On linear systems and τ functions associated with Lamé's equation and Painlevé's equation VI

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ABSTRACT Painlevé's transcendental differential equation P_{VI} may be expressed as the consistency condition for a pair of linear differential equations with 2×2 matrix coefficients with rational entries. By a construction due to Tracy and Widom, this linear system is associated with certain kernels which give trace class operators on Hilbert space. This paper expresses such operators in terms of Hankel operators Γ_ϕ of linear systems which are realised in terms of the Laurent coefficients of the solutions of the differential equations. For such, the Fredholm determinant $\det(I-\Gamma_\phi)$ gives rise to the τ function, which is expressed in terms of the solution of a matrix Gelfand Levitan equation. For suitable values of the parameters, solutions of the hypergeometric equation give a linear system with similar properties. For meromorphic transfer functions $\hat{\phi}$ that have poles on an arithmetic progression, the corresponding Hankel operator has a simple form with respect to an exponential basis in $L^2(0,\infty)$; so $\det(I-\Gamma_\phi P_{(t,\infty)})$ can be expressed as a series of finite determinants. This applies to elliptic functions of the second kind, such as satisfy Lamé's differential equation.

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1. Introduction

Tracy and Widom [29] observed that many important kernels in random matrix theory arise from solutions of linear differential equations with rational coefficients. In this paper, we extend the scope of their investigation by analysing kernels associated with Lamé's equation and Painlevé's equation VI. As these differential equations have solutions which may be expressed in terms of elliptic functions, we begin by reviewing and extending the definitions from [29].

Let $P(x,y)$ be an irreducible complex polynomial, and n the degree of $P(x,y)$ as a polynomial in y. Then we introduce the curve $\mathcal{E}=\{(\lambda,\mu)\in\mathbf{C}:P(\lambda,\mu)=0\}$, and observe that $\mathcal{E}\cup\{(\infty,\infty)\}$ gives a compact Riemann surface which is the n-sheeted branched cover of Riemann's sphere $\mathbf{P}^1$. Let $\mathbf{K}$ be splitting field of $P(x,y)$ over $\mathbf{C}(x)$, so we can regard $\mathbf{K}$

as the space of functions of rational character on $\mathcal{E}$. Let g be the genus of $\mathcal{E}$, and introduce the Jacobi variety $\mathbf{J}$ of $\mathcal{E}$, which is the quotient of $\mathbf{C}^g$ by some lattice $\mathbf{L}$ in $\mathbf{C}^g$.

Definition. By a Tracy–Widom system [29] we mean a differential equation

$$\frac{d}{dx}\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}\begin{bmatrix} f \\ g \end{bmatrix} \tag{1.1}$$

where α, β, γ belong to $\mathbf{K}$ or more generally are locally rational functions on $\mathbf{J}$. Then for solutions with $f, g \in L^\infty((0,\infty);\mathbf{R})$, we introduce an integrable operator on $L^2(0,\infty)$ by the kernel

$$K(x,y) = \frac{f(x)g(y) - f(y)g(x)}{x-y} \qquad (x \neq y; x, y \in \mathbf{R}) \tag{1.2}$$

The kernel K compresses to give an integral operator K_S on $L^2(S;dx)$ for any subinterval S of $(0,\infty)$ and it is important to identify those K_S such that K_S is of trace class and $0 \leq K_S \leq I$. In such cases, the Fredholm determinant $\det(I + \lambda K_S)$ is defined and K_S is associated with a determinantal random point field on S. In particular, $\det(I - K_{(t,\infty)})$ gives the probability that there are no random points on $(0,\infty)$.

Definition *(τ-function).* Suppose that $K : L^2(0,\infty) \to L^2(0,\infty)$ is a self-adjoint operator such that $K \leq I$, K is trace class and $I - K$ is invertible. For a measurable subset S of $(0,\infty)$, let $P_S : L^2(0,\infty) \to L^2(S)$ be the orthogonal projection given by $f \mapsto f\mathbf{I}_S$, where $\mathbf{I}_S$ is the indicator function of S. Then the τ function is

$$\tau(t) = \det(I - KP_{[t,\infty)}) \qquad (t > 0). \tag{1.3}$$

The purpose of this paper is to compute τ for certain kernels that are given by Tracy–Widom systems, and variants thereof. Our technique involves linear systems, and extends ideas developed in [6].

Let H be a complex separable Hilbert spaces, known as the state space, and let $(e^{-tA})_{t>0}$ a bounded C_0-semigroup of linear operators on H; so that $\mathcal{D}(A)$ is a dense linear subspace of H, and $\|e^{-tA}\| \leq M$ for all $t > 0$ and some $M < \infty$. Then let $B : \mathbf{C} \to \mathcal{D}(A)$ and $C : \mathcal{D}(A) \to \mathbf{C}$ be bounded linear operators, and introduce the linear system

$$\begin{aligned} \frac{dX}{dx} &= -AX + BU \qquad (X(0) = 0), \\ Y &= CX \end{aligned} \tag{1.4}$$

known as $(-A, B, C)$. Under further conditions to be discussed below, the integral

$$R_x = \int_x^\infty e^{-tA}BCe^{-tA}\,dt \tag{1.5}$$

converges and defines a trace class operator on H. The notation suggests that R_x is a resolvent operator.

In section 2, we consider integral operators on $L^2(0,\infty)$ with kernels of the form

$$R_0 \leftrightarrow \frac{c(s)b(t)}{s+t} \qquad (s,t>0). \tag{1.6}$$

When $c(s)=b(t)=1$ we obtain the Carleman operator on $L^2(0,\infty)$, which is bounded with continuous spectrum $[0,\pi]$ of multiplicity two; see [26, p. 56]. If b and c satisfy the hypotheses of Theorem 2.1, then R is a trace class operator, and we can define

$$\Phi = \log\det(I+R_0) - \log\det(I-R_0). \tag{1.7}$$

In quantum field theory, Φ arises in the Thermodynamic Bethe Ansatz and was considered in [30]. We introduce a linear system such that R_x^2 is an integrable kernel, and we derive an integral equation for Φ. This integral equation resembles the Gelfand–Levitan equation that arises in the scattering problem for the sinh-Gordon equation. Some additional information about Φ emerges when we introduce an appropriate Hankel operator.

Definition *(Hankel operator).* For a linear system as above, we introduce the *symbol* $\phi(x)=Ce^{-xA}B$, which gives a bounded function $\phi:(0,\infty)\to\mathbf{C}$; this term should not be confused with the different usage in [26, p6]. Generally, for E a separable complex Hilbert space and $\phi\in L^2((0,\infty);E)$, let Γ_ϕ be the Hankel operator

$$\Gamma_\phi h(x) = \int_0^\infty \phi(x+y)h(y)\,dy \tag{1.8}$$

defined on a suitable domain in $L^2(0,\infty)$ into $L^2((0,\infty);E)$.

By forming orthogonal sums of the state space and block operators, we can form sums of symbol functions. Likewise, by forming tensor products of state spaces and operators, we can from products of symbol function. Using these two basic constructions, we can form some apparently complicated symbol functions, starting from the basic multiplication operator $A: f(t)\mapsto tf(t)$ in $L^2(0,\infty)$. Thus we extend the method of section 2 to a more intricate problem.

In section 3, we consider operators related to the solution of Painlevé's transcendental equation VI. Jimbo, Miwa and Ueno [15, 16] showed that the nonlinear differential equation P_{VI} is the compatibility condition for the pair of linear differential equations

$$\frac{d\Phi}{d\lambda} = \Big(\frac{W_0}{\lambda} + \frac{W_1}{\lambda-1} + \frac{W_t}{\lambda-t}\Big)\Phi \tag{1.9}$$

$$\frac{d\Phi}{dt} = \frac{-W_t}{\lambda-t}\Phi \tag{1.10}$$

on the punctured Riemann sphere with 2×2 complex matrices W_0, W_1, W_t depending upon t; see (3.8) for the entries. Using the Laurent series of $\Phi(\lambda)$, we introduce a linear system $(-A, B, C)$ that realises Φ and deduce information about the Hankel operator Γ_Φ. In previous papers [5,6], we have considered kernels that factorize as $K = \Gamma_\phi^\dagger \Gamma_\phi$ where Γ_ϕ is Hilbert-Schmidt, so that $K \geq 0$ and K is trace class. In the context of P_{VI}, we show that the prescription (1.2) gives a kernel K that admits a factorization $K = \Gamma_\phi^\dagger \sigma \Gamma_\phi$, where σ is a constant signature matrix. In section 5 we introduce a suitable τ function and express it as the solution of an integral equation of Gelfand–Levitan type, which we can solve in terms of the linear system. A similar approach works for suitable solutions of Gauss's hypergeometric equation with a restricted choice of parameters, as we show in section 5.

Definition *(Transfer function).* Given a Hilbert space E, for $\phi \in L^2((0,\infty); dt; E)$ let

$$\hat{\phi}(s) = \int_0^\infty e^{-st}\phi(t)\,dt \tag{1.11}$$

be the transfer function of ϕ, otherwise known as the Laplace transform, which gives an analytic function from $\{s : \Re s > 0\}$ into E.

We assume that $\hat{\phi}$ is meromorphic, and that, by virtue of the Mittag-Leffler theorem, one can express ϕ as a series

$$\phi(x) = \sum_{j=1}^\infty \xi_j e^{-\lambda_j x} \tag{1.12}$$

in which we shall always assume that $\Re\lambda_j > 0$ and that the $e^{-\lambda_j x}$ are linearly independent in $L^2(0,\infty)$. We wish to express various τ functions in terms of the determinants

$$D_{S\times T} = \det\Bigl[\frac{1}{\lambda_j + \bar{\lambda}_k}\Bigr]_{(j,k)\in S\times T} \tag{1.13}$$

where S and T are finite subsets of $\mathbf{N}$ of equal cardinality. In sections 6, we consider Hankel operators with symbols as in (1.11), and establish basic results about the expansions of $\det(I - \Gamma_\phi)$ in terms of the bases.

In section 7, we consider the Bessel kernel, which arises in random matrix theory as the hard edge of the eigenvalue distribution from the Jacobi ensemble [28]. Let J_ν be Bessel's function of the first kind of order ν, and let $u(x) = \sqrt{x}J_\nu(2\sqrt{x})$, which satisfies

$$\frac{d^2u}{dx^2} + \Bigl(\frac{1}{x} + \frac{1-\nu^2}{4x^2}\Bigr)u(x) = 0. \tag{1.14}$$

We introduce $\phi(e) = u(e^{-x})$, and the Hankel operator Γ_ϕ with symbol ϕ. The transfer function $\hat{\phi}$ is meromorphic with poles on an arithmetic progression on the positive real axis, so we are able to obtain a simple expansion for $\tau(t) = \det(I - \Gamma_\phi^2 P_{[t,\infty)})$, and identify the determinants $D_{N\times N}$ with combinatorial objects.

In section 8 we consider solutions of Lamé's equation

$$-\Bigl(2\sqrt{(x-e_1)(x-e_2)(x-e_3)}\frac{d}{dx}\Bigr)^2\psi(x) + 2x\psi(x) = B\psi(x) \tag{1.15}$$

which is associated with the elliptic curve $Z^2 = 4(X-e_1)(X-e_2)(X-e_3)$. The solution gives rise to an elliptic function ϕ such that $\hat\phi$ has poles on a bilateral arithmetic procession parallel to the imaginary axis in $\mathbf{C}$. Hence we can prove results concerning the Fredholm determinant of Γ_ϕ.

2. The τ function associated with the Thermodynamic Bethe Ansatz

Following the terminology of [9], we recall a class of operators which includes the kernels K of (1.2) as a special case.

Definition *(Integrable operators).* Let $f_1,\dots,f_N,g_1,\dots,g_N \in L^\infty(0,\infty)$ satisfy

$$\sum_{j=1}^N f_j(x)g_j(x) = 0.$$

Then the bounded linear operator K that has kernel

$$K \leftrightarrow \frac{\sum_{j=1}^N f_j(x)g_j(y)}{x-y} \tag{2.1}$$

is said to be an integrable operator.

Let $\mathcal{D}(A) = \{f \in L^2(0,\infty) : tf(t) \in L^2(0,\infty)\}$ and for $b, c \in \mathcal{D}(A)$ introduce the operators:

$$\begin{array}{rr} A : \mathcal{D} \subset L^2(0,\infty) \to L^2(0,\infty) : & f(x) \mapsto xf(x) \\ B : \mathbf{C} \to \mathcal{D}(A) : & \alpha \mapsto b\alpha; \\ C : \mathcal{D}(A) \to \mathbf{C} : & f \mapsto \int_0^\infty f(s)c(s)\,ds \\ \Theta_x : L^2(0,\infty) \to L^2(0,\infty) : & \Theta_x f(t) = e^{-xt}\bar c(t)\hat f(t) \\ \Xi_x : L^2(0,\infty) \to L^2(0,\infty) : & \Xi_x f(t) = e^{-xt}b(t)\hat f(s) \end{array} \tag{2.2}$$

Then we introduce $\phi(s) = Ce^{-sA}B$ and $\phi_{(x)}(s) = \phi(s+2x)$, and the Hankel integral operator $\Gamma_{\phi_{(x)}}$ with kernel $\phi(s+t+2x)$. Then we introduce $R_x = \int_x^\infty e^{-tA}BCe^{-tA}\,dt$ which has kernel

$$R_x \leftrightarrow \frac{b(t)c(s)e^{-x(s+t)}}{s+t} \qquad (s,t>0). \tag{2.3}$$

Lemma 2.1. *Suppose that $c(t)/\sqrt{t}$ and $b(t)/\sqrt{t}$ belong to $L^2(0,\infty)$, and that c and b belong to $L^\infty(0,\infty)$.*

(i) Then $\Gamma_{\phi_{(x)}}$ and R_x are trace class operators for all $x \geq 0$.

