On the tau function associated with the generalized unitary ensemble

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Abstract. Let v be a real polynomial of even degree, and let ρ be the equilibrium probability measure for v with support S; so that, $v(x) \geq 2\int \log|x-y|\,\rho(dy) + C_v$ for some constant C_v with equality on S. Then S is the union of finitely many bounded intervals with endpoints δ_j, and ρ is given by an algebraic weight $w(x)$ on S. Then the system of orthogonal polynomials for w gives rise to a system of differential equations, known as the Magnus–Schlesinger equations. This paper identifies the τ function of this system with the Hankel determinant $\det[\int x^{j+k}\rho(dx)]_{j,k=0}^{n-1}$ of ρ. The solutions of the Magnus–Schlesinger equation are realised by a linear system, which is used to compute the tau function in terms of a Gelfand–Levitan equation. The tau function is associated with a potential q and a scattering problem for the Schrödinger operator with potential q. The paper describes cases where this is integrable in terms of the nonlinear Fourier transform.

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1. Introduction

This paper concerns systems of orthogonal polynomials that arise in random matrix theory, specifically in the theory of the generalized unitary ensemble, as in [27].

Let X be a $n\times n$ complex Hermitian matrix, and let $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n)$ be the corresponding eigenvalues, listed according to multiplicity. Let $v(x) = \sum_{j=0}^{2N} a_j x^j$ be a real polynomial such that $a_{2N} > 0$; we then consider the potential $V(X) = n^{-1}\sum_{j=1}^{n} v(\lambda_j)$. Now let dX be the product of Lebesgue measure on the entries that are on or above the leading diagonal of X; then there exists $0 < Z_n < \infty$ such that

$$\nu_n^{(2)}(dX) = Z_n^{-1}\exp\big(-n^2 V(X)\big)\,dX \tag{1.1}$$

defines a probability measure on the $n\times n$ complex Hermitian matrices. There is a natural action of the unitary group $U(n)$ on M_n given by $(U, X) \mapsto UXU^\dagger$, which leaves $\nu_n^{(2)}$ invariant. Hence $\nu_n^{(2)}$ is the generalized unitary ensemble with potential v.

Definition. The integrated density of states ρ_n is the probability measure ρ_n on $\mathbf{R}$ such that

$$\int_{-\infty}^{\infty} f(x)\rho_n(dx) = \frac{1}{n}\int_{M_n} \operatorname{trace} f(X)\nu_n^{(2)}(dX) \tag{1.2}$$

for all continuous and compactly supported real functions f. The equilibrium measure ρ is the probability measure with support S that arises as the weak limit of the ρ_n so

$$\int_S f(x)\rho(dx) = \lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\rho_n(dx). \tag{1.3}$$

Boutet de Monvel *et al* [5] prove the existence of this weak limit under general conditions which include the above v. They prove that there exists a constant C_v such that

$$v(x) \geq 2\int \log|x-y|\,\rho(dy) + C_v \qquad (x\in\mathbf{R}) \tag{1.4}$$

and that equality holds if and only if x belongs to S. Furthermore, there exists $g \geq 0$ and

$$-\infty < \delta_1 < \delta_2 \leq \delta_3 < \ldots < \delta_{2g+2} < \infty \tag{1.5}$$

such that

$$S = \cup_{j=1}^{g+1}[\delta_{2j-1}, \delta_{2j}] \tag{1.6}$$

It is a tricky problem, to find S for a given v, and [9] contains some significant results including the bound $g+1 \leq N+1$ on the number of intervals; see [9, Theorem 1.46 and p. 408]. When v is convex, a relatively simple argument shows that $g = 0$ so there is a single interval [20].

Definition The n^{th} order Hankel determinant for ρ is

$$D_n = \det\Big[\int_S x^{j+k}\rho(dx)\Big]_{j,k=0}^{n-1}. \tag{1.7}$$

We regard D_n as a function of $\delta = (\delta_1, \ldots, \delta_{2g+2})$, and derive a system for differential equations for $\log D_n$, known as the Schlesinger equations. In so doing, we follow the analysis of Chen and Its [7], who considered the ρ that is analogous to the Chebyshev distribution on multiple intervals. They used the Akhiezer polynomials, and likewise we will introduce a system of orthogonal polynomials for the measure ρ.

Let $A(z)$ be a proper rational 2×2 matrix function with simple poles at δ_j; let α_j be the residue at δ_j, and suppose that the eigenvalues of α_j are distinct modulo the integers. Consider the differential equation

$$\frac{d}{dz}\Phi = A(z)\Phi(z), \tag{1.8}$$

and introduce the 1-form

$$\Omega(\delta) = \frac{1}{2}\sum_j \operatorname{trace}\operatorname{Residue}(A(z)^2 : z = \delta_j)\, d\delta_j \tag{1.9}$$

to describe its deformations. Then Ω turns out to be closed by [18], so we look for a function τ such that $d\log\tau = \Omega$. This defines the τ function of the deformation equations associated with (1.8). The purpose of this paper is to show that D_n defines the tau function of a system of differential equations associated with ρ by means of the system of orthogonal polynomials for ρ, as we introduce in section 2.

In section 4, we derive the differential equations explicitly, and in section 5 we conclude the proof that D_n gives the appropriate τ function. As an illustration which is of importance in random matrix theory, we calculate the tau function explicitly when ρ is the semicircular law. When S is the union of two intervals, the Schlesinger equations reduce to the Painlevé VI equation, as we discuss on section 6.

In random matrix theory, tau functions are introduced alongside special kernels that describe the distribution of eigenvalues of random matrices; see [13, 27]. For a sequence of real symmetric 2×2 matrices $J\beta_k(n)$, we consider solutions of the differential equation

$$J\frac{dZ}{dx} = \sum_{k=1}^{2g+2} \frac{J\beta_k(n)}{x-\delta_k} Z, \tag{1.10}$$

$$Z(x) \to 0 \qquad (x \to \delta_j),$$

and form the kernel

$$K(x,y) = \frac{Z(y)^\dagger J Z(x)}{y-x}. \tag{1.11}$$

We show that the properties of K depend crucially upon that sequence of signatures of the matrices

$$(\delta_j - \delta_k) J\beta_k(n) \qquad (k = 1, \dots, 2g+2). \tag{1.12}$$

The tau function is associated with a potential $q(x) = -2\frac{d^2}{dx^2}\log\tau(2x)$ and hence the Schrödinger differential operator $\Delta_q = -\frac{d^2}{dx^2} + q(x)$. In scattering theory, one associates with each smooth and compactly supported real q a scattering function ϕ; then one analyses the spectrum of Δ_q in terms of ϕ, with a view to recovering q. The Gelfand–Levitan integral equation links ϕ with q. The Laplace transform $\hat\phi$ is known as the transfer function Conversely, it is known that any bounded and analytic operator-valued function $\hat\phi$ on the unit circle is th transfer function of some discrete time linear system, and there are versions of the result for analytic operator-valued functions on the right half plane.

In this paper we introduce a vector-valued function ϕ from the a a linear system $(-A, B, C)$ so as to realise $\phi(x) = Ce^{-xA}B$, and use the operators A, B, C to solve the Gelfand–Levitan equation.

In section 8 we realise ϕ from a suitable linear system and introduce a matrix Hamiltonian $H(x)$ such that

$$\tau(2x) = \exp\Bigl(-\int_x^\infty \mathrm{trace}\, H(u)\, du\Bigr), \tag{1.13}$$

and prove that $q(x)$ is meromorphic on a region. We regard Δ as integrable if $\Delta f = \lambda f$ can be solved by quadratures for typical λ. This imposes severe restrictions upon q; indeed, Gelfand, Dikij and Its [11, 6] showed that the integrable cases of the Schrödinger equation arise from finite-dimensional Hamiltonian systems. In sections 9 and 10 we consider cases in which the differential equation $-f'' + qf = \lambda f$ has a meromorphic general solution for all λ, and q satisfies one of the following:

(a) q rational and bounded at infinity;

(b) q elliptic;

(c) $q(x)$ real and one periodic, and such that the corresponding Schrödinger equation has finitely many spectral gaps.

In each case, we introduce an appropriate linear system to realise the corresponding transfer function.

2. The equilibrium measure

Given the special form of the potential, the equilibrium measure and its support satisfy special properties. To describe these, we introduce the polynomial u of degree $2N-2$ by

$$u(z) = \int_S \frac{v'(z) - v'(x)}{z - x} \rho(dx) \tag{2.1}$$

and the Cauchy transform of ρ by

$$R(z) = \int_S \frac{\rho(dx)}{x - z} \qquad (z \in \mathbf{C} \setminus S) \tag{2.2}$$

and the weight

$$w(x) = 2Na_{2N} \sqrt{-Q(x) \prod_{j=1}^{2g+2} (x - \delta_{2j-1})(x - \delta_{2j})} \tag{2.3}$$

where $Q(x)$ is a product of monic irreducible quadratic factors such that $w(x)^2 = 4u(x) - v'(x)^2$.

Proposition 2.1 *(i) The Cauchy transform is the algebraic function that satisfies*

$$R(z)^2 + v'(z)R(z) + u(z) = 0 \tag{2.4}$$

and $R(z) \to 0$ as $z \to \infty$. There exist nonzero polynomials u_0, u_1, u_2 such that $u_0 R' = u_1 R + u_2$.

(ii) The support of ρ is

$$S = \{x \in \mathbf{R} : 4u(x) - v'(x)^2 \geq 0\}. \tag{2.5}$$

(iii) The equilibrium measure is absolutely continuous and the Radon–Nikodym derivative satisfies

$$\frac{d\rho}{dx} = \frac{1}{2\pi}\mathbf{I}_S(x)w(x) \tag{2.6}$$

where $2\pi = \int_S w(t)dt$ and $w(x) \to 0$ as x tends to an endpoint of S.

Proof. (i) The quadratic equation is due to Bessis, Itzykson and Zuber, and is proved in the required form in [29]. One can easily deduce that R satisfies a first order linear differential equation with polynomial coefficients.

(ii) Pastur shows that the support is those real x such that

$$|v'(x) + \sqrt{v'(x)^2 - 4u(x)}|^2 = 4u(x), \tag{2.7}$$

and this condition reduces to $4u(x) \geq v'(x)^2$ and $u(x) \geq 0$, where the former inequality implies the latter. The polynomial $4u(x) - v'(x)^2$ has real zeros $\delta_1, \dots, \delta_{2g+2}$, and may additionally have pairs of complex conjugate roots, which we list as $\delta_{2g+3}, \dots, \delta_{4N-2}$ with regard to multiplicity. Hence we can introduce w as above such that $4u(x) - v'(x)^2 = w(x)^2$.

(iii) From (i) we deduce that

$$R(\lambda) = \frac{1}{2\pi i}\int_S \frac{\sqrt{4u(t) - v'(t)^2}}{t - \lambda}dt \tag{2.8}$$

since both sides are holomorphic on $\mathbf{C} \setminus S$, vanish at infinity and have the same jump across S. By Plemelj's formula, we deduce that

$$v'(\lambda) = 2\text{p.v.}\int_S \frac{\sqrt{4u(t) - v'(t)^2}}{\lambda - t}\frac{dt}{2\pi} \qquad (\lambda \in S). \tag{2.9}$$

See [29, 5]. This gives the required expression for ρ.

$\square$

Hence it is natural to introduce the compact Riemann surface

$$\mathcal{E} = \Big\{(x,w) \in \mathbf{C}^2 : w^2 = (2Na_{2N})^2 Q(x) \prod_{j=1}^{g+1} (x-\delta_{2j-1})(x-\delta_{2j})\Big\} \cup \{\infty\}. \tag{2.10}$$

The algebraic function w has real branch points at the δ_j and extends to define a rational function on $\mathcal{E}$.

