Some remarks on James–Schreier spaces

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Abstract

The James–Schreier spaces V_p, where $1 \leqslant p < \infty$, were recently introduced by Bird and Laustsen [5] as an amalgamation of James' quasi-reflexive Banach space on the one hand and Schreier's Banach space giving a counterexample to the Banach–Saks property on the other. The purpose of this note is to answer some questions left open in [5]. Specifically, we prove that (i) the standard Schauder basis for the first James–Schreier space V_1 is shrinking, and (ii) any two Schreier or James–Schreier spaces with distinct indices are non-isomorphic. The former of these results implies that V_1 does not have Pełczyński's property (u) and hence does not embed in any Banach space with an unconditional Schauder basis.

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1 Introduction

Let $1 \leqslant p < \infty$. By the p^{th} *Schreier space*, denoted S_p, we understand the Banach space obtained by completing c_{00} (the vector space of finitely supported scalar sequences) with respect to the norm

$$\|x\|_{S_p} := \sup\left\{ \left(\sum_{j=1}^{k} |\alpha_{n_j}|^p \right)^{\frac{1}{p}} : k, n_1, \ldots, n_k \in \mathbb{N},\ k \leqslant n_1 < n_2 < \cdots < n_k \right\}, \tag{1.1}$$

where $x = (\alpha_n)_{n \in \mathbb{N}} \in c_{00}$. The space S_1 is the one which is usually known as *the* Schreier space in the literature; it was formally introduced by Beauzamy and Lapresté [3], building on ideas from Baernstein's thesis [2], which in turn were inspired by Schreier's seminal construction [9].

The Schreier spaces have recently been amalgamated with James' quasi-reflexive Banach spaces [6] by Bird and Laustsen [5]. More precisely, for $1 \leqslant p < \infty$, the p^{th} *James–Schreier space,* denoted V_p, is the completion of c_{00} with respect to the norm

$$\|x\|_{V_p} := \sup\left\{ \left(\sum_{j=1}^{k} |\alpha_{n_j} - \alpha_{n_{j+1}}|^p \right)^{\frac{1}{p}} : k, n_1, \ldots, n_{k+1} \in \mathbb{N},\ k \leqslant n_1 < n_2 < \cdots < n_{k+1} \right\}, \tag{1.2}$$

where $x = (\alpha_n)_{n \in \mathbb{N}} \in c_{00}$. We refer to [5] for the background and motivation behind these spaces, as well as a thorough study of their fundamental properties. The purpose of this paper is to resolve two problems left open in [5].

First, it was shown in [5] that $(e_n)_{n\in\mathbb{N}}$, where $e_n \in c_{00}$ is the sequence with 1 in position n and 0 elsewhere, is a Schauder basis for V_p for each $p \geqslant 1$ and, moreover, that this basis is *shrinking* (meaning that the associated sequence of biorthogonal functionals $(e'_n)_{n\in\mathbb{N}}$ is a Schauder basis for the dual space V'_p) whenever $p > 1$. The question of whether or not the basis $(e_n)_{n\in\mathbb{N}}$ is shrinking for $p = 1$ was left open; in Section 2 we answer this question in the positive. As a consequence, we deduce that V_1 does not have Pełczyński's property (u) and hence does not embed in a Banach space with an unconditional Schauder basis.

Second, regarding embeddings and isomorphisms of Schreier and James–Schreier spaces, it was proved in [5] that:

(i) for each $p \geqslant 1$, S_p is isomorphic to a complemented subspace of V_p;

(ii) for each $p > 1$, V_p does not embed in S_q for any $q \geqslant 1$; this result extends to the case $p = 1$ by the conclusions of Section 2 of the present paper.

We complete this picture in Section 3 by proving that, for $q > p \geqslant 1$, no subspace of V_q is isomorphic to S_p, and consequently no subspace of S_q is isomorphic to S_p, and no subspace of V_q is isomorphic to V_p. In particular, $S_p \not\cong S_q$ and $V_p \not\cong V_q$ whenever $p \neq q$.

2 The standard basis for the first James–Schreier space is shrinking

As the title indicates, the aim of this section is to prove the following result.

