Some remarks on James–Schreier spaces

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Abstract

The James–Schreier spaces $V_p$, where $1 \leq p < \infty$, were recently introduced by Bird and Laustsen [5] as an amalgamation of James’ quasi-reflexive Banach space on the one hand and Schreier’s Banach space giving a counterexample to the Banach–Saks property on the other. The purpose of this note is to answer some questions left open in [5]. Specifically, we prove that (i) the standard Schauder basis for the first James–Schreier space $V_1$ is shrinking, and (ii) any two Schreier or James–Schreier spaces with distinct indices are non-isomorphic. The former of these results implies that $V_1$ does not have Pełczyński’s property (u) and hence does not embed in any Banach space with an unconditional Schauder basis.

2000 Mathematics Subject Classification: primary 46B03, 46B45; secondary 46B15.

Key words and phrases: Banach space, James–Schreier space, Schreier space, shrinking Schauder basis, Pełczyński’s property (u).


1 Introduction

Let $1 \leq p < \infty$. By the $p^{th}$ Schreier space, denoted $S_p$, we understand the Banach space obtained by completing $c_{00}$ (the vector space of finitely supported scalar sequences) with respect to the norm

$$
\|x\|_{S_p} := \sup \left\{ \left( \sum_{j=1}^{k} |\alpha_{n_j}|^p \right)^{1/p} : k, n_1, \ldots, n_k \in \mathbb{N}, k \leq n_1 < n_2 < \cdots < n_k \right\},
$$

where $x = (\alpha_n)_{n \in \mathbb{N}} \in c_{00}$. The space $S_1$ is the one which is usually known as the Schreier space in the literature; it was formally introduced by Beauzamy and Lapresté [3], building on ideas from Baernstein’s thesis [2], which in turn were inspired by Schreier’s seminal construction [9].

The Schreier spaces have recently been amalgamated with James’ quasi-reflexive Banach spaces [6] by Bird and Laustsen [5]. More precisely, for $1 \leq p < \infty$, the $p^{th}$ James–Schreier space, denoted $V_p$, is the completion of $c_{00}$ with respect to the norm

$$
\|x\|_{V_p} := \sup \left\{ \left( \sum_{j=1}^{k} |\alpha_{n_j} - \alpha_{n_{j+1}}|^p \right)^{1/p} : k, n_1, \ldots, n_{k+1} \in \mathbb{N}, k \leq n_1 < n_2 < \cdots < n_{k+1} \right\},
$$

where $x = (\alpha_n)_{n \in \mathbb{N}} \in c_{00}$. We refer to [5] for the background and motivation behind these spaces, as well as a thorough study of their fundamental properties. The purpose of this paper is to resolve two problems left open in [5].
First, it was shown in [5] that \((e_n)_{n \in \mathbb{N}}\), where \(e_n \in c_{00}\) is the sequence with 1 in position \(n\) and 0 elsewhere, is a Schauder basis for \(V_p\) for each \(p \geq 1\) and, moreover, that this basis is shrinking (meaning that the associated sequence of biorthogonal functionals \((e'_n)_{n \in \mathbb{N}}\) is a Schauder basis for the dual space \(V'_p\)) whenever \(p > 1\). The question of whether or not the basis \((e_n)_{n \in \mathbb{N}}\) is shrinking for \(p = 1\) was left open; in Section 2 we answer this question in the positive. As a consequence, we deduce that \(V_1\) does not have Pelczyński’s property (u) and hence does not embed in a Banach space with an unconditional Schauder basis.

Second, regarding embeddings and isomorphisms of Schreier and James–Schreier spaces, it was proved in [5] that:

(i) for each \(p \geq 1\), \(S_p\) is isomorphic to a complemented subspace of \(V_p\);

(ii) for each \(p > 1\), \(V_p\) does not embed in \(S_q\) for any \(q \geq 1\); this result extends to the case \(p = 1\) by the conclusions of Section 2 of the present paper.