3 Orthogonal polynomials

To introduce the differential equations, we introduce the orthogonal polynomials that are associated with ρ. On account of Proposition 2.1, the orthogonal polynomials are semi classical in Magnus's sense [24], although the weight typically live on several intervals.

Let $(p_j)_{j=0}^\infty$ be the sequence of real monic orthogonal polynomials, where p_j has degree j and let h_j be the constants such that

$$\int_S p_j(x)p_k(x)\rho(dx) = h_j\delta_{jk}; \tag{3.1}$$

and let $(q_j)_{j=1}^\infty$ be the monic polynomials of the second kind, where

$$q_j(z) = \int_S \frac{p_j(z)-p_j(x)}{z-x}\rho(dx) \tag{3.2}$$

has degree $j-1$. The following result is standard in the theory of orthogonal polynomials.

Lemma 3.1. *Let $c_n = h_n/h_{n-1}$ and $b_n = h_n^{-1}\int_S xp_n(x)^2\rho(dx)$. Then*

(i) the polynomials $(p_n)_{n=0}^\infty$ satisfy the recurrence relation

$$xp_n(x) = p_{n+1}(x) + b_{n+1}p_n(x) + c_np_{n-1}(x); \tag{3.3}$$

(ii) the polynomials $(q_j)_{j=1}^\infty$ likewise satisfy (3.3);

(iii) the Hankel determinant satisfies

$$D_n = h_0h_1\dots h_{n-1}. \tag{3.4}$$

We introduce also

$$Y_n(z) = \begin{bmatrix} p_n(z) & \int_S \frac{p_n(t)\rho(dt)}{z-t} \\ \frac{p_{n-1}(z)}{h_{n-1}} & \frac{1}{h_{n-1}}\int_S \frac{p_{n-1}(t)\rho(dt)}{z-t} \end{bmatrix} \tag{3.5}$$

and

$$V_n(z) = \begin{bmatrix} z - b_{n+1} & -h_n \\ 1/h_n & 0 \end{bmatrix}. \tag{3.6}$$

Proposition 3.2 *(i) The matrices satisfy the recurrence relation*

$$Y_{n+1}(z) = V_n(z) Y_n(z). \tag{3.7}$$

(ii) The matrix $Y_n(z)$ is invertible, and $\det Y_n(z) = 1$.

Proof. (i) This follows from (i) and (ii) of the Lemma 3.1.

(ii) This follows by induction, where the induction step follows from the recurrence relation in (i).

□

Let

$$\mu_j(t) = \int_{S \cap (-\infty, t)} x^j \rho(dx) \tag{3.8}$$

be the j^{th} moment of ρ restricted to $(-\infty, t) \cap S$; the corresponding Hankel determinant is

$$D_{n+1}(t) = \det[\mu_{j+k}(t)]_{j,k=0}^{n+1}. \tag{3.9}$$

Let $E_n : L^2(\rho) \to \operatorname{span}\{x^k : k = 0, \dots, n\}$ be the orthogonal projection; we also introduce the projection $P_{(t,\infty)}$ on $L^2(\rho)$ given by multiplication $f \mapsto \mathbf{I}_{(t,\infty)} f$.

Proposition 3.3. *The tau function satisfies*

$$\det(I - E_n P_{(t,\infty)}) = \frac{D_{n+1}(t)}{D_{n+1}}. \tag{3.10}$$

Proof. We introduce an upper triangular matrix $[a_{\ell,j}]_{j,\ell=0}^n$ with ones on the leading diagonal such that $p_j(x) = \sum_{\ell=0}^n a_{\ell j} x^\ell$. Then we can compute

$$\begin{aligned} \det[\mu_{j+k}(t)]_{j,k=0}^n &= \det[a_{\ell,j}]_{\ell,j=0}^n \det[\mu_{j+k}(t)]_{j,k=0}^n \det[a_{k,m}]_{k,m=0}^m \\ &= \det\Big[\int_{-\infty}^t p_j(x) p_k(x) \rho(dx)\Big]_{j,k=0}^n \end{aligned} \tag{3.11}$$

We can also express the operators on $L^2(\rho)$ as matrices with respect to the orthonormal basis $(p_j/\sqrt{h_j})_{j=0}^n$, and we find

$$E_n - E_n P_{(t,\infty)} E_n \leftrightarrow \Big[\frac{1}{\sqrt{h_j h_k}} \int_{-\infty}^{\dagger} p_j(z) p_k(z) \rho(dz)\Big]_{j,k=0}^n \tag{3.12}$$

so that

$$\det\Big[\int_{-\infty}^{\dagger} p_j(x)p_k(x)\rho(dx)\Big]_{j,k=0}^{n} = \det(E_n - E_nP_{(t,\infty)}E_n)h_0\dots h_n. \tag{3.13}$$

We deduce that

$$\det[\mu_{j+k}(t)]_{j,k=0}^{n} = \det(E_n - E_nP_{(t,\infty)}E_n)D_n, \tag{3.14}$$

hence the result.

□

4 Schlesinger's equations and three Lax pairs

We introduce the matrix function

$$A_n(z) = Y_n'(z)Y_n(z)^{-1} + Y_n(z)\begin{bmatrix} 0 & 0 \\ 0 & -w'(z)/w(z) \end{bmatrix} Y_n(z)^{-1}. \tag{4.1}$$

The basic properties of $A_n(z)$ are stated in (i) of the following Lemma, while (ii) gives detailed information that we need in a the subsequent proof of Theorem 5.1.

Lemma 4.1 *(i) Let $v'(z)^2 - 4u(z)$ have zeros at δ_j for $j = 1, \dots, 4N-2$. Then $A_n(z)$ is a proper rational function so that*

$$A_n(z) = \sum_{j=1}^{4N-2} \frac{\alpha_j(n)}{z - \delta_j}, \tag{4.2}$$

where the residue matrices $\alpha_j(n)$ depend implicitly upon δ.

(ii) The $(1,2)$ and diagonal entries of the residue matrices satisfy

$$\sum_{k=1}^{4N-2} \alpha_k(n)_{12} = 0; \tag{4.3}$$

$$\sum_{k=1}^{4N-2} \big(\alpha_k(n)_{11} - \alpha_k(n)_{22}\big) = 2(n+N) - 1; \tag{4.4}$$

$$\sum_{k=1}^{4N-2} \delta_k \alpha_k(n)_{12} = -2h_n(n+N). \tag{4.5}$$

Proof. The defining equation for $A_n(z)$ may be written more explicitly as

$$\begin{bmatrix} p_n'(z) & -\int_S \frac{p_n(t)w(t)dt}{(z-t)^2} \\ \frac{p_{n-1}'(z)}{h_{n-1}} & -\frac{1}{h_{n-1}}\int_S \frac{p_{n-1}(t)w(t)dt}{(z-t)^2} \end{bmatrix}$$

$$= A_n(z)\begin{bmatrix} p_n(z) & \int_S \frac{p_n(t)w(t)dt}{z-t} \\ \frac{p_{n-1}(z)}{h_{n-1}} & \frac{1}{h_{n-1}}\int_S \frac{p_{n-1}(t)w(t)dt}{z-t}\end{bmatrix} + \begin{bmatrix} 0 & \frac{w'(z)}{w(z)}\int_S \frac{p_{n-1}(t)w(t)dt}{z-t} \\ 0 & \frac{w'(z)}{h_{n-1}w(z)}\int_S \frac{p_{n-1}(t)w(t)dt}{z-t}\end{bmatrix}. \tag{4.6}$$

By considering the entries, we see that $A_n(z)$ is a proper rational function with possible simple poles at the δ_j, as in (3.2). Hence we have a Laurent expansion

$$A_n(z) = \frac{1}{z}\sum_{k=1}^{4N-2}\alpha_k(n) + \frac{1}{z^2}\sum_{k=1}^{4N-2}\delta_k\alpha_k(n) + O\Big(\frac{1}{z^3}\Big) \qquad (z\to\infty). \tag{4.7}$$

First we compute the $(1,2)$ entry of $A_n(z)$, namely

$$\begin{aligned} A_n(z)_{12} &= -p_n'(z)\int_S \frac{p_n(t)w(t)dt}{z-t} - p_n(z)\int_S \frac{p_n(t)w(t)dt}{(z-t)^2} - \frac{w'(z)}{w(z)}p_n(z)\int_S \frac{p_n(t)w(t)dt}{z-t} \\ &= -\frac{n}{z^2}\int_S t^n p_n(t)w(t)dt - \frac{(n+1)p_n(z)}{z^{n+2}}\int_S t^n p_n(t)w(t)\,dt \\ &\quad - \frac{w'(z)}{w(z)}\frac{p_n(z)}{z^{n+1}}\int_S t^n p_n(t)w(t)dt + O\Big(\frac{1}{z^3}\Big) \end{aligned} \tag{4.8}$$

and we can reduce these terms to

$$A_n(z)_{12} = \frac{-h_n n}{z^2} - \frac{h_n(n+1)}{z^2} - \frac{h_n(2N-1)}{z^2} + O\Big(\frac{1}{z^3}\Big), \tag{4.9}$$

which gives (4.3) and (4.5).

Next, the $(2,2)$ entry of $A_n(z)$ is

$$\begin{aligned} A_n(z)_{22} &= -\frac{p_{n-1}'(z)}{h_{n-1}}\int_S \frac{p_n(t)w(t)dt}{z-t} - \frac{p_n(z)}{h_{n-1}}\int_S \frac{p_{n-1}(t)w()dt}{(z-t)^2} \\ &\quad - \frac{w'(z)}{w(z)}\frac{p_n(z)}{h_{n-1}}\int_S \frac{p_{n-1}(t)w(t)dt}{z-t} \\ &= -\frac{(n-1)z^{n-2}}{h_{n-1}z^{n+1}}\int_S t^n p_n(t)w(t)dt - \frac{p_n(z)}{h_{n-1}z^{n+1}}\int_S nt^{n-1}p_{n-1}(t)w(t)dt \\ &\quad - \frac{2N-1}{z}\frac{p_n(z)}{z^n}\frac{1}{h_{n-1}}\int_S t^{n-1}p_{n-1}(t)w(t)dt + O(1/z^2) \\ &= \frac{1-n-2N}{z} + O(1/z^2) \end{aligned} \tag{4.10}$$

Similarly, the $(1,1)$ entry is

$$\begin{aligned} A_n(z)_{11} &= \frac{p_n'(z)}{h_{n-1}}\int_S \frac{p_{n-1}(t)w(t)dt}{z-t} + \frac{p_{n-1}(z)}{h_{n-1}}\int_S \frac{p_n(t)w(t)dt}{(z-t)^2} \\ &\quad + \frac{w'(z)}{w(z)}\frac{p_{n-1}(z)}{h_{n-1}}\int_S \frac{p_n(t)w(t)dt}{z-t} \end{aligned}$$

$$
\begin{aligned}
&= \frac{p_n'(z)}{h_{n-1}z^n}\int_S t^{n-1}p_{n-1}(t)w(t)dt + \frac{(n+1)p_{n-1}(z)}{h_{n-1}z^{n+2}}\int_S t^n p_n(t)w(t)dt \\
&\quad + \frac{w'(z)}{w(z)}\frac{p_{n-1}(z)}{h_{n-1}z^{n+1}}\int_S t^n p_n(t)w(t)dt + O\Big(\frac{1}{z^2}\Big) \\
&= \frac{n}{z} + O\Big(\frac{1}{z^2}\Big) \qquad (z \to \infty). \qquad (4.11)
\end{aligned}
$$

By comparing the coefficients of $1/z$ in (4.7) with (4.9), (4.10) and (4.11), we obtain

$$
\sum_{k=1}^{4N-2} \alpha_k(n) = \begin{bmatrix} n & 0 \\ 0 & 1-n-2N \end{bmatrix}, \qquad (4.12)
$$

which leads to (4.4).