(ii) Suppose further that $I + \lambda R_x$ is invertible. Then

$$T_\lambda(x,y) = -\lambda C e^{-xA}(I+\lambda R_x)^{-1} e^{-yA} B \tag{2.4}$$

gives the solution to the equation

$$\lambda\phi(x+y) + T_\lambda(x,y) + \lambda \int_x^\infty T_\lambda(x,z)\phi(z+y)\,dz = 0 \qquad (0<x<y) \tag{2.5}$$

and

$$T_\lambda(x,x) = \frac{d}{dx}\log\det(I+\lambda\Gamma_{\phi_{(x)}}). \tag{2.6}$$

(iii) The operator R_x^2 is an integrable operator with kernel

$$R_x^2 = e^{-xu} b(u) \frac{f_x(u) - f_x(t)}{t-u} c(t) e^{-xt} \tag{2.7}$$

where

$$f_x(u) = \int_0^\infty \frac{b(t)c(t)e^{-tx}}{u+t}\,dt. \tag{2.8}$$

(iv) If $I+\lambda R_x$ and $I-\lambda R_x$ are invertible, then there exists an integrable operator $L_x(\lambda)$ such that

$$I + L_x(\lambda) = (I - \lambda^2 R_x)^{-1}. \tag{2.9}$$

Proof. (i) One checks that Θ_x has kernel $e^{-st}e^{-xt}\bar{c}(t)$ and that Ξ_x has kernel $e^{-st-xs}b(s)$; hence $\Theta_x^\dagger$ and Ξ_x are Hilbert–Schmidt operators. One verifies that their products are $R_x = \Xi_x\Theta_x^\dagger$ and $\Gamma_{\phi_x} = \Theta^\dagger\Xi_x$, and hence R_x and Γ_x are trace class.

(ii) Using (i), we can check that $\det(I+\lambda R_x) = \det(I+\lambda\Gamma_{\phi_{(x)}})$. Then one verifies the remainder by using Lemma 5.1(iii) of [6].

(iii) This result is essentially contained in lemma 2.18 of [9], but we give a proof for completeness. The kernel of R_x^2 is

$$b(s)e^{-sx}c(u)e^{-ux}\int_0^\infty \frac{b(t)c(t)e^{-2tx}}{(s+t)(u+t)}dt \qquad (u,s>0), \tag{2.10}$$

and one can decompose this expression by using partial fractions. By the Cauchy–Schwarz inequality, $|f_x(u)|^2 \leq \int_0^\infty t^{-1}b(t)^2dt\int_0^\infty t^{-1}c(t)^2dt$, so f_x is bounded.

(iv) Furthermore,

$$(I-\lambda^2R_x^2)^{-1} = (I-\lambda R_x)^{-1}(I+\lambda R_x)^{-1} \tag{2.11}$$

is a bounded linear operator; so by Lemma 2.8 of [9], there exists an integrable operator L_x such that $(I + L_x(\lambda))(I - \lambda^2 R_x^2) = I$.

$\square$

Our first application is to the study of the integral operator on $L^2(\mathbf{R})$ that has kernel

$$\frac{E(x)E(y)}{\cosh \frac{1}{2}(x-y)}. \tag{2.12}$$

By changes of variable, we replace this by the integral operator on $L^2(0, \infty)$ with kernel

$$R_x \leftrightarrow \frac{e^{-(s+t)x} b(s) b(t)}{s+t} \qquad (s, t > 0), \tag{2.13}$$

and we introduce $\phi(s) = \int_0^\infty b(t)^2 e^{-st}\, dt$. In applications to physics associated with the Thermodynamic Bethe Ansatz [30], the following quantity is important.

Definition *(Φ-function).* Given R as above, the Φ function is

$$\Phi = \log\det(I + R_0) - \log\det(I - R_0). \tag{2.14}$$

For ϕ depending upon t, let

$$u(x,t) = -2\big(\log\det(I + \Gamma_{\phi_{(x)}}) - \log\det(I - \Gamma_{\phi_{(x)}})\big). \tag{2.15}$$

As an application of Theorem 2.1, we recover and extend some results due to Tracy and Widom [30], and show how to calculate Φ by the techniques of linear systems. Specifically, R_x appears in the scattering theory of the Zakharov–Shabat system of ordinary differential equations, and b is part of the scattering data. One can allow the scattering data to evolve with respect to time t as u evolves under the sine-Gordon equations.

Theorem 2.2. *Suppose that u satisfies the sinh-Gordon equation*

$$\frac{\partial^2 u}{\partial x \partial t} = \sinh u. \tag{2.16}$$

Suppose that b is real valued and that $b(x)/\sqrt{x}$ is in $L^2(0,\infty)$; suppose that $(I - R_x)$ is invertible for all $x \geq 0$.

(i) Let $V(x,y) = -2Ce^{-Ax}(I - R_x^2)^{-1} e^{-Ay} C^\dagger$ be the solution of the integral equation

$$V(x,y) + 2\phi(x+y) - \int_x^\infty V(x,w) \int_x^\infty \phi(w+z)\phi(z+y) dz dw = 0. \tag{2.17}$$

Then

$$\Phi = -\int_0^\infty V(x,x)\,dx. \tag{2.18}$$

(ii) Then R_0 and Γ_ϕ are of trace class and

$$\Phi = \log\det(I+\Gamma_\phi) - \log\det(I-\Gamma_\phi) = -\frac{1}{2}u(0,0), \tag{2.19}$$

thus $u(x,t)$ determines the evolution of Φ through time.

(iii) The function

$$f(\lambda) = \frac{\det(I+\lambda R_0)}{\det(I-\lambda R_0)} \tag{2.20}$$

is meromorphic, and there exists $\varepsilon > 0$ such that all the poles of f are simple and lie in (ε,∞), whereas all the zeros of f are simple and lie in $(-\infty,-\varepsilon]$.

Proof. (i) We observe that $V(x,y) = T_1(x,y) - T_{-1}(x,y)$ satisfies the integral equation (2.14), and one verifies that the given V is such a solution. We recall that $A = A^\dagger$ and $B = C^\dagger$, so the calculations here are simpler than the versions in section 6 of [6]. One checks that

$$V(x,x) = \frac{d}{dx}\Big(\log\det(I+R_x) - \log\det(I-R_x)\Big) \tag{2.21}$$

and $R_x \to 0$ in the trace class operators as $x\to\infty$, so we can recover Φ by integrating. Observe that $x \mapsto R_x$ is decreasing, and log is operator monotone increasing on $(0,\infty)$, so $V(x,x) \leq 0$.

(iii) Consider the differential equations

$$\frac{d\Psi}{dt} = \frac{i}{4\zeta}\begin{bmatrix} \cosh u & -\sinh u \\ \sinh u & -\cosh u \end{bmatrix}\Psi \tag{2.22}$$

and

$$\frac{d\Psi}{dx} = \begin{bmatrix} -i\zeta & \frac{1}{2}\frac{\partial u}{\partial x} \\ \frac{1}{2}\frac{\partial u}{\partial x} & i\zeta \end{bmatrix}\Psi. \tag{2.23}$$

Since u satisfies the sine-Gordon equations, one can easily check that $\frac{\partial^2}{\partial x\partial t}\Psi = \frac{\partial^2}{\partial t\partial x}\Psi$ follows so the systems are consistent.

We recover a result of Tracy and Widom that Φ satisfies the sinh-Gordon hierarchy. To solve this equation by the method of inverse scattering, we introduce scattering data $\phi(x) = Ce^{-xA}C^\dagger$ and solve the Gelfand–Levitan equation

$$T(x,y) + \Phi(x+y) + \int_x^\infty T(x,z)\Phi(z+y)\,dz = 0 \qquad (0<x<y) \tag{2.24}$$

where

$$T = \begin{bmatrix} U & W \\ W & U \end{bmatrix} \qquad \Phi = \begin{bmatrix} 0 & \phi \\ \phi & 0 \end{bmatrix}. \tag{2.25}$$

This reduces to the pair on scalar integral equations

$$\begin{aligned} U(x,y) + \int_x^\infty W(x,z)\phi(z+y)\,dz = 0 \\ W(x,y) + \phi(x+y) + \int_x^\infty U(x,z)\phi(z+y)\,dz = 0 \qquad (0 < x < y) \end{aligned} \tag{2.26}$$

with solutions

$$\begin{aligned} W(x,y) &= -Ce^{-xA}(I - R_x^2)^{-1}e^{-Ay}C^\dagger \\ U(x,y) &= Ce^{-xA}(I - R_x^2)^{-1}R_x e^{-yA}C^\dagger. \end{aligned} \tag{2.27}$$

The differential equation

$$-\frac{d^2\Psi}{dx^2} + \frac{1}{4}\begin{bmatrix} u_x^2 & 2u_{xx} \\ 2u_{xx} & u_x^2 \end{bmatrix}\Psi = \zeta^2\Psi \tag{2.28}$$

has a solution of the form

$$\Psi(x) = \begin{bmatrix} \alpha e^{i\zeta x} \\ \beta e^{-i\zeta x} \end{bmatrix} + \int_x^\infty T(x,y)\begin{bmatrix} \alpha e^{i\zeta y} \\ \beta e^{-i\zeta y} \end{bmatrix} dy, \tag{2.29}$$

where $\frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T}{\partial y^2} - 2(\frac{d}{dx}T(x,x))T$, and from the integral equation (), we obtain

$$-2\frac{d}{dx}\begin{bmatrix} U(x,x) & W(x,x) \\ W(x,x) & U(x,x) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} u_x^2 & 2u_{xx} \\ 2u_{xx} & u_x^2 \end{bmatrix}, \tag{2.30}$$

so

$$\frac{d}{dx}V(x,x) = 2\frac{d}{dx}W(x,x) = -\frac{1}{2}\frac{\partial^2 u}{\partial x^2}. \tag{2.31}$$

Hence the Φ function appears as $\Phi = -2^{-1}u(0,0)$ where

$$\frac{-1}{2}u(x,t) = \Big(\log\det(I + R_x) - \log\det(I - R_x)\Big). \tag{2.32}$$

(iii) The spectrum of R_0 equals the spectrum of Γ_ϕ, where Γ_ϕ is a compact and self-adjoint Hankel operator such that $\Gamma_\phi \geq 0$. Ober has shown that Γ_ϕ all the positive eigenvalues are simple. See [26].

□

3. A linear system associated with Painlevé's equation VI

Suppose that $F(x,w,p)$ is a rational function of (w,p) that is analytic as a function of z; consider the differential equation

$$w''(z) = F(z,w,w'). \tag{3.1}$$

Painlevé considered those differential equations such that the solutions have all their branch points and essential singularities in $\Sigma = \{a_1,\dots,a_m\}$ so that w is meromorphic on $\mathbf{P}^1 \setminus \Sigma$. Subsequently, there emerged six new equations that cannot generally be solved in terms of closed form expressions involving only standard functions that were previously known; these are the Painlevé transcendental differential equations. R. Fuchs discovered the equation

$$\begin{aligned} P_{VI}: \qquad \frac{d^2y}{dt^2} &= \frac{1}{2}\Big(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\Big)\Big(\frac{dy}{dt}\Big)^2 - \Big(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\Big)\frac{dy}{dt} \\ &\quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\Big(\alpha+\frac{\beta t}{y^2}+\frac{\gamma(t-1)}{(y-1)^2}+\frac{\delta t(t-1)}{(y-t)^2}\Big) \end{aligned} \tag{3.2}$$

with constants

$$\alpha = \frac{1}{2}(\theta_\infty-1)^2, \quad \beta = -\frac{1}{2}\theta_0^2, \quad \gamma = \frac{1}{2}\theta_1^2, \qquad \delta = \frac{1}{2}(1-\theta_t^2) \tag{3.3}$$

and

$$\theta_\infty = -2(z_0+z_1+z_t) - (\theta_0+\theta_1+\theta_t). \tag{3.4}$$

The Painlevé equations can be expressed as Hamiltonian systems in the canonical variables (λ,μ), where the Hamiltonian is a rational function of (λ,μ); see [24] for a list. In particular,

$$\begin{aligned} H_{VI}(\lambda,\mu;t) &= \frac{1}{t(t-1)}\Big(\lambda(\lambda-1)(\lambda-t)\mu^2 - \big(\theta_0(\lambda-1)(\lambda-t)+\theta_1\lambda(\lambda-t) \\ &\quad + (\theta_t-1)\lambda(\lambda-1)\big) + \kappa(\lambda-t)\Big) \end{aligned} \tag{3.5}$$

where

$$\alpha = \frac{1}{2}\theta_\infty^2; \quad \beta = -\frac{1}{2}\theta_0^2 \quad \gamma = \frac{1}{2}\theta_1^2 \quad \delta = \frac{1}{2}(1-\theta_t^2), \quad \kappa = \frac{1}{4}\big((\theta_0+\theta_1+\theta_t-1)^2-\theta_\infty^2\big). \tag{3.6}$$

Okamoto [24] showed that there exists a holomorphic function τ on the universal covering surface of $\mathbf{P}^1 \setminus \Sigma$ such that $H(t,\lambda(t),\mu(t)) = \frac{d}{dt}\log\tau(t)$.