We complete this picture in Section 3 by proving that, for \(q > p \geq 1\), no subspace of \(V_q\) is isomorphic to \(S_p\), and consequently no subspace of \(S_q\) is isomorphic to \(S_p\), and no subspace of \(V_q\) is isomorphic to \(V_p\). In particular, \(S_p \not\cong S_q\) and \(V_p \not\cong V_q\) whenever \(p \neq q\).

2 The standard basis for the first James–Schreier space is shrinking

As the title indicates, the aim of this section is to prove the following result.

2.1 Theorem. The standard Schauder basis \((e_n)_{n \in \mathbb{N}}\) for \(V_1\) is shrinking.

The proof of Theorem 2.1 relies on two lemmas. Before presenting these, we recall some notation and terminology from [5]. Throughout, \(K\) denotes the scalar field; either \(K = \mathbb{R}\) or \(K = \mathbb{C}\). We write \(\text{card} A\) for the cardinality of a (typically finite) set \(A\). Suppose that \(A\) is a subset of \(\mathbb{N}\). We then write \(A = \{n_1 < n_2 < \cdots < n_k\}\) to indicate that \(A\) is finite and non-empty and that \(\{n_1, n_2, \ldots, n_k\}\) is the increasing ordering of \(A\). We say that \(A\) is admissible if \(1 \leq \text{card} A \leq \text{min} A\) and permissible if \(2 \leq \text{card} A \leq 1 + \text{min} A\). Thus a typical admissible set has the form \(\{n_1 < n_2 < \cdots < n_k\}\), where \(1 \leq k \leq n_1\), while a typical permissible set can be written as \(\{n_1 < n_2 < \cdots < n_{k+1}\}\), again with \(1 \leq k \leq n_1\).

Now let \(1 \leq p < \infty\). For \(x = (\alpha_n)_{n \in \mathbb{N}} \in c_{00}\) and \(A \subseteq \mathbb{N}\), let \(\mu_p(x, A) := (\sum_{n \in A} |\alpha_n|^p)^{\frac{1}{p}}\). The \(p^{\text{th}}\) Schreier norm of \(x \in c_{00}\) defined by (1.1) can then be expressed as

\[\|x\|_{s_p} = \text{sup}\{\mu_p(x, A) : A \subseteq \mathbb{N} \text{ is admissible}\} .\]

Similarly, for \(x = (\alpha_n)_{n \in \mathbb{N}} \in c_{00}\) and \(A = \{n_1 < n_2 < \cdots < n_{k+1}\} \subseteq \mathbb{N}\), where \(k \in \mathbb{N}\), let \(\nu_p(x, A) := (\sum_{j=1}^{k} |\alpha_{n_j} - \alpha_{n_{j+1}}|^p)^{\frac{1}{p}}\). The \(p^{\text{th}}\) James–Schreier norm of \(x \in c_{00}\) from (1.2) is then given by

\[\|x\|_{v_p} = \text{sup}\{\nu_p(x, A) : A \subseteq \mathbb{N} \text{ is permissible}\} .\]

We are now ready to embark on the proof of Theorem 2.1. The Schreier counterpart of this theorem — that the standard unit vector basis for \(S_1\) is shrinking — is well-known; a proof can be found in [5, Proposition 3.10]. We shall follow a similar strategy here; more care is, however, required to construct a suitable embedding of \(V_1\) into a space of the form \(C(\Omega)\).

It should be noted that our proof (specifically, Lemma 2.3) relies on the fact that the standard unit vector basis for \(S_1\) is shrinking.
2.2 Lemma. (i) Let \( A = \{n_1 < n_2 < \cdots < n_{2k}\} \) be a permissible subset of \( \mathbb{N} \) of even cardinality. Then the functional

\[
\eta_A: (\alpha_n)_{n \in \mathbb{N}} \mapsto \sum_{j=1}^{k} (\alpha_{n_{2j-1}} - \alpha_{n_{2j}}), \quad c_{00} \to \mathbb{K},
\]

extends to a contractive functional on \( V_1 \).