□

Let

$$
\Phi_n(z) = \begin{bmatrix} \sqrt{2\pi i}\,p_n(z) & -\frac{i\pi w(z)p_n(z)+q_n(z)}{w(z)\sqrt{2\pi i}} \\ \frac{\sqrt{2\pi i}\,p_{n-1}(z)}{h_{n-1}} & -\frac{i\pi w(z)p_{n-1}(z)+q_{n-1}(z)}{w(z)h_{n-1}\sqrt{2\pi i}} \end{bmatrix}, \qquad (4.13)
$$

which is a matrix function with entries in $\mathbf{C}(z)[w]$; note that Φ_n also depends upon the δ_j.

Lemma 4.2 *The functions Φ_n satisfy*

(i) the basic differential equation

$$
\frac{d\Phi_n(z)}{dz} = A_n(z)\Phi_n(z); \qquad (4.14)
$$

(ii) the deformation equation

$$
\frac{\partial \Phi_n}{\partial \delta_j} = -\frac{\alpha_j(n)}{z-\delta_j}\Phi_n(z); \qquad (4.15)
$$

(iii) the recurrence relation $\Phi_{n+1}(z) = V_n(z)\Phi_n(z)$;

(iv) and Φ_n is invertible since $\det \Phi_n(z) = 1/w(z)$.

Proof. (i) We can write

$$
\Phi_n(z) = Y_n(z)\begin{bmatrix} \sqrt{2\pi i} & 0 \\ 0 & 1/(w(z)\sqrt{2\pi i}) \end{bmatrix}, \qquad (4.16)
$$

and then the property (i) follows from (3.1).

(ii) This follows from (i) by standard results in the theory of Fuchsian differential equations as in [12, 15].

(iii) The recurrence relation from Proposition 3.2(i).

(iv) Given (iii), this identity follows from Proposition 3.2(ii).

□

Lemma 4.2 states several properties that the Φ_n satisfy simultaneously, and hence gives several consistency conditions. By taking (i), (ii) and (iii) pairwise, we obtain three Lax pairs, which we state in the following three propositions.

Proposition 4.3 *The residue matrices satisfy Schlesinger's equations*

$$\frac{\partial \alpha_k(n)}{\partial \delta_j} = \frac{[\alpha_j(n), \alpha_k(n)]}{\delta_j - \delta_k} \qquad (j \neq k) \tag{4.17}$$

and

$$\frac{\partial \alpha_j(n)}{\partial \delta_j} = -\sum_{k=1; j\neq k}^{4N-2} \frac{[\alpha_j(n), \alpha_k(n)]}{\delta_j - \delta_k}. \tag{4.18}$$

Proof. We can express the consistency condition $\frac{\partial^2 \Phi_n(z)}{\partial \delta_j \partial z} = \frac{\partial^2 \Phi_n(z)}{\partial z \partial \delta_j}$ as the Lax pair

$$\frac{\partial A_n(z)}{\partial \delta_j} - A_n(z)\frac{\alpha_j(n)}{z - \delta_j} = \frac{\alpha_j(n)}{(z-\delta_j)^2} - \frac{\alpha_j(n)A_n(z)}{z - \delta_j} \tag{4.19}$$

and then one can simplify the resulting system of differential equations.

□

Proposition 4.4 *The basic differential equation (4.13) and the recurrence relation (4.15) are consistent, so*

$$A_{n+1}(z)V_n(z) - V_n(z)A_n(z) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.20}$$

Proof. This is the Lax pair associated with the condition

$$A_{n+1}(z)\Phi_{n+1}(z) = \frac{d}{dz}\Phi_{n+1}(z) = \frac{d}{dz}\Big(V_n(z)\Phi_n(z)\Big). \tag{4.21}$$

Proposition 4.5 *(i) The deformation equation (4.14) and the recurrence relation (4.15) are consistent, so*

$$-\frac{\alpha_j(n+1)}{z-\delta_j}V_n(z) + V_n(z)\frac{\alpha_j(n)}{z-\delta_j} = \frac{\partial V_n(z)}{\partial \delta_j}. \tag{4.22}$$

(ii) In particular, the $(1,2)$ *entry satisfies*

$$\frac{\partial}{\partial \delta_j}\log h_n = -h_n^{-1}\alpha_j(n)_{12} \tag{4.23}$$

$\square$

Proof. (i) This is the Lax pair associated with (4.13) and (4.15).

(ii) By letting $z \to \infty$ in (4.24), we deduce

$$-\alpha_j(n+1)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\alpha_j(n) = \begin{bmatrix}-\frac{\partial b_{n+1}}{\partial \delta_j} & -\frac{\partial h_n}{\partial \delta_j}\\ -\frac{1}{h_n^2}\frac{\partial h_n}{\partial \delta_j} & 0\end{bmatrix} \tag{4.24}$$

which implies that $\alpha_j(n)_{12} = -\frac{\partial h_n}{\partial \delta_j}$.

$\square$

Remark. The Chebyshev distribution $(1/\pi)(1-x^2)^{-1/2}$ is the equilibrium distribution on $[-1,1]$ in the absence of a potential. Hence it is unsurprising that our results reduce to those of Chen and Its [7] when we let $N=0$, so that v becomes a constant. In this case, the orthogonal polynomials are the Chebyshev polynomials of the first kind.

5. The tau function

We introduce the differential 1-form on $\mathbf{C}^{4N-2} \setminus \{\text{diagonals}\}$ by

$$\Omega_n = \sum_{j,k=1;j\neq k}^{4N-2} \mathrm{trace}\Big(\frac{\alpha_j(n)\alpha_k(n)}{\delta_j - \delta_k}\Big) d\delta_j. \tag{5.1}$$

Theorem 5.1 *The Hankel determinant D_n gives the tau function, so*

$$\Omega_n = d\log D_n. \tag{5.2}$$

Proof. By Proposition 4.3 and results of Jimbo *et al* [19], the differential form is exact, so $d\Omega_n = 0$; hence there exists a function τ_n such that $d\log\tau_n = \Omega_n$. We proceed to identify this function. By Lemma 3.1(iii), we have $\log h_n = \log D_{n+1}/D_n$, so we consider

$$\Omega_{n+1} - \Omega_n = \sum_{j\neq k:j,k=1}^{4N-2} \mathrm{trace}\Big(\frac{\alpha_j(n+1)\alpha_k(n+1) - \alpha_j(n)\alpha_k(n)}{\delta_j - \delta_k}\Big) d\delta_j \tag{5.3}$$

where by Proposition $\alpha_j(n+1) = V_n(\delta_j)\alpha_j(n)V_n(\delta_j)^{-1}$ so

$$\begin{aligned}&\mathrm{trace}\big(\alpha_j(n+1)\alpha_k(n+1) - \alpha_j(n)\alpha_k(n)\big)\\ &\quad = \mathrm{trace}\big(\alpha_j(n)V_n(\delta_j)^{-1}V_n(\delta_k)\alpha_k(n)V_n(\delta_k)_n^{-1}(\delta_j) - \alpha_j(n)\alpha_k(n)\big).\end{aligned} \tag{5.4}$$

We have

$$V_n(\delta_j)^{-1}V_n(\delta_k) = \begin{bmatrix} 1 & 0 \\ \frac{\delta_j-\delta_k}{h_n} & 1 \end{bmatrix} \tag{5.5}$$

so by direct calculation

$$\begin{aligned}\Omega_{n+1}-\Omega_n = \sum_{j\neq k:j,k=1}^{4N-2} \Big(& h_n^{-1}\alpha_j(n)_{12}\big(\alpha_k(n)_{11}-\alpha_k(n)_{22}\big) \\ & + h_n^{-1}\alpha_k(n)_{12}\big(\alpha_j(n)_{22}-\alpha_j(n)_{11}\big) \\ & - h_n^{-2}(\delta_j-\delta_k)\alpha_j(n)_{12}\alpha_k(n)_{12}\Big)d\delta_j \end{aligned} \tag{5.6}$$

In this sum we have taken $j \neq k$, but the expression is unchanged if we include the corresponding terms for $j = k$; hence the coefficient of $d\delta_j$ is

$$\begin{aligned} & \alpha_j(n)_{12}\sum_{k=1}^{4N-2} h_n^{-1}(\alpha_k(n)_{11}-\alpha_k(n)_{22}) \\ & \quad -(\alpha_j(n)_{11}-\alpha_j(n)_{22})\sum_{k=1}^{4N-2} h_n\alpha_k(n)_{12} \\ & \quad -\frac{\delta_j\alpha_j(n)_{12}}{h_n^2}\sum_{k=1}^{4N-2}\alpha_k(n)_{12}+\frac{\alpha_j(n)_{12}}{h_n^2}\sum_{k=1}^{4N-2}\delta_k\alpha_k(n)_{12}. \end{aligned} \tag{5.7}$$

We use Lemma 4.1 to reduce this to $-h_n^{-1}\alpha_j(n)_{12}$, so

$$\begin{aligned}\Omega_{n+1}-\Omega_n &= -\sum_{j=1}^{4N-2} h_n^{-1}\alpha_j(n)_{12}d\delta_j \\ &= \sum_{j=1}^{4N-2}\frac{\partial}{\partial\delta_j}\log h_n d\delta_j. \end{aligned} \tag{5.8}$$

□

Following [18], we interpret (5.1) in terms of integrable systems and Hamiltonian mechanics. Let $M = M_2(\mathbf{R})^{4N-2}$ be the product space of matrices, and let $G = GL_2(\mathbf{R})$ act on M by conjugating each matrix in the list

$$(X_1,\dots,X_n)\mapsto(UX_1U^{-1},\dots,UX_nU^{-1}).$$

The Lie algebra g of G has dual g*, and for each $\xi\in$ g* the symplectic structure at ξ on g $\times$ g is given by $\omega_\xi(X,Y)=\xi([X,Y])$. Given

$$A(z)=\sum_{k=1}^{2N-2}\frac{\alpha_k}{z-\delta_k} \tag{5.9}$$

as in (4.2), we introduce

$$\omega(X,Y)=\sum_{k=1}^{2N-2}\mathrm{trace}\big(\alpha_k[X_k,Y_k]\big) \tag{5.10}$$

for $X=(X_k)_{k=1}^{2N-2}$ and $Y=(Y_k)_{k=1}^{2N-2}$ in g^{2N-2}. Given $f,g:M\to\mathbf{C}$, their Poisson bracket is $\{f,g\}=X_f(g)$, and the corresponding vector field satisfies

$$X_{\{f,g\}}=[X_f,X_g]. \tag{5.11}$$

The spectral curve of $A(z)$ is the algebraic variety

$$\Sigma_A=\Big\{(z,w)\in\mathbf{C}^2:\det\big(wI-A(z)\big)=0\Big\}. \tag{5.12}$$

As suggested by (5.1), we introduce the Hamiltonian $H_j:M\to\mathbf{C}$ by

$$H_j=\sum_{k:k\neq j}\mathrm{trace}\Big(\frac{\alpha_j(n)\alpha_k(n)}{\delta_j-\delta_k}\Big) \tag{5.13}$$

so that $\frac{\partial}{\partial\delta_j}\log\tau(\delta)=H_j$. We observe that H_j is a polynomial in the entries of $\alpha_j(n)$ and $\alpha_k(n)$, and is a rational function of δ_j and δ_k. To lighten the notation, we temporarily suppress the variable n.

Theorem 5.2. *(i) The Hamiltonian H_j gives a vector field $(X^{(k)}_{H_j})_{k=1}^{2N-2}$ which is associated with the differential equation*

$$\frac{dA}{dt}=\Big[A,\frac{\alpha_j}{z-\delta_j}\Big]. \tag{5.14}$$

(ii) The Poisson brackets of the flows commute, so that $\{H_j,H_k\}=0$.

