(x-\lambda_j) \tag{8.4}$$

of genus $g$. As in [18], we introduce a homology basis for $\mathcal{C}$; we choose a two-sheeted cover of $\mathbf{C}$ with cuts along $S$, and introduce the canonical homology basis consisting of:

- loops $\alpha_j$ that start from $[\lambda_{2g},\infty)$, pass along the top sheet to $[\lambda_{2j-2},\lambda_{2j-1}]$, then return along the bottom sheet to the start on $[\lambda_{2g},\infty)$;
- loops $\beta_j$ that go around the intervals of stability $[\lambda_{2j-2},\lambda_{2j-1}]$ that do not intersect with one another, for $j=1,\dots,g$.

Let $\Omega_0=[\int_{\alpha_k}x^{g-j}dx/w]_{j,k=1}^{n}$, which is invertible, and then form the $g\times 1$ vector of holomorphic one-forms

$$d\zeta = \begin{bmatrix} d\zeta_1 \\ \vdots \\ d\zeta_g \end{bmatrix} = \Omega_0^{-1}\begin{bmatrix} x^{g-1}dx/w \\ \vdots \\ dx/w \end{bmatrix}. \tag{8.5}$$

Then as in [18, p 61], we form the $g\times 2g$ Riemann matrix $[I;\Omega]$ from the $g\times g$ blocks

$$I=\Big[\int_{\alpha_k} d\zeta_j\Big]_{j,k=1}^{g}, \quad \text{and} \quad \Omega=\Big[\int_{\beta_k} d\zeta_j\Big]_{j,k=1}^{g}. \tag{8.6}$$

Let $\Lambda$ be the lattice generated by the columns of $[I;\Omega]$, and note that $\mathbf{C}^g/\Lambda$ is a complex torus, called the Jacobi variety of $\mathcal{C}$. Let $\mathcal{C}_g$ be the space of integral divisors of degree $g$ on $\mathcal{C}$, and let $p_0$ be a fixed point in $\mathcal{C}$, and $\mathbf{p}=p_1\dots p_g$ be a variable in $\mathcal{C}_g$. Then there is a holomorphic and surjective map $\mathcal{C}_g\to\mathbf{C}^g/\Lambda$ given by $p_1\dots p_g\mapsto\sum_{j=1}^{g}\int_{p_0}^{p_j}d\zeta$, or $\mathbf{x}\mapsto Z_{\mathbf{x}}$.

**Definition.** Let $S_x$ be the spectrum of $-d^2/dt^2+q(x+t)$ on the domain $\{f\in L^2[0,1]: f''\in L^2[0,1], f(0)=f(1)=0\}$. Then $S_x$ consists of the double zeros of $\Delta(\lambda)^2-4=0$, which do not depend upon $x$, together with the auxiliary spectrum $\delta_j(x)\in[\lambda_{2j-1},\lambda_{2j}]$ for $j=1,\dots,g$.

Then the auxiliary spectrum determines $q$, up to translation and some constants, as follows from [31]. Let

$$R(x,\lambda)=(\lambda-\delta_1(x))\dots(\lambda-\delta_g(x)), \tag{8.7}$$

and introduce $\sigma_k(x)$ such that $R(x,\lambda) = \sum_{k=0}^{g} \sigma_k(x)\lambda^k$. The function $\sqrt{R(x,\lambda)}$ has a strong formal resemblance to (2.5).

**Theorem 8.1.** *(i) Let $\tau$ be a rational function on $\mathcal{C}$ such that $\tau(\bar z) = \overline{\tau(z)}$. Then there exists a periodic linear system $(-A, B, C; E)$, where $A, B, C$ and $E$ are infinite block diagonal matrices with $2 \times 2$ blocks, such that*

$$\frac{\tau'(x)}{\tau(x)} = \frac{1}{2}\mathrm{trace}\, W(x,x). \tag{8.8}$$

*(ii) In particular, such a periodic linear system exists for each $\sigma_k(x)$.*

*(iii) Let $\lambda \in \mathbf{R}$ satisfy $\Delta(\lambda)^2 < 4$ and let $\mathbf{F}$ be a differential field that contains all of the $\sigma_k(x)$. Then there exists a Liouvillian extension $\mathbf{F}_\lambda$ of $\mathbf{F}$ and an eigenfunction $f_\lambda \in \mathbf{F}_\lambda$ such that $-f_\lambda'' + qf_\lambda = \lambda f_\lambda$.*

**Proof.** (i) Koebe proved the retrosection theorem that every compact Riemann surface of genus $g$ is conformal to the quotient of a planar domain $D$ under the action of a discrete group $\Gamma$ of Möbius transformations, so $T_r : z \mapsto z_r$ for $r \in \Gamma$ is given by $T_r = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $ad - bc = 1$; see [3]. Hence $\mathcal{C}$ may be identified with a fundamental cell $\mathcal{C}_0$, where the images of $\mathcal{C}_0$ under $\Gamma$ tessellate $D$. The group $\Gamma$ has identity $T_0$ and free generators $T_j$ for $j = 1, \dots, g$. Choosing $\gamma_0$ to be the unit circle, and $\gamma_1, \dots, \gamma_g$ to be Jordan curves inside $\gamma_0$ that are exterior to one another, we then let $\gamma_{-j}$ be the Jordan curves in the exterior of $\gamma_0$ so that the $\gamma_{\pm 1}, \dots, \gamma_{\pm g}$ are mutually exterior. The generators may be so chosen that $T_j(\gamma_j) = \gamma_{-j}$ and $T_j$ maps the exterior of $\gamma_j$ with respect to $\mathbf{C}_\infty$ onto the interior of $\gamma_{-j}$ for $j = 1, \dots, g$. Diagrams of the appropriate $D$ appear in [1, 3].

