Proceedings of the Royal Society of Edinburgh, **131A**, 1065–1089, *2001*

Spectral properties of the equation $(\nabla + \mathrm{i}e\boldsymbol{A}) \times \boldsymbol{u} = \pm m\boldsymbol{u}$

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(MS received 7 April 2000; accepted 6 December 2000)

We develop a spectral theory for the equation $(\nabla + \mathrm{i}eA) \times u = \pm mu$ on Minkowski 3-space (one time variable and two space variables); here, A is a real vector potential and the vector product is defined with respect to the Minkowski metric. This equation was formulated by Elton and Vassiliev, who conjectured that it should have properties similar to those of the two-dimensional Dirac equation. Our equation contains a large parameter c (speed of light), and this motivates the study of the asymptotic behaviour of its spectrum as $c \to +\infty$. We show that the essential spectrum of our equation is the same as that of Dirac (theorem 3.1), whereas the discrete spectrum agrees with Dirac to a relative accuracy $\delta\lambda/mc^2 \sim O(c^{-4})$ (theorem 3.3). In other words, we show that our equation has the same accuracy as the two-dimensional Pauli equation, its advantage over Pauli being relativistic invariance.

1. Introduction

The purpose of this paper is to study the spectral properties of (the stationary form of) the equation

$$(\nabla + \mathrm{i}eA) \times u = \pm mu \tag{1.1}$$

in Minkowski 3-space. Here,[1] $\nabla_\mu = \partial/\partial x^\mu$ is the (covariant) space time derivative, A is a given electromagnetic potential (real vector valued function), u is an unknown complex vector-valued function, and we are using the relativistic system of units, i.e. $\hbar = 1$, $c = 1$ and $e \approx -1/\sqrt{137}$. The Minkowski metric is assumed to be $g_{\mu\nu} = \mathrm{diag}(+1, -1, -1)$ and the vector product is defined as $(v \times w)^\lambda := e^{\lambda\mu\nu} v_\mu w_\nu$, where $e^{\lambda\mu\nu}$ is the totally antisymmetric tensor with $e^{012} = +1$. Equation (1.1) was suggested in [3] as part of a general programme of finding possible tensor alternatives to the Dirac equation.

Switching to atomic units, i.e. taking $e = -1$, $\hbar = 1$, $m = 1$ and $c \approx 137 \gg 1$, we can rewrite (1.1) explicitly as

$$\begin{pmatrix} 0 & -P_2 & P_1 \\ -P_2 & 0 & c^{-1}P_0 \\ P_1 & -c^{-1}P_0 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} = \pm \mathrm{i}c \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}, \tag{1.2}$$

where $P = \mathrm{i}\nabla - eA = \mathrm{i}\nabla + A$ is the electromagnetic energy momentum vector.

[1] In what follows Minkowski tensor indices will be denoted by Greek letters and take the values 0, 1, 2.

Remark 1.1. We have included a factor of $1/c$ into the magnetic potential (A^1, A^2), which is customary in mathematical literature on the Dirac equation (see, for example, remark 2 in § 6.1 of [9]). Physically, this convention means that the magnetic field is assumed to be quite strong, so that in the first approximation the energy levels are described by the Pauli equation and not by the Schrödinger equation. This simplifies the asymptotic analysis by reducing the chain of successive approximations (Schrödinger, Pauli, Dirac) to (Pauli, Dirac).

The stationary form of (1.2) is obtained by assuming A is independent of x^0 and $u = u(x^1, x^2)\mathrm{e}^{-\mathrm{i}\lambda x^0}$, where λ is a spectral parameter. Formally, this allows us to replace P_0 with $\lambda + \varPhi$, where we write $\varPhi = eA_0 = -A_0$ for the electric potential. Furthermore, we can use the first row of (1.2) to eliminate u_0 from the remaining two rows. Taking the equation with the upper sign (from now on the other equation will be discussed only for the purpose of comparison), we get

$$\left(\tilde{\mathcal{A}} - \lambda \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\right)\boldsymbol{u} = 0, \tag{1.3}$$

where $\boldsymbol{u} = (u_1, u_2)^{\mathrm{T}}$ is some function from $\mathbb{R}^2$ into $\mathbb{C}^2$ and

$$\tilde{\mathcal{A}} = \begin{pmatrix} P_2^2 & -P_2P_1 \\ -P_1P_2 & P_1^2 \end{pmatrix} + \begin{pmatrix} c^2 & \mathrm{i}\varPhi \\ -\mathrm{i}\varPhi & c^2 \end{pmatrix}.$$

Notation. Having reduced the problem to one on $\mathbb{R}^2$, we will now need to work with the 2-vector part of various 3-vectors. Following tradition, we shall use the contravariant form of the 3-vector for these purposes, i.e. if v is a 3-vector with a 2-vector part $\boldsymbol{u}$, then the components of $\boldsymbol{u}$ are given by $u_i = v^i = -v_i$ for $i = 1, 2$. In particular, denoting the magnetic potential and the momentum 2-vectors by $\boldsymbol{A}$ and $\boldsymbol{P}$, respectively, and relabelling the original electromagnetic potential and energy momentum 3-vectors as A' and P', respectively, we have

$$\boldsymbol{A} = (A_1, A_2) := (A'^1, A'^2) = -(A'_1, A'_2)$$

and

$$\boldsymbol{P} = (P_1, P_2) := (P'^1, P'^2) = -(P'_1, P'_2).$$

For $i = 1, 2$, it follows that $P_i = -\mathrm{i}\partial_i - eA_i = -\mathrm{i}\partial_i + A_i$, where $\partial_i = \partial/\partial x_i$ is the partial derivative with respect to x_i, the ith coordinate. From now on, we shall use A_i and P_i to refer to the components of the 2-vectors $\boldsymbol{A}$ and $\boldsymbol{P}$, respectively.

A computationally more convenient form of the spectral problem given by (1.3) is obtained by applying the constant unitary transformation

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}.$$

Setting $x_\pm = x_1 \pm \mathrm{i}x_2$ for any vector $\boldsymbol{x} = (x_1, x_2)$, we have

$$U^*\tilde{\mathcal{A}}U = \frac{1}{2}\begin{pmatrix} P_-P_+ & -\mathrm{i}P_-^2 \\ \mathrm{i}P_+^2 & P_+P_- \end{pmatrix} + \begin{pmatrix} c^2 - \varPhi & 0 \\ 0 & c^2 + \varPhi \end{pmatrix} = \begin{pmatrix} H_P^+ & B^* \\ B & H_P^- \end{pmatrix} + c^2 I,$$

where

$$\left.\begin{aligned} H_P^{\pm} &= \tfrac{1}{2}P_{\mp}P_{\pm} \mp \varPhi = \tfrac{1}{2}(P_1^2 + P_2{}^2) \pm \tfrac{1}{2}H \mp \varPhi, \\ H &= \partial_1 A_2 - \partial_2 A_1, \qquad B = \tfrac{1}{2}\mathrm{i}P_+^2 \end{aligned}\right\} \tag{1.4}$$

and I is the 2×2 identity matrix. Define another constant 2×2 matrix J by

$$J = U^* \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore, equation (1.3) is equivalent to the spectral problem given by

$$(\mathcal{A} - \lambda J)\boldsymbol{u} = 0, \tag{1.5}$$

where

$$\mathcal{A} = \begin{pmatrix} H_P^+ + c^2 & B^* \\ B & H_P^- + c^2 \end{pmatrix}. \tag{1.6}$$

Remark 1.2. The operators $H_P^{\pm}$ are just the Pauli operators for the electron and positron, respectively, and cH is the magnetic field strength (see also remark 1.1).

The structure of the paper is as follows. In § 2 we give a rigorous mathematical statement of our spectral problem and in § 3 we state our main results regarding the essential spectrum (theorem 3.1) and the behaviour of the discrete spectrum as $c \to +\infty$ (theorem 3.3). The former result is proved in §§ 4 and 5, while the latter is proved in §§ 6 and 7.

The results stated in § 3 are naturally motivated by the structure of our equation (1.5), but their proofs are quite technical. This is related to the fact that the problem (1.5) is not elliptic. Indeed, a straightforward calculation of the principal symbol of the operator $\mathcal{A}$ gives

$$\frac{1}{2}\begin{pmatrix} \xi_1^2 + \xi_2^2 & -\mathrm{i}(\xi_1 - \mathrm{i}\xi_2)^2 \\ \mathrm{i}(\xi_1 + \mathrm{i}\xi_2)^2 & \xi_1^2 + \xi_2^2 \end{pmatrix},$$

and it is easy to see that the determinant of this matrix is zero. Consequently, equation (1.5) cannot be viewed as an analytic perturbation of the Pauli (or Dirac) equation.

2. Mathematical statement of the problem

Equation (1.5) gives rise to a linear spectral pencil problem. Various parts of the spectrum of such a problem can be defined by analogy with the definitions for standard spectral problems.

Definition 2.1. Suppose $\mathcal{A}$ is defined as a closed (unbounded) operator on some dense domain $\operatorname{Dom}\mathcal{A} \subset L^2(\mathbb{R}^2, \mathbb{C}^2)$. We define the *$J$-spectrum* of $\mathcal{A}$, which will be denoted by $\sigma_J(\mathcal{A})$, to be the complement of the set of all $z \in \mathbb{C}$ for which $\mathcal{A} - zJ$ is boundedly invertible. The *J-essential spectrum* (denoted by $\sigma_{J\,\mathrm{Ess}}(\mathcal{A})$) is defined to be the set of all $z \in \mathbb{C}$ for which $\mathcal{A} - zJ$ is not Fredholm (where a closed densely defined operator $\mathcal{B}$ on L^2 is said to be Fredholm if $\operatorname{Ran}\mathcal{B}$ is closed and $\operatorname{Ker}\mathcal{B}$ and $L^2/\operatorname{Ran}\mathcal{B}$ are both finite dimensional). If $\operatorname{Ker}(\mathcal{A} - \lambda J) \neq 0$, then $\lambda \in \sigma_J(\mathcal{A})$ will be called a *J-eigenvalue* of $\mathcal{A}$.