Borodin and Deift [7] have identified a kernel K such that $\tau(t) = \det(I - P_{(t,\infty)}K)$. The methods used in [11, 15, 16] involve complex analysis and differential geometry, and are not intended to address the properties of the operator K. Here obtain a suitable K by introducing a linear system.

The Painlevé equation P_{VI} is associated with the system

$$\frac{d\Phi}{d\lambda} = \Bigl(\frac{W_0}{\lambda} + \frac{W_1}{\lambda - 1} + \frac{W_t}{\lambda - t}\Bigr)\Phi \tag{3.7}$$

$$\frac{d\Phi}{dt} = \frac{-W_t}{\lambda - t}\Phi \tag{3.8}$$

where the fixed singular points are $\{0, 1, \infty\}$ and

$$W_\nu = W_\nu(t) = \begin{bmatrix} z_\nu + \theta_\nu/2 & -u_\nu z_\nu \\ u_\nu^{-1}(z_\nu + \theta_\nu) & -z_\nu - \theta_\nu/2 \end{bmatrix} \qquad (\nu = 0, 1, t) \tag{3.9}$$

with parameters θ_ν and z_ν satisfying various conditions specified in [16]. The consistency condition for the system (3.7) and (3.8)

$$\frac{\partial W}{\partial t} - \frac{\partial \Omega}{\partial \lambda} = W\Omega - \Omega W. \tag{3.10}$$

reduces to the identity

$$\frac{1}{\lambda}\frac{\partial W_0}{\partial t} + \frac{1}{(\lambda - 1)}\frac{\partial W_1}{\partial t} + \frac{1}{(\lambda - t)}\frac{\partial W_t}{\partial t} = \frac{[W_0, W_t]}{\lambda(\lambda - t)} + \frac{[W_1, W_t]}{(\lambda - 1)(\lambda - t)}, \tag{3.11}$$

which leads, after a lengthy computation given in Appendix C of [16], to the equation P_{VI}.

Jimbo *et al* [15, 16, 17] introduced pairs of differential equations (1.1) and such that (3.11) reduces to one of the Painlevé equations. In the present context (3.7) are known as the deformation equations and (3.11) is associated with the names of Schlesinger and Garnier [10]. Note that trace $W = 0$ if and only if JW is symmetric; also W is nilpotent if and only if JW is symmetric and $\det(JW) = 0$.

First we introduce a linear system for the differential equation (3.16); later we introduce a linear system that realises the kernel most naturally associated with P_{VI}. The following result is a consequence of results of Turrittin [31], who clarified certain facts about the Birkhoff canonical form for matrices.

Lemma 3.1. *Let $W_\infty = -(W_0 + W_1 + W_t)$ and suppose that the eigenvalues of W_∞ are $\mp\theta_\infty/2$ where $\pm\theta_\infty$ is not a positive integer. Then there exist 2×2 complex matrices $C_0 = I$ and C_j for $j = 1, 2, \dots$ such that*

$$\Phi(x) = \Bigl(I + \sum_{j=1}^\infty \frac{C_j}{x^j}\Bigr)x^{-W_\infty}\Phi_0 \qquad (|x| > t) \tag{3.12}$$

satisfies the differential equation (3.7).

Proof. We can define $x^{-W_\infty} = \exp(-W_\infty \log x)$ as a convergent power series. By considering terms in the convergent Laurent series, one requires to show that there exist coefficients C_j that satisfy the recurrence relation

$$\begin{aligned} C_n(-W_\infty - nI) = -W_\infty C_n + W_1(C_0 + \ldots + C_{n-1}) \\ + tW_t(t^{n-1}C_0 + t^{n-2}C_1 + \ldots + C_{n-1}), \end{aligned} \tag{3.13}$$

where $W_\infty + nI$ and W_∞ have no common eigenvalues. Sylvester showed that, given square matrices V, W and Z such that V and W have no eigenvalues in common, the matrix equation $CV - WC = Z$ has a unique solution C; see [31, Lemma 1]. Hence unique C_n exist, and one shows by induction that $\|C_n\|$ is at most of geometric growth in n. In particular, if $\|W_\infty\| < 1$, then the solution of $W_\infty C_n - C_n(W_\infty + nI) = D_n$ is

$$C_n = -\int_0^\infty e^{sW_\infty} D_n e^{-s(W_\infty + nI)}\, ds. \tag{3.14}$$

□

We have proved that (3.7) has a solution in a neighbourhood of infinity, and one can show that it extends to an analytic solution on the universal cover of the punctured Riemann sphere $\mathbf{P}^1 \setminus \{0, 1, t, \infty\}$. (Jimbo, Miwa and Ueno [15] have shown that any C^2 solution of the pair (3.7) and (3.8) on $\mathbf{R}$ extends to a meromorphic solution on $\mathbf{C}$; see [10, Remark 4.7].)

Extending the construction of (2.2), we realise this solution via a linear system. We introduce the output space $H_0 = \mathbf{C}^2$, then the Hilbert space $H_1 = \ell^2(H_0)$, the state space $H = L^2((t, \infty); ds; H_1)$ and then let $\mathcal{D}(A) = \{f \in H : sf(s) \in H\}$; then we choose

$$b_j(s) = \Gamma(jI + W_\infty)^{-1} s^{j-1+W_\infty} \qquad (j = 0, 1, \ldots), \tag{3.15}$$

recalling that $\Gamma(z)^{-1}$ is entire. With this choice and some convergence factor $\kappa_0 > 1$, we introduce linear maps

$$\begin{aligned} A : \mathcal{D}(A) \to H : &\qquad f(s) \mapsto sf(s); \\ B_W : &\qquad \beta \mapsto (\kappa_0^j b_j(s)\beta)_{j=0}^\infty; \\ C : \mathcal{D}(A) \to \mathbf{C}^2 : &\qquad (f_j)_{j=0}^\infty \mapsto \sum_{j=0}^\infty \int_0^\infty \kappa_0^{-j} C_j f_j(s)\, ds. \end{aligned} \tag{3.16}$$

As usual, we introduce $\Xi_x : L^2(0,\infty) \to H$ such that

$$\Xi_x f = \int_x^\infty e^{-sA} B_W f(s)\, ds \tag{3.17}$$

and the observability operator $\Theta_x : L^2((0,\infty); H_0) \to L^2((t,\infty); H_1)$ by

$$\Theta_x f = \int_x^\infty e^{-sA^\dagger} C_W^\dagger f(s)\, ds. \tag{3.18}$$

Proposition 3.2. *(i) There exist $\kappa_0, x_0 > 0$ such that the operators $\Theta_x : L^2((0,\infty); H_0) \to H$ and $\Xi_x : L^2((0,\infty); H_0) \to H$ are Hilbert–Schmidt for $x > x_0$.*

(ii) For $x > x_0$, the linear system $(-A, B_W, C_W)$ realises the solution Φ of (3.7), so that

$$\Phi(x;t) = C_W e^{-xA} B_W \Phi_0. \tag{3.19}$$

(iii) Let $\phi_W(x;t) = C_W e^{-xA} B_W$. Then the Hankel operator on $L^2((x_0,\infty); H_0)$ with symbol ϕ_W is trace class.

Proof. We note that Θ_x has kernel $(e^{-su}\kappa_0^{-j} C_j^\dagger)_{j=0}^\infty$, and hence the Hilbert–Schmidt norm satisfies

$$\begin{aligned}\|\Theta_x\|_{HS}^2 &= \sum_{j=0}^\infty \int_t^\infty \int_x^\infty e^{-2su}\kappa_0^{-2j}\, dsdu\, \|C_j^\dagger\|_{HS}^2 \\ &\le \sum_{j=0}^\infty \frac{\|C_j^\dagger\|_{HS}^2 e^{-2xt}}{\kappa_0^{2j} 4xt};\end{aligned} \tag{3.20}$$

so we choose κ_0 so that this series converge. Next we observe that $\Xi_x : L^2((x,\infty); H_0) \to L^2((t,\infty); H_1)$ has kernel $(e^{-su}\kappa_0^j b_j(u))_{j=0}^\infty$, and hence has Hilbert–Schmidt norm

$$\begin{aligned}\|\Xi_x\|_{HS}^2 &= \sum_{j=0}^\infty \int_x^\infty \int_t^\infty e^{-2su}\kappa_0^{2j} \|b_j(u)\|_{HS}^2\, duds \\ &\le \kappa_W \sum_{j=0}^\infty \Gamma(j)^{-2} \int_t^\infty \kappa_0^{2j} u^{2j-1} e^{-2xu}\, du \\ &\le \kappa_W \sum_{j=0}^\infty \frac{\kappa_0^{2j}\Gamma(2j)}{\Gamma(j)^2 (2x)^{2j}}\end{aligned} \tag{3.21}$$

for some $\kappa_W > 0$. Having chosen κ_0, we then select x_0 so that the series converges for all $x > x_0$; then both Θ_x and Ξ_x are Hilbert–Schmidt.

(ii) Hence we can calculate

$$\begin{aligned} C_W e^{-xA} B_W &= \sum_{j=0}^{\infty} \int_0^{\infty} C_j e^{-xs} b_j(s)\, ds \\ &= \sum_{j=0}^{\infty} C_j \Gamma(jI + W_\infty)^{-1} \int_0^{\infty} s^{j+W_\infty -1} e^{-sx}\, ds \\ &= \sum_{j=0}^{\infty} C_j x^{-W_\infty - j}. \end{aligned} \tag{3.22}$$

Clearly the C_j and hence the operators C_W and B_W depend upon t. By differentiating (3.20), one can obtain a recurrence relation for the derivatives $\frac{dC_j}{dt}$. We have $\Phi(x;t) = C_W e^{-tA} B_W \Phi_0$ and (3.7) reduces to

$$\frac{dC_W}{dt} e^{-tA} B_W + C_W e^{-xA} \frac{dB_W}{dt} = \frac{-W_t}{x-t} C_W e^{-xA} B_W.$$

(iii) By (i), the operator $\Theta_x^\dagger \Xi_x$ is trace class on $L^2((0,\infty); H_0)$ for all $x > x_0$.

Furthermore, the operator $R_x = \int_x^\infty e^{-sA} B_W C_W e^{-sA}\, ds$ on H may be represented as a kernel with values in a doubly infinite block matrix with 2×2 matrix entries, namely

$$R_x \leftrightarrow \Bigl[\frac{\kappa_0^{j-k} b_j(u) C_k e^{-x(u+v)}}{u+v} \Bigr]_{j,k=0,1,\dots}; \tag{3.23}$$

this generalises (2.3). Consequently one can in principle compute the kernel

$$G_W(x,y) = -C_W e^{-xA} (I - R_x)^{-1} e^{-yA} B_W, \tag{3.24}$$

which satisfies

$$G_W(x,y) + \phi_W(x+y) + \int_x^\infty G_W(x,w)\phi_W(w+y)\, dw = 0 \qquad (t < x < y) \tag{3.25}$$

where $\phi_W(x;t) = C_W e^{-xA} B_W$.

$\square$

We also introduce

$$\sigma_{j,k} = \begin{bmatrix} I_j & 0 \\ 0 & -I_k \end{bmatrix} \tag{3.33}$$

which has rank $j+k$ and signature $j-k$.

Theorem 3.3. *Suppose that W_∞ is as in Lemma 3.1. Let $\Phi(\lambda;t)$ be a bounded solution of (3.7) in $L^2((t,\infty);d\lambda;\mathbf{R}^2)$ such that $\int_t^\infty \lambda^{-1}\|\Phi(\lambda;t)\|^2 d\lambda<\infty$, and let*

$$K(\lambda,\mu;t)=\frac{\langle J\Phi(\lambda;t),\Phi(\mu;t)\rangle}{\lambda-\mu}. \tag{3.26}$$

(i) Then there exists $\phi\in L^2((0,\infty);\lambda d\lambda;\mathbf{R}^6)$ such that

$$K(\lambda,\mu;t)=\int_0^\infty \langle \sigma_{3,3}\phi(\lambda+s;t),\phi(\mu+s;t)\rangle\, ds \qquad (\lambda,\mu>t;\lambda\neq\mu). \tag{3.27}$$

and hence K defines a trace class operator on $L^2((t,\infty);d\lambda)$.
(ii) The kernel $\frac{\partial}{\partial t}K(\lambda,\mu;t)$ is of finite rank in (λ,μ).

Proof. Jimbo [14] has shown that the fundamental solution matrix to (3.16) satisfies

$$Y(x,t)=\big(1+O(x^{-1})\big)\begin{bmatrix} x^{-\theta_\infty/2} & 0\\ 0 & x^{\theta_\infty/2}\end{bmatrix}; \tag{3.28}$$

hence there exist solutions that satisfy the hypotheses.