Given $u, v \in \mathcal{C}_0$, we introduce

$$S_{u,v}(\zeta,\xi) = \prod_{r\in\Gamma} \frac{(u_r - \zeta)(v_r - \xi)}{(u_r - \xi)(v_r - \zeta)}, \tag{8.9}$$

a variant of the Schottky–Klein prime function. Baker [1, section 230] shows that $\zeta \mapsto S_{u,v}(\zeta,\xi)$ defines a meromorphic function on $\mathbf{C}$ which is invariant under the action of $\Gamma$, and hence defines a meromorphic function on $\mathcal{C}$; moreover, $S_{u,v}(\zeta,\xi)$ has only a simple zero at $\zeta = u$ and a simple pole at $\zeta = v$ inside $\mathcal{C}_0$. Baker proves convergence of the product, but we do not have an effective estimate on the rate of convergence.

Let $\zeta = e^{is}, \xi = e^{it}$ and $u_r = q_r e^{i\psi_r}, v_r = p_r e^{i\phi_r}$ where $q_r, p_r > 0$ and $s, t, \psi_r, \phi_r$ are real. For each $r \in \Gamma$, we introduce a periodic linear system by $(-A_r, B_r, C_r; E_r)$ where $A_r = C_r = J/2$, $E_r = q_r \exp(\psi_r J)$ and $B_r = 2E_r$; likewise, we introduce $(-A_r, \tilde B_r, C_r, \tilde E_r)$ where $A_r = C_r = J/2$, $\tilde E_r = p_r \exp(\phi_r J)$ and $\tilde B_r = -2\tilde E_r$; then

$$\frac{\det(I - e^{-sA_r}E_r e^{-sA_r})}{\det(I - e^{-sA_r}\tilde E_r e^{-sA_r})} = \left|\frac{e^{is} - u_r}{e^{is} - v_r}\right|^2. \tag{8.10}$$

We sum these linear systems over $r \in \Gamma$ and obtain the tau function

$$\left|S_{u,v}(e^{is}, e^{it})\right|^2 = \prod_{r \in \Gamma} \frac{\det(I - e^{-sA_r} E_r e^{-sA_r}) \det(I - e^{-tA_r} \tilde{E}_r e^{-tA_r})}{\det(I - e^{-sA_r} \tilde{E}_r e^{-sA_r}) \det(I - e^{-tA_r} E_r e^{-tA_r})}. \tag{8.11}$$

Now each $\sigma_k(x)$ is meromorphic on $\mathcal{C}$, hence has zeros $a_1, \dots, a_m$ and poles $b_1, \dots, b_m$ such that $Z_{a_1 \dots a_m} = Z_{b_1 \dots b_m}$ modulo $\Lambda$. Therefore by Liouville's theorem we can write

$$\sigma_k(\zeta) = c(\xi) \prod_{j=1}^{m} S_{a_j, b_j}(\zeta, \xi) \tag{8.12}$$

for some meromorphic $c(\xi)$ and then express $|\sigma_k(\zeta)|^2$ as a product of determinants.

Note that $(\zeta, w)$ lies on $\mathcal{C}$ if and only if $(\bar{\zeta}, \bar{w})$ lies on $\mathcal{C}$, so the operation $\tau(\zeta) \mapsto \overline{\tau(\bar{\zeta})}$ is well defined. Finally, we take the derivative with respect to $x \in \mathbf{R}$ of the real function $\tau(x)$ and find $(\log |\tau(x)|^2)' = 2\tau'(x)/\tau(x)$.

(ii) Let $M$ be the set of $C^2$ potentials $q : \mathbf{R} \to \mathbf{R}$ that are 1-periodic and such that the simple periodic spectrum consists of $\{\lambda_1 < \dots < \lambda_g\}$; then $M$ is diffeomorphic to a real torus $\mathbf{R}^g/\mathbf{Z}^g$, and is known as the isospectral torus. In particular, the set $\{q(t+x) : 0 < x < 1\}$ of translations of $q$ gives a torus $\mathbf{T}$ inside $M$, so it is natural to parametrize $\mathbf{T}$ by $e^{2\pi i x}$ on the circle. Differentiation along $[0,1]$ transforms to differentiation along a direction in $\mathbf{R}^g/\mathbf{Z}^g$.

McKean and van Moerbeke [31] showed that $q$ is the restriction to a straight line in $\mathbf{C}^g/\Lambda$ of a function of rational character. Likewise, each $\sigma_k(x)$ is meromorphic on $\mathcal{C}$, and real for $x \in \mathbf{R}$, so we can obtain an expression analogous to (7.19) by (i).

(iii) Note that $R(x, \lambda)$ belongs to $\mathbf{F}$, and as a function of $x$ satisfies Drach's equation [9, p. 927]

$$\frac{1}{4} R'(x, \lambda)^2 - \frac{1}{2} R(x, \lambda) R''(x, \lambda) + (q(x) - \lambda) R(x, \lambda)^2 = w(\lambda)^2 \tag{8.13}$$

as [31, p. 235]; where, crucially, the right-hand side is a polynomial in $\lambda$ independent of $x$. In particular, if $\lambda$ is a simple periodic eigenvalue, then $w(\lambda) = 0$ and $f_\lambda(x) = \sqrt{R(x, \lambda)}$ gives a corresponding eigenfunction as in [31, p. 235]. When $\Delta(\lambda)^2 < 4$, $R(x, \lambda)$ is never zero for $0 \leq x \leq 1$, and we can form

$$f_\lambda(x) = \sqrt{R(x, \lambda)} \sin\Bigl(iw(\lambda) \int_0^x \frac{dy}{R(y, \lambda)}\Bigr), \tag{8.14}$$

as in [9, p. 932]. Evidently this is obtained from $R(\lambda, x)$ by Liouvillian operations and by using (8.13) one shows that $f_\lambda$ is an eigenfunction as in [31, p 237].