Remark 2.2. The relationship

$$\mathcal{A} - zJ = J(J\mathcal{A} - zI) = (\mathcal{A}J - zI)J \tag{2.1}$$

allows us to reformulate statements regarding the J-spectrum of $\mathcal{A}$ in terms of the (regular) spectrum of $J\mathcal{A}$ or $\mathcal{A}J$. In particular, $\sigma_J(\mathcal{A}) = \sigma(J\mathcal{A}) = \sigma(\mathcal{A}J)$ (with a similar relationship holding for the essential spectra), while λ is a J-eigenvalue of $\mathcal{A}$ if and only if it is an eigenvalue of $J\mathcal{A}$ or $\mathcal{A}J$. The alternative points of view given by (2.1) will be used repeatedly below, especially in §§ 6 and 7 to appropriately modify standard properties of resolvents to the case of linear spectral pencils.

Suppose $\lambda \in \sigma_J(\mathcal{A})$ is a J-eigenvalue of $\mathcal{A}$. We define the *geometric multiplicity* of λ to be $\dim \operatorname{Ker}(\mathcal{A} - zJ)$. Clearly, this is the same as $\dim \operatorname{Ker}(J\mathcal{A} - zI)$ or $\dim \operatorname{Ker}(\mathcal{A}J - zI)$, which are just the geometric multiplicities of λ regarded as a eigenvalue of $J\mathcal{A}$ or $\mathcal{A}J$, respectively. Following the general definition for spectral pencils (see [4], for example), we can define the *algebraic multiplicity* of λ to be the sum of the lengths of a canonical set of Jordan chains corresponding to λ. It is straightforward to see that this is just the algebraic multiplicity of λ regarded as an eigenvalue of $J\mathcal{A}$ or $\mathcal{A}J$. We say that λ is *semi-simple* if its geometric and algebraic multiplicities are equal.

Notation. For any $p \in [1, \infty]$ and $k \in \mathbb{Z}$, we shall use L^p_k with norm $\|\cdot\|_{L^p_k}$ to denote the usual Sobolev space on $\mathbb{R}^2$; here, k is the 'number' of p-integrable derivatives. Depending on the context, elements of L^p_k will take values in either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^2$ or $\mathbb{C}^2$. The omission of k will imply its value is 0. We shall also use L^p_∞ to denote the space $\bigcap_{k\in\mathbb{Z}} L^p_k$ (without any topology).

In order to define $\mathcal{A}$ as a closed (unbounded) operator, we impose some conditions on the potentials Φ and $\boldsymbol{A}$.

(A1) $\Phi = \Phi_0 + \Phi_1$ for some $\Phi_0 \in L^\infty$ and $\Phi_1 \in L^1$ which has compact support and satisfies $(1 + |\xi|^2)^{k/2}\hat{\Phi}_1(\xi) \in L^p$ for some $p \in [1, \infty]$ and $k \geqslant 2(1 - 1/p)$.

(A2) $\boldsymbol{A} \in L^\infty_{\mathrm{loc}} \cap L^2_{1\,\mathrm{loc}}$.

Henceforth we shall assume these conditions are always satisfied. It follows that for any $\boldsymbol{u} \in C_0^\infty$ the formal operator given by (1.6) defines some $\mathcal{A}\boldsymbol{u} \in L^2$, i.e. (1.6) defines an operator on L^2 with domain C_0^∞, which we shall denote by $\mathcal{A}'$. Now

$$\begin{pmatrix} H_P^+ & B^* \\ B & H_P^- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} P_-P_+ & -\mathrm{i}P_-{}^2 \\ \mathrm{i}P_+{}^2 & P_+P_- \end{pmatrix} - \Phi J = T^*T - \Phi J,$$

where T is the operator given by $T = (1/\sqrt{2})\begin{pmatrix}\mathrm{i}P_+ & P_-\end{pmatrix}$. It follows easily that $\mathcal{A}'$ is symmetric. Furthermore, the quadratic form associated to $\mathcal{A}'$, $K_{\mathcal{A}'}$, is given by

$$K_{\mathcal{A}'}(\boldsymbol{u}) = \langle \mathcal{A}'\boldsymbol{u}, \boldsymbol{u}\rangle = \|T\boldsymbol{u}\|^2 + c^2\|\boldsymbol{u}\|^2 - K_\Phi(\boldsymbol{u}),$$

where

$$K_\Phi(\boldsymbol{u}) = \langle \Phi J\boldsymbol{u}, \boldsymbol{u}\rangle = \langle u_1, \Phi u_1\rangle - \langle u_2, \Phi u_2\rangle$$

for all $\boldsymbol{u} \in C_0^\infty$. In § 4 we shall prove the following.

PROPOSITION 2.3. *Let* $\kappa > 0$. *Then there exists a constant* C *such that the estimate* $|K_{\Phi}(\boldsymbol{u})| \leqslant \kappa\|T\boldsymbol{u}\|^2 + C\|\boldsymbol{u}\|^2$ *holds for all* $\boldsymbol{u} \in C_0^\infty$.

Therefore, $K_{\mathcal{A}'}$ and hence $\mathcal{A}'$ are semi-bounded. From standard results (see § 10.3 in [1], for example), it follows that $\mathcal{A}'$ has a self-adjoint extension (the Friedrichs extension), which we shall denote by $\mathcal{A}$. Furthermore, $C_0^\infty \subset \operatorname{Dom}\mathcal{A} \subset L^2$ and the quadratic form associated to $\mathcal{A}$, $K_{\mathcal{A}}$, is the closure of $K_{\mathcal{A}'}$. More precisely, $C_0^\infty \subset \operatorname{Dom}\mathcal{A} \subset \operatorname{Dom} K_{\mathcal{A}} \subset L^2$,

$$\operatorname{Dom} K_{\mathcal{A}} \text{ is the closure of } C_0^\infty \text{ with respect to the norm } (\|T\boldsymbol{u}\|^2 + \|\boldsymbol{u}\|^2)^{1/2} \tag{2.2}$$

and, for all $\boldsymbol{u} \in \operatorname{Dom} K_{\mathcal{A}}$,

$$K_{\mathcal{A}}(\boldsymbol{u}) = \|T\boldsymbol{u}\|^2 + c^2\|\boldsymbol{u}\|^2 - K_{\Phi}(\boldsymbol{u}), \qquad K_{\Phi}(\boldsymbol{u}) = \langle u_1, \Phi u_1\rangle - \langle u_2, \Phi u_2\rangle. \tag{2.3}$$

Conditions A1 and A2 thus allow us to define $\mathcal{A}$ as a self-adjoint operator. Since $(\mathcal{A}J)^* = J^*\mathcal{A}^* = J\mathcal{A}$ and $\sigma_J(\mathcal{A}) = \sigma(J\mathcal{A}) = \sigma(\mathcal{A}J)$ (see remark 2.2), we immediately have that $\sigma_J(\mathcal{A})$ is symmetric about the real axis. However, the operators $J\mathcal{A}$ and $\mathcal{A}J$ are not self-adjoint (or even normal) so, in general, $\sigma_J(\mathcal{A})$ will contain non-real points and non-semi-simple eigenvalues. There are several extra conditions we can impose on Φ and/or $\boldsymbol{A}$ that allow us to proceed further. The next result gives one such approach (essentially the approach used in [3]).

THEOREM 2.4. *Suppose there exists* $\delta > 0$ *such that*

$$|K_{\Phi}(\boldsymbol{u})| \leqslant \|T\boldsymbol{u}\|^2 + (c^2 - \delta)\|\boldsymbol{u}\|^2 \tag{2.4}$$

for all $\boldsymbol{u} \in C_0^\infty$ *(note that, in particular, this is satisfied if* $\|\Phi\|_{L^\infty} \leqslant c^2 - \delta$*). Then* $\sigma_J(\mathcal{A}) \subseteq (-\infty, -\delta] \cup [\delta, \infty)$ *and contains no non-semi-simple* J*-eigenvalues.*

Proof. Using (2.2), it immediately follows that (2.4) holds for all $\boldsymbol{u} \in \operatorname{Dom} K_{\mathcal{A}} \supset \operatorname{Dom}\mathcal{A}$. Thus, for all $\boldsymbol{u} \in \operatorname{Dom}\mathcal{A}$,

$$\langle \mathcal{A}\boldsymbol{u}, \boldsymbol{u}\rangle = \|T\boldsymbol{u}\|^2 + c^2\|\boldsymbol{u}\|^2 - K_{\Phi}(\boldsymbol{u}) \geqslant \delta\|\boldsymbol{u}\|^2,$$

and so $\mathcal{A} \geqslant \delta I$. Therefore, $\mathcal{A}$ has a boundedly invertible positive self-adjoint square root $\mathcal{A}^{1/2}$, which allows us to rewrite the spectral problem given by (1.5) as

$$(I - z\mathcal{A}^{-1/2}J\mathcal{A}^{-1/2})\boldsymbol{v} = 0.$$

It follows that $\sigma_J(\mathcal{A}) = \{z \mid 1/z \in \sigma(\mathcal{A}^{-1/2}J\mathcal{A}^{-1/2})\}$. However, $\mathcal{A}^{-1/2}J\mathcal{A}^{-1/2}$ is a self-adjoint operator bounded by $1/\delta$, so $\sigma(\mathcal{A}^{-1/2}J\mathcal{A}^{-1/2}) \subseteq [-1/\delta, 1/\delta]$ and contains no non-semi-simple eigenvalues. The result then follows. □

Some other conditions that allow us to obtain useful results about $\sigma_J(\mathcal{A})$ are as follows.