2.1 Theorem. *The standard Schauder basis $(e_n)_{n\in\mathbb{N}}$ for V_1 is shrinking.*

The proof of Theorem 2.1 relies on two lemmas. Before presenting these, we recall some notation and terminology from [5]. Throughout, $\mathbb{K}$ denotes the scalar field; either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We write $\operatorname{card} A$ for the cardinality of a (typically finite) set A. Suppose that A is a subset of $\mathbb{N}$. We then write $A = \{n_1 < n_2 < \cdots < n_k\}$ to indicate that A is finite and non-empty and that $\{n_1, n_2, \ldots, n_k\}$ is the increasing ordering of A. We say that A is *admissible* if $1 \leqslant \operatorname{card} A \leqslant \min A$ and *permissible* if $2 \leqslant \operatorname{card} A \leqslant 1 + \min A$. Thus a typical admissible set has the form $\{n_1 < n_2 < \cdots < n_k\}$, where $1 \leqslant k \leqslant n_1$, while a typical permissible set can be written as $\{n_1 < n_2 < \cdots < n_{k+1}\}$, again with $1 \leqslant k \leqslant n_1$.

Now let $1 \leqslant p < \infty$. For $x = (\alpha_n)_{n\in\mathbb{N}} \in c_{00}$ and $A \subseteq \mathbb{N}$, let $\mu_p(x, A) := \big(\sum_{n\in A} |\alpha_n|^p\big)^{\frac{1}{p}}$. The p^{th} Schreier norm of $x \in c_{00}$ defined by (1.1) can then be expressed as

$$\|x\|_{S_p} = \sup\big\{\mu_p(x, A) : A \subseteq \mathbb{N} \text{ is admissible}\big\}.$$

Similarly, for $x = (\alpha_n)_{n\in\mathbb{N}} \in c_{00}$ and $A = \{n_1 < n_2 < \cdots < n_{k+1}\} \subseteq \mathbb{N}$, where $k \in \mathbb{N}$, let $\nu_p(x, A) := \big(\sum_{j=1}^{k} |\alpha_{n_j} - \alpha_{n_{j+1}}|^p\big)^{\frac{1}{p}}$. The p^{th} James–Schreier norm of $x \in c_{00}$ from (1.2) is then given by

$$\|x\|_{V_p} = \sup\big\{\nu_p(x, A) : A \subseteq \mathbb{N} \text{ is permissible}\big\}.$$

We are now ready to embark on the proof of Theorem 2.1. The Schreier counterpart of this theorem — that the standard unit vector basis for S_1 is shrinking — is well-known; a proof can be found in [5, Proposition 3.10]. We shall follow a similar strategy here; more care is, however, required to construct a suitable embedding of V_1 into a space of the form $C(\Omega)$. It should be noted that our proof (specifically, Lemma 2.3) relies on the fact that the standard unit vector basis for S_1 is shrinking.

2.2 Lemma. (i) *Let $A = \{n_1 < n_2 < \cdots < n_{2k}\}$ be a permissible subset of $\mathbb{N}$ of even cardinality. Then the functional*

$$\eta_A\colon\ (\alpha_n)_{n\in\mathbb{N}} \mapsto \sum_{j=1}^{k}(\alpha_{n_{2j-1}} - \alpha_{n_{2j}}), \quad c_{00} \to \mathbb{K},$$

extends to a contractive functional on V_1.

(ii) *For each $x \in c_{00}$, there is a permissible subset A of $\mathbb{N}$ of even cardinality such that $\bigl|\langle x, \eta_A\rangle\bigr| \geqslant \varepsilon\|x\|_{V_1}$, where*

$$\varepsilon := \begin{cases} \frac{1}{2} & \text{for } \mathbb{K} = \mathbb{R} \\ \frac{1}{4} & \text{for } \mathbb{K} = \mathbb{C}. \end{cases} \tag{2.1}$$

Proof. (i). Linearity of η_A is clear, while contractivity follows from the fact that

$$\bigl|\langle x, \eta_A\rangle\bigr| \leqslant \sum_{j=1}^{k}|\alpha_{n_{2j-1}} - \alpha_{n_{2j}}| \leqslant \nu_1(x, A) \leqslant \|x\|_{V_1} \qquad \bigl(x = (\alpha_n)_{n\in\mathbb{N}} \in c_{00}\bigr)$$

because the set A is permissible.