(ii) For each \( x \in c_{00} \), there is a permissible subset \( A \) of \( \mathbb{N} \) of even cardinality such that

\[
\left| \langle x, \eta_A \rangle \right| \geq \varepsilon \|x\|_{V_1},
\]

where we have introduced \( \varepsilon := \frac{1}{2} \) for \( \mathbb{K} = \mathbb{R} \) and \( \varepsilon := \frac{1}{4} \) for \( \mathbb{K} = \mathbb{C} \).

Proof. (i). Linearity of \( \eta_A \) is clear, while contractivity follows from the fact that

\[
\left| \langle x, \eta_A \rangle \right| \leq \sum_{j=1}^{k} |\alpha_{n_{2j-1}} - \alpha_{n_{2j}}| \leq \nu_1(x, A) \leq \|x\|_{V_1} \quad (x = (\alpha_n)_{n \in \mathbb{N}} \in c_{00})
\]

because the set \( A \) is permissible.

(ii). Suppose that \( x = (\alpha_n)_{n \in \mathbb{N}} \in c_{00} \) is non-zero. We shall first consider the case where \( \alpha_n \in \mathbb{R} \) for each \( n \in \mathbb{N} \). Choose a permissible set \( B = \{n_1 < n_2 < \cdots < n_k\} \subseteq \mathbb{N} \) of minimal cardinality such that \( \|x\|_{V_1} = \nu_1(x, B) \). The minimality of \( \text{card } B \) ensures that:

(a) \( \alpha_{n_j} \neq \alpha_{n_{j+1}} \) for each \( j \in \{1, \ldots, k\} \), because if \( \alpha_{n_j} = \alpha_{n_{j+1}} \) for some \( j \), then

\[
\nu_1(x, B) = \nu_1(x, B \setminus \{n_j\});
\]

(b) if \( \alpha_{n_j} > \alpha_{n_{j+1}} \) for some \( j \in \{1, \ldots, k-1\} \), then \( \alpha_{n_{j+1}} < \alpha_{n_{j+2}} \); the reason is that the assumption \( \alpha_{n_j} > \alpha_{n_{j+1}} > \alpha_{n_{j+2}} \) would imply that

\[
\nu_1(x, B) = \sum_{\ell=1}^{j-1} |\alpha_{n_\ell} - \alpha_{n_{\ell+1}}| + (\alpha_{n_j} - \alpha_{n_{j+1}}) + (\alpha_{n_{j+1}} - \alpha_{n_{j+2}}) + \sum_{\ell=j+2}^{k} |\alpha_{n_\ell} - \alpha_{n_{\ell+1}}| = \nu_1(x, B \setminus \{n_j+1\});
\]

(c) similarly, if \( \alpha_{n_j} < \alpha_{n_{j+1}} \) for some \( j \in \{1, \ldots, k-1\} \), then \( \alpha_{n_{j+1}} > \alpha_{n_{j+2}} \).

Since \( \nu_1(x, B) = \nu_1(-x, B) \), we may suppose that \( \alpha_{n_1} > \alpha_{n_2} \); observations (b)–(c) then imply that \( \alpha_{n_1} > \alpha_{n_2} < \alpha_{n_3} > \alpha_{n_4} < \cdots \).