(B1) In addition to the requirements imposed by condition A1, we assume that $\Phi_0 \in L^\infty \cap L^2_{1\,\mathrm{loc}}$, $\Phi_0(x) \to 0$ as $|x| \to \infty$ and, if $p = \infty$, then $|\xi|^2\hat{\Phi}_1(\xi) \to 0$ as $|\xi| \to \infty$,

(B2) In addition to the requirements imposed by condition A2, we assume there exists a disc $B_n \subset \mathbb{R}^2$ of radius r_n for each $n \in \mathbb{N}$ such that $r_n \to \infty$ and $\|\Phi\|_{L^2(B_n)}, \|H\|_{L^2(B_n)} = o(r_n)$ as $n \to \infty$ (where H is defined in (1.4)).

(C) Φ and $\boldsymbol{A}$ are smooth with values and derivatives of all orders bounded (in the L^∞ norm). Furthermore, Φ and H decay at infinity.

REMARK 2.5. Condition A1 forces $\Phi_1 \in L^p$ for all $p \in [1, \infty)$; however, $\Phi_1 \notin L^\infty$ in general (unless $p = 1$, in which case Φ_1 must be continuous). The same is true for condition B1.

Examples of functions Φ_1 satisfying condition A1 but not the last part of condition B1 include $\Phi_1(x) = \phi(x)\log(|x|)$ and $\Phi_1(x) = \phi(x)H(x_1)H(x_2)$, where $\phi \in C_0^\infty$ is some cut-off function with $\phi(0) \neq 0$ and $H : \mathbb{R} \to \mathbb{R}$ is the Heaviside function.

Since $\|f\|_{L^2(B_n)} \leqslant \sqrt{\pi} r_n \|f\|_{L^\infty(B_n)}$ for any function f, it is enough to have that $\|\Phi\|_{L^\infty(B_n)}, \|H\|_{L^\infty(B_n)} \to 0$ in condition B2 (see remark 5.10 for further technical details regarding this condition).

Clearly, condition C implies conditions B1 and B2, while all of the above conditions are satisfied if Φ and $\boldsymbol{A}$ are Schwartz class functions.

3. Main results

Our main result concerning the J-essential spectrum of $\mathcal{A}$ is the following.

THEOREM 3.1. *If Φ satisfies condition B1, then $\sigma_{J\,\mathrm{Ess}}(\mathcal{A}) \subseteq (-\infty, -c^2] \cup [c^2, \infty)$. Furthermore, $\sigma_J(\mathcal{A}) \setminus \sigma_{J\,\mathrm{Ess}}(\mathcal{A})$ consists of isolated J-eigenvalues of finite algebraic (and hence geometric) multiplicity.*

If Φ and $\boldsymbol{A}$ satisfy condition B2, then $\sigma_{J\,\mathrm{Ess}}(\mathcal{A}) \supseteq (-\infty, -c^2] \cup [c^2, \infty)$.

This result was proved in [3] under the assumptions that Φ and $\boldsymbol{A}$ are smooth, their values and derivatives of all orders vanish at infinity and $\|\Phi\|_{L^\infty} < c^2$.

It is natural to compare the spectral properties of our equation (1.1) with those of the Dirac equation. Using atomic units, the stationary form of Dirac's equation in $\mathbb{R}^2$ can be written as

$$(\mathcal{D} - \lambda I)\psi = 0,$$

where

$$\mathcal{D} = \begin{pmatrix} -\Phi + c^2 & cP_- \\ cP_+ & -\Phi - c^2 \end{pmatrix}, \tag{3.1}$$

λ is the spectral parameter and ψ is a spinor (i.e. a function from $\mathbb{R}^2$ into $\mathbb{C}^2$). Assuming Φ and $\boldsymbol{A}$ satisfy conditions B1 and B2, it is possible to define $\mathcal{D}$ as a self-adjoint operator on a dense domain in L^2 and show that

$$\sigma_{\mathrm{Ess}}(\mathcal{D}) = (-\infty, -c^2] \cup [c^2, \infty)$$

(see [9], for example). Thus the essential spectra of the operators $\mathcal{A}$ and $\mathcal{D}$ coincide.

If we assume that Φ and $\boldsymbol{A}$ satisfy condition C, we can also compare the discrete spectra of these operators in the gap $(-c^2, c^2)$. To do this, it is easiest to compare both operators to the Pauli operator H_P^+ (after we have shifted the former spectra by $-c^2$ to allow for the rest mass of the electron). For the Dirac operator, we have the following result (see [6] or [9]).

THEOREM 3.2. *Let $\lambda < 0$ be an eigenvalue of Pauli's operator H_P^+ with multiplicity k and let $\{v_1, \dots, v_k\}$ be an orthonormal basis for the corresponding eigenspace.*

Then, for all sufficiently large c, there exist k (not necessarily distinct) isolated eigenvalues $\lambda_1(c), \dots, \lambda_k(c)$ *in* $\sigma(\mathcal{D})$ *which admit the asymptotic expansion*

$$\lambda_i(c) = c^2 + \lambda - \mu_i c^{-2} + O(c^{-4}), \quad i = 1, \dots, k,$$

where $\mu_1, \dots, \mu_k$ *are the (repeated) eigenvalues of the* $k \times k$ *matrix with entries* $\langle v_i, \frac{1}{4} P_-(\varPhi + \lambda) P_+ v_j \rangle$ *for* $i, j = 1, \dots, k$. *Furthermore, there exists a c-independent neighbourhood U of* 0 *such that these are the only points of* $\sigma(\mathcal{D}) \cap (c^2 + U)$.

We establish a similar result for the operator $\mathcal{A}$ (note that if $c^2 > \|\varPhi\|_{L^\infty}$, then theorem 2.4 shows that all J-eigenvalues of $\mathcal{A}$ are real and semi-simple).

THEOREM 3.3. *Let* $\lambda < 0$ *be an eigenvalue of Pauli's operator* H_P^+ *with multiplicity k and let* $\{v_1, \dots, v_k\}$ *be an orthonormal basis for the corresponding eigenspace. Then, for all sufficiently large c, there exist k (not necessarily distinct) isolated J-eigenvalues* $\lambda_1(c), \dots, \lambda_k(c)$ *in* $\sigma_J(\mathcal{A})$ *which admit the asymptotic expansion*

$$\lambda_i(c) = c^2 + \lambda - \mu_i c^{-2} + O(c^{-4}), \quad i = 1, \dots, k,$$

where $\mu_1, \dots, \mu_k$ *are the (repeated) eigenvalues of the* $k \times k$ *matrix with entries* $\langle v_i, \frac{1}{2} B^* B v_j \rangle$ *for* $i, j = 1, \dots, k$. *Furthermore, there exists a c-independent neighbourhood U of* 0 *such that these are the only points of* $\sigma_J(\mathcal{A}) \cap (c^2 + U)$.

Therefore, the discrete spectrum of $\mathcal{A}$ agrees with that of the Dirac operator to a relative accuracy of $O(c^{-4})$, although we cannot expect to better than this in general.

REMARK 3.4. The operator (3.1) is the Dirac operator corresponding to 'spin-up electrons' in $\mathbb{R}^2$; it is also possible to consider the Dirac operator corresponding to 'spin-down electrons',

$$\mathcal{D} = \begin{pmatrix} -\varPhi + c^2 & cP_+ \\ cP_- & -\varPhi - c^2 \end{pmatrix}.$$

In this case, comparison should be made with the operator similar to $\mathcal{A}$ obtained by taking the lower sign in (1.2) (and an appropriately modified Pauli operator).

Making the appropriate basic changes, it is also possible to compare points of the discrete spectra of $\mathcal{A}$ and $\mathcal{D}$ just above $-c^2$ (i.e. when dealing with positrons); the spectra have to be shifted by c^2 and comparison made to positive eigenvalues of the Pauli operator $-H_P^-$.

4. Some technical results

In this section we deal with some technical results that are needed for the proofs of proposition 2.3 and the first part of theorem 3.1; the former is given at the end of this section and the later in the next section.

Define a function $\varLambda$ on $\mathbb{R}^2$ by $\varLambda(\xi) = (1 + |\xi|^2)^{1/2}$ and let T_0 be the operator defined in the same way as T except with $\boldsymbol{A} = 0$, i.e. $T_0 = (1/\sqrt{2})\big(\partial_+ \quad -\mathrm{i}\partial_-\big)$. Also, throughout this section, let $\varPhi_1$, p and k be as given by condition A1.