$\square$

**Remarks 8.2.** (i) Whereas Theorem 8.1 enables us to express all rational functions on a hyperelliptic curve $\mathcal{C}$ in terms of block diagonal linear systems, the tau function $D_n(t)$ of Theorem

5.1 could involve elements of $\mathbf{F}_m(\mathcal{C})$ for $m = 2g + 1$, which are generally rather complicated. Nevertheless, in [2], the authors give criteria in terms of the action of the symplectic group on $\Omega$ for a special hyperelliptic curve $\mathcal{C}$ to be a finite cover of an elliptic curve.

(ii) For integers $\ell \geq 2$, Lamé's operator $-d^2/du^2 + \ell(\ell+1)\wp$ has $S$ with $\ell$ gaps, and the hyperelliptic spectral curve $\mathcal{C}$ has genus $\ell$. Using this, Maier [29, Theorem 4.1] has constructed a cover of degree $N = \ell(\ell+1)/2$ of the elliptic curve by $\mathcal{C}$, and thus obtains hyperelliptic integrals that reduce to elliptic integrals.

(iii) The elliptic potentials that are algebro–geometric are characterized in [22].

## 9. Kernels associated with rational matrix ODE

In this section, we introduce kernels that are associated with the basic differential equation (4.6), and then factorize them in terms of Hankel operators; then in section 10 we derive a Gelfand–Levitan equation that produces $(\log\tau)''$. We wish to introduce a differential equation that is equivalent to (4.6), but which involves a rational matrix function $B_n(x)$ with $\mathrm{trace}\, B_n(x) = 0$. First we let $\nu_j = -2^{-1}\mathrm{trace}\, \alpha_j(n)$ and observe that $\nu_j$ does not depend upon $n$. Indeed, by multiplying the recurrence relation (4.20) by $V_n^{-1}$, one deduces that $\mathrm{trace}\, A_{n+1}(z) = \mathrm{trace}\, A_n(z)$, and since $\mathrm{trace}\, \alpha_j(n) = \lim_{z\to\delta_j}(z-\delta_j)\mathrm{trace}\, A_n(z)$, we deduce that $\mathrm{trace}\, \alpha_j(n)$ is constant with respect to $n$. By following calculations in [12], one can show that $\sum_{j=1}^{4N-2}\mathrm{trace}\, \alpha_j(n) = 1-2N$. Now, given $\Phi_n$ as in (4.5), let

$$\Psi_n(z) = \Phi_n(z)\prod_{j=1}^{4N-2}(z-\delta_j)^{\nu_j}. \tag{9.1}$$

We next introduce the matrix valued kernel

$$M_n(z,\zeta) = \frac{\Psi_n(z)^t J \Psi_n(\zeta)}{-2\pi i(z-\zeta)}, \tag{9.2}$$

where $^t$ denotes the matrix transpose and analyse the top left entry of $M_n$ as an integral operator on $L^2(S;\mathbf{C}^2)$.

**Proposition 9.1.** *Let $E_n(z,\zeta)$ be the kernel of the orthogonal projection in $L^2(w)$ onto* $\mathrm{span}\{x^j : j = 0,\dots,n-1\}$. *Then the top left entry of $M_n(z,\zeta)$ equals*

$$M_n(z,\zeta)_{11} = \frac{h_n}{h_{n-1}}\prod_{j=1}^{4N-2}(z-\delta_j)^{\nu_j}\prod_{j=1}^{4N-2}(\zeta-\delta_j)^{\nu_j}E_n(z,\zeta). \tag{9.3}$$

**Proof.** The Christoffel–Darboux formula [33] gives

$$E_n(z,\zeta) = \frac{p_n(z)p_{n-1}(\zeta) - p_{n-1}(z)p_n(\zeta)}{h_n(z-\zeta)}. \tag{9.4}$$

One can find $\Psi_n(z)^t J\Psi_n(\zeta)$ by direct calculation, and compare (9.2) with (9.4).

$\square$

Let $\beta_j(n) = \alpha_j(n) + \nu_j I_2$, which has zero trace. Furthermore, if $\Phi_n$ is a solution of the basic differential equation (4.6), then

$$\frac{d}{dz}\Psi_n(z) = B_n(z)\Psi_n(z) \tag{9.5}$$

where

$$B_n(z) = \sum_{j=1}^{4N-2} \frac{\beta_j(n)}{z-\delta_j}. \tag{9.6}$$

We pause to note an existence result for solutions of the matrix system (9.5).

**Lemma 9.2.** *Suppose that $\beta_j(n)$ has eigenvalues $\pm\kappa_j(n)$ where $2\kappa_j(n)$ is not an integer. Let $C_j$ be the connection matrix associated with $\delta_j$. Then on a neighbourhood of $\delta_j$, there exists an analytic matrix function $\Xi_{n,j}$ such that*

$$\Psi_n(z) = \Xi_{n,j}(z)(z-\delta_j)^{\beta_j(n)}C_j \tag{9.7}$$

*satisfies (9.5).*

**Proof.** This follows from Turrittin's theorem; see [5, 42]. The matrix $C_j$ diagonalizes the monodromy matrix corresponding to a circuit round the pole $\delta_j$; see [24, p. 308].

$\square$

In this section we are concerned with local behaviour of the kernels, and how they operate on single intervals, so we can assume that $C_j = I$ for this particular interval in $S$.