(ii). Suppose that $x = (\alpha_n)_{n\in\mathbb{N}} \in c_{00}$ is non-zero. We shall first consider the case where $\alpha_n \in \mathbb{R}$ for each $n \in \mathbb{N}$. Choose a permissible set $B = \{n_1 < n_2 < \cdots < n_{k+1}\} \subseteq \mathbb{N}$ of minimal cardinality such that $\|x\|_{V_1} = \nu_1(x, B)$. The minimality of $\operatorname{card} B$ ensures that:

(a) $\alpha_{n_j} \neq \alpha_{n_{j+1}}$ for each $j \in \{1, \ldots, k\}$, because if $\alpha_{n_j} = \alpha_{n_{j+1}}$ for some j, then

$$\nu_1(x, B) = \nu_1\bigl(x, B \setminus \{n_j\}\bigr);$$

(b) if $\alpha_{n_j} > \alpha_{n_{j+1}}$ for some $j \in \{1, \ldots, k-1\}$, then $\alpha_{n_{j+1}} < \alpha_{n_{j+2}}$; the reason is that the assumption $\alpha_{n_j} > \alpha_{n_{j+1}} > \alpha_{n_{j+2}}$ would imply that

$$\begin{aligned}\nu_1(x, B) = \sum_{\ell=1}^{j-1}|\alpha_{n_\ell} - \alpha_{n_{\ell+1}}| + (\alpha_{n_j} - \alpha_{n_{j+1}}) + (\alpha_{n_{j+1}} - \alpha_{n_{j+2}}) \\ + \sum_{\ell=j+2}^{k}|\alpha_{n_\ell} - \alpha_{n_{\ell+1}}| = \nu_1\bigl(x, B \setminus \{n_{j+1}\}\bigr);\end{aligned}$$

(c) similarly, if $\alpha_{n_j} < \alpha_{n_{j+1}}$ for some $j \in \{1, \ldots, k-1\}$, then $\alpha_{n_{j+1}} > \alpha_{n_{j+2}}$.

Since $\nu_1(x, B) = \nu_1(-x, B)$, we may suppose that $\alpha_{n_1} > \alpha_{n_2}$; observations (b)–(c) then imply that $\alpha_{n_1} > \alpha_{n_2} < \alpha_{n_3} > \alpha_{n_4} < \cdots$.

We now split in two cases, depending on the parity of k. For k even, we see that

$$\|x\|_{V_1} = \nu_1(x, B) = \sum_{j=1}^{k/2}\bigl((\alpha_{n_{2j-1}} - \alpha_{n_{2j}}) + (\alpha_{n_{2j+1}} - \alpha_{n_{2j}})\bigr) = \bigl|\langle x, \eta_C\rangle\bigr| + \bigl|\langle x, \eta_D\rangle\bigr|, \tag{2.2}$$

where we have introduced $C := \{n_1 < n_2 < \cdots < n_k\}$ and $D := \{n_2 < n_3 < \cdots < n_{k+1}\}$. Each of these two sets is permissible and has even cardinality, and (2.2) implies that either $A := C$ or $A := D$ must satisfy $\bigl|\langle x, \eta_A\rangle\bigr| \geqslant \|x\|_{V_1}/2$.

When k is odd, a similar calculation shows that $\|x\|_{V_1} = \bigl|\langle x, \eta_B\rangle\bigr| + \bigl|\langle x, \eta_E\rangle\bigr|$, where $E := \{n_2 < n_3 < \cdots < n_k\}$. Hence either $A := B$ or $A := E$ satisfies $\bigl|\langle x, \eta_A\rangle\bigr| \geqslant \|x\|_{V_1}/2$, and in both cases A is permissible and has even cardinality. This completes the proof in the real case.

Now suppose that $\mathbb{K} = \mathbb{C}$, and define $y := (\operatorname{Re}\alpha_n)_{n\in\mathbb{N}}$ and $z := (\operatorname{Im}\alpha_n)_{n\in\mathbb{N}}$. Then we have $x = y + \mathrm{i}z$, so that $\|x\|_{V_1} \leqslant \|y\|_{V_1} + \|z\|_{V_1}$ and thus either $\|y\|_{V_1} \geqslant \|x\|_{V_1}/2$ or $\|z\|_{V_1} \geqslant \|x\|_{V_1}/2$. We consider the first case only; the second is similar. As y has real coordinates, the first part of the argument applies, yielding a permissible set A of even cardinality such that $\bigl|\langle y, \eta_A\rangle\bigr| \geqslant \|y\|_{V_1}/2$, and consequently we have

$$\bigl|\langle x, \eta_A\rangle\bigr| = \bigl|\langle y, \eta_A\rangle + \mathrm{i}\langle z, \eta_A\rangle\bigr| \geqslant \bigl|\langle y, \eta_A\rangle\bigr| \geqslant \frac{\|y\|_{V_1}}{2} \geqslant \frac{\|x\|_{V_1}}{4},$$

as required. □

2.3 Lemma. *For each bounded functional f on V_1, the set $\mathbb{E}(f) := \bigl\{n \in \mathbb{N} : \langle e_n, f\rangle = 1\bigr\}$ is finite.*