(i) We suppress the parameter t to simplify notation. From the differential equation (3.7), we have

$$\Big(\frac{\partial}{\partial\lambda}+\frac{\partial}{\partial\mu}\Big)\frac{\langle J\Phi(\lambda),\Phi(\mu)\rangle}{\lambda-\mu}=\Big(\frac{1}{\lambda-\mu}\Big)\sum_{\nu=0,1,t}\Big\langle\Big(\frac{JW_\nu}{\lambda-\nu}+\frac{W_\nu^\dagger J}{\mu-\nu}\Big)\Phi(\lambda),\Phi(\mu)\Big\rangle. \tag{3.29}$$

Now

$$JW_\nu=\begin{bmatrix} -(z_\nu+\theta_\nu)/u_\nu & z_\nu+\theta_\nu/2\\ z_\nu+\theta_\nu/2 & -u_\nu z_\nu\end{bmatrix} \qquad (\nu=0,1,t) \tag{3.30}$$

which have rank two and signature zero since $\det W_\nu=-\theta_\nu^2/4<0$. Hence $JW_\nu=V_\nu^\dagger\sigma_{1,1}V_\nu$ for some 2×2 real matrix V_ν, and $JW_\nu=V_\nu^\dagger\sigma_{1,1}V_\nu$. Thus we find that (3.30) reduces to

$$-\frac{\langle\sigma_{1,1}V_0\Phi(\lambda),V_0\Phi(\mu)\rangle}{\lambda\mu}-\frac{\langle\sigma_{1,1}V_1\Phi(\lambda),V_1\Phi(\mu)\rangle}{(\lambda-1)(\mu-1)}-\frac{\langle\sigma_{1,1}V_t\Phi(\lambda),V_t\Phi(\mu)\rangle}{(\lambda-t)(\mu-t)}. \tag{3.31}$$

Let

$$\phi(\lambda)=\begin{bmatrix}\frac{V_0\Phi(\lambda)}{\lambda}\\ \frac{V_1\Phi(\lambda)}{\lambda-1}\\ \frac{V_t\Phi(\lambda)}{\lambda-t}\end{bmatrix}, \tag{3.32}$$

which satisfies, after we permute the coordinates in the obvious way,

$$\begin{aligned}-\sum_{\nu=0,1,t}\frac{\langle\sigma_{1,1}V_\nu\Phi(\lambda),V_\nu\Phi(\mu)\rangle}{(\lambda-\nu)(\mu-\nu)}&=-\langle\sigma_{3,3}\phi(\lambda),\phi(\mu)\rangle\\ &=\Big(\frac{\partial}{\partial\lambda}+\frac{\partial}{\partial\mu}\Big)\int_0^\infty\langle\sigma_{3,3}\phi(\lambda+s),\phi(\mu+s)\rangle\, ds.\end{aligned} \tag{3.33}$$

We observe that both sides of (3.33) converge to zero as $\lambda \to \infty$ and as $\mu \to \infty$. By comparing the derivatives as in (3.27) and (3.33), we deduce (3.35).

Then $K = \Gamma_\phi^\dagger \sigma_{3,3}\Gamma_\phi$. We observe that the Hilbert–Schmidt norm of Γ_ϕ satisfies

$$\begin{aligned} \|\Gamma_\phi\|_{HS}^2 &= \int_t^\infty (\lambda - t)\|\phi(\lambda)\|^2\, d\lambda \\ &\leq \kappa \int_t^\infty \frac{\|\Phi(\lambda)\|^2}{\lambda}\, d\lambda \end{aligned} \tag{3.34}$$

for some $\kappa > 0$, so K gives a trace class operator on $L^2(t,\infty)$.

(ii) By a similar calculation, one can compute the derivative of K with respect to the position of the critical point, and find

$$\frac{\partial}{\partial t}K(\lambda,\mu;t) = \frac{1}{(\lambda - t)(\mu - t)}\Big\langle \begin{bmatrix} -(z_t+\theta_t)/u_t & z_t+\theta_t/2 \\ z_t+\theta_t/2 & -u_t z_t \end{bmatrix} \Phi(\lambda;t), \Phi(\mu;t)\Big\rangle; \tag{3.35}$$

evidently this is a finite sum of products of functions of λ and functions of μ for each t.

□

4. The τ function associated with Painlevé's equation VI

We now derive an integral equation for $\det(I - KP_{(x,\infty)})$. From Proposition 3.2, we recall the linear system $(-A_W, B_W, C_W)$ that realises ϕ_W, and likewise we introduce a linear system $(-A_V, B_V, C_V)$ that realises $\phi_V = \mathrm{diagonal}(V_0/x, V_1/(x-1), V_t/(x-t))$; then by considering

$$(-(A_V \otimes I + I \otimes A_W), B_V \otimes B_W, C_V \otimes C_W)$$

we introduce a new linear system that realises ϕ from Theorem 3.3, so that $\phi(x) = Ce^{-xA}B$.

Next we let Γ_ϕ be the Hankel integral operator with symbol ϕ; also let $\phi_{(x)}(y) = \phi(y+2x)$ and let L_x be observability Gramian

$$L_x = \int_x^\infty e^{-tA}BB^\dagger e^{-tA^\dagger}\, ds = \Xi_x \Xi_x^\dagger. \tag{4.1}$$

To take account of the signature, we introduce the the modified controllability Gramian

$$Q_x^\sigma = \int_x^\infty e^{-sA^\dagger}C^\dagger \sigma_{3,3} C e^{-sA}\, ds. \tag{4.2}$$

We also introduce the $(6+1)\times(6+1)$ block matrices

$$G(x,y) = \begin{bmatrix} U(x,y) & V(x,y) \\ T(x,y) & \zeta(x,y) \end{bmatrix} \tag{4.3}$$

and

$$\Phi(x) = \begin{bmatrix} 0 & \phi(x) \\ \phi(x)^\dagger & 0 \end{bmatrix}, \tag{4.4}$$

and the integral equation

$$G(x,y) + \Phi(x+y) + \int_x^\infty G(x,w) * \Phi(w+y)\, dw = 0, \tag{4.5}$$

where we have introduced a special matrix product to incorporate the signature, namely

$$\begin{aligned} &\int_x^\infty G(x,w) * \Phi(w+y)\, dw \\ &= \begin{bmatrix} \int_x^\infty V(x,w)\phi(w+y)^\dagger \sigma_{3,3} dw & \int_x^\infty U(x,w)\phi(w+y) dw \\ \int_x^\infty \zeta(x,y)\phi(w+y)^\dagger dw & \int_x^\infty T(x,w)\sigma_{3,3}\phi(w+y) dw \end{bmatrix}. \end{aligned} \tag{4.6}$$

Theorem 4.1. *Suppose that $x > t$ is such that Q_x and L_x are trace-class operators with operator norms less than one. Then there exists a solution to the integral equation (4.5) such that $\tau_K(x) = \det(I - P_{(x,\infty)}K)$ satisfies*

$$\frac{d}{dx} \log \tau_K(x) = \operatorname{trace} G(x,x). \tag{4.7}$$

Proof. By Theorem 3.3, we have $K = \Gamma_\phi^\dagger \sigma_{3,3} \Gamma_\phi$, and so

$$\begin{aligned} \tau_K(x) &= \det(I - P_{(x,\infty)} \Gamma_\phi^\dagger \sigma_{3,3} \Gamma_\phi) \\ &= \det(I - \Xi_x^\dagger \Theta_x \sigma_{3,3} \Theta_x^\dagger \Xi_x) \\ &= \det(I - \Xi_x \Xi_x^\dagger \Theta_x \sigma_{3,3} \Theta_x^\dagger) \\ &= \det(I - Q_x^\sigma L_x). \end{aligned} \tag{4.8}$$

One can verify that

$$\begin{aligned} &\begin{bmatrix} U(x,y) & V(x,y) \\ T(x,y) & \zeta(x,y) \end{bmatrix} \\ &= \begin{bmatrix} Ce^{-xA}(I - L_x Q_x^\sigma)^{-1} L_x e^{-yA^\dagger} C^\dagger \sigma_{3,3} & -Ce^{-xA}(I - L_x Q_x^\sigma)^{-1} e^{-yA} B \\ -B^\dagger e^{-xA^\dagger}(I - Q_x^\sigma L_x)^{-1} e^{-yA^\dagger} C^\dagger & B^\dagger e^{-xA^\dagger}(I - Q_x^\sigma L_x)^{-1} Q_x^\sigma e^{-yA} B \end{bmatrix} \end{aligned} \tag{4.9}$$

gives a solution to (4.6), so that

$$\begin{aligned} \operatorname{trace} U(x,x) &= \operatorname{trace}\Big((I - L_x Q_x^\sigma)^{-1} L_x e^{-xA^\dagger} C^\dagger \sigma_{3,3} C e^{-xA}\Big) \\ &= -\operatorname{trace}\Big((I - L_x Q_x^\sigma)^{-1} L_x \frac{dQ_x^\sigma}{dx}\Big). \end{aligned} \tag{4.10}$$

Likewise we have

$$\begin{aligned}\zeta(x,x) &= \mathrm{trace}\Big((I-Q_xL_x)^{-1}Q_x^\sigma e^{-xA}BB^\dagger e^{-xA^\dagger}\Big)\\ &= -\mathrm{trace}\Big((I-Q_x^\sigma L_x)^{-1}Q_x^\sigma \frac{dL_x}{dx}\Big).\end{aligned}\tag{4.11}$$

Adding and rearranging, we obtain

$$\begin{aligned}\mathrm{trace}\, G(x,x) &= \zeta(x,x) + \mathrm{trace}\, U(x,x)\\ &= -\mathrm{trace}\Big((I-L_xQ_x^\sigma)^{-1}L_x\frac{dQ_x^\sigma}{dx}\Big)\\ &\qquad - \mathrm{trace}\Big((I-L_xQ_x^\sigma)^{-1}\frac{dL_x}{dx}Q_x^\sigma\Big)\\ &= \frac{d}{dx}\mathrm{trace}\log(I-L_xQ_x^\sigma)\\ &= \frac{d}{dx}\log\tau_K(x).\end{aligned}\tag{4.12}$$

□

We introduce the new variable u by the elliptic integral

$$u=\int_\infty^y \frac{d\lambda}{\sqrt{\lambda(\lambda-1)(\lambda-t)}},\tag{4.13}$$

then we let $Z=\frac{\partial y}{\partial u}$ and $Y=y$, so (Y,Z) lies on the elliptic curve $Z^2=Y(Y-1)(Y-t)$ which depends upon the parameter t. Soon after his discovery of P_{VI}, R. Fuchs showed that if $y(t)$ satisfies P_{VI}, then $u(t)$ satisfies

$$\begin{aligned}&-t(1-t)\frac{d^2u}{dt^2}+(2t-1)\frac{du}{dt}+\frac{u}{4}\\ &\quad = -\frac{\sqrt{y(y-1)(y-t)}}{t(1-t)}\Big(2\alpha+\frac{2\beta t}{y^2}+\frac{\gamma(t-1)}{(y-1)^2}+(\delta-1/2)\frac{t(t-1)}{(y-t)^2}\Big),\end{aligned}\tag{4.14}$$

where we recognise Legendre's differential operator on the left-hand side. By analysing these solutions, Guzzetti [13] obtains various series representations and bounds on the growth of $y(t)$.

5. Kernels associated with the hypergeometric equation

The P_{VI} equation is closely related to Gauss's hypergeometric equation

$$\lambda(1-\lambda)\frac{d^2f}{d\lambda^2}+(c-(a+b+1)\lambda)\frac{df}{d\lambda}-abf(\lambda)=0.\tag{5.1}$$

We introduce $c_0 = c$ and $c_1 = a + b - c + 1$, then the matrix

$$W(\lambda) = \begin{bmatrix} 0 & \lambda^{-c_0}(\lambda-1)^{-c_1} \\ -ab\lambda^{c_0-1}(\lambda-1)^{c_1-1} & 0 \end{bmatrix} \tag{5.2}$$

so that we can express (5.1) in the form of a first order linear differential equation as in (5.4). For special choices of the parameters a, b, c, we can obtain a factorization For a separable Hilbert space H we introduce the identity operator I_H and

$$\sigma_{H,H} = \begin{bmatrix} I_H & 0 \\ 0 & -I_H \end{bmatrix}. \tag{5.3}$$

Theorem 5.1. *Suppose that $0 \leq c \leq 1$ and $a + b = 0$, that $2\sqrt{-ab}$ is not an integer, and that $-ab - (2c-1)^2/4 > 1$, and let Ψ be a bounded solution for the equation*

$$\frac{d\Psi}{d\lambda} = W(\lambda)\Psi(\lambda), \tag{5.4}$$

such that $\int_1^\infty x\|\Psi(x)\|^2 dx < \infty$; then let

$$K(x,y) = \frac{\langle J\Psi(x), \Psi(y)\rangle}{x-y} \qquad (x \neq y; x, y > 1). \tag{5.5}$$

(i) Then there exists a separable Hilbert space H and $\phi : (1,\infty) \to H^2$ such that $\int_{1+\delta}^\infty x\|\phi(x)\|^2_{H^2}\, dx < \infty$ and $K = \Gamma_\phi^\dagger \sigma_{H,H}\Gamma_\phi$ so that K defines a trace class kernel on $L^2((1+\delta,\infty); dx)$ for all $\delta > 0$.
(ii) The statement of Theorem 4.1 applies to

$$\tau_K(s) = \det(I - KP_{(s,\infty)}) = \det(I - \Gamma^\dagger_{\phi_{(s/2)}}\sigma_{H,H}\Gamma_{\phi_{(s/2)}}), \tag{5.6}$$

with obvious changes to notation.
(iii) If moreover c is rational, then K arises from a Tracy–Widom system as in (1.1).