Let $\boldsymbol{u} \in C_0^\infty$. Thus the Fourier transform $\hat{\boldsymbol{u}}$ is Schwartz class. Now define maps F and G on C_0^∞ by setting

$$\begin{pmatrix} F\boldsymbol{u} \\ G\boldsymbol{u} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i}\xi_-/\xi_+ \\ \mathrm{i}\xi_+/\xi_- & 1 \end{pmatrix} \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix}.$$

It is easy to see that $\boldsymbol{Fu}$ and $\boldsymbol{Gu}$ are bounded rapidly decreasing functions. Furthermore,

$$\text{the Fourier transform of } T_0\boldsymbol{u} \text{ is } \xi_- \boldsymbol{Gu}, \tag{4.1}$$

while Parseval's identity gives

$$\|\boldsymbol{Fu}\|^2, \|\boldsymbol{Gu}\|^2 \leqslant \|\boldsymbol{Fu}\|^2 + \|\boldsymbol{Gu}\|^2 = \|\boldsymbol{u}\|^2. \tag{4.2}$$

Setting $f = \boldsymbol{Fu}$ and $g = \boldsymbol{Gu}$, Parseval's identity also gives

$$\begin{aligned}
&K_{\Phi_1}(\boldsymbol{u}) \\
&\quad = \iint \hat{\Phi}_1(\xi-\eta)(\hat{u}_1(\eta)\bar{\hat{u}}_1(\xi) - \hat{u}_2(\eta)\bar{\hat{u}}_2(\xi))\,\mathrm{d}\eta\mathrm{d}\xi \\
&\quad = \frac{1}{2}\iint \hat{\Phi}_1(\xi-\eta)\biggl[\biggl(1-\frac{\eta_+\xi_-}{\eta_-\xi_+}\biggr)f(\eta)\bar{f}(\xi) - \biggl(1-\frac{\eta_-\xi_+}{\eta_+\xi_-}\biggr)g(\eta)\bar{g}(\xi)\biggr]\mathrm{d}\eta\mathrm{d}\xi \\
&\qquad + \frac{1}{2}\iint \hat{\Phi}_1(\xi-\eta)\biggl[\biggl(\frac{\mathrm{i}\xi_+}{\xi_-}+\frac{\mathrm{i}\eta_+}{\eta_-}\biggr)f(\eta)\bar{g}(\xi) - \biggl(\frac{\mathrm{i}\xi_-}{\xi_+}+\frac{\mathrm{i}\eta_-}{\eta_+}\biggr)g(\eta)\bar{f}(\xi)\biggr]\mathrm{d}\eta\mathrm{d}\xi.
\end{aligned}$$

Now $|\hat{\Phi}_1|$ is symmetric about 0 (since Φ_1 is real valued) and $|\xi_\pm/\xi_\mp| = |\eta_\pm/\eta_\mp| = 1$. It follows that

$$|K_{\Phi_1}(\boldsymbol{u})| \leqslant I_{\Phi_1}(\boldsymbol{Fu}) + I_{\Phi_1}(\boldsymbol{Gu}) + 2J_{\Phi_1}(\boldsymbol{Fu}, \boldsymbol{Gu}), \tag{4.3}$$

where $I_{\Phi_1}(\cdot)$ and $J_{\Phi_1}(\cdot,\cdot)$ are defined by

$$I_{\Phi_1}(f) = \frac{1}{2}\iint \biggl|\hat{\Phi}_1(\xi-\eta)\biggl(1-\frac{\eta_+\xi_-}{\eta_-\xi_+}\biggr)f(\eta)\bar{f}(\xi)\biggr|\,\mathrm{d}\eta\mathrm{d}\xi$$

and

$$J_{\Phi_1}(f,g) = \iint |\hat{\Phi}_1(\xi-\eta)f(\eta)g(\xi)|\,\mathrm{d}\eta\mathrm{d}\xi.$$

Now, for any $\xi, \eta \neq 0$, it can be checked that

$$\frac{1}{2}\biggl|1-\frac{\eta_+\xi_-}{\eta_-\xi_+}\biggr| \leqslant \min\{1, \Lambda(\xi-\eta)\Lambda^{-1}(\xi)\}. \tag{4.4}$$

We shall use $a(\xi,\eta)$ to denote the expression on the left-hand side of (4.4) in the proof of the following result.

Lemma 4.1. *There exists a constant C and a non-negative function $\sigma \in C^\infty$ with $\|\sigma\|_{L^\infty} \leqslant 1$ such that $I_{\Phi_1}(f) \leqslant C\|\sigma f\|^2 \leqslant C\|f\|^2$ for any bounded rapidly decreasing function f. Furthermore, if $p \neq \infty$ or if $p = \infty$ and $|\xi|^2\hat{\Phi}_1(\xi) \to 0$ as $|\xi| \to \infty$, then we can choose σ so that $\sigma(\xi) \to 0$ as $|\xi| \to \infty$.*

Proof. We have $|\hat{\Phi}_1(\xi-\eta)|a(\xi,\eta) = |\hat{\Phi}_1(\eta-\xi)|a(\eta,\xi)$ (note that Φ_1 is real valued, so $|\hat{\Phi}_1|$ is symmetric about 0). Together with Hölder's inequality, we then get

$$I_{\Phi_1}(f) = \iint |\hat{\Phi}_1(\xi-\eta)a(\xi,\eta)f(\eta)\bar{f}(\xi)|\,\mathrm{d}\eta\mathrm{d}\xi \leqslant \int \mu(\xi)|f(\xi)|^2\,\mathrm{d}\xi,$$

where

$$\mu(\xi) = \int \frac{\Lambda^\alpha(\xi)}{\Lambda^\alpha(\eta)} |\hat{\Phi}_1(\xi - \eta)| a(\xi, \eta) \, \mathrm{d}\eta$$

and $\alpha = \frac{3}{2}(1 - 1/p)$. Defining $q \in [1, \infty]$ by $1/p + 1/q = 1$ and using Hölder's inequality once again, we then have $\mu(\xi) \leqslant M(\xi) N(\xi)$, where

$$M(\xi) = \left[\int |a^{1/2}(\xi, \eta) \Lambda^k(\xi - \eta) \hat{\Phi}_1(\xi - \eta)|^p \, \mathrm{d}\eta \right]^{1/p} \tag{4.5}$$

and

$$N(\xi) = \left[\int \left(\frac{\Lambda^\alpha(\xi) a^{1/2}(\xi, \eta)}{\Lambda^\alpha(\eta) \Lambda^k(\xi - \eta)} \right)^q \mathrm{d}\eta \right]^{1/q}.$$

If $p = 1$, then $q = \infty$ and $\alpha = k = 0$ so $N(\xi) = \|a^{1/2}(\xi, \cdot)\|_{L^\infty} \leqslant 1$ by (4.4). On the other hand, if $p > 1$, then $\alpha q = \frac{3}{2}$ and $kq \geqslant 2$, while (4.4) gives

$$a^{q/2}(\xi, \eta) \leqslant \Lambda^{1/2}(\xi - \eta) \Lambda^{-1/2}(\xi).$$

Therefore,

$$N(\xi) \leqslant \left[\Lambda(\xi) \int \frac{\mathrm{d}\eta}{\Lambda^{3/2}(\eta) \Lambda^{3/2}(\xi - \eta)} \right]^{1/q}.$$

A scaling argument shows that this expression is bounded. Recombining the two cases, it follows that we can find a constant C_1 such that $N(\xi) \leqslant C_1$ for all ξ and p. However, equation (4.4) also gives $M(\xi) \leqslant \|\Lambda^k \hat{\Phi}_1\|_{L^p}$. Hence $\mu(\xi) \leqslant C_1 \|\Lambda^k \hat{\Phi}_1\|_{L^p}$ and the first part of the result follows with $\sigma = 1$.

Now suppose $p \neq \infty$ or $p = \infty$ and $|\xi|^2 \hat{\Phi}_1(\xi) \to 0$ as $|\xi| \to \infty$. Choose $\varepsilon > 0$ and set $\delta = \varepsilon / 2C_1 > 0$. Our assumptions on $\hat{\Phi}_1$ then allow us to find some $r > 0$ such that $\|(1 - \chi) \Lambda^k \hat{\Phi}_1\|_{L^p} < \delta$, where χ is the characteristic function of the disc of radius r and centre 0 in $\mathbb{R}^2$. Now (4.4) gives

$$(1 - \chi(\xi - \eta)) a^{1/2}(\xi, \eta) \leqslant 1 - \chi(\xi - \eta) \quad \text{and} \quad \chi(\xi - \eta) a^{1/2}(\xi, \eta) \leqslant (1 + r^2)^{1/4} \Lambda^{-1/2}(\xi)$$

(where the last estimate follows from the fact that $\chi(\xi - \eta) = 0$ for $|\xi - \eta| > r$). If we now assume that $|\xi| > \delta^{-2} \|\Lambda^k \hat{\Phi}_1\|_{L^p}^2 (1 + r^2)^{1/2}$, equation (4.5) gives us

$$M(\xi) \leqslant \|(1 - \chi) \Lambda^k \hat{\Phi}_1\|_{L^p} + (1 + r^2)^{1/4} \Lambda^{-1/2}(\xi) \|\Lambda^k \hat{\Phi}_1\|_{L^p} \leqslant 2\delta.$$

Since $\mu(\xi) \leqslant M(\xi) N(\xi)$ and $N(\xi) \leqslant C_1$ from above, we then have $\mu(\xi) \leqslant \varepsilon$. Thus $\mu(\xi) \to 0$ as $|\xi| \to \infty$. The second part of the result now follows if we choose σ to be any suitably scaled smooth majoritant of μ that decays at infinity. □

Lemma 4.2. *There exists a constant C such that for any bounded rapidly decreasing functions f and g we have $J_{\Phi_1}(f, g) \leqslant 2C \|f\| \|\Lambda^{3/4} g\| \leqslant C(\|f\|^2 + \|\Lambda^{3/4} g\|^2)$.*

Proof. Using Hölder's inequality, we get

$$J_{\Phi_1}(f, g) = \iint |\hat{\Phi}_1(\xi - \eta) f(\eta) g(\xi)| \, \mathrm{d}\eta \mathrm{d}\xi \leqslant \|\hat{\Phi}_1\|_{L^{4/3}} \|f\| \|g\|_{L^{4/3}}$$

and $\|g\|_{L^{4/3}} \leqslant \|\Lambda^{-3/4}\|_{L^4}\|\Lambda^{3/4}g\| = C\|\Lambda^{3/4}g\|$, where C is a positive constant. It remains to show that $\|\hat{\Phi}_1\|_{L^{4/3}}$ is bounded; this can be done by considering two cases.

CASE 1 ($p > \frac{4}{3}$). Hölder's inequality gives $\|\hat{\Phi}_1\|_{L^{4/3}} \leqslant \|\Lambda^k\hat{\Phi}_1\|_{L^p}\|\Lambda^{-k}\|_{L^q}$, where q is defined by $\frac{3}{4} = 1/p + 1/q$. Therefore, $kq \geqslant (8p-8)/(3p-4) > 2$, which implies $\|\Lambda^{-k}\|_{L^q} < +\infty$. Hence $\|\hat{\Phi}_1\|_{L^{4/3}} < +\infty$.