Proof. For clarity, we write $(d_n)_{n\in\mathbb{N}}$ for the standard unit vector basis for S_1 in this proof, while $(e_n)_{n\in\mathbb{N}}$ denotes the standard basis for V_1, as usual; thus $d_n = e_n$ as vectors, but we regard the former as an element of S_1, while the latter belongs to V_1.

It suffices to prove that each of the sets $\mathbb{E}(f) \cap 2\mathbb{N}$ and $\mathbb{E}(f) \cap (2\mathbb{N} - 1)$ is finite. To verify the first of these assertions, we note that, by [5, Proposition 4.10], we have a bounded operator $\Phi\colon S_1 \to V_1$ given by $\Phi d_n := e_{2n}$ for each $n \in \mathbb{N}$. Denoting by Φ' the adjoint of this operator, we find

$$\langle e_{2n}, f\rangle = \langle \Phi d_n, f\rangle = \langle d_n, \Phi' f\rangle \to 0 \quad \text{as} \quad n \to \infty$$

because the basis $(d_n)_{n\in\mathbb{N}}$ for S_1 is shrinking, and consequently the set $\mathbb{E}(f) \cap 2\mathbb{N}$ is finite.

The second assertion is an easy consequence of this. Indeed, by [5, Proposition 4.18(i)], the left shift given by $\Lambda e_1 := 0$ and $\Lambda e_{n+1} := e_n$ for each $n \in \mathbb{N}$ defines a contractive operator on V_1. Since $\langle e_{2n-1}, f\rangle = \langle \Lambda e_{2n}, f\rangle = \langle e_{2n}, \Lambda' f\rangle$ for each $n \in \mathbb{N}$, we see that $\mathbb{E}(f) \cap (2\mathbb{N} - 1) = \bigl(\mathbb{E}(\Lambda' f) \cap 2\mathbb{N}\bigr) - 1$, and the latter set is finite by the first part of the proof (applied to the functional $\Lambda' f$ instead of f). □

Proof of Theorem 2.1. By a standard characterization of shrinking bases (*e.g.*, see [1, Proposition 3.2.7]), we must prove that every normalized block basic sequence $(u_n)_{n\in\mathbb{N}}$ of the standard basis $(e_n)_{n\in\mathbb{N}}$ for V_1 is weakly null.

The Banach–Alaoglu Theorem implies that the set

$$\Omega := \bigl\{f \in V_1' : \|f\|_{V_1'} \leqslant 1 \text{ and } \langle e_n, f\rangle \in \{0, \pm 1\}\ (n \in \mathbb{N})\bigr\}$$

is a compact Hausdorff space when equipped with the weak*-topology inherited from the dual space V_1' of V_1. By the definition of this topology, the mapping $Ux\colon \Omega \to \mathbb{K}$ given by $(Ux)f := \langle x, f\rangle$ for each $f \in \Omega$ is continuous for each $x \in V_1$, so it induces a mapping $U\colon V_1 \to C(\Omega)$ which is easily seen to be linear and contractive. Moreover, the functional η_A from Lemma 2.2(i) belongs to Ω whenever A is a permissible subset of $\mathbb{N}$ of even cardinality, so Lemma 2.2(ii) implies that U is bounded below by the number ε given by (2.1). In other words, U is an isomorphism of V_1 onto its image inside $C(\Omega)$. Since the weak topology on the image of U is just the restriction of the weak topology on $C(\Omega)$, we conclude that the sequence $(u_n)_{n\in\mathbb{N}}$ is weakly null in V_1 if and only if $(Uu_n)_{n\in\mathbb{N}}$ is weakly null in $C(\Omega)$.