Proof. Let

$$q(\lambda) = \frac{-ab}{\lambda(\lambda-1)} - \frac{1}{4}\Big(\frac{c}{\lambda} - \frac{1-c}{\lambda-1}\Big)^2, \tag{5.7}$$

which is positive for large λ. By the Liouville–Green transformation [25, p.229] , we can obtain solutions to (5.1) of the form

$$f_\pm(\lambda) \asymp \lambda^{-c/2}(\lambda-1)^{-(1-c)/2}q(\lambda)^{-1/4}\exp\Big(\pm\int_2^\lambda q(x)^{1/2}\, dx\Big) \qquad (\lambda\to\infty), \tag{5.8}$$

and one can deduce that $\int_2^\infty xf_-(x)^2\, dx < \infty$. Hence there exist solutions that satisfy the hypotheses.

(i) We observe that $c_1 + c_0 = 1$, so $0 \leq c_0, c_1, 1 - c_0, 1 - c_1 \leq 1$; we assume that $0 < c_0, c_1 < 1$, as the cases of equality are easier. Evidently the functions $\lambda^{-c_0}(\lambda-1)^{c_0-1}$ and $\lambda^{c_1-1}(\lambda-1)^{-c_1}$ are operator monotone decreasing on $(1,\infty)$ in Loewner's sense and by [1, p.577]

$$\lambda^{-c_0}(\lambda-1)^{c_0-1} = \frac{\sin \pi c_0}{\pi}\int_{-1}^{0} \frac{(-u)^{-c_0}(1+u)^{c_0-1}\,du}{\lambda+u} \qquad (x>1); \tag{5.9}$$

clearly a similar representation holds for $\lambda^{-c_1}(\lambda-1)^{c_1-1}$ with c_1 instead of c_0. Hence there exist positive measures ω_1 and ω_0 on $[-1,0]$ such that

$$\frac{JW(x)+W(y)^\dagger J}{x-y} = \begin{bmatrix} ab\frac{x^{-c_1}(x-1)^{c_1-1}-y^{-c_1}(y-1)^{c_1-1}}{x-y} & 0\\ 0 & \frac{x^{-c_0}(x-1)^{c_0-1}-y^{-c_0}(y-1)^{c_0-1}}{x-y}\end{bmatrix}$$

$$= \int_{-1}^{0} \frac{1}{(x+u)(y+u)}\begin{bmatrix} -ab\omega_1(du) & 0\\ 0 & -\omega_0(du)\end{bmatrix} \tag{5.10}$$

in which $-ab \geq 0$. The matrix kernel $(JW(x)+W(y)^\dagger J)/(x-y)$ operates as a Schur multiplier on the rank one tensor $\Psi(x)\otimes\Psi(y)$ in $L^2((1+\delta,\infty);\mathbf{R}^2)$; hence for each $\delta>0$, there exists $\kappa_\delta>0$ such the Schur multiplier norm is bounded by κ_δ. Since $\Psi(x+s)$ gives a Hilbert–Schmidt kernel, the operator $\int_0^\infty \Psi(x+s)\otimes\Psi(y+s)\,ds$ is trace class, and it follows that

$$K(x,y) = \int_0^\infty \Big\langle \frac{JW(x+s)+W(y+s)^\dagger J}{x-y}\Psi(x+s), \Psi(y+s)\Big\rangle ds \tag{5.11}$$

is also trace class on $L^2((1+\delta,\infty);dx)$. As in Theorem 1.1 of [4], we can introduce the Hilbert space H, $\phi\in L^2((1+\delta,\infty);xdx;H^2)$ and the Hankel operator Γ_ϕ with symbol ϕ such that $K=\Gamma_\phi^\dagger\sigma_{H,H}\Gamma_\phi$, so

$$K(x,y) = \int_0^\infty \langle \sigma_{H,H}\phi(x+s), \phi(y+s)\rangle_{H^2}\,ds \tag{5.12}$$

where $\sigma_{H,H}$ takes account of the fact that the Schur multiplier is positive on the top left matrix block and negative on the bottom right matrix block.

(ii) We observe that

$$W(\lambda) = \frac{1}{\lambda}\begin{bmatrix} 0 & 1\\ -ab & 0\end{bmatrix} + O(\lambda^{-2}) \qquad (|\lambda|\to\infty), \tag{5.13}$$

is analytic at infinity and the residue matrix has eigenvalues $\pm\sqrt{-ab}$ which do not differ by a positive integer. Hence we can repeat the proof of Lemma 3.1 and realise the solution

Ψ of (5.4) by a linear system involving the coefficients in the Laurent series of Ψ. Then we can realise $\phi \in L^2((0,\infty); H^2)$ by means of a linear system $(-A, B, C)$, where the state space is $L^2((0,\infty); H^2)$. We can now follow through the proof in section 4 as express τ in terms of the Gelfand–Levitan equation.

(iii) Let $c = k/n$; then $\{(X, Z) : Z^n = X^k (X-1)^{n-k}\}$ gives a n-sheeted cover of $\mathbf{P}^1$, ramified at $0, 1$. On this compact Riemann surface, the functions $\lambda^{-c_0}(\lambda-1)^{c_0-1}$ and $\lambda^{c_0-1}(\lambda-1)^{-c_0}$ are rational.

□

6. The τ function associated with a Hankel operator on exponential bases

We wish to find a more explicit expression for τ and for $\sigma(t) = \frac{d}{dt}\log\tau(t)$ for suitable K, especially those K that factor as $K = \Gamma_\phi^\dagger \Gamma_\phi$. We can obtain an explicit formula for τ when ϕ has the exponential expansion

$$\phi(x) = \sum_{j=1}^\infty \xi_j e^{-\lambda_j x} \tag{6.1}$$

where the coefficients ξ_j lie in some Hilbert space E. In this section we establish the existence of such expansions by using the theory of approximation of compact Hankel operators, whereas in subsequent sections we consider the transfer function $\hat\phi(s)$ of ϕ and use the Mittag-Leffler expansion to give explicit formulas. The Hankel operator with symbol ϕ can be expressed in terms of the exponential basis as a relatively simple matrix, so we can derive expressions for its Fredholm determinant. Our applications in sections 6 and 7 are to cases in which the poles lie on an arithmetic progression, which occurs when ϕ is a theta function.

We suppose that $\lambda_j \in \mathbf{C}$ with $\Re\lambda_j > 0$ are such that $(e^{-t\lambda_j})_{j=1}^\infty$ are linearly independent exponentials, so that

$$D_N = \det\Big[\frac{1}{\lambda_j + \bar\lambda_k}\Big]_{j,k=1}^N > 0 \qquad (N = 1, 2, \dots). \tag{6.2}$$

Suppose that $\xi = (\xi_j)_{j=1}^\infty \in \ell^1$ and introduce the operators

$$\begin{matrix} B : \mathbf{C} \to \ell^1 \subset \ell^2 : & a \mapsto a\xi \\ e^{-tA} : \ell^2 \to \ell^2 : & (\alpha_j)_{j=1}^\infty \mapsto (e^{-t\lambda_j}\alpha_j)_{j=1}^\infty \\ C : \ell^1 \subset \ell^2 \to \mathbf{C} : & (\alpha_j)_{j=1}^\infty \mapsto \sum_{j=1}^\infty \alpha_j \\ \Theta : L^2(0,\infty) \to \ell^2 : & f \mapsto (\int_0^\infty e^{-\bar\lambda_j s} f(s)\, ds)_{j=1}^\infty. \end{matrix} \tag{6.3}$$

Theorem 6.1. *Suppose that Θ is bounded and that there exist constants $\delta, M > 0$ such that $\Re\lambda_j \geq \delta$ and $\sum_{k=1}^\infty |\lambda_j + \lambda_k|^{-2} \leq M$ for all j; let $\xi \in \ell^1$.*

(i) Then $\phi(x) = Ce^{-xA}B$ gives rise to a Hankel operator $\Gamma_\phi : L^2(0,\infty) \to L^2(0,\infty)$ with symbol ϕ, which is trace class.

(ii) The operator

$$R_x = \int_x^\infty e^{-sA}BCe^{-sA}\,ds \tag{6.4}$$

on ℓ^2 is trace class, and for μ is an open neighbourhood of zero, the kernel $T_\mu(x,y) = -\mu Ce^{-xA}(I+\mu R_x)^{-1}e^{-yA}B$ gives a solution to the integral equation

$$T_\mu(x,y) + \mu\phi(x+y) + \mu\int_x^\infty T_\mu(x,z)\phi(z+y)\,dz = 0 \qquad (0 < x \le y). \tag{6.5}$$

(iii) Suppose that $(I - R_t)$ is invertible for all $t > 0$. Then the Hankel operator $\Gamma_{\phi_{(t)}}$ with kernel $\phi(x+y+2t)$ satisfies

$$\det(I - \Gamma_{\phi_{(t)}}) = \exp\Bigl(-\int_t^\infty T_{-1}(u,u)\,du\Bigr). \tag{6.6}$$

Proof. (i) The kernel may be expressed as a sum of rank-one kernels

$$\Gamma_\phi \leftrightarrow \sum_{j=1}^\infty \xi_j e^{-\lambda_j(x+y)} \tag{6.7}$$

where $\sum_{j=1}^\infty |\xi_j|/\Re\lambda_j$ converges, so Γ_ϕ is trace class.

(ii) By considering the rows of the matrix

$$R_x \leftrightarrow \Bigl[\frac{\xi_j e^{-(\lambda_j+\lambda_k)x}}{\lambda_j+\lambda_k}\Bigr]_{j,k=1}^\infty \tag{6.8}$$

we see that R_x is also trace class. When $|\mu|\|R_x\| < 1$, the kernel $T_\mu(x,y)$ is well defined, and one verifies the identity (6.5) by substituting.

(iii) The operators

$$C : \ell^1 \to \mathbf{C}, \qquad e^{-tA} : \ell^1 \to \ell^1, \qquad R_x : \ell^1 \to \ell^1, \qquad B : \mathbf{C} \to \ell^1 \tag{6.9}$$

are all bounded, and $\xi \mapsto R_x$ is continuous from ℓ^1 to the trace class; hence $T(x,y)$ depends continuously on ξ in a neighbourhood of 0 in ℓ^1. Suppose that $(\xi^{(n)})_{n=1}^\infty$ is a sequence of vectors in ℓ^1 that have only finitely many nonzero terms, and that $\xi^{(n)} \to \xi$ as $n \to \infty$. Denoting the operators corresponding to $\xi^{(n)}$ by $R_x^{(n)}$ etcetera, we can manipulate the finite matrices and deduce that

$$T_{-1}^{(n)}(x,x) = \frac{d}{dx}\log\det(I - R_x^{(n)}) \tag{6.10}$$

and hence

$$\int_s^t T_{-1}^{(n)}(x,x)\,dx = \log\det(I - R_t^{(n)}) - \log\det(I - R_s^{(n)}); \tag{6.11}$$

so letting $n \to \infty$, we deduce that

$$\int_s^t T_{-1}(x,x)\,dx = \log\det(I - R_t) - \log\det(I - R_s). \tag{6.12}$$

The operator $\Xi : L^2(0,\infty) \to \ell^2$ given by

$$\Xi f = \int_0^\infty e^{-tA} B f(t)\,dt \tag{6.13}$$

has matrix representation

$$\Xi\Xi^\dagger \leftrightarrow \Big[\frac{\xi_j \bar\xi_k}{\lambda_j + \bar\lambda_k}\Big]_{j,k=1}^\infty \tag{6.14}$$

with respect to the standard basis (e_j), and hence Ξ is Hilbert–Schmidt since $\sum_{j=1}^\infty \|\Xi^\dagger e_j\|^2 < \infty$. The operator Θ is bounded by hypothesis, hence $\Theta^\dagger$ is also bounded; so $R_0 = \Xi\Theta^\dagger$ is also Hilbert–Schmidt.

The operator Γ_ϕ is trace class by (ii), and the non-zero eigenvalues of $\Gamma_\phi = \Theta^\dagger\Xi$ and $R_0 = \Xi\Theta^\dagger$ are equal, hence

$$\det(I - \Gamma_{\phi_{(x)}}) = \det(I - R_x) \tag{6.15}$$

which when combined with (6.12), implies that

$$\log\det(I - \Gamma_{\phi_{(t)}}) - \log\det(I - \Gamma_{\phi_{(s)}}) = \int_t^s T_{-1}(u,u)\,du. \tag{6.16}$$

Evidently $\Gamma_{\phi_{(s)}} \to 0$ as $s \to \infty$, and hence (6.6) follows from (6.16).

□

Theorem 6.2. *Let K be an integral operator on $L^2((0,\infty); dt; \mathbf{C})$ such that:*

(i) $0 \le K \le I$ and $I - K$ is invertible;

(ii) there exists a separable Hilbert space E and $\phi \in L^2((0,\infty); tdt; E)$ such that $K = \Gamma_\phi^\dagger \Gamma_\phi$.

Then K has a τ-function τ_K and there exists a sequence $(K_n)_{n=1}^\infty$ of finite rank integral operators with corresponding τ-functions τ_{K_n} such that:

(1) $K_n \to K$ in trace class norm;

(2) $\tau_{K_n}(x) \to \tau_K(x)$ uniformly on compact sets as $n \to \infty$;

(3) $\tau_{K_n}(x)=\sum_{j=1}^{N_n} a_{jn}e^{-\mu_{jn}x}$ for some $a_{jn},\mu_{jn}\in\mathbf{C}$ with $\Re\mu_{jn}>0$ that are given in Proposition 6.4 below.