Let $\mathrm{J}_\nu$ be the Bessel function of the first kind of order $\nu$. Tracy and Widom showed that the Bessel kernel, often used to describe eigenvalue distributions near to hard edges, may be expressed as the square of a Hankel operator on $(0,1)$ so that

$$\frac{\sqrt{x}\mathrm{J}_{\nu+1}(\sqrt{x})\mathrm{J}_\nu(\sqrt{y}) - \mathrm{J}_\nu(\sqrt{x})\sqrt{y}\mathrm{J}_{\nu+1}(\sqrt{y})}{x-y} = \frac{1}{2}\int_0^1 \mathrm{J}_\nu(\sqrt{xt})\mathrm{J}_\nu(\sqrt{yt})\,dt; \tag{9.8}$$

see [40]. The following result extends this idea. For notational simplicity, we consider the interval $(\delta_1,\delta_2)$ and assume that $\delta_1 = 0$ and $1 < \delta_2$; the general case follows by scaling and translating. For a continuous function $\phi : (0,1) \to \mathbf{R}^{8N-6}$, the Hankel operator $\Gamma_\phi : L^2((0,1); dy/y; \mathbf{R}) \to L^2((0,1); dy/y; \mathbf{R}^{8N-6})$ is given by

$$\Gamma_\phi f(x) = \int_0^1 \phi(xy)f(y)\frac{dy}{y}. \tag{9.9}$$

Since $\beta_k(n)$ has zero trace, the matrix $(-\delta_k)J\beta_k(n)$ is real and symmetric, hence is congruent to either

$$\sigma_k = \pm\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad \pm\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \quad \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \quad \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}. \tag{9.10}$$

**Definition.** Let the signature matrix of $B_n(z)$ be $\sigma = \mathrm{diagonal}[\sigma_k]_{k=2}^{4N-2}$ in $M_{8N-6}(\mathbf{R})$ be the block diagonal sum of these matrices over the various $k$.

**Theorem 9.3.** *(i) Let $\beta_1(n)$ be as in Lemma 9.2; assume henceforth that $\kappa_1(n) > 0$. Then there exists $Z_n$, a $2\times 1$ real vector solution of (9.5) such that $Z_n(x) \to 0$ as $x \to 0$.*

*(ii) The integral operator on $L^2((0,1); dx/x)$ with kernel*

$$K_n(z,\zeta) = \frac{\sqrt{z\zeta}Z_n(\zeta)^\dagger J Z_n(z)}{z-\zeta} \tag{9.11}$$

*is of trace class; moreover, there exists a vector Hankel operator $\Gamma_{\psi_n} : L^2((0,1); \mathbf{R}; dy/y)$ $\to L^2((0,1); \mathbf{R}^{8N-6}; dy/y)$ such that*

$$K_n = \Gamma_{\psi_n}^\dagger \sigma \Gamma_{\psi_n}. \tag{9.12}$$

*(iii) If $\sigma \geq 0$, then $K_n \geq 0$.*

**Proof.** (i) There exists an invertible constant $2\times 2$ matrix $S_n$ such that

$$S_n z^{\beta_1(n)} S_n^{-1} = \begin{bmatrix} z^{\kappa_1(n)} & 0\\ 0 & z^{-\kappa_1(n)}\end{bmatrix}, \tag{9.13}$$

where $\kappa_1(n) > 0$. Hence by Lemma 9.2, there exists a constant $2\times 1$ matrix $C$ such that $Z_n(z) = \Psi_n(z)C$ is a solution of (9.5), and $Z_n(z) = O(|z|^{\kappa_1(n)})$ as $z \to 0$.

(ii) Hence we can introduce $K_n$ by (9.11), and next we prove that the kernel satisfies

$$\Bigl(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\Bigr)K_n(x,y) = \sum_{k=2}^{4N-2} \frac{-\delta_k\sqrt{xy}}{(x-\delta_k)(y-\delta_k)} Z_n(y)^\dagger J\beta_k(n) Z_n(x). \tag{9.14}$$

First note that by homogeneity $(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})(\sqrt{xy}/(x-y)) = 0$. Since the $\beta_k(n)$ have zero trace, we have $J\beta_k(n) + \beta_k(n)^\dagger J = 0$ and hence the differential equation (9.5) gives

$$\begin{aligned}
&\Bigl(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\Bigr)Z_n(y)^\dagger J Z_n(x)\\
&\quad = Z_n(y)^\dagger B_n(y)^\dagger J Z_n(x) + Z_n(y)^\dagger J B_n(x) Z_n(x)\\
&\quad = \sum_{k=2}^{4N-2} Z_n(y)^\dagger J\beta_k(n) Z_n(x)\Bigl(\frac{x}{x-\delta_k} - \frac{y}{y-\delta_k}\Bigr);
\end{aligned} \tag{9.15}$$

note that the term $k=1$ gives zero contribution. On dividing by $x-y$ and multiplying by $\sqrt{xy}$, we obtain

$$\Big(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\Big)\frac{\sqrt{xy}Z_n(y)^\dagger JZ_n(x)}{x-y}=\sum_{k=2}^{4N-2}\frac{-\delta_k\sqrt{xy}}{(x-\delta_k)(y-\delta_k)}Z_n(y)^\dagger J\beta_k(n)Z_n(x) \tag{9.16}$$

as in (9.14). Noting the shape of the final factor in (9.16), we choose

$$\phi_n(x)=\mathrm{column}\Big[\frac{\sqrt{x}Z_n(x)}{x-\delta_k}\Big]_{k=2,\dots,4N-2} \tag{9.17}$$

which has a $2\times 1$ entry for each endpoint $\delta_k$ of $S$ to the right of 0, and the block diagonal matrix

$$\beta(n)=\mathrm{diagonal}\Big[-\delta_k J\beta_k(n)\Big]_{k=2,\dots,4N-2} \tag{9.18}$$

with $2\times 2$ blocks, and we consider

$$\tilde K_n(x,y)=\int_0^1\phi_n(yz)^\dagger\beta(n)\phi_n(zx)\frac{dz}{z}. \tag{9.19}$$

Next note that since $\kappa_1(n)>0$, we have $\tilde K(x,y)\to 0$ as $x,y\to 0$. Then

$$\begin{aligned}\Big(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\Big)\tilde K_n(x,y)&=\int_0^1\big(y\phi_n'(yz)^\dagger\beta(n)\phi_n(zx)+x\phi_n(yz)^\dagger\beta(n)\phi_n'(xz)\big)\,dz\\&=\phi_n(y)^\dagger\beta(n)\phi_n(x)-\phi_n(0)^\dagger\beta(n)\phi_n(0),\end{aligned} \tag{9.20}$$

and we have $\phi_n(0)=0$, so

$$K_n(x,y)=\tilde K_n(x,y)+\xi(x/y) \tag{9.21}$$

for some function $\xi$. But $Z_n(z)/z^{\kappa_1(n)}$ is analytic on a neighbourhood of 0, so it is clear that $K_n(x,y)\to 0$ and $\tilde K_n(x,y)\to 0$ as $x\to 0$ or $y\to 0$; hence $\xi=0$.

