Dykmak et al. Reply: The phase shift between the periodic response of a system (the signal) and the periodic driving force that gives rise to it is defined uniquely in statistical physics. The ensemble-averaged signal \( \langle q(t) \rangle = \sum_n a(n) \cos[n \Omega t + \phi(n)] \), where \( 2\pi/\Omega \) is the period of the force. All of the phases \( \phi(n) \) can be measured experimentally. In the commonly considered case of a cosine force \( A \cos(\Omega t) \) and a nearly cosine signal, the term “phase shift” refers to \( \phi \equiv \phi(1) \). There is no ambiguity about this; neither are there two different phase shifts [1]. It was \( \phi \) that was investigated both in [2] and in earlier theoretical papers [3,4]. Provided the periodic force is weak, \( \phi \) can be expressed in terms of a linear susceptibility [5]. It was the finiteness of the phase \( \phi \) that Gammaitoni et al. claimed [6], wrongly [2], to have been “ruled out as apparently spurious” in stochastic resonance.

The topic of our Letter [2] was phase shifts in stochastic resonance (SR), a noise-induced enhancement of the signal-to-noise ratio \( R \) that is significant when [4] \( \Omega \ll \tau_r^{-1} \), where \( \tau_r^{-1} (= 1 \text{ for the overdamped bistable system of [2]}) \) is the reciprocal intrawell relaxation time (not “librational frequency”). In the range \( \Omega \tau_r > 1 \) [1], on the other hand (actually, \( \Omega \tau_r > 0.5 \) for the system of [2]), SR does not occur; see Fig. 1, inset. In contrast to the exponentially fast rise of \( -\phi(D) \) with increasing \( D \) (followed by a slower decrease) observed [2] for small \( A \) and \( \Omega \), \( -\phi(D) \) for large \( \Omega \) (Fig. 1) displays a much shallower maximum (but nonetheless increases, rather than decreases [1], for small \( D \); we have noted that the signal, too, initially increases with \( D \)). The steep initial rise of \( -\phi(D) \) for small \( \Omega \) [2] is associated with the onset of the noise-induced interwell transitions that are responsible for SR; moreover, \( \phi \) is evidently a more sensitive indicator of these transitions than \( R \). The monotonic decrease of \( -\phi(D) \) in [7] does not contradict this result, because the signal from the experiment had apparently [8] been passed through a two-state filter prior to analysis, thus removing the effect of the intrawell vibrations and mimicking the two-state approximation of earlier theories [3,4].

Applied consistently to the SR problem—which involves more than merely a linearization of the transition probabilities as suggested in [9]—our linear response theory (LRT) approach has been shown [2,10] to yield good agreement with experimental measurements of the amplitude and phase of the signal over a wide range of parameters for small amplitudes of the force.

We acknowledge valuable discussions with N. D. Stein.

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Received 2 November 1992
PACS numbers: 05.40.+j, 02.50.-r

[8] L. Gammaitoni, in reply to a question at NATO ARW on SR in San Diego, April 1992; (private communication).

FIG. 1. Phase shift \( -\phi \) of the signal induced by a weak periodic force of frequency \( \Omega \) in the overdamped bistable system (1) of [2], as a function of noise intensity \( D \). Inset: Signal-to-noise ratio \( R \) as a function of \( D \) at large \( \Omega \). The data points are by digital simulation; the curves are LRT based on Eqs. (6)–(10) of [2], to first order in \( L_n(\omega) \).