Proof. (1) For $\phi\in L^2((0,\infty);tdt;E)$, the operator Γ_ϕ is Hilbert–Schmidt and hence K is trace class. By the Adamyan–Arov–Krein theorem [26], there exists a sequence $(\Gamma_{\phi^{(n)}})_{n=1}^\infty$ of finite-rank Hankel operators such that $\Gamma_{\phi^{(n)}}\to\Gamma_\phi$ in Hilbert–Schmidt norm.

Kronecker showed that a Hankel operator $\Gamma_{\phi^{(n)}}$ has finite rank if and only if the transfer function $\hat\phi^{(n)}(s)$ is rational; see [26]. Hence the typical form for $\phi^{(n)}$ is a finite sum

$$\phi^{(n)}(t)=\sum_{j,k}\xi_{k,j}t^ke^{-\lambda_jt} \tag{6.17}$$

where $\xi_{k,j}\in E$ and $\Re\lambda_j>0$; the terms with factor t^k give poles of order $k+1$. To resolve the poles of order greater than one into sums of simple poles, we introduce the difference operator Δ_ε by $\Delta_\varepsilon g(\lambda)=\varepsilon^{-1}(g(\lambda+\varepsilon)-g(\lambda))$, which satisfies $\lim_{\varepsilon\to0}\Delta_\varepsilon^k g(\lambda)=g^{(k)}(\lambda)$ whenever g is k-times differentiable with respect to λ. By the dominated convergence theorem,

$$\int_0^\infty t|k!(-\Delta_\varepsilon)^ke^{-\lambda_jt}-t^ke^{-\lambda_jt}|^2\,dt\to0 \tag{6.18}$$

as $\varepsilon\to0$, so we can replace $t^ke^{-\lambda_jt}$ by $k!(-\Delta_\varepsilon)^ke^{-\lambda_jt}$ at the cost of a small change in the operator $\Gamma_{\phi^{(n)}}$ in Hilbert–Schmidt norm. Thus we eliminate poles of order greater than one, and we can ensure that $0\le\Gamma^\dagger_{\phi^{(n)}}\Gamma_{\phi^{(n)}}\le I$, with $I-\Gamma^\dagger_{\phi^{(n)}}\Gamma_{\phi^{(n)}}$ invertible. Let $K_n=\Gamma^\dagger_{\phi^{(n)}}\Gamma_{\phi^{(n)}}$ so that K_n has finite rank and $K_n\to K$ as in trace norm as $n\to\infty$.

(2) Let $\phi_{(x)}(t)=\phi(t+2x)$ and $\phi^{(n)}_{(x)}(t)=\phi^{(n)}(t+2x)$. We have $\Gamma^\dagger_{\phi^{(n)}_{(x)}}\Gamma_{\phi^{(n)}_{(x)}}\to\Gamma_{\phi_{(x)}}\Gamma_{\phi_{(x)}}$ in trace class norm as $n\to\infty$ so

$$\begin{aligned}\tau(x)&=\det(I-KP_{(x,\infty)})\\&=\det(I-\Gamma^\dagger_{\phi_{(x)}}\Gamma_{\phi_{(x)}})\\&=\lim_{n\to\infty}\det(I-\Gamma^\dagger_{\phi^{(n)}_{(x)}}\Gamma_{\phi^{(n)}_{(x)}})\\&=\lim_{n\to\infty}\tau_{K_n}(x)\end{aligned} \tag{6.19}$$

since the Fredholm determinant is a continuous functional on the trace class operators.

(3) To calculate the function $\tau_{K_n}(x)$ in (3) of Theorem 6.2, we assume that $\phi^{(n)}$ has the form

$$\phi^{(n)}(t)=\sum_{j=1}^N\xi_j^\dagger e^{-\bar\lambda_jt}\qquad(t>0) \tag{6.20}$$

where $\xi_j \in E$ and $\Re\lambda_j > 0$. Without loss of generality we can replace E by the subspace $\mathrm{span}(\xi_j)_{j=1}^N$ and for notational simplicity we take $\xi_j \in M_{1,\nu}(\mathbf{C})$ where $\nu \leq N$.

We introduce

$$a_j = \mathrm{row}\Big[\frac{\xi_j e^{-2\lambda_j x}}{\lambda_j + \bar\lambda_k}\Big] \in M_{1,\nu N}(\mathbf{C}) \tag{6.21}$$

and

$$b_m = \mathrm{column}\Big[\frac{\xi_k^\dagger e^{-2\bar\lambda_k x}}{\bar\lambda_k + \lambda_m}\Big]_{k=1}^N \in M_{\nu N,1}(\mathbf{C}). \tag{6.22}$$

□

Lemma 6.3. *The matrix*

$$K = [a_j b_m]_{j,m=1}^N \tag{6.23}$$

represents the operator $\Gamma^\dagger_{\phi^{(n)}_{(x)}}\Gamma_{\phi^{(n)}_{(x)}}$ *with respect to the (non-orthogonal) basis* $(e^{-\lambda_j s})_{j=1}^N$ *of* $\mathrm{span}(e^{-\lambda_j s})_{j=1}^N$.

Proof. We observe the Laplace transform of $\phi^{(n)}_{(x)}$ is the rational function

$$\hat\phi^{(n)}_{(x)}(s) = \sum_{j=1}^{\nu} \frac{\xi_j^\dagger e^{-2\lambda_j x}}{s+\lambda_j}. \tag{6.24}$$

The operator $\Gamma^\dagger_{\phi^{(n)}_{(x)}}\Gamma_{\phi^{(n)}_{(x)}}$ has kernel in the variables (s,t)

$$\int_0^\infty \langle \phi^{(n)}(2x+s+u), \phi^{(n)}(2x+t+u)\rangle \, du \tag{6.25}$$

and hence one computes

$$\Gamma^\dagger_{\phi^{(n)}_{(x)}}\Gamma_{\phi^{(n)}_{(x)}} : e^{-\lambda_m s} \mapsto \sum_{j,k=1}^N \frac{\langle \xi_j, \xi_m\rangle e^{-2(\bar\lambda_k+\lambda_j)x}}{(\lambda_j+\bar\lambda_k)(\bar\lambda_k+\lambda_m)} e^{-\lambda_j s}. \tag{6.26}$$

Recalling the definitions (6.21) and (6.22), one computes

$$a_j b_m = \sum_{j=1}^N \frac{\langle \xi_j, \xi_k\rangle e^{-2(\lambda_j+\bar\lambda_k)x}}{(\lambda_j+\bar\lambda_k)(\bar\lambda_k+\lambda_m)} \tag{6.27}$$

and by comparing this with (6.23), one obtains the stated identity.

□

We can proceed to compute the τ function when $\phi^{(n)}$ is as in Theorem 6.2. For $S, T \subseteq \{1, \dots, N\}$, let $K_{S,K}$ be the submatrix of K_n that is indexed by $(j,k) \in S \times T$, and let $\sharp S$ be the number of elements of S.

Proposition 6.4. *(i) Suppose that $\phi^{(n)} : (0,\infty) \to \mathbf{C}$ is as in (6.20). Then*

$$\tau_{K_n}(x) = \sum_{\ell=0}^{N} (-1)^\ell \sum_{T,S:\sharp S=\sharp T=\ell} \prod_{j\in S} \xi_j e^{-2\lambda_j x} \prod_{k\in T} \bar\xi_k e^{-2\bar\lambda_k x} \det\Bigl[\frac{1}{\lambda_j+\bar\lambda_k}\Bigr]^2_{j\in S,k\in T}. \tag{6.28}$$

(ii) Suppose that $\phi^{(n)} : (0,\infty) \to E$ where E has orthonormal basis $(e_r)_{r=1}^{\nu}$ and let $\xi_j^{(r)} = \langle \xi_j, e_r\rangle$. Then

$$\tau_{K_n}(x) = \sum_{S,T:\sharp S=\sharp T} (-1)^{\sharp S} \det\Bigl[\frac{\xi_j^{(r)} e^{-2\lambda_j x}}{\lambda_j+\bar\lambda_k}\Bigr]_{j\in S;(k,r)\in T} \det\Bigl[\frac{\bar\xi_k^{(r)} e^{-2\bar\lambda_k x}}{\lambda_m+\bar\lambda_k}\Bigr]_{m\in S;(k,r)\in T} \tag{6.29}$$

and the sum is over all pairs of subsets $S \subseteq \{1,\dots,N\}$ and $T \subseteq \{1,\dots,N\}\times\{1,\dots,\nu\}$ that have equal cardinality.

Proof. (i) By the Lemma we have $\tau_{K_n}(x) = \det(I-K_n)$, and by expansion of the determinant we have

$$\det(I-K_n) = \sum_{S:S\subseteq\{1,\dots,N\}} (-1)^{\sharp S} \det K_{S,S} \tag{6.30}$$

where $\det K_{\emptyset,\emptyset} = 1$ and otherwise

$$\det K_{S,S} = \det\Bigl[\sum_{k=1}^{N} \frac{\xi_j\bar\xi_k e^{-2(\lambda_j+\bar\lambda_k)x}}{(\lambda_j+\bar\lambda_k)(\bar\lambda_k+\lambda_m)}\Bigr]_{j,m\in S} \tag{6.31}$$

which reduces by the Cauchy–Binet formula to

$$\begin{aligned} &\sum_{T:\sharp T=\sharp S} \det\Bigl[\frac{\xi_j e^{-2\lambda_j x}}{\lambda_j+\bar\lambda_k}\Bigr]_{j\in S,k\in T} \det\Bigl[\frac{\bar\xi_k e^{-2\bar\lambda_k x}}{\bar\lambda_k+\lambda_m}\Bigr]_{k\in T,m\in S} \\ &\quad= \sum_{T:\sharp T=\sharp S} \Bigl(\prod_{j\in S} \xi_j e^{-2\lambda_j x} \prod_{k\in T} \bar\xi_k e^{-2\bar\lambda_k x}\Bigr) \det\Bigl[\frac{1}{\lambda_j+\bar\lambda_k}\Bigr]_{j\in S,k\in T} \det\Bigl[\frac{1}{\lambda_m+\bar\lambda_k}\Bigr]_{m\in S,k\in T}. \end{aligned} \tag{6.32}$$

By taking the sums over both S and T, we obtain the stated formula.

(ii) To prove (ii) one follows a similar route until line (6.32), except that we have $\langle \xi_j,\xi_k\rangle = \sum_{r=1}^{\nu} \xi_j^{(r)}\bar\xi_k^{(r)}$, so the indices in the Cauchy–Binet formula are over the product set $T \subseteq \{1,\dots,N\}\times\{1,\dots,\nu\}$.

□

7. The τ function for the hard edge

Our first application of section 5 is to the hard edge ensemble. The Jacobi polynomials arise when one applies the Gram–Schmidt process to $(x^k)_{k=0}^{\infty}$ with respect to the weight

$(1-x)^\alpha(1+x)^\beta$ on $[-1,1]$ for $\alpha,\beta>-1$. The zeros of the polynomials of high degree tend to accumulate at the so-called hard edges $1-$ and $(-1)+$. According to [28], the kernel that describes the limiting behaviour of the joint distribution of the scaled zeros near to the hard edges is given by

$$\frac{J_\nu(2\sqrt{x})\sqrt{y}J_\nu'(2\sqrt{y})-\sqrt{x}J_\nu'(2\sqrt{x})J_\nu(2\sqrt{y})}{x-y}=\int_0^1 J_\nu(2\sqrt{tx})J_\nu(2\sqrt{ty})\,dt \tag{7.1}$$

on $L^2((0,1);dt)$; here J_ν is Bessel's function of the first kind of order ν. Hence we change variables and introduce the Hankel operators on $L^2((0,\infty);dt)$.

Proposition 7.1. *For $\nu>-1$, let $\phi(x)=e^{-x/2}J_\nu(2e^{-x/2})$ and let Γ_ϕ be the Hankel integral operator on $L^2(0,\infty)$ with symbol ϕ. Then Theorem 3.2 applies to Γ_ϕ.*

Proof. From the power series for J_ν, we obtain a rapidly convergent series

$$\phi(x)=\sum_{n=0}^\infty \frac{(-1)^n e^{-(2n+\nu+1)x/2}}{n!\Gamma(\nu+n+1)}\qquad (x>0) \tag{7.2}$$

giving a meromorphic transfer function

$$\hat\phi(s)=\sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(\nu+n+1)(s+n+(\nu+1)/2)}, \tag{7.3}$$

for which the poles form an arithmetic progression along the negative real axis.

We choose $\lambda_n=(2n+\nu+1)/2$, which gives an arithmetic progression along the positive real axis, starting at $(\nu+1)/2>0$, and $\sum_{n=0}^\infty \lambda_n^{-2}<\infty$. The operator $\Theta:\ell^2\to L^2(0,\infty)$ is bounded by duality since

$$\begin{aligned}\int_0^\infty \Bigl|\sum_{n=0}^\infty a_n e^{-\lambda_n x}\Bigr|^2\,dx &= \sum_{n,m=0}^\infty \frac{a_n\bar a_m}{\lambda_n+\lambda_m}\\ &\leq C\sum_{n=0}^\infty |a_n|^2\end{aligned} \tag{7.4}$$

by Hilbert's inequality. Hence Γ_ϕ is a self-adjoint trace class operator, and Theorem 7.2 applies.

We can now compute some of the finite determinants that appear in the expansion of $\det(I-\Gamma^2_{\phi_{(x)}})$ from Proposition 7.4.