By the choice of $\sigma$, there exists a block diagonal matrix $\gamma(n)$ such that $\gamma(n)^\dagger\sigma\gamma(n)=\beta(n)$, so we can introduce $\psi_n(x)=\gamma(n)\phi_n(x)$ such that $\phi_n(x)^\dagger\beta(n)\phi_n(y)=\psi_n(x)^\dagger\sigma\psi_n(y)$. For this symbol function $\psi_n$ we have

$$K_n(x,y)=\int_0^1\psi_n(yz)^\dagger\sigma\psi_n(zx)\frac{dz}{z}, \tag{9.22}$$

or in terms of Hankel operators $K_n=\Gamma_{\psi_n}^\dagger\sigma\Gamma_{\psi_n}$. We have

$$\int_0^1\Big(\log\frac{1}{u}\Big)\,\|\psi_n(u)\|^2\,\frac{du}{u}<\infty, \tag{9.23}$$

so $\Gamma_\psi$ is Hilbert–Schmidt and hence $K_n$ is of trace class.

(iii) If $\sigma \geq 0$, or equivalently $\sigma_k \geq 0$ for all $k$, then $K_n \geq 0$ by (ii).

□

## 10. The tau function and scattering function for the basic differential equation

In this section, we introduce a matrix scattering function $\Psi$ for express the tau function $\tau$ of $K_n$ from Theorem 9.3. We then establish the correspondence between the scattering function and $\tau$ by an integral equation of Gelfand–Levitan type. The first step is introduce a scattering function $\psi$ and then to realise this by a linear system as in [5, 6].

The differential equation

$$\frac{dZ_n}{dx} = B_n(x)Z_n(x) \tag{10.1}$$

has a solution from which we constructed a symbol function

$$\psi_n(x) = \mathrm{column}\Big[\frac{\sqrt{x}\gamma(n)Z(x)}{x-\delta_k}\Big]_{k=2}^{4N-2}. \tag{10.2}$$

Recalling that $\delta_1 = 0$ and suppressing $n$ for simplicity, we change $x \in (0,1)$ to $t \in (0,\infty)$ by letting $x = e^{-t}$ and in the new variables write

$$\psi(t) = \sum_{\ell=0}^{\infty} \chi_\ell e^{-(\kappa_1+\ell+1/2)t}, \tag{10.3}$$

where $\sum_{\ell=0}^{\infty} \|\chi_\ell\| < \infty$ and $\kappa_1 > 0$. Likewise, we write $\tau(t)$ for $\tau(e^{-t})$.

Let $\Omega = \{z : \Re z > 0\}$ be the open right half-plane, let

$$\Psi(x) = \begin{bmatrix} 0 & \psi(x) \\ \psi(\bar{x})^\dagger & 0 \end{bmatrix} \tag{10.4}$$

and extend $\Psi$ to an analytic function $\Psi : \Omega \to M_{8N-5}(\mathbf{C})$ such that $\Psi(x) = \Psi(x)^\dagger$ for $x > 0$. Let $\Psi_{(s)} = \Psi(x+2s)$ and $\Psi^*_{(s)}(x) = \Psi(x+2\bar{s})^\dagger$ and let $\sigma$ be a constant matrix; then let $K_s = \Gamma_{\Psi^*_{(s)}}\sigma\Gamma_{\Psi_{(s)}}$ be a family of operators on $L^2(0,\infty)$.

**Proposition 10.1.** *(i) The tau function associated with $K = \Gamma_{\Psi^*}\sigma\Gamma_\Psi$ is $\tau(2s) = \det(I - K_s)$, which gives an analytic function on $\Omega$.*

*(ii) Let $q(s) = -2\frac{d^2}{ds^2}\log\tau(2s)$. Then $q(s)$ is meromorphic on $\Omega$, and analytic where $\int_0^\infty x\|\Psi(x+s)\|^2dx < 1$.*

*(iii) If $0 \leq K \leq I$ as an operator, then $\tau(s)$ is non-negative for $0 < s < \infty$, increasing and converges to one as $s \to \infty$.*

**Proof.** (i) The kernel of the Hankel operator $\Gamma_{\Psi_{(s)}}$ has a nuclear expansion

$$\Gamma_{\Psi_{(s)}} \leftrightarrow \sum_{\ell=0}^{\infty} e^{-(\kappa_1+\ell+1/2)(x+y+2s)} \begin{bmatrix} 0 & \chi_\ell \\ \chi_\ell^\dagger & 0 \end{bmatrix} \tag{10.5}$$

where $\sum_{\ell=0}^\infty \|\chi_\ell\| \int_0^\infty e^{-2(\kappa_1+\ell+1/2)(x+\Re s)}dx < \infty$, so the Fredholm determinants are well defined. As in Schwarz's reflection principle, $s \mapsto \Psi^*_{(s)}$ is analytic, and $\Gamma_{\Psi_{(s)}}$ is Hilbert–Schmidt, so $K_s$ is an analytic trace-class valued function on $\Omega$. Using unitary equivalence, one checks that

$$\det(I - K_s) = \det(I - P_{(2s,\infty)}K) \qquad (s > 0). \tag{10.6}$$

(ii) Except on the discrete set of zeros of $\tau(2s)$, the operator $I - K_s$ is invertible and

$$q(s) = 2\frac{d}{ds}\mathrm{trace}\Big((I - K_s)^{-1}\frac{dK_s}{ds}\Big). \tag{10.7}$$

(iii) This follows from (10.6).