We now split in two cases, depending on the parity of \( k \). For \( k \) even, we see that

\[
\|x\|_{V_1} = \nu_1(x, B) = \sum_{j=1}^{k/2} ((\alpha_{n_{2j-1}} - \alpha_{n_{2j}}) + (\alpha_{n_{2j+1}} - \alpha_{n_{2j}})) = \left| \langle x, \eta C \rangle \right| + \left| \langle x, \eta D \rangle \right|,
\]

where we have introduced \( C := \{n_1 < n_2 < \cdots < n_k\} \) and \( D := \{n_2 < n_3 < \cdots < n_{k+1}\} \). Each of these two sets is permissible and has even cardinality, and (2.2) implies that either \( A := C \) or \( A := D \) must satisfy \( \left| \langle x, \eta A \rangle \right| \geq \|x\|_{V_1}/2 \).
When \( k \) is odd, a similar calculation shows that \( \|x\|_{V_1} = |\langle x, \eta_B \rangle| + |\langle x, \eta_E \rangle| \), where \( E := \{n_2 < n_3 < \cdots < n_k\} \). Hence either \( A := B \) or \( A := E \) satisfies \( |\langle x, \eta_A \rangle| \geq \|x\|_{V_1}/2 \), and in both cases \( A \) is permissible and has even cardinality. This completes the proof in the real case.

Now suppose that \( \mathbb{K} = \mathbb{C} \), and define \( y := (\text{Re} \, \alpha_n)_{n \in \mathbb{N}} \) and \( z := (\text{Im} \, \alpha_n)_{n \in \mathbb{N}} \). Then we have \( x = y + iz \), so that \( \|x\|_{V_1} \leq \|y\|_{V_1} + \|z\|_{V_1} \) and thus either \( \|y\|_{V_1} \geq \|x\|_{V_1}/2 \) or \( \|z\|_{V_1} \geq \|x\|_{V_1}/2 \). We consider the first case only; the second is similar. As \( y \) has real coordinates, the first part of the argument applies, yielding a permissible set \( A \) of even cardinality such that \( |\langle y, \eta_A \rangle| \geq \|y\|_{V_1}/2 \), and consequently we have

\[
|\langle x, \eta_A \rangle| = |\langle y, \eta_A \rangle + i(z, \eta_A)| \geq \|y\|_{V_1}/2 \geq \|x\|_{V_1}/4,
\]

as required. \( \square \)

2.3 Lemma. For each bounded functional \( f \) on \( V_1 \), the set \( \mathcal{E}(f) := \{n \in \mathbb{N} : \langle e_n, f \rangle = 1\} \) is finite.

Proof. For clarity, we write \((d_n)_{n \in \mathbb{N}}\) for the standard unit vector basis for \( S_1 \) in this proof, while \((e_n)_{n \in \mathbb{N}}\) denotes the standard basis for \( V_1 \), as usual; thus \( d_n = e_n \) as vectors, but we regard the former as an element of \( S_1 \), while the latter belongs to \( V_1 \).

It suffices to prove that each of the sets \( \mathcal{E}(f) \cap 2\mathbb{N} \) and \( \mathcal{E}(f) \cap (2\mathbb{N} - 1) \) is finite. To verify the first of these assertions, we note that, by [5, Proposition 4.10], we have a bounded operator \( \Phi : S_1 \to V_1 \) given by \( \Phi d_n := e_{2n} \) for each \( n \in \mathbb{N} \). Denoting by \( \Phi' \) the adjoint of this operator, we find

\[
\langle e_{2n}, f \rangle = \langle \Phi d_n, f \rangle = \langle d_n, \Phi' f \rangle \to 0 \quad \text{as} \quad n \to \infty
\]

because the basis \((d_n)_{n \in \mathbb{N}}\) for \( S_1 \) is shrinking, and consequently the set \( \mathcal{E}(f) \cap 2\mathbb{N} \) is finite.

The second assertion is an easy consequence of this. Indeed, by [5, Proposition 4.18(i)], the left shift given by \( \lambda e_1 := 0 \) and \( \lambda e_{n+1} := e_n \) for each \( n \in \mathbb{N} \) defines a contractive operator on \( V_1 \). Since \((e_{2n-1}, f) = \langle \lambda e_{2n}, f \rangle = \langle e_{2n}, \lambda' f \rangle \) for each \( n \in \mathbb{N} \), we see that \( \mathcal{E}(f) \cap (2\mathbb{N} - 1) = (\mathcal{E}(\lambda' f) \cap 2\mathbb{N}) - 1 \), and the latter set is finite by the first part of the proof (applied to the functional \( \lambda' f \) instead of \( f \)). \( \square \)

Proof of Theorem 2.1. By a standard characterization of shrinking bases (e.g., see [1, Proposition 3.2.7]), we must prove that every normalized block basic sequence \((u_n)_{n \in \mathbb{N}}\) of the standard basis \((e_n)_{n \in \mathbb{N}}\) for \( V_1 \) is weakly null.