To prove the latter statement, by the Jordan Decomposition, it suffices to verify that $\langle Uu_n, \lambda\rangle \to 0$ as $n \to \infty$ for each state λ on $C(\Omega)$. The Riesz Representation Theorem implies that λ is given by

$$\langle g, \lambda\rangle = \int_\Omega g\,\mathrm{d}\rho \qquad \big(g \in C(\Omega)\big)$$

for some probability measure ρ on Ω. Now we observe that:

(a) for each $f \in \Omega$, the sequence $\big((Uu_n)(f)\big)_{n\in\mathbb{N}} = \big(\langle u_n, f\rangle\big)_{n\in\mathbb{N}}$ is 0 eventually; the reason is that, on the one hand, Lemma 2.3 implies that the set $N := \mathbb{E}(f) \cup \mathbb{E}(-f)$ is finite, and by the definition of Ω, $\langle e_n, f\rangle = 0$ for each $n \in \mathbb{N} \setminus N$, while on the other the fact that $(u_n)_{n\in\mathbb{N}}$ is a block basic sequence of $(e_n)_{n\in\mathbb{N}}$ implies that there is a natural number n_0 such that $u_n \in \operatorname{span}\{e_j : j > \max N\}$ whenever $n \geqslant n_0$;

(b) the constant function 1 is ρ-integrable and dominates $\big(|Uu_n|\big)_{n\in\mathbb{N}}$.

In particular, (a) implies that the sequence $(Uu_n)_{n\in\mathbb{N}}$ converges pointwise to 0 on Ω, and so, by Lebesgue's Dominated Convergence Theorem, we have

$$\langle Uu_n, \lambda\rangle = \int_\Omega Uu_n\,\mathrm{d}\rho \to \int_\Omega 0\,\mathrm{d}\rho = 0 \quad \text{as} \quad n \to \infty,$$

as required. $\square$

2.4 Remark. The fact that the basis for V_1 is shrinking is in sharp contrast to the situation for the first James space J_1. Indeed, J_1 is isomorphic to ℓ_1, so no basis for it can be shrinking. Lohman and Casazza [7] have generalized James' construction to produce quasi-reflexive spaces from Banach spaces with a symmetric basis other than ℓ_p for $p \geqslant 1$; however, as in James' classical case, they only establish that the basis for their new spaces is shrinking when $p > 1$ (see [7, Theorem 9]).

Finally in this section we observe that V_1 does not have the property (u) introduced by Pełczyński [8], thus answering another question left open in [5]. Indeed, since we now know that the standard basis for V_1 is shrinking, we can copy the proof of [5, Theorem 6.3] verbatim to reach the desired conclusion.

2.5 Theorem. *The first James–Schreier space V_1 does not have Pełczyński's property* (u) *and hence does not embed in any Banach space with an unconditional basis. In particular, V_1 does not embed in S_p for any $p \geqslant 1$.*

3 Any two Schreier or James–Schreier spaces with distinct indices are non-isomorphic

Rather than establishing the results stated in the title of this section directly, we take a unified approach based on the following, slightly more general, lemma. As in the proof of Lemma 2.3, we write $(d_n)_{n\in\mathbb{N}}$ for the unit vector basis for S_p, while $(e_n)_{n\in\mathbb{N}}$ denotes the standard basis for V_q.

3.1 Lemma. *Let $q > p \geqslant 1$, and let N be an infinite subset of $\mathbb{N}$. Then no subspace of V_q is isomorphic to the subspace $\overline{\operatorname{span}}\,\{d_n : n \in N\}$ of S_p.*

Proof. Assume towards a contradiction that $R\colon \overline{\operatorname{span}}\,\{d_n : n \in N\} \to V_q$ is a bounded operator which is bounded below by some $\delta > 0$. Then, on the one hand, we have $\|Rd_n\|_{V_q} \geqslant \delta$ for each $n \in N$, while on the other the sequence $(d_n)_{n\in N}$ is weakly null because $(d_n)_{n\in\mathbb{N}}$ is a shrinking basis for S_p, and therefore $(Rd_n)_{n\in N}$ is also weakly null. Hence the Bessaga–Pełczyński Selection Principle [4] implies that a subsequence of $(Rd_n)_{n\in N}$ is a basic sequence equivalent to a block basic sequence of $(e_n)_{n\in\mathbb{N}}$; that is, there exist a strictly increasing mapping $\sigma\colon \mathbb{N} \to N$ and a bounded operator $T\colon \overline{\operatorname{span}}\,\{Rd_{\sigma(n)} : n \in \mathbb{N}\} \to V_q$ such that T is bounded below by some $\varepsilon > 0$ and $(TRd_{\sigma(n)})_{n\in\mathbb{N}}$ is a block basic sequence of $(e_n)_{n\in\mathbb{N}}$. By [5, Lemma 4.13], this means in particular that we have a bounded operator $U\colon \ell_q \to V_q$ given by $Uf_n = TRd_{\sigma(n)}/\|TRd_{\sigma(n)}\|_{V_q}$ for each $n \in \mathbb{N}$, where $(f_n)_{n\in\mathbb{N}}$ denotes the standard unit vector basis for ℓ_q. Thus, we conclude that