CASE 2 ($1 \leqslant p \leqslant \frac{4}{3}$). We have $\Phi_1 \in L^1$, so $\hat{\Phi}_1 \in L^\infty$. Also, $\Lambda^k\hat{\Phi}_1 \in L^p$ for some $k \geqslant 0$, so $\hat{\Phi}_1 \in L^p$. However, $L^{4/3} \subseteq L^p \cap L^\infty$, giving $\|\hat{\Phi}_1\|_{L^{4/3}} < +\infty$. □

Proof of proposition 2.3. Write $\Phi = \Phi_0 + \Phi_1$, where Φ_0 and Φ_1 are as given by condition A1, and choose $\phi \in C_0^\infty$ to be a cut-off function with $\operatorname{Ran}\phi \subseteq [0,1]$ and which is equal to 1 on $\operatorname{supp}(\Phi_1)$. Thus $\Phi_1 = \phi^2\Phi_1$. It follows that for any $\boldsymbol{u} \in C_0^\infty$ we have $K_{\Phi_1}(\boldsymbol{u}) = \langle \Phi_1 J\boldsymbol{u}, \boldsymbol{u}\rangle = K_{\Phi_1}(\phi\boldsymbol{u})$ and so

$$|K_\Phi(\boldsymbol{u})| \leqslant |K_{\Phi_0}(\boldsymbol{u})| + |K_{\Phi_1}(\boldsymbol{u})| \leqslant \|\Phi_0\|_{L^\infty}\|\boldsymbol{u}\|^2 + |K_{\Phi_1}(\phi\boldsymbol{u})|. \tag{4.6}$$

On the other hand,

$$T_0(\phi\boldsymbol{u}) = \phi T\boldsymbol{u} - \frac{\phi}{\sqrt{2}}(\mathrm{i}A_+u_1 + A_-u_2) + \frac{1}{\sqrt{2}}((\partial_+\phi)u_1 - \mathrm{i}(\partial_-\phi)u_2).$$

Since ϕ has compact support, $\|\phi\|_{L^\infty} = 1$ and $\boldsymbol{A} \in L^\infty_{\mathrm{loc}}$, we then obtain

$$\|\phi\boldsymbol{u}\| \leqslant \|\boldsymbol{u}\| \quad \text{and} \quad \|T_0(\phi\boldsymbol{u})\|^2 \leqslant C_1\|\boldsymbol{u}\|^2 + 2\|T\boldsymbol{u}\|^2 \tag{4.7}$$

for some constant C_1.

By combining (4.3) with lemmas 4.1 and 4.2, we get

$$|K_{\Phi_1}(\phi\boldsymbol{u})| \leqslant C_2\|F(\phi\boldsymbol{u})\|^2 + C_3\|G(\phi\boldsymbol{u})\|^2 + C_4\big\|\Lambda^{3/4}G(\phi\boldsymbol{u})\big\|^2 \tag{4.8}$$

for some constants C_2, C_3 and C_4. Now $C_4\Lambda^{3/2}(\xi) \leqslant C_5 + \kappa|\xi_-|^2/2$ for some constant C_5. With the help of (4.1) and Parseval's identity, it follows that

$$C_4\big\|\Lambda^{3/4}G(\phi\boldsymbol{u})\big\|^2 \leqslant C_5\|G(\phi\boldsymbol{u})\|^2 + \tfrac{1}{2}\kappa\|T_0(\phi\boldsymbol{u})\|^2.$$

Using this estimate and (4.2), equation (4.8) now gives

$$|K_{\Phi_1}(\phi\boldsymbol{u})| \leqslant (C_2 + C_3 + C_5)\|\phi\boldsymbol{u}\|^2 + \tfrac{1}{2}\kappa\|T_0(\phi\boldsymbol{u})\|^2.$$

Equations (4.6) and (4.7) now complete the proof. □

5. The essential spectrum

In order to prove the first part of theorem 3.1, we will first show that the form $K_\Phi(\boldsymbol{u})$ is 'relatively compact' with respect to the form $\|T\boldsymbol{u}\|^2 + \|\boldsymbol{u}\|^2$. We begin by establishing this for $K_{\Phi_1}(\boldsymbol{u})$.

LEMMA 5.1. *Let Φ_1 be as given by condition B1. Then, for any $\varepsilon > 0$, there exists a finite-dimensional vector space $L \subset C_0^\infty$ such that $|K_{\Phi_1}(\boldsymbol{u})| \leqslant \varepsilon(\|T\boldsymbol{u}\|^2 + \|\boldsymbol{u}\|^2)$ for all $\boldsymbol{u} \in L^\perp \cap C_0^\infty$.*

Proof. Let ϕ and C_1 be as defined in the proof of proposition 2.3. Also choose $\psi \in C_0^\infty$, with ψ equal to 1 on $\operatorname{supp}(\phi)$. Now let $\boldsymbol{u} \in C_0^\infty$. By combining the identity $K_{\Phi_1}(\boldsymbol{u}) = K_{\Phi_1}(\phi\boldsymbol{u})$ with (4.3) and lemmas 4.1 and 4.2, we get

$$|K_{\Phi_1}(\boldsymbol{u})| \leqslant C_2(\|\sigma F(\phi\boldsymbol{u})\|^2 + \|\sigma G(\phi\boldsymbol{u})\|^2) + C_3\|F(\phi\boldsymbol{u})\|\,\|\Lambda^{3/4}G(\phi\boldsymbol{u})\| \tag{5.1}$$

for some constants C_2 and C_3 and a bounded non-negative function $\sigma \in C^\infty$ that satisfies $\sigma(\xi) \to 0$ as $|\xi| \to \infty$ (the existence of which comes from lemma 4.1 now that the extra conditions are satisfied). By Parseval's identity,

$$\|\sigma F(\phi\boldsymbol{u})\|^2 + \|\sigma G(\phi\boldsymbol{u})\|^2 = \|\sigma(-\mathrm{i}\partial)(\phi\boldsymbol{u})\|^2, \tag{5.2}$$

while, with some help from (4.1) and the fact that $T_0(\phi\boldsymbol{u}) = \psi T_0(\phi\boldsymbol{u})$,

$$\begin{aligned}\|\Lambda^{3/4}G(\phi\boldsymbol{u})\|^2 &= \int (|\Lambda^{-1/4}(\xi)G(\phi\boldsymbol{u})(\xi)|^2 + |\Lambda^{-1/4}(\xi)\xi_- G(\phi\boldsymbol{u})(\xi)|^2)\,\mathrm{d}\xi \\ &\leqslant \|\Lambda^{-1/4}(-\mathrm{i}\partial)(\phi\boldsymbol{u})\|^2 + \|\Lambda^{-1/4}(-\mathrm{i}\partial)(\psi T_0(\phi\boldsymbol{u}))\|^2. \end{aligned} \tag{5.3}$$

Set $\delta = \varepsilon/(C_2 + 2C_3 + C_1C_3) > 0$. Now the functions σ and $\Lambda^{-1/4}$ decay at infinity while $\phi \in C_0^\infty$. It follows that the operators $\sigma(-\mathrm{i}\partial)(\phi\,\cdot\,)$ and $\Lambda^{-1/4}(-\mathrm{i}\partial)(\phi\,\cdot\,)$ are compact on L^2 (see Appendix 2 to § XI.3 of [8] for example). Thus we can find a finite collection of functions $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in C_0^\infty$ such that

$$\|\sigma(-\mathrm{i}\partial)(\phi\boldsymbol{u})\| \leqslant \sqrt{\delta}\|\boldsymbol{u}\| \quad \text{and} \quad \|\Lambda^{-1/4}(-\mathrm{i}\partial)(\phi\boldsymbol{u})\| \leqslant \delta\|\boldsymbol{u}\| \tag{5.4}$$

if $\langle \boldsymbol{u}, \boldsymbol{v}_i\rangle = 0$ for $i = 1, \dots, n$. Similarly, the operator $\Lambda^{-1/4}(-\mathrm{i}\partial)(\psi\,\cdot\,)$ is compact on L^2, so we can find another finite collection of functions $\phi_1, \dots, \phi_m \in C_0^\infty$ such that

$$\|\Lambda^{-1/4}(-\mathrm{i}\partial)(\psi T_0(\phi\boldsymbol{u}))\| \leqslant 2\delta\|T_0(\phi\boldsymbol{u})\| \tag{5.5}$$

if $\langle T_0(\phi\boldsymbol{u}), \phi_i\rangle = 0$ for $i = 1, \dots, m$. Now define a finite-dimensional vector space by

$$L = \operatorname{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n, \phi T_0^*\phi_1, \dots, \phi T_0^*\phi_m\} \subset C_0^\infty$$

and suppose $\boldsymbol{u} \in L^\perp \cap C_0^\infty$. It follows that the conditions for (5.4) and (5.5) are satisfied. We also have $\|F(\phi\boldsymbol{u})\| \leqslant \|\boldsymbol{u}\|$ by (4.2) and (4.7). Combining these observations with (5.1)–(5.5), we then get

$$\begin{aligned}|K_{\Phi_1}(\boldsymbol{u})| &\leqslant C_2\delta\|\boldsymbol{u}\|^2 + C_3\|\boldsymbol{u}\|(\delta\|\boldsymbol{u}\| + 2\delta\|T_0(\phi\boldsymbol{u})\|) \\ &\leqslant (C_2 + 2C_3)\delta\|\boldsymbol{u}\|^2 + C_3\delta\|T_0(\phi\boldsymbol{u})\|^2.\end{aligned}$$

Estimate (4.7), together with the definition of δ, then completes the result. □

We now extend the previous result to deal with Φ (rather than just Φ_1).