Definition *(Partition).* By a partition λ we mean a list $n_1\geq n_2\geq\dots\geq n_\ell$ of positive integers, so that the sum $|\lambda|=\sum_{j=1}^\ell n_j$, is split into $\ell=\ell(\lambda)$ parts. For each λ, the symmetric group on $|\lambda|$ letters has an irreducible unitary representation on a complex inner product space S_λ, known as the Specht module. For notational convenience, we introduce a null partition with $\ell(\emptyset)=0$ and write $\dim(S_\emptyset)=1$.

Proposition 7.2. *Suppose that $\nu=0$. Let $K=\Gamma_\phi^2$ and $\tau(x)=\mathrm{trace}(I-KP_{[x,\infty)})$. Then K is a trace class operator on $L^2(0,\infty)$ such that $0\leq K\leq I$ and*

$$\tau(x)=\sum_\lambda(-1)^{\ell(\lambda)}\frac{\dim(S_\lambda)^2}{(|\lambda|!)^2}e^{-2|\lambda|x} \tag{7.5}$$

where the sum is over all partitions.

Let $E_n=\mathrm{span}\{e^{-(2j+\nu+1)x}:j=0,\dots,n\}$ and let $Q_n:L^2(0,\infty)\to E_n$ be the orthogonal projection; likewise we introduce the closure E_∞ of the subspace $\cup_{n=1}^\infty E_n$ and the corresponding orthogonal projection $Q_\infty:L^2(0,\infty)\to E_\infty$. Observe that $Q_n\to Q_\infty$ in the strong operator topology as $n\to\infty$ and that $\Gamma_{\phi_{(x)}}Q_\infty=Q_\infty\Gamma_{\phi_{(x)}}$; hence $\det(I-\Gamma^2_{\phi_{(x)}})=\lim_{n\to}\det(I-Q_n\Gamma^2_{\phi_{(x)}}Q_n)$.

The matrix of $Q_n\Gamma^2_{\phi_{(x)}}Q_n$ with respect to $(e^{-(2j+\nu+1)s})_{j=0}^n$ satisfies

$$Q_n\Gamma^2_{\phi_{(x)}}Q_n\sim\Bigl[\frac{(-1)^{j+m}e^{-2x(j+m+\nu+1)}}{j!m!\Gamma(\nu+j+1)\Gamma(m+\nu+1)}\sum_{k=0}^\infty\frac{1}{(j+k+\nu+1)(m+k+\nu+1)}\Bigr]_{j,m=0}^n. \tag{7.6}$$

We observe that the corresponding infinite matrix for $Q_\infty\Gamma^2_{\phi_{(x)}}$ has entries that summable with respect to j and m over $j,m=0,1,\dots$; thus $\det(I-\Gamma^2_{\phi_{(x)}})$ is a determinant of Hill's type.

We consider the determinant in (3.14). We change notation so as to allow the running indices in sums to be $j,k=0,1,\dots$, and we let S and T be subsets of $\{0,1,2,\dots\}$ that are finite and of equal cardinality. Suppose that the elements of S are $m_1>m_2>\dots>m_\ell$, while the elements of T are $k_1>k_2>\dots>k_\ell$; then let $N=\ell+\sum_{i=1}^\ell(m_i+k_i)$. Then in Frobenius's coordinates [9, 21], there is a partition $\lambda\leftrightarrow(m_1,\dots,m_\ell;k_1,\dots,k_\ell)$ with $|\lambda|$ with a corresponding Specht module S_λ such that

$$\det\Bigl[\frac{1}{m!\Gamma(m+1)(m+k+1)}\Bigr]_{m\in S,k\in T}\frac{\prod_{k\in T}k!}{\prod_{m\in S}m!}\frac{\dim(S_\lambda)}{(|\lambda|)!} \tag{7.7}$$

as in the hook length formula of representation theory; see in [21]. Hence the pair of sets S and T, each with $\ell(\lambda)$ elements give rise to the product of determinants

$$\det\Bigl[\frac{1}{j!\Gamma(j+1)(j+k+1)}\Bigr]_{j\in S,k\in T}\det\Bigl[\frac{1}{m!\Gamma(m+1)(m+k+1)}\Bigr]_{m\in S,k\in T}=\frac{\dim(S_\lambda)^2}{(|\lambda|!)^2} \tag{7.8}$$

and the exponential

$$e^{-\sum_{j\in S}(2j+1)x-\sum_{k\in T}(2k+1)x}=e^{-2|\lambda|x}. \tag{7.9}$$

Conversely, each partition λ of some positive integer gives a Ferrers diagram and we can introduce subsets $S,T\subset\{0,1,\dots\}$ that are finite and of equal cardinality which gives a contribution to the sum (3.14) from the prescription of (7.9) and (7.10). By summing over all partitions, or equivalently all pairs of sets S and T, we obtain the series (7.6).

□

Remark. Borodin, Okounkov and Olshanski [8] have computed a Fredholm determinant for the discrete Bessel kernel, and derived a result vaguely similar to (7.6). The determinant $\det(I-KP_{(0,s)})$ was computed by Forrester, and Forrester and Witte have considered various circular ensembles [11].

8. A τ function related to Lamé's equation

To conclude this paper, we consider Hankel operators related to Lamé's equation. First we review some ideas that originate with Hochstadt and are developed by McKean and van Moerbecke in [23].

Let $\mathcal{E}$ be a compact Riemann surface of genus g, and $\mathbf{J}$ the Jacobi variety of $\mathcal{E}$, which we identify with $\mathbf{C}^g/\mathbf{L}$ for some lattice $\mathbf{L}$ in $\mathbf{C}^g$. An abelian function is a locally rational function on $\mathbf{J}$, or equivalently a periodic meromorphic function on $\mathbf{C}^g$ with $2g$ complex periods. A theta function (or elliptic function of the second kind) $\theta:\mathbf{C}^g\to\mathbf{P}^1$ with respect to $\mathbf{L}$ is a meromorphic function, not identically zero, such that there exists a linear map $x\mapsto L(x,u)$ for $x\in\mathbf{C}^g$ and $u\in\mathbf{L}$ and a function $\eta:\mathbf{L}\to\mathbf{C}$ such that $\theta(x+u)=\theta(x)e^{2\pi i(L(x,u)+\eta(u))}$ for all $x\in\mathbf{C}^g$ and $u\in\mathbf{L}$. The pair (L,η) is called the type of θ, as in [20].

Suppose that $q:\mathbf{R}\to\mathbf{R}$ is infinitely differentiable and periodic with period one. Let U_λ be the fundamental solution matrix for Hill's equation

$$-\frac{d^2}{dt^2}f+q(t)f(t)=\lambda f(t) \tag{8.1}$$

so that $U_\lambda(0)=I$, and let $\Delta(\lambda)=\operatorname{trace}U_\lambda(1)$ be the discriminant. Suppose in particular that λ lies inside the Bloch spectrum of $-\frac{d^2}{dt^2}+q(t)$, but that $4-\Delta(\lambda)^2\neq 0$. Then any nontrivial solution of (8.1) is bounded but not periodic.

We suppose that $4-\Delta(\lambda)^2$ has only finitely many simple zeros $0<\lambda_0^{(1)}<\lambda_1^{(1)}<\dots<\lambda_{2g}^{(1)}$, and let $\lambda_k^{(2)}$ be double zeros for $k=1,2,\dots$; then

$$4-\Delta(\lambda)^2=c_1\prod_{j=0}^{2g}\Bigl(1-\frac{\lambda}{\lambda_j^{(1)}}\Bigr)\prod_{k=1}^{\infty}\Bigl(1-\frac{\lambda}{\lambda_k^{(2)}}\Bigr)^2. \tag{8.2}$$

Proposition 8.1. *Suppose that the discriminant has this form. Then Hill's equation gives a Tracy–Widom system on a hyperelliptic curve of genus g.*

Proof. The equation (8.1) has nontrivial bounded solutions if and only if $|\Delta(\lambda)| < 2$, so that λ lies in an interval of stability. Hence the spectrum of $-\frac{d^2}{dt^2} + q$ in $L^2(\mathbf{R})$ has the form

$$[\lambda_0^{(1)}, \lambda_1^{(2)}] \cup [\lambda_2^{(1)}, \lambda_3^{(1)}] \cup \ldots \cup [\lambda_{2g}, \infty). \tag{8.3}$$

The zeros of $\Delta'(\lambda)$ consist of all the $\lambda_k^{(2)}$ together with zeros λ_j' that interlace the simple zeros of $4 - \Delta(\lambda)^2$, so $\lambda_{2j-1}^{(1)} < \lambda_j' < \lambda_{2j}^{(1)}$ for $j = 1, \ldots, g$; hence

$$\frac{\Delta'(\lambda)}{\sqrt{4 - \Delta(\lambda)^2}} = \frac{\prod_{j=1}^{g}\Big(1 - \frac{\lambda}{\lambda_j'}\Big)}{\sqrt{\prod_{j=0}^{2g}\Big(1 - \frac{\lambda}{\lambda_j^{(1)}}\Big)}}. \tag{8.4}$$

We introduce a new variable by the integral

$$t = -\int \frac{\Delta'(X)dX}{\sqrt{4 - \Delta(X)^2}} \tag{8.5}$$

so that $2\cos t = \Delta(X)$. We invert this relation by introducing a hyperelliptic function $Q(t)$ so that $2\cos t = \Delta(Q(t))$. We introduce the hyperelliptic curve

$$\mathcal{E}: \qquad Z^2 = \prod_{j=0}^{2g}\Big(1 - \frac{X}{\lambda_j^{(1)}}\Big), \tag{8.6}$$

which has genus g. After a little reduction, Hill's equation becomes

$$-\Big(\frac{Z}{\prod_{j=1}^{g}(1 - X/\lambda_j')}\frac{d}{dX}\Big)^2 f + q(Q^{-1}(X))f = \lambda f. \tag{8.7}$$

Now by results of McKean and van Moerbeke, $q(Q^{-1}(X))$ is an abelian function on $\mathcal{E}$ and may be viewed as a locally rational function on the Jacobian variety $\mathbf{J}$ over $\mathcal{E}$. Hence we can express (8.6) as a matrix differential equation with coefficients in the field of locally rational functions on $\mathbf{J}$.

Suppose in particular that q is elliptic with periods $2K$ and $2K'i$ where $K, K' > 0$. Gesztesy and Weikard [12] have shown that the spectrum has only finitely many gaps if and only if $z \mapsto U_\lambda(z)$ is meromorphic for all $\lambda \in \mathbf{C}$. By a classical result of Picard, there exists a nonsingular matrix A_λ such that $U_\lambda(z + 2K) = U_\lambda(z)A_\lambda$. If A_λ has distinct eigenvalues, then there exists a solution f to (8.1) that is a theta function with respect to the lattice $\mathbf{L} = \{2Km + 2K'in : m, n \in \mathbf{Z}\}$.

Next we describe in more detail the case of genus one. We recall Jacobi's sinus amplitudinus of modulus k is $\mathrm{sn}(x \mid k) = \sin\psi$ where

$$x = \int_0^\psi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}. \tag{8.8}$$

For $0<k<1$, let $K(k)$ be the complete elliptic integral

$$K(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}}; \tag{8.9}$$

next let $K'(k) = K(\sqrt{1-k^2})$; then $\mathrm{sn}(z \mid k)^2$ has real period K and complex period $2iK'$. For $\ell = 1, 2, \dots$, the standard form of Lamé's equation is

$$\Big(-\frac{d^2}{dz^2} + \ell(\ell+1)k^2\mathrm{sn}(z \mid k)^2\Big)\Phi(z) = \lambda\Phi(z). \tag{8.10}$$

We introduce

$$(e_1, e_2, e_3) = \Big(\frac{2-k^2}{3}, \frac{2k^2-1}{3}, -\frac{k^2+1}{3}\Big), \tag{8.11}$$

and

$$g_2 = \frac{4(k^4-k^2+1)}{3}, \quad g_3 = \frac{4(k^2-2)(2k^2-1)(k^2+1)}{27}; \tag{8.12}$$

then let the Weierstrass function $\mathcal{P}(z) = \mathcal{P}(z; g_2, g_3)$ be

$$\mathcal{P}(z) = e_3 + (e_1-e_2)\big(\mathrm{sn}(z \mid k)\big)^{-2}. \tag{8.13}$$

Likewise, $\mathcal{P}(z)$ has periods $2K$ and $2iK'$, and $\mathcal{P}(x+iK')$ is bounded, real and $2K$-periodic. In terms of the new variable $x = z+iK'$ and the constant $B = -\lambda(e_1-e_2) - \ell(\ell+1)e_3$, the differential equation becomes

$$\Big(-\frac{d^2}{dx^2} + \ell(\ell+1)\mathcal{P}(x)\Big)\Phi(x) + B\Phi(x) = 0. \tag{8.14}$$

For $\ell = 1$ and $\lambda \in [k^2, 1] \cup [k^2+1, \infty)$, all solutions to (8.10) are bounded; however, except for the countable subset of values of λ that gives the periodic spectrum, these solutions are not K or $2K$ periodic; see [22]. Suppose that $\ell = 1$ and write $B = \mathcal{P}(\alpha)$ where a is the spectral parameter. We introduce Weierstrass's functions

$$\sigma(z) = z\prod_{\omega\in\mathbf{L}^*}\Big(1-\frac{z}{\omega}\Big)\exp\Big(\frac{z}{\omega} + \frac{1}{2}\Big(\frac{z}{\omega}\Big)^2\Big)$$

where $\mathbf{L}^* = \mathbf{L} \setminus \{(0,0)\}$, and $\zeta(z) = \sigma'(z)/\sigma(z)$ so that $\mathcal{P} = -\zeta'$. Then by [19, (13)] the equation (8.10) has a nontrivial solution

$$\Phi(x;\alpha) = -\frac{\sigma(x-\alpha)}{\sigma(\alpha)\sigma(x)} e^{\zeta(\alpha)x} \tag{8.15}$$

such that $\Phi(x;\alpha)\Phi(-x;\alpha) = \mathcal{P}(\alpha) - \mathcal{P}(x)$ and $\alpha \mapsto \Phi(x;\alpha)$ is doubly periodic.