(iii) Under this flow, the spectral curve of A is invariant.

Proof. (i) For each $Y=(Y_k)_{k=1}^{2N-2}$, we introduce a flow on M by $\dot\alpha_k=[Y_k,\alpha_k]$. We can differentiate H_j in the direction of Y and obtain

$$Y(H_j)=\sum_{k:k\neq j}\frac{\mathrm{trace}\left([Y_k,\alpha_k]\alpha_j\right)}{\delta_j-\delta_k}+\frac{\mathrm{trace}\left(\alpha_k[Y_k,\alpha_j]\right)}{\delta_j-\delta_k} \tag{5.15}$$

With H_k, we associate the Hamiltonian vector field $X_{H_j}=(X^{(k)}_{H_j})_{k=1}^{2N-2}$ such that

$$Y(H_j)=\omega(X_{H_j},Y)=\sum_{k=1}^{2N-2}\mathrm{trace}\big(\alpha_k[(X^{(k)}_{H_j},Y_k]\big). \tag{5.16}$$

We deduce that

$$X^{(k)}_{H_j} = \frac{\alpha_j}{\delta_j - \delta_k} \qquad (k \neq j) \tag{5.17}$$

$$X^{(j)}_{H_j} = \sum_{k:k\neq j} \frac{\alpha_k}{\delta_j - \delta_k}. \tag{5.18}$$

It is then a simple calculation to check that $\dot{\alpha}_k = [X^{(k)}_{H_j}, \alpha_k]$ extends to give () for $(d/dt)A(z)$.

(ii) Given the vector fields $(X^{(j)}_{H_k})_j$ corresponding to H_k and $(X^{(j)}_{H_\ell})_j$ corresponding to H_ℓ from (), one can compute

$$\{H_k, H_\ell\} = \sum_j \mathrm{trace}\big([X^{(j)}_{H_k}, A_j] X^{(j)}_{H_\ell}\big) \tag{5.19}$$

and reduce the expression to zero by an elementary calculation.

(iii) One can check that for each positive integer m, the $\frac{d}{dt}\mathrm{trace}A(z)^m = 0$, and hence $\det(wI - A(z))$ is invariant under the flow.

The spectral curve of $A(z)$ is Σ_A, and the eigenspace $\{\xi : A(z)^t\xi = w\xi\}$ gives a line bundle on Σ_A. Since $A(z)$ is rational on $\mathbf{P}^1$, the Riemann surface Σ_A is algebraic and hence compact and of finite genus g. Associated with Σ_A, there is a Jacobian J^g and a complex torus $\mathbf{T} = J^g/\mathbf{Z}^{2g}$ for the lattice $\mathbf{Z}^{2g}$. By Theorem 5.2, each H_j gives a constant vector field on $\mathbf{T}$ and the associated flow in $\mathbf{T}$ is linear.

Example 5.3. *(The tau function for the semicircular distribution.)* For $a < b$, let

$$v(z) = \frac{8}{(b-a)^2}\Big(z - \frac{a+b}{2}\Big)^2, \tag{5.20}$$

so, by standard results used in random matrix theory [27], the equilibrium measure is the semicircular law

$$\rho(dx) = \frac{8}{\pi(b-a)^2}\sqrt{(b-x)(x-a)}\,\mathbf{I}_{[a,b]}(x)dx \tag{5.21}$$

on $[a, b]$. Let U_n be the Chebyshev polynomial of the second kind of degree n, which satisfies

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \tag{5.22}$$

and let

$$p_n(x) = 2^{-2n}(b-a)^n U_n\Big(\frac{2x-(a+b)}{b-a}\Big) \tag{5.23}$$

which is monic and of degree n, and the p_n are orthogonal with respect to the measure ρ. By elementary calculations involving trigonometric functions, one can show that $h_n = 2^{-4n}(b-a)^{2n}$ and

$$A_n(x) = \frac{1}{(x-b)(x-a)}\begin{bmatrix} n(x-(a+b)/2) & -(n+1)(b-a)^2h_{n-1}/8 \\ n(b-a)^2/(2h_{n-1}) & -(n+1)(x-(a+b)/2)\end{bmatrix}, \tag{5.24}$$

which has poles at a and b, as expected. One verifies that

$$\Omega_n = \Big(\frac{n^2+(n+1)^2}{4}\Big)\frac{da-db}{a-b} + \frac{n(n+2)}{16}(a-b)(da-db) \tag{5.25}$$

so that

$$\tau = (a-b)^{(2n^2+2n+1)/4}e^{n(n+2)(a-b)^2/32}. \tag{5.26}$$

6. Painlevé equations for pairs of intervals

Akhiezer considered a generalization of the Chebyshev polynomials to the pair of intervals $[-1,\alpha]\cup[\beta,1]$, and investigated their properties by conformal mapping. Chen and Lawrence [8] used the theory of elliptic functions to investigate these polynomials. In this section we obtain the differential equation in the where S is two intervals, and obtain a differential equation for the endpoints that is related to the one from [8].

Let v be a polynomial of degree $2N\geq 4$ such that $S=[\delta_1,\delta_2]\cup[\delta_3,\delta_4]$. There exists a Möbius transformation φ such that $\varphi(\delta_1)=0$, $\varphi(\delta_2)=1$ and $\varphi(\delta_4)=\infty$; then we let $t=\varphi(\delta_3)$. Having fixed three of the endpoints, we can introduce the differential equations from section 4 that describe the effect of varying the endpoint t, namely

$$\frac{d}{dx}\Phi(x) = \Big(\frac{\alpha_0}{x}+\frac{\alpha_1}{x-1}+\frac{\alpha_t}{x-t}\Big)\Phi \tag{6.1}$$

and

$$\frac{\partial\Phi}{\partial t} = \frac{-\alpha_t}{x-t}\Phi. \tag{6.2}$$

Let $A(x,t)$ be the matrix $(\alpha_0/x+\alpha_1/(x-1)+\alpha_t/(x-t))$ and let $A(x,t)_{12}$ be its top right entry. Then we introduce $x=\lambda(t)$ such that $A(x,t)_{12}=0$; then by [17, p. 1333], the corresponding Schlesinger equations give a version of the nonlinear Painlevé equation P_{VI} in terms of λ, namely

$$\begin{aligned}\frac{d^2\lambda}{dt^2}&+\Big(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t}\Big)\frac{d\lambda}{dt}-\frac{1}{2}\Big(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t}\Big)\Big(\frac{d\lambda}{dt}\Big)^2\\ &=\frac{1}{2}\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\Big(k_\infty-\frac{k_0t}{\lambda^2}+\frac{k_1(t-1)}{(\lambda-1)^2}-\frac{(k_t-1)t(t-1)}{(\lambda-t)^2}\Big).\end{aligned} \tag{6.3}$$

Having transformed S to $[0,1]\cup[t,\infty]$ we can lift this to the portions of the real axis that are covered by the elliptic curve $\mathcal{E}=\{(\lambda,w): w^2=4\lambda(\lambda-1)(\lambda-t)\}$; hence we transform to the dependent variables

$$u=\int_0^\lambda \frac{ds}{\sqrt{s(s-1)(s-t)}}, \tag{6.4}$$

where $\lambda=\mathcal{P}(u/2)$ and $w=\mathcal{P}'(u/2)$ gives a point on $\mathcal{E}$ in terms of Weierstrass's function $\mathcal{P}$ with $e_1=t$, $e_2=1$ and $e_3=0$. Using the substitution $u=u(\lambda(t),t)$, Fuchs [15] showed how P_{VI} reduces in some cases to the linear Legendre equation.

The tau function satisfies

$$\frac{d}{dt}\log\tau=\mathrm{trace}\Big(\frac{\alpha_0\alpha_t}{t}+\frac{\alpha_1\alpha_t}{t-1}\Big). \tag{6.5}$$

7. Kernels associated with Schlesinger's equations

Let J_α be Bessel's function of the first kind of order α. Then the kernel

$$F_\alpha(x,y)=\frac{\mathrm{J}_\alpha(\sqrt{x})\sqrt{y}\mathrm{J}'_\alpha(\sqrt{y})-\sqrt{x}\mathrm{J}'_\alpha(\sqrt{x})\mathrm{J}_\alpha(\sqrt{y})}{2(x-y)}$$

is known as the hard edge kernel for $(0,1)$. Using simple estimates on the series, one can show that if $\gamma+\alpha/2>3/4$. then $(xy)^\gamma F_\alpha(x,y)\to 0$ as $x,y\to 0$. series The system of differential equations

$$x\frac{d}{dx}\begin{bmatrix} f\\ g\end{bmatrix}=\begin{bmatrix}0 & x\\ -1/4 & -(\alpha+1)\end{bmatrix}\begin{bmatrix} f\\ g\end{bmatrix}$$

has solution $f(x)=x^{-\alpha/2}\mathrm{J}_\alpha(\sqrt{x})$ and $g(x)=f'(x)$. Tracy and Widom show that the Hankel integral operator

$$T_\alpha f(x)=\int_0^1\sqrt{xy}\mathrm{J}_\alpha(\sqrt{xy})f(y)\frac{dy}{y}$$

is closely related to $T_\alpha^2=F_\alpha$. This kernel has universal properties which apply to the edge distributions of eigenvalues from the modified Jacobi ensemble.

In this section we obtain analogous results for the basic differential equation (4.7). First we let $\nu_j=-2^{-1}\mathrm{trace}\,\alpha_j(n)$ and observe that ν_j does not depend upon n. Indeed, by multiplying (4.22) by V_n^{-1}, one deduces that $\mathrm{trace}\,A_{n+1}(z)=\mathrm{trace}\,A_n(z)$, and since $\mathrm{trace}\,\alpha_j(n)=\lim_{z\to\delta_j}(z-\delta_j)\mathrm{trace}\,A_n(z)$, we deduce that $\mathrm{trace}\,\alpha_j(n)$ is constant with respect to n. By (4.12), we have $\sum_{j=1}^{4N-2}\mathrm{trace}\,\alpha_j(n)=1-2N$. Now, given Φ_n as in (), let

$$\Psi_n(z)=\prod_{j=1}^{2g+2}(z-\delta_j)^{\nu_j}\Phi_n(z), \tag{7.1}$$

and introduce the matrix

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{7.2}$$

We next introduce the matrix valued kernel

$$M_n(z, \zeta) = \frac{\Psi_n(z)^\dagger J \Psi_n(\zeta)}{-2\pi i(z - \zeta)}; \tag{7.3}$$

we aim to show that M_n is positive definite as an integral operator on $L^2(S)$, and we observe that this property does not change if we introduce weights on S.

Proposition 7.1. *Let $E_{n-1}(z, \zeta)$ be the kernel of the orthogonal projection onto span$\{x^j : j = 0, \dots, n-1\}$ in $L^2(\rho)$. Then the top left entry of $M_n(z, \zeta)$ equals*

$$M_n(z, \zeta)_{11} = \frac{h_n}{h_{n-1}} \prod_{j=1}^{2g+2} (z - \delta_j)^{\nu_j} \prod_{j=1}^{2g+2} (\zeta - \delta_j)^{\nu_j} E_{n-1}(z, \zeta). \tag{7.4}$$

Proof. The Christoffel–Darboux formula gives

$$E_n(z, \zeta) = \frac{p_n(z)p_{n-1}(\zeta) - p_{n-1}(z)p_n(\zeta)}{h_n(z - \zeta)}. \tag{7.5}$$

One can find $\Psi_n(z)^\dagger J \Psi_n(\zeta)$ by direct calculation, and compare with this.