LEMMA 5.2. *Suppose Φ satisfies condition B1. Then, for any $\varepsilon > 0$, there exists a finite-dimensional vector space $L \subset C_0^\infty$ such that $|K_\Phi(\boldsymbol{u})| \leqslant \varepsilon(\|T\boldsymbol{u}\|^2 + \|\boldsymbol{u}\|^2)$ for all $\boldsymbol{u} \in L^\perp \cap \operatorname{Dom}\mathcal{A}$.*

Proof. If $\Psi \in L^2_{1\,\mathrm{loc}}$ and $\operatorname{supp}(\Psi)$ is compact, then $\Lambda\hat{\Psi} \in L^2$. It follows that we can write $\Phi_0 = \Phi_0' + \Phi_1'$, where Φ_0' and Φ_1' satisfy condition B1 (with $p = 2$) and $\|\Phi_0'\|_{L^\infty} < \frac{1}{3}\varepsilon$. Hence

$$|K_{\Phi_0'}(\boldsymbol{u})| \leqslant \tfrac{1}{3}\varepsilon\|\boldsymbol{u}\|^2 \leqslant \tfrac{1}{3}\varepsilon(\|T\boldsymbol{u}\|^2 + \|\boldsymbol{u}\|^2)$$

for all $\boldsymbol{u} \in C_0^\infty$. On the other hand, lemma 5.1 gives us a finite-dimensional vector space $L \subset C_0^\infty$ such that

$$|K_{\Phi_1}(\boldsymbol{u})|, |K_{\Phi_1'}(\boldsymbol{u})| \leqslant \tfrac{1}{3}\varepsilon(\|T\boldsymbol{u}\|^2 + \|\boldsymbol{u}\|^2)$$

for all $\boldsymbol{u} \in L^\perp \cap C_0^\infty$. However, $K_\Phi(\boldsymbol{u}) = K_{\Phi_0'}(\boldsymbol{u}) + K_{\Phi_1}(\boldsymbol{u}) + K_{\Phi_1'}(\boldsymbol{u})$, which, combined with (2.2), now completes the result. □

PROPOSITION 5.3. *Let $z \in \mathbb{C} \setminus ((-\infty, -c^2] \cup [c^2, \infty))$ and suppose Φ satisfies condition B1. Then* $\operatorname{Ker}(\mathcal{A} - zJ)$ *is finite dimensional and* $\operatorname{Ran}(\mathcal{A} - zJ)$ *is closed.*

Proof. Write $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$. Firstly, suppose $y = 0$ (so $z = x \in (-c^2, c^2)$) and set $\delta = \operatorname{dist}(x, \{-c^2, c^2\}) > 0$. It follows that, for all $\boldsymbol{u} \in \operatorname{Dom}\mathcal{A}$,

$$\begin{aligned} K_{\mathcal{A}}(\boldsymbol{u}) - \langle zJ\boldsymbol{u}, \boldsymbol{u}\rangle &= \|T\boldsymbol{u}\|^2 + c^2\|\boldsymbol{u}\|^2 - x\langle J\boldsymbol{u}, \boldsymbol{u}\rangle - K_\Phi(\boldsymbol{u}) \\ &\geqslant \|T\boldsymbol{u}\|^2 + \delta\|\boldsymbol{u}\|^2 - K_\Phi(\boldsymbol{u}). \end{aligned}$$

Hence we can choose $\varepsilon > 0$ so that

$$|K_{\mathcal{A}}(\boldsymbol{u}) - \langle zJ\boldsymbol{u}, \boldsymbol{u}\rangle| \geqslant \varepsilon\|T\boldsymbol{u}\|^2 + 2\varepsilon\|\boldsymbol{u}\|^2 - |K_\Phi(\boldsymbol{u})|.$$

Now suppose $y \neq 0$ and set $w = x(x^2 + y^2)^{-1/2}$. Therefore, $|w| < 1$, which implies $\delta = (1 - w^2)^{1/2} > 0$. Let $\boldsymbol{u} \in \operatorname{Dom}\mathcal{A}$ and set $d = \|T\boldsymbol{u}\|^2 + c^2\|\boldsymbol{u}\|^2$. Then

$$\begin{aligned} &|\|T\boldsymbol{u}\|^2 + c^2\|\boldsymbol{u}\|^2 - \langle zJ\boldsymbol{u}, \boldsymbol{u}\rangle|^2 \\ &\qquad = \delta^2 d^2 + w^2 d^2 - 2w(x^2 + y^2)^{1/2} d\langle J\boldsymbol{u}, \boldsymbol{u}\rangle + (x^2 + y^2)\langle J\boldsymbol{u}, \boldsymbol{u}\rangle^2 \geqslant \delta^2 d^2. \end{aligned}$$

It follows that, for all $\boldsymbol{u} \in \operatorname{Dom}\mathcal{A}$,

$$\begin{aligned} |K_{\mathcal{A}}(\boldsymbol{u}) - \langle zJ\boldsymbol{u}, \boldsymbol{u}\rangle| &\geqslant \delta(\|T\boldsymbol{u}\|^2 + c^2\|\boldsymbol{u}\|^2) - |K_\Phi(\boldsymbol{u})| \\ &\geqslant \varepsilon\|T\boldsymbol{u}\|^2 + 2\varepsilon\|\boldsymbol{u}\|^2 - |K_\Phi(\boldsymbol{u})| \end{aligned}$$

for some suitably chosen $\varepsilon > 0$. By recombining the two cases (i.e. when $y = 0$ and $y \neq 0$) and applying lemma 5.2, we can now find a finite-dimensional subspace $L \subset C_0^\infty$ such that, for all $\boldsymbol{u} \in L^\perp \cap \operatorname{Dom}\mathcal{A}$, we have

$$|\langle(\mathcal{A} - zJ)\boldsymbol{u}, \boldsymbol{u}\rangle| = |K_{\mathcal{A}}(\boldsymbol{u}) - \langle zJ\boldsymbol{u}, \boldsymbol{u}\rangle| \geqslant \varepsilon\|\boldsymbol{u}\|^2 \quad \Rightarrow \quad \|(\mathcal{A} - zJ)\boldsymbol{u}\| \geqslant \varepsilon\|\boldsymbol{u}\|. \tag{5.6}$$

Now suppose we have a sequence satisfying

$$\{\boldsymbol{u}_i\}_{i\in\mathbb{N}} \subset \operatorname{Dom}\mathcal{A}, \qquad \|\boldsymbol{u}_i\| = 1, \qquad (\mathcal{A} - zJ)\boldsymbol{u}_i \to 0. \tag{5.7}$$

Write $\boldsymbol{u}_i = \boldsymbol{u}_i^0 + \boldsymbol{u}_i^1$, where $\boldsymbol{u}_i^0 \in L$ and $\boldsymbol{u}_i^1 \in L^\perp$. Thus $\boldsymbol{u}_i^0 \in C_0^\infty \subset \operatorname{Dom}\mathcal{A}$ and so $\boldsymbol{u}_i^1 \in \operatorname{Dom}\mathcal{A}$ as well. Now $\{\boldsymbol{u}_i^0\}_{i\in\mathbb{N}}$ is a bounded sequence in the finite-dimensional space L, so it contains a convergent subsequence $\{\boldsymbol{u}_{i(j)}^0\}_{j\in\mathbb{N}}$. However, $\mathcal{A} - zJ$ is bounded on L (as $L \subset \operatorname{Dom}\mathcal{A}$ is finite dimensional), so (5.7) implies that $\{(\mathcal{A} - zJ)\boldsymbol{u}_{i(j)}^1\}_{j\in\mathbb{N}}$ is convergent and hence Cauchy. Since $\boldsymbol{u}_{i(j)}^1 \in L^\perp \cap \operatorname{Dom}\mathcal{A}$, equation (5.6) then implies that $\{\boldsymbol{u}_{i(j)}^1\}_{j\in\mathbb{N}}$ is also Cauchy and hence convergent. By adding $\{\boldsymbol{u}_{i(j)}^0\}_{j\in\mathbb{N}}$ and $\{\boldsymbol{u}_{i(j)}^1\}_{j\in\mathbb{N}}$, it follows that we can find a convergent subsequence of any sequence satisfying (5.7). A standard argument (see theorems IV.5.10

and IV.5.11 in [7], for example) shows that this implies that $\mathcal{A} - zJ$ has a finite-dimensional kernel and a closed range. $\square$

The next two results are needed to help control the behaviour of the non-real part of $\sigma_J(\mathcal{A})$. To do this, we will need to consider the standard spectral problem associated to the (self-adjoint) operator $\mathcal{A}$.