□

We wish to express $\tau'/\tau$ as a rational expression in certain infinite matrices with entries from $\mathbf{C}(e^{-t}, e^{-(\kappa_1+1/2)t})$. To do so, we realise $\Psi$ via a linear system suggested by the inverse scattering transform as in [6]. Let $H_0 = \mathbf{C}^{8N-6}$ be the column vectors, $H = \ell^2$ be Hilbert sequence space, written as infinite columns, and introduce an infinite row of column vectors $C \in \ell^2(H_0)$ by $C = (\chi_\ell/\|\chi_\ell\|^{1/2})_{\ell=0}^\infty$ and a column $B \in \ell^2$ by $B = (\|\chi_\ell\|^{1/2})_{\ell=0}^\infty$ and the infinite square matrix $A = \mathrm{diagonal}\,[\ell + \kappa_1 + 1/2]_{\ell=0}^\infty$. Whereas $A$ is real and diagonal, we shall write $A^\dagger$ in some subsequent formulas, so as to emphasize their symmetry.

In the following result we use the $(8N-5)\times(8N-5)$ block matrices

$$W(x,y) = \begin{bmatrix} U(x,y) & v(x,y) \\ w(x,y)^\dagger & z(x,y)\end{bmatrix}, \quad \Psi(x) = \begin{bmatrix} 0 & \psi(x) \\ \psi(\bar x)^\dagger & 0\end{bmatrix}, \tag{10.8}$$

so that $\Psi(\bar x) = \Psi(x)^\dagger$ and the matrix Hamiltonian

$$H(x) = \begin{bmatrix} U(x,x)\sigma & v(x,x) \\ w(x,x)^\dagger\sigma & z(x,x)\end{bmatrix} \tag{10.9}$$

where $v, w \in H_0$, $U$ operates upon $H_0$ and $z$ is a scalar. To simplify the statements of results, we use a special non-associative product $*$, involving $\sigma$, that is defined by

$$\begin{bmatrix} U & v \\ w^\dagger & z\end{bmatrix} * \begin{bmatrix} 0 & \psi \\ \psi^\dagger & 0\end{bmatrix} = \begin{bmatrix} v\psi^\dagger & U\sigma\psi \\ z\psi^\dagger & w^\dagger\sigma\psi\end{bmatrix}. \tag{10.10}$$

**Theorem 10.2.** *(i) The linear system $(-A, B, C)$ realises $\psi(t) = Ce^{-tA}B$ of (10.3).*

*(ii) There exists a solution of the Gelfand–Levitan equation*

$$W(x,y) + \Psi(x+y) + \int_x^\infty W(x,s) * \Psi(s+y)\,ds = 0 \qquad (0 < x < y) \tag{10.11}$$

*such that $\tau$ of Proposition 10.1(i) is holomorphic on $\{t : \Re t > 0\}$ and satisfies*

$$\frac{d}{dx}\log\tau(2x) = \mathrm{trace}\,H(x) \qquad (x > 0). \tag{10.12}$$

*(iii) Suppose moreover that $\int_0^\infty x\|\Psi(x)\|^2 dx < 1$. Then*

$$\Big(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\Big) W(x,y) = -2\frac{dH}{dx}\, W(x,y). \tag{10.13}$$

**Proof.** (i) This identity follows from (10.3). Since $\kappa_1+\ell+1/2>0$, the semigroup $e^{-tA} = \mathrm{diagonal}\,[e^{-t(\kappa_1+\ell+1/2)}]_{\ell=0}^\infty$ consists of trace-class operators, and the integrals in the remainder of the proof are convergent.

(ii) To obtain an expression for the solution, we introduce auxiliary operators $L_x$ and $Q_x^\sigma$, then express $\tau$ as a Fredholm determinant of $L_xQ_x^\sigma$, and finally write $\tau'/\tau$ as a rational expression in the various operators. We introduce the observability Gramian with its matrix representation

$$Q_x^\sigma = \int_x^\infty e^{-sA^\dagger} C^\dagger \sigma C e^{-sA}\, ds \leftrightarrow \Big[\frac{e^{-(\ell+m+2\kappa_1+1)x}\chi_\ell^\dagger\sigma\chi_m}{(\ell+m+2\kappa_1+1)\|\chi_\ell\|^{1/2}\|\chi_m\|^{1/2}}\Big]_{\ell,m=0,1,\dots}, (x>0), \tag{10.14}$$

modified to take account of $\sigma$, and the usual controllability Gramian

$$L_x = \int_x^\infty e^{-sA}BB^\dagger e^{-sA^\dagger}\, ds \leftrightarrow \Big[\frac{e^{-(\ell+m+2\kappa_1+1)x}\|\chi_\ell\|^{1/2}\|\chi_m\|^{1/2}}{\ell+m+2\kappa_1+1}\Big]_{\ell,m=0,1,\dots}, \tag{10.15}$$

both of which define trace-class operators on $\ell^2$, and where $L_x\geq 0$. (The matrix expressions resemble the soliton determinants of [24, (6.24)] and [25, p. 409].) The controllability operator $\Xi_x : L^2((0,\infty);H_0)\to H$ is

$$\Xi_x f = \int_x^\infty e^{-sA}Bf(s)\,ds \tag{10.16}$$

while the observability operator is $\Theta_x : L^2((0,\infty);H_0)\to H$ is

$$\Theta_x f = \int_x^\infty e^{-sA^\dagger}C^\dagger f(s)\,ds. \tag{10.17}$$

Finally, we let $\psi_{(x)}(s)=\psi(s+2x)$, so that $\psi_{(x)}$ is realised by $(-A, e^{-xA}B, Ce^{-xA})$. In terms of these operators, we have the basic identities

$$\Gamma_{\psi_{(x)}} = \Theta_x^\dagger\Xi_x, \qquad \Gamma_{\psi_{(x)}}^\dagger = \Xi_x^\dagger\Theta_x \tag{10.18}$$

while

$$L_x = \Xi_x\Xi_x^\dagger \qquad \text{and} \qquad Q_x^\sigma = \Theta_x\sigma\Theta_x^\dagger. \tag{10.19}$$