□

Let $\beta_j(n) = \alpha_j(n) + \nu_j I_2$, which has zero trace. Furthermore, if Φ_n is a solution of the basic differential equation (4.14), then

$$\frac{d}{dz}\Psi_n(z) = B_n(z)\Psi_n(z) \tag{7.6}$$

where

$$B_n(z) = \sum_{j=1}^{2g+2} \frac{\beta_j(n)}{z - \delta_j}. \tag{7.7}$$

We pause to note an existence result for solutions of the matrix system (7.6).

Lemma 7.2. *Suppose that $\beta_j(n)$ has eigenvalues $\pm\lambda_j(n)$ where $2\lambda_j(n)$ is not an integer. Then on a neighbourhood of δ_j, there exists an analytic matrix function $\Xi_{n,j}$ such that*

$$\Psi_n(x) = \Xi_{n,j}(z)(x - \delta_j)^{\beta_j(n)} \tag{7.8}$$

satisfies (7.6).

Proof. This follows from Turrittin's theorem.

□

For notational simplicity, we consider the interval (δ_1, δ_2) and assume that $\delta_1 = 0$ and $1 < \delta_2$; the general case follows by scaling. For a continuous function $\phi : (0,1) \to \mathbf{R}^{4g+2})$, the Hankel operator $\Gamma_\phi : L^2((0,1); dy/y; \mathbf{R}) \to L^2((0,1); dy/y; \mathbf{R}^{4g+2}(\mathbf{R}))$ is given by

$$\Gamma_\phi f(x) = \int_0^1 \phi(xy) f(y) \frac{dy}{y}. \tag{7.9}$$

Since $\beta_k(n)$ has zero trace, the matrix $(\delta_1 - \delta_k) J\beta_k(n)$ is real symmetric and hence is congruent to either

$$\sigma_k = \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \pm \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; \tag{7.10}$$

let $\sigma = \mathrm{diagonal}[\sigma_k]_{k=2}^{2g+2}$ be the block diagonal sum of these matrices.

Theorem 7.3. *Suppose that $\beta_1(n)$ is as in Lemma 7.2. Then there exists Z_n, a 2×1 real vector solution of (7.8) such that $Z_n(x) \to 0$ as $x \to \delta_1$ and*

$$K_n(z, \zeta) = \frac{Z_n(\zeta)^\dagger J Z_n(z)}{z - \zeta} \tag{7.11}$$

defines an integral operator on $L^2(0,1)$. Moreover, there exists a real vector Hankel operator Γ_{ψ_n} such that

$$K_n = \Gamma_{\psi_n}^\dagger \sigma \Gamma_{\psi_n} \tag{7.12}$$

as operators on $L^2((0,1), dy/y)$. If $\sigma \geq 0$, then $K_n \geq 0$.

Proof. There exists an invertible constant 2×2 matrix S_n such that

$$S_n z^{\beta_1(n)} S_n^{-1} = \begin{bmatrix} z^{\lambda_1(n)} & 0 \\ 0 & z^{-\lambda_1(n)} \end{bmatrix}. \tag{7.13}$$

where $\lambda_1(n) > 0$. Hence by Lemma 7.2, there exists a constant 2×1 matrix C such that $Z_n(z) = \Psi_n(z) C$ is a solution of (7.6), and $Z_n(z) = O(|z - \delta_1|^{\lambda_1(n)})$ as $z \to \delta_1$. Hence we can introduce K_n, and next we prove that the kernel satisfies

$$\Big((x - \delta_j)\frac{\partial}{\partial x} + (y - \delta_j)\frac{\partial}{\partial y}\Big) K_n(x,y) = \sum_{k : k \neq j} \frac{\delta_j - \delta_k}{(x - \delta_k)(y - \delta_k)} Z_n(y)^\dagger J \beta_k(n) Z_n(x). \tag{7.14}$$

Since the $\beta_k(n)$ have zero trace, we have $J\beta_k(n)+\beta_k(n)^\dagger J=0$ and hence the differential equation gives

$$\begin{aligned}\Bigl((x-\delta_j)\frac{\partial}{\partial x}+(y-\delta_j)\frac{\partial}{\partial y}\Bigr)&Z_n(y)^\dagger JZ_n(x)\\ &=Z_n(y)^\dagger B_n(y)^\dagger JZ_n(x)+Z_n(y)^\dagger JB_n(x)Z_n(x)\\ &=\sum_{k:k\neq j}Z_n(y)^\dagger J\beta_k(n)Z_n(x)\Bigl(\frac{x-\delta_j}{x-\delta_k}-\frac{y-\delta_j}{y-\delta_k}\Bigr).\end{aligned}\tag{7.15}$$

on dividing by $x-y$, we obtain

$$\Bigl((x-\delta_j)\frac{\partial}{\partial x}+(y-\delta_j)\frac{\partial}{\partial y}\Bigr)\frac{Z_n(y)^\dagger JZ_n(x)}{x-y}=\sum_{k:k\neq j}\frac{\delta_j-\delta_k}{(x-\delta_k)(y-\delta_k)}Z_n(y)^\dagger J\beta_k(n)Z_n(x)\tag{7.16}$$

as in (7.11).

Now we focus attention on $j=1$. Noting the shape of the final factor in (7.13), we choose

$$\phi_n(x)=\mathrm{column}\Bigl[\frac{Z_n(x)}{x-\delta_k}\Bigr]_{k=2,\dots,2g+2}\tag{7.17}$$

which has a 2×1 entry for each endpoint δ_k of S after δ_1, and the block diagonal matrix

$$\beta(n)=\mathrm{diagonal}\Bigl[-\delta_kJ\beta_k(n)\Bigr]_{k=2,\dots,2g+2}\tag{7.18}$$

with 2×2 blocks, and we consider

$$\tilde K_n(x,y)=\int_0^1\phi_n(yz)^\dagger\beta(n)\phi_n(zx)\frac{dz}{z}.\tag{7.19}$$

Then

$$\begin{aligned}\Bigl(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\Bigr)\tilde K_n(x,y)&=\int_0^1\bigl(y\phi_n'(yz)^\dagger\beta(n)\phi_n(zx)+x\phi_n(yz)^\dagger\beta(n)\phi_n'(xy)\bigr)\,dz\\ &=\phi_n(y)^\dagger\beta(n)\phi_n(x)-\phi_n(0)^\dagger\beta(n)\phi_n(0).\end{aligned}\tag{7.20}$$

By hypothesis (i), we have $\phi_n(0)=0$, so

$$K_n(x,y)=\tilde K_n(x,y)+\xi(x/y)\tag{7.21}$$

for some function ξ. But $Z_n(z)/(z-\delta_1)^{\lambda_1(n)}$ is analytic on a neighbourhood of δ_1, so it is clear that $K_n(x,y)\to 0$ and $\tilde K_n(x,y)\to 0$ as $x\to\delta_1$ or $y\to\delta_1$; hence $\xi=0$.

By the choice of σ, there exists a block diagonal matrix $\gamma(n)$ such that $\gamma(n)^\dagger\sigma\gamma(n) = \beta(n)$, so we can introduce $\psi_n(x) = \gamma(n)\phi_n(x)$ such that $\phi_n(x)^\dagger\beta(n)\phi_n(y) = \psi_n(x)^\dagger\sigma\psi_n(y)$. For this symbol function ψ_n we have

$$K_n(x,y) = \int_0^1 \psi_n(yz)^\dagger\sigma\psi_n(zx)\frac{dz}{z}, \tag{7.22}$$

or in terms of Hankel operators $K_n = \Gamma_{\psi_n}^\dagger\sigma\Gamma_{\psi_n}$.

□

Corollary 7.4 *Suppose that Z is a solution of*

$$\frac{d}{dx}Z = \Big(\frac{\beta_0}{x} + \frac{\beta_1}{x-1} + \frac{\beta_t}{x-t}\Big)Z \tag{7.23}$$

such that $Z(\bar{x}) = \bar{Z}(x)$ and where (i) β_0 is as in Lemma 7.2, and $Z(x) \to 0$ as $x \to 0$;

(ii) $J\beta_1$ is positive definite;

(iii) $J\beta_t \geq 0$.

Then there exist an invertible real matrix S and a real diagonal matrix D such that

$$\psi(x) = \begin{bmatrix} \frac{SZ(x)}{x-1} \\ \frac{DSZ(x)}{x-t} \end{bmatrix} \tag{7.24}$$

satisfies $\psi(\bar{x}) = \overline{\psi(x)}$ and

$$\frac{Z(y)^\dagger JZ(x)}{x-y} = \int_0^1 \psi(xz)^\dagger\psi(zy)\frac{dz}{z}. \tag{7.25}$$

Proof. We can simultaneously reduce the quadratic forms associated with $J\beta_1$ and $J\beta_t$, and introduce an invertible real matrix S such that $J\beta_1 = SS^\dagger$ and $J\beta_t = SD^2S^\dagger$, where D is a real diagonal matrix such that the diagonal entries of D^2 satisfy $\det(\beta_1 - \lambda\beta_t) = 0$. Then we can write

$$-\psi(y)^\dagger\psi(x) = Z(y)^\dagger\Big(\frac{J\beta_1}{(x-1)(y-1)} + \frac{J\beta_t t}{(x-t)(y-t)}\Big)Z(x). \tag{7.26}$$

Now we can follow the proof of Theorem 7.3, and deduce that

$$-\psi(y)^\dagger\psi(x) = \Big(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\Big)\frac{Z(y)^\dagger JZ(x)}{x-y}; \tag{7.27}$$

hence we can obtain the result by integrating and using (i).

□

Example 7.5 For the semicircle law on $[a, b]$, as in example 5.2, we have

$$J\beta_a(n) = \begin{bmatrix} \frac{n(b-a)}{2h_{n-1}} & \frac{2n+1}{4} \\ \frac{2n+1}{4} & \frac{(n+1)(b-a)h_{n-1}}{8} \end{bmatrix} \tag{7.28}$$

and

$$J\beta_b(n) = \begin{bmatrix} \frac{-n(b-a)}{2h_{n-1}} & \frac{2n+1}{4} \\ \frac{2n+1}{4} & \frac{-(n+1)(b-a)h_{n-1}}{8} \end{bmatrix}, \tag{7.29}$$

so that

$$\det J\beta_a(n) = \det J\beta_b(n) = \frac{n(n+1)(b-a)^2}{16} - \frac{(2n+1)^2}{16}. \tag{7.30}$$

In particular, when $a = -1$ and $b = 1$, the matrices $J\beta_{-1}(n)$ and $J\beta_1(n)$ are indefinite.

8. The tau function realised by a linear system

In this section, we express the tau function of K_n from Theorem 7.3 as a Fredholm determinant, and then obtain this from the solution of an integral equation of Gelfand–Levitan type. The first step is to realise the solution of a basic differential equation () by a linear system, and then we can apply methods of scattering theory.

The differential equation

$$\frac{dZ_n}{dx} = B_n(x)Z_n(x) \tag{8.1}$$

has a solution from which we constructed a symbol function

$$\psi_n(x) = \mathrm{column}\Big[\frac{\gamma(n)Z(x)}{x - \delta_k}\Big]_{k=2}^{2g+2}. \tag{8.2}$$

Suppressing n for simplicity, we change $x \in (0, 1)$ to $t \in (0, \infty)$ by letting $x = \delta_1 + e^{-t}$ and in the new variables write

$$\psi(t) = \sum_{\ell=0}^{\infty} \chi_\ell e^{-(\lambda_1+\ell)t}. \tag{8.3}$$

where $\sum_{\ell=0}^{\infty} \|\chi_\ell\| < \infty$. Likewise, we write $\tau(t)$ for $\tau(\delta_1 + e^{-t})$.

Definition. Let K be a trace class operator on $L^2(0, \infty)$. Then we define the tau function by

$$\tau(t) = \det(I - KP_{(t,\infty)}). \tag{8.4}$$

This definition is motivated by Proposition 3.3 and Theorem 5.1.