$$\|U\|\,\|T\|\,\|R\|\,n^{\frac{1}{q}} \geqslant \left\| U\Bigl(\sum_{j=n}^{2n-1} \|TRd_{\sigma(j)}\|_{V_q} f_j\Bigr)\right\|_{V_q} \geqslant \varepsilon\delta \left\|\sum_{j=n}^{2n-1} d_{\sigma(j)}\right\|_{S_p} = \varepsilon\delta n^{\frac{1}{p}}, \tag{3.1}$$

where the final equality follows from the admissibility of the set $\sigma\bigl([n, 2n-1] \cap \mathbb{N}\bigr)$ on which the vector $\sum_{j=n}^{2n-1} d_{\sigma(j)}$ is supported. Rearranging (3.1), we obtain

$$\frac{\|U\|\,\|T\|\,\|R\|}{\varepsilon\delta} \geqslant n^{\frac{1}{p}-\frac{1}{q}},$$

which is a contradiction because the left-hand side is independent of n, while the right-hand side tends to infinity as $n \to \infty$. $\square$

3.2 Theorem. *Let $q > p \geqslant 1$. Then:*

(i) *no subspace of V_q is isomorphic to V_p or S_p;*

(ii) *no subspace of S_q is isomorphic to S_p or V_p.*

Proof. (i). Taking $N = \mathbb{N}$ in Lemma 3.1, we see that no subspace of V_q is isomorphic to S_p. Since V_p contains a subspace isomorphic to S_p by [5, Proposition 4.10], this in turn implies that no subspace of V_q can be isomorphic to V_p.

(ii). If S_q contained a subspace isomorphic to S_p, then by the above-mentioned result from [5], V_q would also contain a subspace isomorphic to S_p, contradicting (i). Finally, for similar reasons S_q cannot contain a subspace isomorphic to V_p. $\square$

3.3 Corollary. *Let $p, q \geqslant 1$ be distinct. Then $V_p \not\cong V_q$ and $S_p \not\cong S_q$.*

3.4 Remark. (i) The fact stated in Theorem 3.2(ii) that no subspace of S_q is isomorphic to V_p when $q > p \geqslant 1$ is actually true without any restrictions on $p, q \geqslant 1$. The reason is that the James–Schreier spaces all fail to have Pełczyński's property (u) by Theorem 2.5 and [5, Theorem 6.3], while each Schreier space has an unconditional basis.

(ii) The conclusion of Lemma 3.1 (and thus that of Theorem 3.2) actually holds whenever $p, q \geqslant 1$ are distinct. As this was not needed to prove our main result, Corollary 3.3, we just give a brief sketch of the argument, which is by contradiction. As in the proof of Lemma 3.1, we obtain a normalized basic sequence (u_n) in V_q which is equivalent to a subsequence of the unit vector basis for S_p. It follows that (u_n) is *Schreier ℓ_p* which means that there is a constant $C > 0$ such that $(u_i)_{i\in A}$ is C-equivalent to the unit vector basis of ℓ_p^k for every admissible subset A of $\mathbb{N}$, where $k := \operatorname{card} A$.

There are now two cases. If, after passing to a subsequence, we have $||u_n||_{c_0} \to 0$ as $n \to \infty$, then a further subsequence of (u_n) is equivalent to the unit vector basis of c_0; the proof of this is similar to that of V_q being c_0-saturated given in [5, Theorem 5.2]. Otherwise there exists $\delta > 0$ such that $||u_n||_{c_0} > \delta$ for each $n \in \mathbb{N}$. An easy computation then shows that $(u_{2i})_{i \in A}$ is $\frac{3}{\delta}$-equivalent to the unit vector basis of ℓ_q^k for every admissible subset A of $\mathbb{N}$, where $k := \operatorname{card} A$. Both cases contradict the fact that (u_n) is a Schreier ℓ_p sequence.

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