The Banach–Alaoglu Theorem implies that the set

\[
\Omega := \{f \in V_1' : \|f\|_{V_1'} \leq 1 \text{ and } \langle e_n, f \rangle \in \{0, \pm 1\} \quad (n \in \mathbb{N})\}
\]

is a compact Hausdorff space when equipped with the weak*-topology inherited from the dual space \( V_1' \) of \( V_1 \). By the definition of this topology, the mapping \( U : \Omega \to \mathbb{K} \) given by \( (Ux)f := \langle x, f \rangle \) for each \( f \in \Omega \) is continuous for each \( x \in V_1 \), so it induces a mapping \( U : V_1 \to C(\Omega) \) which is easily seen to be linear and contractive. Moreover, the functional \( \eta_A \) from Lemma 2.2(i) belongs to \( \Omega \) whenever \( A \) is a permissible subset of \( \mathbb{N} \) of even cardinality, so Lemma 2.2(ii) implies that \( U \) is bounded above by the number \( \varepsilon \) given by (2.1). In other words, \( U \) is an isomorphism of \( V_1 \) onto its image inside \( C(\Omega) \). Since the weak topology on the image of \( U \) is just the restriction of the weak topology on \( C(\Omega) \), we conclude that the sequence \((u_n)_{n \in \mathbb{N}}\) is weakly null in \( V_1 \) if and only if \((Uu_n)_{n \in \mathbb{N}}\) is weakly null in \( C(\Omega) \).
To prove the latter statement, by the Jordan Decomposition, it suffices to verify that \( \langle U u_n, \lambda \rangle \to 0 \) as \( n \to \infty \) for each state \( \lambda \) on \( C(\Omega) \). The Riesz Representation Theorem implies that \( \lambda \) is given by
\[
\langle g, \lambda \rangle = \int_\Omega g \, d\rho \quad (g \in C(\Omega))
\]
for some probability measure \( \rho \) on \( \Omega \). Now we observe that:

(a) for each \( f \in \Omega \), the sequence \( \left( \langle U u_n(f) \rangle_{n \in \mathbb{N}} = \left( \langle u_n, f \rangle \right)_{n \in \mathbb{N}} \right) \) is 0 eventually; the reason is that, on the one hand, Lemma 2.3 implies that the set \( N := \mathbb{E}(f) \cup \mathbb{E}(-f) \) is finite, and by the definition of \( \Omega \), \( \langle e_n, f \rangle = 0 \) for each \( n \in \mathbb{N} \setminus N \), while on the other the fact that \( (u_n)_{n \in \mathbb{N}} \) is a block basic sequence of \( (e_n)_{n \in \mathbb{N}} \) implies that there is a natural number \( n_0 \) such that \( u_n \in \text{span} \{ e_j : j > \max N \} \) whenever \( n \geq n_0 \);

(b) the constant function 1 is \( \rho \)-integrable and dominates \( \left( |U u_n| \right)_{n \in \mathbb{N}} \).

In particular, (a) implies that the sequence \( (U u_n)_{n \in \mathbb{N}} \) converges pointwise to 0 on \( \Omega \), and so, by Lebesgue’s Dominated Convergence Theorem, we have
\[
\langle U u_n, \lambda \rangle = \int_\Omega U u_n \, d\rho \to \int_\Omega 0 \, d\rho = 0 \quad \text{as} \quad n \to \infty,
\]
as required.