Let $\Omega = \{z : \Re z \geq 0\}$ be the open right half-plane, and suppose that $\Psi : \Omega \to M_n(\mathbf{C})$ is an analytic function such that $\Psi(x) = \Psi(x)^\dagger$ for $x > 0$ and

$$\int_0^\infty x\|\Psi(x+s)\|^2 dx < \infty \qquad (s \in \Omega). \tag{8.5}$$

Let $\Psi_{(s)} = \Psi(x+2s)$ and $\Psi^*_{(s)}(x) = \Psi(x+2\bar{s})^\dagger$ and let σ be a constant matrix; then let $K_s = \Gamma_{\Psi^*_{(s)}}\sigma\Gamma_{\Psi_{(s)}}$ be a family of operators on $L^2(0,\infty)$.

Proposition 8.1. *(i) The τ function associated with $K = \Gamma_{\Psi^*}\sigma\Gamma_\Psi$ is $\tau(2s) = \det(I - K_s)$, which gives an analytic function on Ω.*

(ii) Let $q(s) = -2\frac{d^2}{ds^2}\log\tau(2s)$. Then $q(s)$ is meromorphic on Ω, and analytic where $\int_0^\infty x\|\Psi(x+s)\|^2 dx < 1$.

(iii) Suppose that Ψ is periodic with imaginary period p. Then q also has period p.

(iv) If $0 \leq K \leq I$, then $\tau(s)$ is non-negative for $0 < s < \infty$, increasing and converges to one as $s \to \infty$.

Proof. (i) As in Schwarz's reflection principle, $s \mapsto \Psi^*_{(s)}$ is analytic, and $\Gamma_{\Psi_{(s)}}$ is Hilbert–Schmidt, so K_s is analytic trace-class valued function on Ω. Using unitary equivalence, one checks that

$$\det(I - K_s) = \det(I - P_{(2s,\infty)}K) \tag{8.6}$$

for $s > 0$.

(ii) Except on the discrete set of zeros of $\tau(2s)$, the operator $I - K_s$ is invertible and

$$q(s) = 2\frac{d}{ds}\mathrm{trace}\Big((I - K_s)^{-1}\frac{dK_s}{ds}\Big). \tag{8.7}$$

(iii) Since $\bar{p}$ is also a period, we have $\Psi^*_{(s+p)}(x) = \Psi_{(s)}(x)$ and hence $K_s = K_{s+p}$.

(iv) This follows from (8.7).

□

Next, we obtain an alternative formula for q by realising Ψ via a linear system. The technique is suggested by the inverse scattering transform.

Let $H_0 = \mathbf{C}^{4g+2}$ be the column vectors, $H = \ell^2$ be Hilbert sequence space, written as infinite columns, and introduce an infinite row of column vectors $C \in \ell^2(H_0)$ by $C = (\chi_\ell/\|\chi_\ell\|^{1/2})_{\ell=0}^\infty$ and a column $B \in \ell^2$ by $B = (\|\chi_\ell\|^{1/2})_{\ell=0}^\infty$ and the infinite square matrix $A = \mathrm{diagonal}\,(\ell + \lambda_1)_{\ell=0}^\infty$. While A is real and symmetric, we will write $A^\dagger$ in some subsequent formulas, so as to emphasize their symmetry.

In the following result we use the $(4g+3)\times(4g+3)$ block matrices

$$W(x,y) = \begin{bmatrix} U(x,y) & v(x,y) \\ w(x,y)^\dagger & z(x,y) \end{bmatrix}, \Psi(x) = \begin{bmatrix} 0 & \psi(x) \\ \psi(\bar{x})^\dagger & 0 \end{bmatrix}, \tag{8.8}$$

so that $\Psi(\bar{x}) = \Psi(x)^\dagger$ and the matrix Hamiltonian

$$H(x) = \begin{bmatrix} U(x,x)\sigma & v(x,x) \\ w(x,x)^\dagger\sigma & z(x,x) \end{bmatrix} \tag{8.9}$$

where $v, w \in H_0$, U operates upon H_0 and z is a scalar. To simplify the statements of results, we use a special non-associative product $*$ defined by

$$\begin{bmatrix} U & v \\ w^\dagger & z \end{bmatrix} * \begin{bmatrix} 0 & \phi \\ \phi^\dagger & 0 \end{bmatrix} = \begin{bmatrix} v\phi^\dagger & U\sigma\phi \\ z\phi^\dagger & w^\dagger\sigma\phi \end{bmatrix}. \tag{8.10}$$

Theorem 8.2 *(i) The symbol ψ is realised by the linear system $(-A, B, C)$, so*

$$\psi(t) = Ce^{-tA}B. \tag{8.11}$$

(ii) There exists a solution of the Gelfand–Levitan equation

$$W(x,y) + \Psi(x+y) + \int_x^\infty W(x,s) * \Psi(s+y)\, ds = 0 \qquad (0 < x < y) \tag{8.12}$$

such that the tau function of Proposition 8.1(i) satisfies

$$\frac{d}{dx}\log\tau(2x) = \operatorname{trace} H(x) \qquad (x > 0). \tag{8.13}$$

Proof. (i) This follows from (8.7).

(ii) We introduce the modified observability Gramian

$$Q_x^\sigma = \int_x^\infty e^{-sA^\dagger} C^\dagger \sigma C e^{-sA}\, ds \qquad (x > 0) \tag{8.14}$$

to take account of σ, and the usual controllability Gramian

$$L_x = \int_x^\infty e^{-sA} BB^\dagger e^{-sA^\dagger}\, ds \tag{8.15}$$

which define trace class operators on ℓ^2 such that $L_x \geq 0$. The controllability operator $\Xi_x : L^2((0,\infty); H_0) \to H$ is

$$\Xi_x f = \int_x^\infty e^{-tA} B f(s)\, ds \tag{8.16}$$

while the observability operator is $\Theta_x : L^2((0,\infty); H_0) \to H$ is

$$\Theta_x f = \int_x^\infty e^{-sA^\dagger} C^\dagger f(s)\, ds. \tag{8.17}$$

Finally, we let $\psi_{(x)}(s) = \psi(s+2x)$, so that $\psi_{(x)}$ is realised by $(-A, e^{-xA}B, e^{-xA}C)$. In terms of these operators, we have the basic identities

$$\Gamma_{\psi_{(x)}} = \Theta_x^\dagger \Xi_x, \qquad \Gamma^\dagger_{\psi_{(x)}} = \Xi_x^\dagger \Theta_x \tag{8.18}$$

while

$$L_x = \Xi_x \Xi_x^\dagger \qquad \text{and} \qquad Q_x^\sigma = \Theta_x \sigma \Theta_x^\dagger. \tag{8.19}$$

Hence we can rearrange the factors in the Fredholm determinants

$$\begin{aligned}\det(I - \lambda\Gamma_{\psi_{(x)}}^\dagger \sigma \Gamma_{\psi_{(x)}}) &= \det(I - \lambda \Xi_x^\dagger \Theta_x \sigma \Theta_x^\dagger \Xi_x) \\ &= \det(I - \lambda \Xi_x \Xi_x^\dagger \Theta_x \sigma \Theta_x^\dagger) \\ &= \det(I - \lambda L_x Q_x^\sigma). \end{aligned} \tag{8.20}$$

We deduce that

$$\begin{aligned}\log\tau(2x) &= \log\det(I - \Gamma_\psi^\dagger \sigma \Gamma_\psi P_{(2x,\infty)}) \\ &= \log\det(I - \sigma\Gamma_\psi P_{(2x,\infty)} \Gamma_\psi^\dagger) \\ &= \log\det(I - \sigma\Gamma_{\psi_{(x)}} \Gamma_{\psi_{(x)}}^\dagger) \\ &= \log\det(I - \Gamma_{\psi_{(x)}}^\dagger \sigma \Gamma_{\psi_{(x)}}) \\ &= \operatorname{trace} \log(I - L_x Q_x^\sigma), \end{aligned} \tag{8.21}$$

and hence

$$\begin{aligned}\frac{d}{dx}\log\tau(2x) &= \operatorname{trace}\Big((I - L_x Q_x^\sigma)^{-1}\big(e^{-xA}BB^\dagger e^{-xA^\dagger} Q_x^\sigma + L_x e^{-xA^\dagger} C^\dagger \sigma C e^{-xA}\big)\Big) \\ &= B^\dagger e^{-xA^\dagger} Q_x^\sigma (I - L_x Q_x^\sigma)^{-1} e^{-xA} B \\ &\quad + \operatorname{trace} \sigma C e^{-xA}(I - L_x Q_x^\sigma)^{-1} L_x e^{-xA^\dagger} C^\dagger. \end{aligned} \tag{8.22}$$

The integral equation

$$\begin{aligned}&\begin{bmatrix} U(x,y) & v(x,y) \\ w(x,y)^\dagger & z(x,y)\end{bmatrix} + \begin{bmatrix} 0 & \psi(x+y) \\ \psi(x+y)^\dagger & 0 \end{bmatrix} \\ &+ \int_x^\infty \begin{bmatrix} U(x,s) & v(x,s) \\ w(x,s)^\dagger & z(x,s)\end{bmatrix} * \begin{bmatrix} 0 & \psi(s+y) \\ \psi(s+y)^\dagger & 0 \end{bmatrix} ds = 0 \end{aligned} \tag{8.23}$$

reduces to the identities

$$\begin{aligned} U(x,y) &= -\int_x^\infty v(x,s)\psi(s+y)^\dagger\, ds \\ z(x,y) &= -\int_x^\infty \psi(s+y)^\dagger \sigma w(x,s)\, ds \end{aligned} \tag{8.24}$$

and the pair of integral equations

$$v(x,y) + \psi(x+y) - \int_x^\infty \int_x^\infty v(x,t)\psi(t+s)^\dagger \sigma \psi(s+y)\, dsdt = 0 \tag{8.25}$$

and

$$w(x,y)+\psi(x+y)-\int_x^\infty\int_x^\infty \psi(s+y)\psi(t+s)^\dagger\sigma w(x,t)\,dtds=0. \tag{8.26}$$

To solve these integral equations, we let

$$v(x,y)=-Ce^{-xA}(I-L_xQ_x^\sigma)^{-1}e^{-yA}B \tag{8.27}$$

and

$$w(x,y)=-Ce^{-yA}(I-L_xQ_x^\sigma)^{-1}e^{-xA}B; \tag{8.28}$$

then by substituting these into (8.21) we obtain the diagonal blocks of the solution, namely

$$U(x,y)=Ce^{-xA}(I-L_xQ_x^\sigma)^{-1}L_xe^{-yA^\dagger}C^\dagger \tag{8.29}$$

and

$$z(x,y)=B^\dagger e^{-yA^\dagger}Q_x^\sigma(I-L_xQ_x^\sigma)^{-1}e^{-xA}B. \tag{8.30}$$

Hence we can identify the trace of the solution as

$$\begin{aligned}\operatorname{trace}H(x)&=\operatorname{trace}\sigma U(x,x)+z(x,x)\\&=\operatorname{trace}\sigma Ce^{-xA}(I-L_xQ_x^\sigma)^{-1}e^{-xA^\dagger}C^\dagger\\&\qquad+B^\dagger e^{-xA^\dagger}Q_x^\sigma(I-L_xQ_x^\sigma)^{-1}e^{-xA}B\\&=\frac{d}{dx}\log\tau(2x).\end{aligned} \tag{8.31}$$

Proposition 8.3. *Suppose that $W(x,y)$ is a solution of the Gelfand–Levitan equation (8.11) and $\int_0^\infty x\|\Psi(x)\|^2dx<1$. Then*

$$\Big(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\Big)W(x,y)=-2\frac{dH}{dx}\,W(x,y). \tag{8.32}$$

Proof. By integrating by parts, we obtain the identity

$$\Big(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\Big)W(x,y)-2\frac{dH}{dx}\Psi(x+y)+\int_x^\infty\Big(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial s^2}\Big)W(x,s)*\Psi(s+y)\,ds=0 \tag{8.33}$$

for $0<x<y$. One can easily verify that the product $*$ and the standard matrix multiplication satisfy $(QW)*\Psi=Q(W*\Psi)$, hence the formula

$$-2\frac{dH}{dx}W(x,y)-2\frac{dH}{dx}\Psi(x+y)-\int_x^\infty\Big(2\frac{dH}{dx}W(x,s)\Big)*\Phi(s+y)\,ds=0 \tag{8.34}$$

follows from multiplying (8.33) by $-2\frac{dH}{dx}$, and this shows that both $-2\frac{dH}{dx}W(x,y)$ and $(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2})W(x,y)$ are solutions of the same integral equation. By uniqueness of solutions, they are equal.