Hence we can rearrange the factors in the Fredholm determinants

$$\begin{aligned}\det(I-\lambda\Gamma_{\psi_{(x)}}^\dagger\sigma\Gamma_{\psi_{(x)}}) &= \det(I-\lambda\Xi_x^\dagger\Theta_x\sigma\Theta_x^\dagger\Xi_x)\\ &= \det(I-\lambda\Xi_x\Xi_x^\dagger\Theta_x\sigma\Theta_x^\dagger)\\ &= \det(I-\lambda L_xQ_x^\sigma).\end{aligned} \tag{10.20}$$

We deduce that

$$\begin{aligned}
\log\tau(2x) &= \log\det(I-\Gamma_\psi^\dagger\sigma\Gamma_\psi P_{(2x,\infty)})\\
&= \log\det(I-\sigma\Gamma_\psi P_{(2x,\infty)}\Gamma_\psi^\dagger)\\
&= \log\det(I-\sigma\Gamma_{\psi_{(x)}}\Gamma^\dagger_{\psi_{(x)}})\\
&= \log\det(I-\Gamma^\dagger_{\psi_{(x)}}\sigma\Gamma_{\psi_{(x)}})\\
&= \operatorname{trace}\log(I-L_xQ_x^\sigma), \qquad (10.21)
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{d}{dx}\log\tau(2x) &= \operatorname{trace}\Bigl((I-L_xQ_x^\sigma)^{-1}\bigl(e^{-xA}BB^\dagger e^{-xA^\dagger}Q_x^\sigma + L_xe^{-xA^\dagger}C^\dagger\sigma Ce^{-xA}\bigr)\Bigr)\\
&= B^\dagger e^{-xA^\dagger}Q_x^\sigma(I-L_xQ_x^\sigma)^{-1}e^{-xA}B\\
&\quad + \operatorname{trace}\sigma Ce^{-xA}(I-L_xQ_x^\sigma)^{-1}L_xe^{-xA^\dagger}C^\dagger. \qquad (10.22)
\end{aligned}$$

The integral equation

$$\begin{aligned}
&\begin{bmatrix} U(x,y) & v(x,y)\\ w(x,y)^\dagger & z(x,y)\end{bmatrix} + \begin{bmatrix} 0 & \psi(x+y)\\ \psi(x+y)^\dagger & 0\end{bmatrix}\\
&+\int_x^\infty \begin{bmatrix} U(x,s) & v(x,s)\\ w(x,s)^\dagger & z(x,s)\end{bmatrix} * \begin{bmatrix} 0 & \psi(s+y)\\ \psi(s+y)^\dagger & 0\end{bmatrix} ds = 0 \qquad (10.23)
\end{aligned}$$

reduces to the identities

$$U(x,y) = -\int_x^\infty v(x,s)\psi(s+y)^\dagger\, ds, \qquad (10.24)$$

$$z(x,y) = -\int_x^\infty w(x,s)^\dagger\sigma\psi(s+y)\, ds, \qquad (10.25)$$

and the pair of integral equations

$$v(x,y)+\psi(x+y)-\int_x^\infty\int_x^\infty v(x,t)\psi(t+s)^\dagger\sigma\psi(s+y)\, dsdt = 0 \qquad (10.26)$$

and

$$w(x,y)+\psi(x+y)-\int_x^\infty\int_x^\infty \psi(s+y)\psi(t+s)^\dagger\sigma w(x,t)\, dtds = 0. \qquad (10.27)$$

To solve these integral equations, we let

$$v(x,y) = -Ce^{-xA}(I-L_xQ_x^\sigma)^{-1}e^{-yA}B \qquad (10.28)$$

and

$$w(x,y) = -Ce^{-yA}(I-L_xQ_x^\sigma)^{-1}e^{-xA}B; \qquad (10.29)$$

then by substituting these into (10.24) and (10.25) we obtain the diagonal blocks of the solution $W$, namely

$$U(x,y) = Ce^{-xA}(I - L_x Q_x^\sigma)^{-1} L_x e^{-yA^\dagger} C^\dagger \tag{10.30}$$

and

$$z(x,y) = B^\dagger e^{-yA^\dagger} Q_x^\sigma (I - L_x Q_x^\sigma)^{-1} e^{-xA} B; \tag{10.31}$$

note that these are rational operator expressions in $e^{-xA}, e^{-xA^\dagger}, L_x, Q_x^\sigma, B, B^\dagger, C$ and $C^\dagger$. Hence we can identify the trace of the solution (10.9) as

$$\begin{aligned}\operatorname{trace} H(x) &= \operatorname{trace} \sigma U(x,x) + z(x,x)\\ &= \operatorname{trace} \sigma C e^{-xA}(I - L_x Q_x^\sigma)^{-1} e^{-xA^\dagger} C^\dagger\\ &\qquad + B^\dagger e^{-xA^\dagger} Q_x^\sigma (I - L_x Q_x^\sigma)^{-1} e^{-xA} B\\ &= \frac{d}{dx} \log \tau(2x).\end{aligned} \tag{10.32}$$

(iii) By integrating by parts, we obtain the identity

$$\Big(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\Big) W(x,y) - 2\frac{dH}{dx}\Psi(x+y) + \int_x^\infty \Big(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial s^2}\Big) W(x,s) * \Psi(s+y)\, ds = 0 \tag{10.33}$$

for $0 < x < y$. One can easily verify that the product $*$ and the standard matrix multiplication satisfy $(QW) * \Psi = Q(W * \Psi)$, hence the formula

$$-2\frac{dH}{dx} W(x,y) - 2\frac{dH}{dx}\Psi(x+y) - \int_x^\infty \Big(2\frac{dH}{dx} W(x,s)\Big) * \Psi(s+y)\, ds = 0 \tag{10.34}$$

follows from multiplying (10.33) by $-2\frac{dH}{dx}$, and this shows that both $-2\frac{dH}{dx}W(x,y)$ and $(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2})W(x,y)$ are solutions of the same integral equation (10.11). By uniqueness of solutions, they are equal.