2.4 Remark. The fact that the basis for \( V_1 \) is shrinking is in sharp contrast to the situation for the first James space \( J_1 \). Indeed, \( J_1 \) is isomorphic to \( \ell_1 \), so no basis for it can be shrinking. Lohman and Casazza \cite{7} have generalized James’ construction to produce quasi-reflexive spaces from Banach spaces with a symmetric basis other than \( \ell_p \) for \( p \geq 1 \); however, as in James’ classical case, they only establish that the basis for their new spaces is shrinking when \( p > 1 \) (see \cite[Theorem 9]{7}).

Finally in this section we observe that \( V_1 \) does not have the property \( (u) \) introduced by Pełczyński \cite{8}, thus answering another question left open in \cite{5}. Indeed, since we now know that the standard basis for \( V_1 \) is shrinking, we can copy the proof of \cite[Theorem 6.3]{5} verbatim to reach the desired conclusion.

2.5 Theorem. The first James–Schreier space \( V_1 \) does not have Pełczyński’s property \( (u) \) and hence does not embed in any Banach space with an unconditional basis. In particular, \( V_1 \) does not embed in \( S_p \) for any \( p \geq 1 \).

3 Any two Schreier or James–Schreier spaces with distinct indices are non-isomorphic

Rather than establishing the results stated in the title of this section directly, we take a unified approach based on the following, slightly more general, lemma. As in the proof of Lemma 2.3, we write \( (d_n)_{n \in \mathbb{N}} \) for the unit vector basis for \( S_p \), while \( (e_n)_{n \in \mathbb{N}} \) denotes the standard basis for \( V_q \).

3.1 Lemma. Let \( q > p \geq 1 \), and let \( N \) be an infinite subset of \( \mathbb{N} \). Then no subspace of \( V_q \) is isomorphic to the subspace \( \text{span} \{ d_n : n \in N \} \) of \( S_p \).
Proof. Assume towards a contradiction that \( R : \overline{\text{span}} \{ d_n : n \in N \} \to V_q \) is a bounded operator which is bounded below by some \( \delta > 0 \). Then, on the one hand, we have \( \| R d_n \|_{V_q} \geq \delta \) for each \( n \in N \), while on the other the sequence \( (d_n)_{n \in N} \) is weakly null because \( (d_n)_{n \in N} \) is a shrinking basis for \( S_p \), and therefore \( (Rd_n)_{n \in N} \) is also weakly null. Hence the Bessaga–Pełczyński Selection Principle [4] implies that a subsequence of \((Rd_n)_{n \in N}\) is a basic sequence equivalent to a block basic sequence of \((e_n)_{n \in N}\); that is, there exist a strictly increasing mapping \( \sigma : N \to N \) and a bounded operator \( T : \overline{\text{span}} \{ Rd_{\sigma(n)} : n \in N \} \to V_q \) such that \( T \) is bounded below by some \( \varepsilon > 0 \) and \((TRd_{\sigma(n)})_{n \in N}\) is a block basic sequence of \((e_n)_{n \in N}\). By [5, Lemma 4.13], this means in particular that we have a bounded operator \( U : \ell_q \to V_q \) given by \( Uf_n = TRd_{\sigma(n)}/\|TRd_{\sigma(n)}\|_{V_q} \) for each \( n \in N \), where \((f_n)_{n \in N}\) denotes the standard unit vector basis for \( \ell_q \). Thus, we conclude that

\[
\|U\| \|T\| \|R\| n^\frac{1}{p} \geq \left\| U \left( \sum_{j=n}^{2n-1} \|TRd_{\sigma(j)}\|_{V_q} f_j \right) \right\|_{V_q} \geq \varepsilon \delta \left\| \sum_{j=n}^{2n-1} d_{\sigma(j)} \right\|_{S_p} = \varepsilon \delta n^\frac{1}{p}, \tag{3.1}
\]

where the final equality follows from the admissibility of the set \( \sigma([n,2n-1] \cap N) \) on which the vector \( \sum_{j=n}^{2n-1} d_{\sigma(j)} \) is supported. Rearranging (3.1), we obtain

\[
\frac{\|U\| \|T\| \|R\|}{\varepsilon \delta} \geq n^{\frac{1}{p} - \frac{1}{q}},
\]

which is a contradiction because the left-hand side is independent of \( n \), while the right-hand side tends to infinity as \( n \to \infty \). \( \square \)