$\square$

Suppose that $\varphi_\lambda(x)$ satisfies $-\varphi_\lambda''(x) + q(x)\varphi_\lambda(x) = \lambda\varphi_\lambda(x)$; then $W(x,y) = \cos y\sqrt{\lambda}\varphi_\lambda(x)$ gives a solution of the hyperbolic equation (8.31). In the next section, we consider cases in which we can find such solutions explicitly.

9. Integrability of the tau function of a linear system

In this section we consider the algebraic properties of τ, and how these depend upon the properties of Z in the linear differential equation (8.1) with rational matrix coefficients and ψ from Theorem 8.2. The main connection between τ and ϕ is given by the Gelfand Levitan equation of Theorem 8.2, and the consequent differential equation of Proposition 8.3, which introduces the potential q. To describe these, we recall some terminology from the algebraic theory of differential equations.

Let $\mathbf{F}$ be a field with differential ∂ that contains the subfield $\mathbf{C}$ of constants and adjoin an element h to form $\mathbf{F}(h)$, where either:

(i) $h = \int g$ for some $g \in \mathbf{F}$, so $\partial h = g$;

(ii) $h = \exp \int g$ for some $g \in \mathbf{F}$; or

(iii) h is algebraic over $\mathbf{F}$.

Definition Let $\mathbf{F}_j$ be a field with differential ∂ that contains the subfield $\mathbf{C}$ of constants and suppose that

$$\mathbf{F}_1 \subseteq \mathbf{F}_2 \subseteq \ldots \subseteq \mathbf{F}_n \tag{9.1}$$

where $\mathbf{F}_j$ arises from $\mathbf{F}_{j-1}$ by applying some operation (i), (ii) or (iii). Then $\mathbf{F}_n$ is a Liouvillian extension of $\mathbf{F}_1$.

Example 9.1 The tau function (5.26) is an element of some Liouvillian extension of $\mathbf{C}(x)$. Furthermore, the Chebyshev polynomials of the second kind are described most simply in terms of $U_n(\cos\theta)$

Definition. Let q be meromorphic on $\mathbf{C}$. We say that q is algebro-geometric if there exists $R : \mathbf{C}^2 \to \mathbf{C} \cup \{\infty\}$ such that (i) $x \mapsto R(x;\lambda)$ is meromorphic, (ii) $\lambda \mapsto R(x;\lambda)$ is polynomial; and

$$-R''' + 4(q-\lambda)R' + 2q'R = 0. \tag{9.2}$$

Drach observed that if $R(x;\lambda)$ belongs to some differential field $\mathbf{F}$ for typical λ, then

$$\psi(x) = \sqrt{R(x;\lambda)}\exp\Bigl(-\int \frac{dt}{R(t;\lambda)}\Bigr) \tag{9.3}$$

gives a solution to Schrödinger's equation which lies in some Liouville extension of $\mathbf{F}$. The following result summarizes various sufficient conditions for a potential to be algebro-geometric.

Theorem 9.2 [16] *Suppose that $-f''+qf=\lambda f$ has a meromorphic fundamental solution for each λ and that either*

(a) $q(x)$ is rational, and $q(x)$ is bounded as $|x|\to\infty$; or

(b) q is elliptic, that is, doubly periodic;

(c) q is periodic, with period one, and there exists $R>0$ such that q is bounded on $\{z\in\mathbf{C}:|\Im z|>R\}$.

Then q is algebro-geometric.

Let $q(x) = -2\frac{d}{dx}\mathrm{trace}H(x)$ and suppose theat q is meromorphic on $\mathbf{C}$. This is a reasonable assumption in view of Proposition 8.1. We proceed to consider the cases (a), (b) and (c) of Theorem 9.2, and the linear systems $(-A,B,C)$ that give rise to them, and the corresponding τ functions.

(a) Rational potential

Lemma 9.3. *Suppose that q is rational and bounded at infinity, and that the general solution of $-f''+qf=\lambda f$ is meromorphic. Then f satisfies a linear system with rational transfer function.*

Proof. By a theorem of Halphen, the general solution of $-f''+qf=\lambda f$ has the form $f(x)=\sum_{j=1}^n q_j(x)e^{-\lambda_j x}$, where $q_j(x)$ are polynomials. Hence there exist constants a_j, not all zero such that $\sum_{k=0}^N a_k f^{(k)}(x)=0$; so by taking the Laplace transform, we can recover the rational transfer function for this linear differential equation.

□

We consider the tau function that corresponds to a linear system with a stable rational transfer function.

Theorem 9.4. *Suppose that $\psi(t)$ is realised as $\psi(t)=Ce^{-tA}B$, where A is a finite rank matrix, such that $NA+A^\dagger N$ is positive definite for some positive definite N. Then $\tau(x)$ belongs to some Liouvillian extension field $\mathbf{F}$ of $\mathbf{C}(t)$ which depends on the spectrum of A.*

Proof. We recall that any proper rational function arises as the transfer function of a linear system that has a finite matrix A, so $\hat\phi(\lambda)=D+C(\lambda I+A)^{-1}B$.

Now we consider $(-A, B, C)$ as in the theorem. By Lyapunov's criterion, all of the the eigenvalues λ_j of A satisfy $\Re\lambda_j > 0$, hence $\|e^{-tA}\|$ is of exponential decay as $t \to \infty$. By considering the Jordan canonical form of A, we obtain matrix polynomials $p_j(t)$ such that $e^{-tA} = \sum_{j=1}^{n} p_j(t)e^{-\lambda_j t}$. Let $\mathbf{F}_0$ be a Liouvillian extension of $\mathbf{C}(t)$ that contains all of the $e^{-t\lambda_j}$ and $e^{-t\bar{\lambda}_j}$, hence $\mathbf{F}_0$ contains all the entries of $e^{-tA}BB^\dagger e^{-tA^\dagger}$ and $e^{-tA^\dagger}C^\dagger\sigma Ce^{-tA}$. The operator L_x is an indefinite integral of $e^{-tA}BB^\dagger e^{-tA^\dagger}$ while the operator Q_x^σ is an indefinite integral of $e^{-tA^\dagger}C^\dagger\sigma Ce^{-tA}$, hence L_x and Q_x^σ have entries in some Liouvillian extension $\mathbf{F}_1$ of $\mathbf{F}_0$; moreover, the entries of $(I - L_xQ_x^\sigma)^{-1}$ are quotients of determinants with elements in $\mathbf{F}_1$. Hence by (8.17), $\frac{d}{dx}\log\tau(2x)$ gives an element of $\mathbf{F}_1$, so $\tau(x)$ itself is in an extension $\mathbf{F}$ of $\mathbf{F}_1$.

□

Theorem 9.4 is applicable in the context of soliton solutions of Schrödinger's equation, as in section 4 of [4].

Continuing the theme of finite matrices, we formulate version of the Gelfand–Levitan equation that is appropriate when $\phi(x) = Ce^{-xA}B$ is a periodic function. A variant of this was used in [10] to solve the matrix nonlinear Schrödinger equation.

Proposition 9.5. *(i) Let A, B, C and E be square matrices such that $\exp(2\pi A) = I$ and $BC = AE + EA$, and let $\phi(x) = Ce^{-xA}B$. Then*

$$W(x, y) = Ce^{-xA}\big(I + E - e^{-xA}Ee^{-xA}\big)^{-1}e^{-yA}B \tag{9.5}$$

satisfies

$$-\phi(x+y) + W(x,y) - \int_x^{2\pi} W(x,z)\phi(z+y)\,dz = 0 \qquad (0 < x < y < 2\pi). \tag{9.6}$$

(ii) Suppose moreover that $ABC = BCA$ and $2\pi\|\phi\|_\infty < 1$, then

$$\frac{\partial^2 W}{\partial x^2} - \frac{\partial^2 W}{\partial y^2} - H(x)W(x,y) = 0 \tag{9.7}$$

with $H(x) = -2\frac{d}{dx}W(x,x)$.

(iii) Let $\mathbf{F}$ be a differential field that contains all the entries of e^{-xA}. Then $\mathbf{F}$ contains all the entries of H.

Proof. (i) One can check that

$$\int_x^{2\pi} e^{-zA}BCe^{-zA}\,dz = e^{-xA}Ee^{-xA} - E \tag{9.8}$$

and it is then a simple matter to verify the Gelfand–Levitan equation (9.6).

(ii) By repeatedly differentiating the integral equation, and using periodicity, one derives the identity

$$\frac{\partial^2 W}{\partial x^2}-\frac{\partial^2 W}{\partial y^2}+2\frac{d}{dx}W(x,x)\phi(x+y)+W(x,0)\phi'(y)-\frac{\partial W}{\partial y}(x,0)\phi(y)$$

$$-\int_x^{2\pi}\Big(\frac{\partial^2 W}{\partial x^2}-\frac{\partial^2 W}{\partial y^2}\Big)\phi(z+y)\,dz=0 \tag{9.9}$$

Since $ABC-CBA=0$, we obtain

$$W(x,0)\phi'(y)-\frac{\partial W}{\partial y}(x,0)\phi(y)=0, \tag{9.10}$$

so (9.9) is a multiple of the original integral equation by $H(x)$. By the assumptions on $\|\Phi\|_\infty$, the solutions are unique, hence the differential equation is satisfied.

(iii) This follows from the definition of W.

□

10. Integrable cases of Hill's equation

In view of Proposition 8.1 and Theorem 9.2(b) and (c), it is natural to consider $\phi(s)$ with s a complex variable and to investigate the case when ϕ is periodic. When $q(s)$ is periodic, the differential equation $-f''+qf=\lambda f$ is known as Hill's equation.

(b) Elliptic potential

Let $\mathcal{E}$ be the elliptic function field of functions of rational character on the complex torus $\mathbf{C}/(\mathbf{Z}+i\mathbf{Z})$, and let $\mathcal{P}$ be Weierstrass's elliptic function. Then $\mathcal{E}$ is equal to $\mathbf{C}(\mathcal{P})[\mathcal{P}']$, and $\mathcal{E}$ has a Liouville extension $\mathcal{E}_\theta$ that Jacobi's elliptic theta function θ_1. By introducing infinite block matrices, we now consider a linear system that has potential $\mathcal{P}$ and one can interpret the following result as saying that Lamé's operator $-\frac{d^2}{dx^2}+\mathcal{P}$ has the scattering function $\sin x$.

Definition. Given square matrices A, B and C, such that $\exp A=I$, we let $\phi(x)=Ce^{-xA}B$ be the scattering function for $(-A,B,C)$ and then

$$W(x,y)=Ce^{-xA}\big(I-e^{-xA}Be^{-xA}\big)^{-1}e^{-yA}B. \tag{10.1}$$

We call $q(x)=-2\frac{d}{dx}\operatorname{trace}W(x,x)$ the potential associated with $(-A,B,C)$ and $\tau(x)=\int\operatorname{trace}W(x,x)dx$ the tau function.