Proposition 5.4. *Suppose Φ satisfies condition B1 and let $\delta > 0$. Then the spectral subspace of $\mathcal{A}$ corresponding to $(-\infty, c^2 - \delta] \cap \sigma(\mathcal{A})$ is finite dimensional.*

Proof. Let $x = c^2 - \delta$ and $\varepsilon = \min\{1, \frac{1}{2}\delta\} > 0$. Therefore, for all $\boldsymbol{u} \in \operatorname{Dom} \mathcal{A}$,

$$\begin{aligned} K_{\mathcal{A}}(\boldsymbol{u}) - x\|\boldsymbol{u}\|^2 &= \|T\boldsymbol{u}\|^2 + c^2\|\boldsymbol{u}\|^2 - x\|\boldsymbol{u}\|^2 - K_{\Phi}(\boldsymbol{u}) \\ &\geqslant \varepsilon\|T\boldsymbol{u}\|^2 + 2\varepsilon\|\boldsymbol{u}\|^2 - K_{\Phi}(\boldsymbol{u}). \end{aligned}$$

By applying lemma 5.2, it follows that we can find a finite-dimensional subspace $L \subset C_0^\infty$ such that

$$\langle (\mathcal{A} - xI)\boldsymbol{u}, \boldsymbol{u} \rangle = K_{\mathcal{A}}(\boldsymbol{u}) - x\|\boldsymbol{u}\|^2 \geqslant \varepsilon\|\boldsymbol{u}\|^2$$

for all $\boldsymbol{u} \in L^\perp \cap \operatorname{Dom} \mathcal{A}$. Therefore, the spectral subspace of $\mathcal{A}$ corresponding to $(-\infty, x] \cap \sigma(\mathcal{A})$ has dimension at most $\dim L$. $\square$

Proposition 5.5. *There can be at most finitely many non-real J-eigenvalues in $\sigma_J(\mathcal{A})$.*

Proof. Let $\mathcal{Q}_-$ and $\mathcal{Q}_+$ be the (self-adjoint) spectral projections of $\mathcal{A}$ corresponding to $(-\infty, 0) \cap \sigma(\mathcal{A})$ and $[0, +\infty) \cap \sigma(\mathcal{A})$, respectively. Therefore, $\pm\mathcal{Q}_\pm\mathcal{A}\mathcal{Q}_\pm \geqslant 0$, and so we can define non-negative self-adjoint operators by $\mathcal{B}_\pm = (\pm\mathcal{Q}_\pm\mathcal{A}\mathcal{Q}_\pm)^{1/2}$. Define further operators by $\mathcal{Q} = \mathcal{Q}_+ - \mathcal{Q}_-$ and $\mathcal{B} = \mathcal{B}_+ - \mathcal{B}_-$. A straightforward calculation gives

$$\mathcal{B}\mathcal{Q}\mathcal{B} = \mathcal{A}. \tag{5.8}$$

Suppose $\{z_i \mid i \in I\}$ is a finite set of non-real J-eigenvalues of $\mathcal{A}$ such that $\{z_i, \bar{z}_i \mid i \in I\}$ is a set of $2|I|$ distinct points. Choose $0 \neq \boldsymbol{u}_i \in \operatorname{Ker}(\mathcal{A} - z_i J)$ for each $i \in I$ and set $\boldsymbol{v}_i = \mathcal{B}\boldsymbol{u}_i$. Define L and L' to be the linear spans of $\{\boldsymbol{u}_i \mid i \in I\}$ and $\{\boldsymbol{v}_i \mid i \in I\}$, respectively. Since the $\boldsymbol{u}_i$ are eigenvectors of the operator $J\mathcal{A}$ corresponding to distinct eigenvalues, the set $\{\boldsymbol{u}_i \mid i \in I\}$ must be linearly independent. Therefore, $\dim L = |I|$.

Claim. $\dim L' \leqslant \dim \mathcal{Q}_-$. *Suppose $\boldsymbol{v} \in \operatorname{Ker} \mathcal{Q}_-$ for some $\boldsymbol{v} = \sum_{i \in I} \lambda_i \boldsymbol{v}_i$, where $\lambda_i \in \mathbb{C}$ for each $i \in I$. Since $\mathcal{Q}_+ + \mathcal{Q}_- = I$ and $\langle \boldsymbol{v}, \mathcal{Q}_-\boldsymbol{v} \rangle = 0$, we have*

$$\begin{aligned} \|\boldsymbol{v}\|^2 &= \langle \boldsymbol{v}, (\mathcal{Q}_+ + \mathcal{Q}_-)\boldsymbol{v} \rangle \\ &= \langle \boldsymbol{v}, (\mathcal{Q}_+ - \mathcal{Q}_-)\boldsymbol{v} \rangle \\ &= \sum_{i,j \in I} \lambda_i \bar{\lambda}_j \langle \mathcal{B}\boldsymbol{u}_i, \mathcal{Q}\mathcal{B}\boldsymbol{u}_j \rangle \\ &= \sum_{i,j \in I} \lambda_i \bar{\lambda}_j \langle \boldsymbol{u}_i, \mathcal{A}\boldsymbol{u}_j \rangle \end{aligned} \tag{5.9}$$

by (5.8) and the self-adjointness of $\mathcal{B}$. Now $\mathcal{A}$ is self-adjoint and $\mathcal{A}\boldsymbol{u}_i = z_i J\boldsymbol{u}_i$ for each $i \in I$. Therefore,

$$z_i\langle J\boldsymbol{u}_i, \boldsymbol{u}_j\rangle = \langle \mathcal{A}\boldsymbol{u}_i, \boldsymbol{u}_j\rangle = \langle \boldsymbol{u}_i, \mathcal{A}\boldsymbol{u}_j\rangle = \bar{z}_j\langle \boldsymbol{u}_i, J\boldsymbol{u}_j\rangle.$$

However, J is also self-adjoint while $z_i \neq \bar{z}_j$ for any $i, j \in I$. Thus $\langle \boldsymbol{u}_i, \mathcal{A}\boldsymbol{u}_j\rangle = 0$, and so $\|\boldsymbol{v}\| = 0$ by (5.9). It follows that the projection $\mathcal{Q}_-$ restricted to L' has a trivial kernel. Hence $\dim L' \leqslant \dim \operatorname{Ran} \mathcal{Q}_- = \dim \mathcal{Q}_-$*.*

Now suppose that $\mathcal{B}\boldsymbol{u} = 0$ for some $\boldsymbol{u} \in L$. Since $\mathcal{A} = \mathcal{B}\mathcal{Q}\mathcal{B}$ (by (5.8)), we have $\boldsymbol{u} \in \operatorname{Ker}\mathcal{A}$; that is, the kernel of the restriction $\mathcal{B}|_L : L \to L'$ is contained in $\operatorname{Ker}\mathcal{A}$. Since L and L' are finite dimensional and $\mathcal{B}|_L : L \to L'$ is surjective, we thus get

$$\begin{aligned} |I| &= \dim L \\ &= \dim L' + \dim \operatorname{Ker} \mathcal{B}|_L \\ &\leqslant \dim \mathcal{Q}_- + \dim \operatorname{Ker} \mathcal{A}, \end{aligned}$$

where the inequality follows with the help of the claim. However, $\dim \mathcal{Q}_-$ and $\dim \operatorname{Ker}\mathcal{A}$ are both finite by proposition 5.4. The result now follows. □

REMARK 5.6. We can define an *indefinite* inner product on L^2 by the expression

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\mathcal{Q}} = \langle \boldsymbol{u}, \mathcal{Q}\boldsymbol{v}\rangle.$$

The pair $(L^2, \langle\cdot,\cdot\rangle_{\mathcal{Q}})$ is then a Pontrjagin space (i.e. a Krein space with a finite rank of indefiniteness (see [2], for example)). In this setting, the operator

$$\mathcal{A}_{\mathcal{Q}} = (\mathcal{B}_+ + \mathcal{B}_-)J(\mathcal{B}_+ - \mathcal{B}_-)$$

is self-adjoint (note that $(\mathcal{B}_+ + \mathcal{B}_-)\mathcal{Q} = (\mathcal{B}_+ - \mathcal{B}_-) = \mathcal{Q}(\mathcal{B}_+ + \mathcal{B}_-)$). Furthermore, the (standard) spectrum of $\mathcal{A}_{\mathcal{Q}}$ and the J-spectrum of $\mathcal{A}$ agree, modulo special consideration of the point 0. Proposition 5.5 now follows from a simplified form of a general result (see theorems IX.4.3 and IX.4.6 in [2]; the relevant part of the proofs of these results forms the basis for the proof of proposition 5.5). Using the full generality of this result, we can show that sum of the algebraic multiplicities of all the non-real and non-semi-simple J-eigenvalues of $\mathcal{A}$ is at most $2m + 1$, m being the dimension of the spectral subspace of $\mathcal{A}$ corresponding to $(-\infty, 0] \cap \sigma(\mathcal{A})$ (which is finite by proposition 5.4).

Proof of the first part of theorem 3.1. Let $\Sigma = \mathbb{C}\backslash((-\infty, -c^2] \cup [c^2, \infty))$ and choose $z \in \Sigma$. By proposition 5.3, we immediately have that $\operatorname{Ker}(\mathcal{A} - zJ)$ is finite dimensional and $\operatorname{Ran}(\mathcal{A} - zJ)$ is closed. It follows (see theorem IV.5.13 of [7], for example) that

$$\operatorname{Ran}(\mathcal{A} - zJ) = (\operatorname{Ker}(\mathcal{A} - zJ)^*)^{\perp}.$$

However, $(\mathcal{A} - zJ)^* = \mathcal{A} - \bar{z}J$ and $\bar{z} \in \Sigma$, so $\operatorname{Ker}(\mathcal{A} - zJ)^*$ must also be finite dimensional by proposition 5.3. Thus $\mathcal{A} - zJ$ is Fredholm, and so $z \notin \sigma_{J\,\mathrm{Ess}}(\mathcal{A})$ by definition. Furthermore,

$$\operatorname{Index}(\mathcal{A} - zJ) = \dim \operatorname{Ker}(\mathcal{A} - zJ) - \dim \operatorname{Ker}(\mathcal{A} - \bar{z}J).$$

Now let $\Sigma' = \mathbb{C} \setminus \sigma_{J\,\mathrm{Ess}}(\mathcal{A}) \supseteq \Sigma$. Therefore, Σ' is connected. Using standard stability theorems for Fredholm operators (in particular, see theorem IV.5.31 of [7]), it follows that $\mathrm{Index}(\mathcal{A} - zJ)$ is constant on Σ' while $\dim\mathrm{Ker}(\mathcal{A} - zJ)$ is constant on $\Sigma' \setminus M$ where M is a (possibly empty) set of isolated points in Σ'. On the other hand, proposition 5.5 implies $\dim\mathrm{Ker}(\mathcal{A} - zJ) = 0$ for all but a finite number of $z \in \Sigma' \setminus \mathbb{R}$. It follows that $\mathrm{Index}(\mathcal{A} - zJ) = 0 = \dim\mathrm{Ker}(\mathcal{A} - zJ)$ for all $z \in \Sigma' \setminus M$, and so $\sigma_J(\mathcal{A}) \setminus \sigma_{J\,\mathrm{Ess}}(\mathcal{A}) \subseteq M$. Finally, let $z \in \sigma_J(\mathcal{A}) \setminus \sigma_{J\,\mathrm{Ess}}(\mathcal{A})$. Thus $J\mathcal{A} - zI = J(\mathcal{A} - zJ)$ is Fredholm and so z must be an eigenvalue of $J\mathcal{A}$ of finite algebraic multiplicity by theorem IV.5.28 in [7]. From remark 2.2, it follows that z is a J-eigenvalue of $\mathcal{A}$ of finite algebraic multiplicity. □

For the remainder of this section, we will use $\mathcal{A}_0$ to denote the operator defined as for $\mathcal{A}$ except with $\Phi = 0$ and $\boldsymbol{A} = 0$. Thus $\mathcal{A}_0 = T_0^* T_0 + c^2 I$, where T_0 is the operator defined at the beginning of § 4. We will prove the second part of theorem 3.1 by using Weyl's criterion (i.e. by constructing a sequence of approximate eigenvectors); the next result essentially does this for $\mathcal{A}_0$.