$\square$

While $\tau'/\tau$ would appear to be transcendental over $\mathbf{C}(e^{-t}, e^{-(\kappa_1+1/2)t})$, we can obtain $\tau'/\tau$ as a limit of elements of this field.

**Corollary 10.3.** *(i) There exists a sequence of finite-rank matrices $(A_n)_{n=1}^\infty$, with corresponding tau functions $\tau_n$, such that $\frac{d}{dt}\log\tau_n(2t)$ is a meromorphic function that belongs to $\mathbf{C}(e^{-(\kappa_1+1/2)t}, e^{-t})$ and $\tau_n(2t) \to \tau(2t)$ as $n \to \infty$, uniformly on compact subsets of $\{t : \Re t > 0\}$.*

*(ii) Suppose further that $\kappa_1$ is rational. Then there exists a positive integer $N_1$ such that $\frac{d}{dt}\log\tau(2t)$ is periodic with period $2\pi i N_1$, and $\tau_n(2t)$ is given by elementary functions as in (10.37) below.*

**Proof.** (i) Let $\mathbf{F} = \mathbf{C}(e^{-(\kappa_1+1/2)t}, e^{-t})$ and observe that the entries of the matrices that represent $e^{-tA}$, $L_t$ and $Q_t^\sigma$ all belong to $\mathbf{F}$. We introduce the finite-rank matrices

$$A_n = \mathrm{diagonal}\,[\kappa_1 + 1/2, \kappa_1 + 3/2, \dots, \kappa_1 + n + 1/2, 0, 0, \dots] \tag{10.35}$$

so that $\|e^{-tA} - e^{-tA_n}\|_{c^1} \leq e^{-(\kappa_1+n+1)\Re t}/(1 - e^{-\Re t})$. Similarly, we cut down $B$ to its first $n$ rows $B_n \in M_{n\times 1}(\mathbf{C})$ and $C$ to its first $n$ columns $C_n \in M_{1\times n}(\mathbf{C})$; then we introduce the corresponding $L_{t,n}$ and $Q^\sigma_{t,n}$ by the formulas (10.14) and (10.15), suitably adjusted, then we follow through the proof of Theorem 10.2 to produce the appropriate choice of $W_n(t,t)$ by the prescription of (10.8). By inspecting matrix entries, we see that $L_{t,n}$, $Q^\sigma_{t,n}$ and hence $\det(I - L_{t,n}Q^\sigma_{t,n})$ are entire, and hence $W_n(t,t)$ is meromorphic on $\mathbf{C}$. Likewise, we observe that $\det(I - L_{t,n}Q^\sigma_{t,n}) \in \mathbf{F}$, and hence we can solve the Gelfand–Levitan equation (10.12) with matrices with entries in $\mathbf{F}$. In particular, from (10.32) we obtain $\frac{d}{dt}\log\tau_n(2t) = \mathrm{trace}\, H_n(t)$ in $\mathbf{F}$, where $\tau_n(2t) \to \tau(2t)$ as $n \to \infty$, uniformly on compact subsets of $\{t : \Re t > 0\}$.

(ii) In this case, the set $\{m\kappa_1 + m/2 + n\ell; m, n \in \mathbf{Z}; \ell = 0, 1, 2, \dots\}$ is a finitely generated subgroup of the rationals, and hence has a smallest positive element $M/N_1$, where $M, N_1 \in \mathbf{N}$ with $M < N_1$. Then all the terms $N_1(\kappa_1 + \ell + 1/2)$ are positive integers, so $\exp(-(t + 2\pi N_1 i)A) = \exp(-tA)$ for all $\Re t > 0$, hence $\tau'(2t)/\tau(2t)$ is periodic.

By (i), there exists a rational function $r_n$ such that

$$\frac{d}{dt}\log\tau_n(2t) = r_n(e^{-t/N_1}). \tag{10.36}$$

Suppose for simplicity that $r_n(z)/z$ has only simple poles; then from the partial fractions decomposition, there exist coefficients $\alpha_j, \beta_j$ and $\gamma_j$ and $b_j, c_j$ such that $b_j^2 < c_j$, real poles $a_k$ and a polynomial $q_n(z)$ such that (10.36) integrates to

$$\log\tau_n(2t) = q_n(e^{-t/N_1}) + \sum_k \alpha_k \log|e^{-t/N_1} - a_k| + \sum_j \beta_j \log(e^{-2t/N_1} + 2b_j e^{-t/N_1} + c_j)$$
$$+ \sum_j \frac{\gamma_j}{\sqrt{c_j - b_j^2}} \tan^{-1}\frac{e^{-t/N_1} + b_j}{\sqrt{c_j - b_j^2}}. \tag{10.37}$$

When $r_n(z)/z$ has higher order poles, one likewise obtains expressions for $\log\tau(2t)$ that involve similar, but more complicated, elementary functions.

$\square$

**Remark 10.4.** Let $q_n(t) = -2\frac{d^2}{dt^2}\log\tau_n(2t)$ be as in Corollary 10.3(ii). Then $-d^2f/dt^2 + q_n(it)f = \lambda f$ has the form of a complex Hills equation. The criteria for integrability are considered in [9, 31]. Typical periodic potentials are not of finite gap and may be associated with curves of infinite genus; see [30, 17].

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