In particular, Let A, B and C be the infinite block diagonal matrices with 2×2 diagonal blocks

$$A = \mathrm{diagonal}[J, J, J, \dots], \tag{10.2}$$

$$B = \mathrm{diagonal}[I, q^2 I, q^4 I, \dots], \tag{10.3}$$

and

$$C = -2\mathrm{diagonal}[\frac{1}{2}J, J, J, \dots]; \tag{10.4}$$

Proposition 10.1 *The matrices satisfy the Gelfand–Levitan equation*

$$-\phi(x+y) + W(x,y) - W(x,x)C^{-1}\phi(x+y) = 0, \tag{10.5}$$

the traces satisfy $\mathrm{trace}\,\phi(x) = -2(1+q^2)(1-q^2)^{-1}\sin x$ *and*

$$\mathcal{P}(x) = \frac{d}{dx}\mathrm{trace} W(x,x) + c \tag{10.6}$$

for some constant c, where $\mathcal{P}$ is Weierstrass's elliptic function.

Proof. One can easily verify that W and ϕ satisfy (10.5).

The Jacobi elliptic function satisfies

$$\theta_1(x) = 2q^{1/4}\sin x \prod_{n=1}^{\infty}\big(1 - 2q^{2n}\cos 2x + q^{4n}\big)(1-q^{2n}) \tag{10.7}$$

so that

$$\frac{\theta_1'(x)}{\theta_1(x)} = \frac{\cos x}{\sin x} + \sum_{n=1}^{\infty}\frac{4q^{2n}\sin 2x}{1 - 2q^{2n}\cos 2x + q^{4n}}. \tag{10.8}$$

Using simple linear algebra, one obtains the identity

$$\frac{2q^{2n}\sin 2x}{1 - 2q^{2n}\cos 2x + q^{4n}} = \mathrm{trace}\Big(J\big(I - q^{2n}\exp(-2xJ)\big)^{-1}\Big) \qquad (n = 1, 2, \dots) \tag{10.9}$$

and the corresponding identity

$$\frac{\cos x}{\sin x} = \mathrm{trace}\Big(J\big(I - \exp(-2xJ)\big)^{-1}\Big) \tag{10.10}$$

for $n = 0$; we deduce that

$$\frac{\theta_1'(x)}{\theta_1(x)} = \mathrm{trace} W(x,x). \tag{10.11}$$

By a standard identity from the theory of elliptic functions [26, p132]

$$\mathcal{P}(x) = -\frac{d^2}{dx^2}\log\theta_1(x) + e_1 + \frac{d^2}{dx^2}\log\theta_1(x)|_{x=1/2} \tag{10.12}$$

and the stated result follows directly.

□

Theorem 10.2 *Let τ be an elliptic function. Then $\frac{d^2}{dx^2}\log\tau(2x)$ is the potential of a linear system $(-A, B, C)$ where A, B and C are infinite block diagonal matrices with 2×2 blocks.*

Proof. We may assume without loss that τ is doubly periodic with respect to the lattice $\mathbf{Z}^2$. Any elliptic function is the ratio of theta functions, so

$$\tau(x) = \prod_{j=1}^{m}\frac{\theta(x-a_j)}{\theta(x-b_j)} \tag{10.13}$$

where $a_1 + \ldots + a_m = b_1 + \ldots + b_m$. We can use the proof of Proposition 10.1 to express $\frac{d^2}{dx^2}\log\theta(2x-a_j)$ in terms of a linear systems, and then we can combine the block diagonal matrices to produce infinite block diagonal matrices A, B and C with 2×2 block diagonal entries such that $\phi(x) = Ce^{-xA}B$ and

$$W(x,y) = Ce^{-xA}\big(I + e^{-xA}Be^{-xA}\big)^{-1}e^{-yA}B \qquad (0 < x < y) \tag{10.14}$$

satisfy

$$-\phi(x+y) + W(x,y) + W(x,x)C^{-1}\phi(x+y) = 0 \tag{10.15}$$

and

$$\frac{d^2}{dx^2}\log\tau(2x) = -\frac{d}{dx}\mathrm{trace}\, W(x,x). \tag{10.16}$$

□

(c) Hyperelliptic potentials

We introduce the Schrödinger differential operator $\Delta = -\frac{d^2}{dx^2} + q(x)$. Then we introduce the Bloch spectrum of Δ, which is

$$\sigma_B = \{\lambda\in\mathbf{C} : \Delta f = \lambda f \quad \text{for some} \quad f\in L^\infty\}. \tag{10.17}$$

One can show that the Bloch spectrum of an algebro-geometric potential has only finitely many gaps. Suppose that

$$\sigma_B = [\lambda_0,\lambda_1]\cup[\lambda_2,\lambda_3]\cup\ldots\cup[\lambda_{2g},\infty) \tag{10.18}$$

with g gaps. The λ_j are the points of the simple periodic spectrum, such that $-f'' + qf = \lambda_j f$ has a unique solution, up to scalar multiples, that is one or two periodic. Let Φ be the 2×2 fundamental solution matrix that satisfies

$$\frac{d}{dx}\Phi(x) = \begin{bmatrix} 0 & 1 \\ -\lambda + q(x) & 0 \end{bmatrix} \Phi(x) \qquad \Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{10.19}$$

and let $D(\lambda) = \operatorname{trace} \Phi(1)$ be the discriminant of Hill's equation. We can characterize the spectral gaps as $\{\lambda \in \mathbf{R} : D(\lambda)^2 < 4\}$.

The hyperelliptic curve $\mathcal{C} : y^2 = -\prod_{j=1}^{2g}(x - \lambda_j)$ has genus g, and we can form the hyperelliptic function field $\mathcal{E}_g = \mathbf{C}(x)[y]$. The torus

$$\mathbf{T}^g = \Big\{ \frac{1}{2}\Big(D(x_j) + \sqrt{D(x_j)^2 - 4}\Big) : \lambda_{2j-1} \le x_j \le \lambda_{2j} : j = 1, \dots, g \Big\} \tag{10.20}$$

has dimension g.

To obtain a model for the Riemann surface of $\mathcal{C}$, we choose a two-sheeted cover of $\mathbf{C}$ with cuts along σ_B, and introduce a homology basis consisting of:

- loops α_j that start from $[\lambda_{2g}, \infty)$, pass along to top sheet to $[\lambda_{2j-2}, \lambda_{2j-1}]$, then return along the bottom sheet to the start on $[\lambda_{2g}, \infty)$;
- loops β_j that go around the intervals of stability $[\lambda_{2j-2}, \lambda_{2j-1}]$ that do not intersect with one another, for $j = 1, \dots, g$.

Then we introduce the differentials

$$d\omega_j = \frac{x^{j-1}\,dx}{y} \qquad (j = 1, \dots, g) \tag{10.21}$$

and then we form the $g \times 2g$ Riemann matrix $[2I, 2\Omega]$ from the $g \times g$ matrix blocks

$$2I = \Big[\int_{\alpha_k} d\omega_j\Big]_{j,k=1}^{g}, \quad \text{and} \quad 2\Omega = \Big[\int_{\beta_k} d\omega_j\Big]_{j,k=1}^{g}. \tag{10.22}$$

Then the corresponding theta function is

$$\Theta(s \mid \Omega) = \sum_{n \in \mathbf{Z}^g} \exp\big(i\pi\langle \Omega n, n\rangle + 2\pi i \langle s, n\rangle\big). \tag{10.23}$$

Example. Suppose that $g = 2$ and let

$$\Omega = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

where $\Im a > 0, \Im d > 0$ and $b \in \mathbf{Q}$. Then choose $p \in \mathbf{N}$ such that $pb \in \mathbf{Z}$. One can easily check that

$$\Theta(s,t \mid \Omega) = \sum_{r,\mu=0}^{p-1} e^{\pi(ar^2+2br\mu+d\mu^2)} e^{2\pi rs} e^{2\pi\mu t} \theta(ps+r \mid p^2 a)\theta(pt+\mu \mid p^2 d). \tag{10.24}$$

Proposition 10.3. *Suppose that q is a periodic potential with g spectral gaps, as above, and that $\Theta(\ \mid \Omega)$ is a finite sum of products of Jacobi elliptic functions. Then there exist*

(i) $N < \infty$, $x_j \in \mathbf{R}$, $\sigma_j \in \mathbf{C}$ with $\Im\sigma_j > 0$;

(ii) block diagonal matrices A_j, B_j and C_j with 2×2 diagonal blocks;

such that $\theta(x - x_j \mid \sigma_j)$ is the tau function of $(-A_j, B_j, C_j)$ and q belongs to the field $\mathbf{C}(\theta(x - x_j \mid \sigma_j); j = 1, \dots, N)$ that is generated by the elliptic theta functions.

Proof. McKean and van Moerbeke [25, p260] considered the manifold $\mathcal{M}$ of all the smooth real one-periodic potentials such that the corresponding Hill's operator has simple spectrum $\{\lambda_1, \dots, \lambda_{2g}\}$, and showed that $\mathcal{M}$ is diffeomorphic to $\mathbf{T}^g$. Let Λ be the lattice generated by the columns of $[I;\Omega]$, and note that $\mathbf{C}^g/\Lambda$ is the Jacobi variety of $\mathcal{C}$. They have shown that q extends to a $2g$-fold periodic function on the complexification of $\mathbf{T}^g$, hence gives an abelian function which is periodic with period lattice Λ. The extended function q belongs to $\mathcal{E}_g$, hence is a theta quotient. Moreover, translation on the potential is equivalent to ((linear motion on $\mathbf{C}^g/\Lambda$ at constant velocity.

Thus they solve the inverse spectral problem explicitly by showing on [25, p.262] that

$$q(x) = \sum_{j=0}^{g} \varepsilon_j \frac{\Theta(X - \omega_j^*/2 \mid \Omega)\Theta(X - \omega_j^{**}/2 \mid \Omega)}{\Theta(X - \omega_\infty^*/2 \mid \Omega)\Theta(X - \omega_\infty^{**}/2 \mid \Omega)} \tag{10.25}$$

where $X = (x_1, \dots, x_{g-1}, ax + b)$ has $a, b, x_1, \dots, x_{g-1}$ fixed, and the constants $\varepsilon_j, \omega_j^*, \omega_j^{**}, \omega_\infty^*$ and ω_∞^{**} are notionally computable.

By hypothesis, each factor $\Theta(X - \omega^*/2 \mid \Omega)$ may be written a a finite sum of products of functions such as $\theta(ax + c_j \mid d_j)$, and we can apply Proposition 10.1 to each such factor.

□

Weierstrass and Poincaré developed a systematic reduction procedure for such elliptic functions of higher genus, so we can describe the scope of Proposition 10.3. The Siegel upper half-space is

$$\mathcal{S}_g = \{\Omega \in M_{g\times g}(\mathbf{C}) : \Omega = \Omega^t; \Im\Omega > 0\}. \tag{10.26}$$

Let X and J be the $2g \times 2g$ rational block matrices

$$X = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \tag{10.27}$$

such that $XJX^t = J$; the set of all such X is the symplectic group $Sp(2g; \mathbf{Q})$. Now X is associated with a the transformation φ_X of $\mathcal{S}_g$ given by

$$\varphi_X(\Omega) = (\alpha + \beta\Omega)^{-1}(\gamma + \delta\Omega), \tag{10.28}$$

thus $Sp(2g; \mathbf{Q})$ acts on $\mathcal{S}_g$.

Proposition 10.4. [1] *(i) Suppose that Ω can be reduced to a diagonal matrix by action of the symplectic group. Then $\Theta(\,|\,\Omega)$ can be expressed as a series of products of Jacobian elliptic theta functions.*

(ii) The orbit of $Sp(2g, \mathbf{Q})$ that contains iI is dense in $\mathcal{S}_g$.

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