3.2 Theorem. Let \( q > p \geq 1 \). Then:

(i) no subspace of \( V_q \) is isomorphic to \( V_p \) or \( S_p \);

(ii) no subspace of \( S_q \) is isomorphic to \( S_p \) or \( V_p \).

Proof. (i). Taking \( N = N \) in Lemma 3.1, we see that no subspace of \( V_q \) is isomorphic to \( S_p \). Since \( V_p \) contains a subspace isomorphic to \( S_p \) by [5, Proposition 4.10], this in turn implies that no subspace of \( V_q \) can be isomorphic to \( V_p \).

(ii). If \( S_q \) contained a subspace isomorphic to \( S_p \), then by the above-mentioned result from [5], \( V_q \) would also contain a subspace isomorphic to \( S_p \), contradicting (i). Finally, for similar reasons \( S_q \) cannot contain a subspace isomorphic to \( V_p \). \( \square \)

3.3 Corollary. Let \( p, q \geq 1 \) be distinct. Then \( V_p \not\cong V_q \) and \( S_p \not\cong S_q \).

3.4 Remark. (i) The fact stated in Theorem 3.2(ii) that no subspace of \( S_q \) is isomorphic to \( V_p \) when \( q > p \geq 1 \) is actually true without any restrictions on \( p, q \geq 1 \). The reason is that the James–Schreier spaces all fail to have Pelczynski’s property (u) by Theorem 2.5 and [5, Theorem 6.3], while each Schreier space has an unconditional basis.

(ii) The conclusion of Lemma 3.1 (and thus that of Theorem 3.2) actually holds whenever \( p, q \geq 1 \) are distinct. As this was not needed to prove our main result, Corollary 3.3, we just give a brief sketch of the argument, which is by contradiction. As in the proof of Lemma 3.1, we obtain a normalized basic sequence \( (u_n) \) in \( V_q \) which is equivalent to a subsequence of the unit vector basis for \( S_p \). It follows that \( (u_n) \) is Schreier \( \ell_p \) which means that there is a constant \( C > 0 \) such that \( (u_i)_{i \in A} \) is \( C \)-equivalent to the unit vector basis of \( \ell_p^k \) for every admissible subset \( A \) of \( N \), where \( k := \text{card} \ A \).
There are now two cases. If, after passing to a subsequence, we have \( \|u_n\|_{c_0} \to 0 \) as \( n \to \infty \), then a further subsequence of \((u_n)\) is equivalent to the unit vector basis of \( c_0 \); the proof of this is similar to that of \( V_q \) being \( c_0 \)-saturated given in [5, Theorem 5.2]. Otherwise there exists \( \delta > 0 \) such that \( \|u_n\|_{c_0} \geq \delta \) for each \( n \in \mathbb{N} \). An easy computation then shows that \((u_{2n})_{n \in A}\) is \( \frac{3}{\delta} \)-equivalent to the unit vector basis of \( \ell^k_q \) for every admissible subset \( A \) of \( \mathbb{N} \), where \( k := \text{card } A \). Both cases contradict the fact that \((u_n)\) is a Schreier \( \ell_p \) sequence.

Acknowledgements

We gratefully acknowledge the financial support from the EPSRC (Bird: grants EP/P500303/1, EP/P501482/1 and EP/P502656X/1; Laustsen and Zsák: grant EP/F023537/1) that has enabled us to carry out the research which this paper is based upon.

We should also like to thank Hung Le Pham for some helpful discussions regarding the results presented in Section 3.

References


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