Writing $X = \mathcal{P}(x)$, $Y = \mathcal{P}(y)$ and $Z = \mathcal{P}'(x)$, the point (X, Z) lies on the elliptic curve

$$\mathcal{E}: \quad Z^2 = 4(X - e_1)(X - e_2)(X - e_3) \tag{8.16}$$

and the elliptic function field $\mathbf{K}$ consists of the field of rational functions of X with Z adjoined and we think of B as a point on $\mathcal{E}$. The solutions give rise to a natural kernel, for after we make the local change of independent variable $x \mapsto X$ and write $f(X) = \Phi(x;\alpha)$ and $g(X) = \Phi'(x;\alpha)$, we have

$$\frac{d}{dX}\begin{bmatrix} f(X) \\ g(X) \end{bmatrix} = \frac{1}{Z}\begin{bmatrix} 0 & 1 \\ 2X + \mathcal{P}(\alpha) & 0 \end{bmatrix}\begin{bmatrix} f(X) \\ g(X) \end{bmatrix} \tag{8.17}$$

and by [19, (18)]

$$\frac{f(X)g(Y) - g(X)f(Y)}{X - Y} = \Phi(x + y;\alpha). \tag{8.18}$$

The right-hand side has the shape of the kernel of Hankel integral operator. In the remainder of this section we introduce this operator, and compute the corresponding Fredholm determinant.

Lemma 8.2. *Let* $\beta = -2K\zeta(\alpha) + \alpha\zeta(\alpha + 2K) - \alpha\zeta(\alpha)$, *suppose that* $\Re\beta > 0$ *and let* $t \in \mathbf{C}$ *such that* $\Phi(x + 2t;\alpha)$ *is analytic for* $x \in [0, 2K]$. *Let* $\phi_{(t)}(x) = \Phi(x + 2t;\alpha)$ *and* $h(s) = \int_0^{2K} e^{-su}\phi_{(t)}(u)\,du$. *Then* $\phi_{(t)}$ *is a theta function and has an exponential expansion*

$$\phi_{(t)}(x) = \sum_{m=-\infty}^{\infty} \frac{1}{2K} h\Big(\frac{2\pi i m - \beta}{2K}\Big) e^{x(2\pi i m - \beta)/(2K)} \qquad (x > 0) \tag{8.19}$$

and $\hat{\phi}_{(t)}$ *is a meromorphic function with poles in an arithmetic progression.*

Proof. We introduce $\eta = \zeta(\alpha + 2K) - \zeta(\alpha)$ and $\eta' = \zeta(\alpha + 2iK') - \zeta(\alpha)$. Then σ is a theta function and satisfies a simple functional equation given in [20, p.109]; from this we deduce that Φ is also a theta function and satisfies the functional equations

$$\Phi(x + 2K;\alpha) = \Phi(x;\alpha)e^{2K\zeta(\alpha) - \alpha\eta}, \quad \Phi(x + 2iK';\alpha) = \Phi(x;\alpha)e^{2iK'\zeta(\alpha) - \alpha\eta'}. \tag{8.20}$$

Hence $x \mapsto \Phi(x + 2t;\alpha)$ is of exponential decay as $x \to \infty$ through real values.

Due to (8.20), the transfer function of $\phi_{(t)}(x)$ is

$$\hat{\phi}_{(t)}(s) = \sum_{k=0}^{\infty} \int_{2Kk}^{2K(k+1)} e^{-su}\Phi(u+2t;\alpha)\, du$$
$$= (1-e^{-2Ks+2K\zeta(\alpha)-\alpha\eta})^{-1} \int_0^{2K} \Phi(u+2t;\alpha)e^{-su}\, du \tag{8.21}$$

which is meromorphic with possible poles at the points $s = (2K)^{-1}(2K\zeta(\alpha) - \alpha\eta + 2\pi mi)$ for $m \in \mathbf{Z}$ which form a vertical arithmetic progression in the left half plane. The position of the poles is determined by the type of the theta function.

We can deduce the exponential expansion by inverting the Laplace transform. Let $T = (2m+1)\pi/(2K)$ let $x > 0$ and consider the contour $[-iT, iT] \oplus S_T$, where S_T is the semicircular arc in the left half plane with centre 0 that goes from $-iT$ to iT; then by Cauchy's Residue Theorem we have

$$\int_{S_T} e^{sx}\hat{\phi}_{(t)}(s)\, ds + \int_{[-iT,iT]} e^{sx}\hat{\phi}_{(t)}(s)\, ds = \frac{\pi i}{K}\sum_{n=-m}^{m} h\Big(\frac{2\pi ni-\beta}{2K}\Big)e^{x(2\pi in-\beta)/(2K)}. \tag{8.22}$$

We integrate $\int_0^{2K} \Phi(u+2t;\alpha)e^{-su}\, du$ by parts and write

$$e^{sx}\hat{\phi}_{(t)}(s) = \frac{e^{sx}}{s(1-e^{-2Ks-\beta})}\Big(-e^{-2Ks}\phi_{(t)}(2K) + \phi_{(t)}(0) + \int_0^{2K} e^{-su}\phi'_{(t)}(u)\, du\Big) \tag{8.23}$$

and then use Jordan's Lemma to show that $\int_{S_T} e^{sx}\hat{\phi}_{(t)}(s)\, ds \to 0$ as $T \to \infty$. Hence

$$\phi_{(t)}(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{sx}\hat{\phi}_{(t)}(s)\, ds = \sum_{n=-\infty}^{\infty} \frac{1}{2K} h\Big(\frac{2\pi ni-\beta}{2K}\Big)e^{x(2\pi in-\beta)/(2K)}. \tag{8.24}$$

□

Theorem 8.3. *Let $\phi_{(t)}(x) = \Phi(x+2t;\alpha)$ and let $\Gamma_{\phi_{(t)}}$ be the Hankel integral operator on $L^2(0,\infty)$ with symbol $\phi_{(t)}$. Then the conclusions of Theorem 8.1 hold for $\Gamma_{\phi_{(t)}}$.*

Proof. Let $\lambda_n = (2\pi in + \beta)/(2K)$ where $\Re\beta > 0$. Then by a standard argument from the calculus of residues, we have

$$\sum_{k=-\infty}^{\infty} \frac{1}{|\lambda_j+\lambda_k|^2} = \frac{K^2}{\beta\tanh\beta} \qquad (j \in \mathbf{Z}); \tag{8.25}$$

The operator $\Theta : L^2(0,\infty) \to \ell^2$ given by

$$f \mapsto \Big(\int_0^{\infty} e^{-\bar{\lambda}_j s} f(s)\, ds\Big)_{j=-\infty}^{\infty} \tag{8.26}$$

is bounded. Indeed, we observe that the sequence $(e^{-\lambda_n x})_{n=-\infty}^{\infty}$ forms a Riesz basic sequence in $L^2(0,\infty)$, in the sense that there exists a constant $C>0$ such that

$$C^{-1}\sum_{n=-\infty}^{\infty}|a_n|^2 \le \int_0^\infty \Big|\sum_{n=-\infty}^{\infty} a_n e^{-\lambda_n x}\Big|^2 dx \le C\sum_{n=-\infty}^{\infty}|a_n|^2 \tag{8.27}$$

for all $(a_n)\in\ell^2$. To prove this, one uses a simple scaling argument and orthogonality of the sequence $(e^{2\pi i n x})_{n=-\infty}^{\infty}$ in $L^2[0,1]$. In particular, this shows that $\Theta^\dagger:\ell^2\to L^2(0,\infty)$ is bounded, so Θ is bounded.

We can now use the general Theorem 8.1. Given this rapid decay and the fact that $\Phi(x+y+2t;a)$ is analytic, one can easily check that $\Gamma_{\phi_{(t)}}$ is trace class.

$\square$

Let

$$D_N=\det\Big[\frac{1}{\lambda_j+\bar\lambda_k}\Big]_{j,k=1}^N.$$

Proposition 8.4. *Suppose that* $\lambda_j=(2\pi i j+\beta)/(2K)$ *where* $\Re\beta>0$ *and* $K>0$. *Let* μ *be the Haar probability measure on the unitary group* $U(N)$, *and let* $\arg e^{i\theta}=\theta$ *for* $0<\theta<2\pi$. *Then*

$$D_N=\Big(\frac{2K}{1-e^{-2\Re\beta}}\Big)^N\int_{U(N)}\exp\Big(-\frac{\Re\beta}{\pi}\mathrm{trace}\arg U\Big)\mu(dU). \tag{8.28}$$

(ii) There exists a constant $c>0$ *such that*

$$\Big(\frac{K}{\sinh\Re\beta}\Big)^N e^{-(2c)^{1/3}N^{2/3}(\Re\beta)^{2/3}}\le D_N\le\Big(\frac{K}{\sinh\Re\beta}\Big)^N e^{(2c)^{1/3}N^{2/3}(\Re\beta)^{2/3}}. \tag{8.29}$$

so

$$D_N^{1/N}\to K\,\mathrm{cosech}\,\Re\beta\qquad (N\to\infty). \tag{8.30}$$

Proof. (i) Let

$$f(u)=\frac{2Ke^{-2\Re\beta u}}{1-e^{-2\Re\beta}}\qquad(0<u<1) \tag{8.31}$$

and let the Fourier coefficients of f be $a_k=\int_0^1 f(u)e^{-2\pi iku}du$, which we compute and find

$$\frac{1}{\lambda_j+\bar\lambda_k}=a_{j-k}. \tag{8.32}$$

Then we can use an identity due to Heine, and express the Toeplitz determinant of $[a_{j-k}]$ as an integral

$$\det[a_{j-k}]_{j,k=1,\dots,N}=\frac{1}{N!}\int_{[0,1]^N}\prod_{1\le j<k\le N}\big|e^{2\pi i\theta_j}-e^{2\pi i\theta_k}\big|^2\prod_{j=1}^N f(\theta_j)\,d\theta_1\dots d\theta_N, \tag{8.33}$$

which we regard as an integral over the maximal torus in $U(N)$, and hence we convert the expression into an integral over the group $U(N)$, obtaining

$$\det\Big[\frac{1}{\lambda_j+\bar\lambda_k}\Big]_{j,k=1}^{N}=\int_{U(N)}\exp\Big(\operatorname{trace}\log f(\arg U/(2\pi))\Big)\mu(dU). \tag{8.34}$$

(ii) Let $U\in U(N)$ have eigenvalues $e^{i\theta_1},\dots,e^{i\theta_N}$ where $0\leq\theta_1\leq\dots\leq\theta_N\leq 2\pi$; then the expression

$$\operatorname{trace}\arg U-\pi N=\theta_1+\dots+\theta_N-N\pi \tag{8.35}$$

satisfies a central limit theorem, but we need to adjust the functions slightly to accommodate the discontinuity of arg. Let $g_1,g_2:\mathbf{R}\to\mathbf{R}$ be Lipschitz functions with Lipschitz constant L, that are periodic with period 2π, and satisfy $g_1(\theta)\leq\theta\leq g_2(\theta)$ for $0\leq\theta<2\pi$, and

$$\pi-\frac{1}{L}\leq\int_0^{2\pi}g_1(\theta)\,d\theta\leq\int_0^{2\pi}g_2(\theta)\,d\theta\leq\pi+\frac{1}{L}. \tag{8.36}$$

By Szegö's asymptotic formula [18], there exists a constant c such that

$$\begin{aligned}\int_{U(N)}\exp\Big(-\frac{\Re\beta}{\pi}\sum_{j=1}^{N}\theta_j\Big)\mu(dU)&\leq\int_{U(N)}\exp\Big(-\frac{\Re\beta}{\pi}\sum_{j=1}^{N}g_1(\theta_j)\Big)\mu(dU)\\&\leq\exp\Big(-N\Re\beta\int_0^{2\pi}g_1(\theta)\frac{d\theta}{\pi}+c(\Re\beta)^2L^2\Big);\end{aligned} \tag{8.37}$$

hence we have an upper bound on D_N of

$$\Big(\frac{2K}{1-e^{-2\Re\beta}}\Big)^N\int_{U(N)}\exp\Big(-\frac{\Re\beta}{\pi}\sum_{j=1}^{N}\theta_j\Big)\mu(dU)\leq\Big(\frac{2K}{e^{\Re\beta}-e^{-\Re\beta}}\Big)^N e^{\Re\beta N/L+c(\Re\beta)^2L^2}. \tag{8.38}$$

Using g_2 instead of g_1, one can likewise obtain a lower bound on D_N. To conclude the proof, we choose $L=N^{1/3}(2c\Re\beta)^{-1/3}$.

□

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