Proposition 5.7. *Suppose $\lambda \in (-\infty, -c^2] \cup [c^2, \infty)$ and we have a disc $B \subset \mathbb{R}^2$ of radius $r > \frac{1}{2}$. Then we can find $0 \neq \boldsymbol{v} \in C_0^\infty$ with $\mathrm{supp}(\boldsymbol{v}) \subseteq B$ such that*

$$\|(\mathcal{A}_0 - \lambda J)\boldsymbol{v}\| \leqslant C_1 \|\boldsymbol{v}\|/r \quad \textit{and} \quad \|\boldsymbol{v}\|_{L_1^\infty} \leqslant C_2 \|\boldsymbol{v}\|/r$$

for some constants C_1 and C_2 that are independent of B and r.

Proof. Since the operator $\mathcal{A}_0$ and all the norms appearing in the statement of the proposition are translation invariant, it suffices to prove the result assuming that B is centred at 0.

Let $\xi \in \mathbb{R}^2$, define $\xi \cdot x = x_1\xi_1 + x_2\xi_2$ for all $x \in \mathbb{R}^2$ and define a function $\boldsymbol{w}$ by $\boldsymbol{w}(x) = \boldsymbol{a}\mathrm{e}^{\mathrm{i}\xi\cdot x}$ for some constant vector $\boldsymbol{a}$. Thus $(\mathcal{A}_0 - \lambda J)\boldsymbol{w}(x) = M\boldsymbol{a}\mathrm{e}^{\mathrm{i}\xi\cdot x}$, where M is the constant matrix

$$M = \frac{1}{2}\begin{pmatrix} |\xi|^2 & -\mathrm{i}\xi_-^2 \\ \mathrm{i}\xi_+^2 & |\xi|^2 \end{pmatrix} + \begin{pmatrix} c^2 - \lambda & 0 \\ 0 & c^2 + \lambda \end{pmatrix}.$$

Now $\det(M) = c^4 - \lambda^2 + c^2|\xi|^2$. Since $\lambda^2 \geqslant c^4$ by assumption, we can choose ξ so that $\det(M) = 0$. Choosing $\boldsymbol{a}$ to be a non-zero null-vector of M, we thus have $(\mathcal{A}_0 - \lambda J)\boldsymbol{w} = 0$.

Let $\phi \in C_0^\infty$ be a non-zero function with $\mathrm{supp}(\phi) \subseteq \{|x| \leqslant 1\} \subset \mathbb{R}^2$ and define $\phi_B \in C_0^\infty$ by $\phi_B(x) = \phi(x/r)$. Set $\boldsymbol{v} = \phi_B \boldsymbol{w}$. Clearly, $\mathrm{supp}(\boldsymbol{v}) \subseteq B$, while

$$\|\partial^\alpha \phi_B\|_{L^\infty} = r^{-|\alpha|}\|\partial^\alpha \phi\|_{L^\infty} \tag{5.10}$$

for any multi-index α. Since $\boldsymbol{a}$ is non-zero, we also have that

$$\|\boldsymbol{w}\|_{L^2(B)} = C_1\|\boldsymbol{v}\| \quad \text{and} \quad \|\boldsymbol{w}\|_{L_1^\infty(B)} = C_2\|\boldsymbol{v}\|/r \tag{5.11}$$

for some positive constants C_1 and C_2. Now

$$\begin{aligned}(\mathcal{A}_0 - \lambda J)\boldsymbol{v} = \phi_B(\mathcal{A}_0 - \lambda J)\boldsymbol{w} &+ \frac{1}{2}\begin{pmatrix} -\partial_-\partial_+\phi_B & \mathrm{i}\partial_-^2\phi_B \\ -\mathrm{i}\partial_+^2\phi_B & -\partial_+\partial_-\phi_B \end{pmatrix}\boldsymbol{w} \\ &+ \frac{1}{2}\left(\begin{pmatrix} \mathrm{i}\xi_- \\ -\xi_+ \end{pmatrix}\begin{pmatrix} -\partial_+\phi_B & \mathrm{i}\partial_-\phi_B \end{pmatrix} - \begin{pmatrix} \partial_-\phi_B \\ \mathrm{i}\partial_+\phi_B \end{pmatrix}\begin{pmatrix} \mathrm{i}\xi_+ & \xi_- \end{pmatrix}\right)\boldsymbol{w}.\end{aligned}$$

However, $(\mathcal{A}_0 - \lambda J)\boldsymbol{w} = 0$, so

$$\|(\mathcal{A}_0 - \lambda J)\boldsymbol{v}\| \leqslant C_3 \sum_{i,j=1,2} (\|\partial_i \phi_B\|_{L^\infty} + \|\partial_i\partial_j\phi_B\|_{L^\infty})\|\boldsymbol{w}\|_{L^2(B)} \leqslant C_4\|\boldsymbol{v}\|/r$$

for some constants C_3 and C_4 using (5.10) and (5.11). Applying these equations again, we also have

$$\|\boldsymbol{v}\|_{L_1^\infty} \leqslant C_5\|\phi_B\|_{L_1^\infty}\|\boldsymbol{w}\|_{L_1^\infty(B)} \leqslant C_6\|\boldsymbol{v}\|/r$$

for some more constants C_5 and C_6. □

Before generalizing the previous result to the operator $\mathcal{A}$, we must consider a technical complication that arises in dealing with $\boldsymbol{A}$ (essentially relating to the fact that condition B2 places some decay requirements on H but not on $\boldsymbol{A}$). The next lemma addresses this issue but before stating it we make the following observation.

Remark 5.8. Suppose $B \subseteq \mathbb{R}^2$ is a disc of radius $r > 1$ and $g \in L^2_{\mathrm{loc}}$. By taking a suitable periodic extension of g and using Fourier series, it is possible to find $f \in L^2_{2\mathrm{loc}}$ such that $\Delta f = g$ on B. Furthermore, by using a scaling argument, it is possible to ensure that $\|f\|_{L_2^2(B)} \leqslant C_1 r^2\|g\|_{L^2(B)}$ for some constant C_1 that is independent of B.

Lemma 5.9. *Suppose $\boldsymbol{A} \in L^2_{1\mathrm{loc}}$, let $B_1 \subset \mathbb{R}^2$ be a disc of radius $r > 1$ and define B_0 to be the disc with the same centre and a radius of $\frac{1}{2}r$. Then, for any $\varepsilon > 0$, there exists $\phi \in C_0^\infty$ such that $\|A_i - \partial_i\phi\|_{L_1^2(B_0)} \leqslant C_1 r^2\|H\|_{L^2(B_1)} + \varepsilon$, for $i = 1, 2$.*

Proof. We have $\partial_1 A_1 + \partial_2 A_2$, $H \in L^2_{\mathrm{loc}}$, so, using remark 5.8, we can find $f, h \in L^2_{2\mathrm{loc}}$ that satisfy the identities $\Delta f = \partial_1 A_1 + \partial_2 A_2$ and $\Delta h = H$ on B_1 and the norm estimate

$$\|h\|_{L_2^2(B_1)} \leqslant C_1 r^2\|H\|_{L^2(B_1)}. \tag{5.12}$$

Now set $\boldsymbol{A}' = \boldsymbol{A} - (\partial_1 f, \partial_2 f) - (-\partial_2 h, \partial_1 h) \in L^2_{1\mathrm{loc}}$. Straightforward calculations give $\partial_1 A_1' + \partial_2 A_2' = 0 = \partial_1 A_2' - \partial_2 A_1'$ on B_1, from which it follows that $\Delta A_i' = 0$ for $i = 1, 2$. Standard regularity results then imply $\boldsymbol{A}'|_{B_1} \in C^\infty(B_1)$. Now fix the centre of B_1 as the origin and define a function g on B_1 by

$$g(x) = \int_0^1 (x_1 A_1'(tx) + x_2 A_2'(tx))\,\mathrm{d}t.$$

Clearly, $g \in C^\infty(B_1)$, while a simple calculation gives $\boldsymbol{A}' = (\partial_1 g, \partial_2 g)$ on B_1. Now let $\sigma \in C_0^\infty$ be a cut-off function equal to 1 on B_0 and with $\mathrm{supp}(\sigma) \subseteq B_1$. Set $\psi = \sigma(f+g)$. Clearly, $\psi \in L_2^2$, while $\boldsymbol{A} - (\partial_1\psi, \partial_2\psi) = (-\partial_2 h, \partial_1 h)$ on B_0. The norm estimate (5.12) immediately gives $\|A_i - \partial_i\psi\|_{L_1^2(B_0)} \leqslant C_1 r^2\|H\|_{L^2(B_1)}$ for $i = 1, 2$. Since C_0^∞ is dense in L_2^2, we can find $\phi \in C_0^\infty$ with $\|\phi - \psi\|_{L_2^2} < \varepsilon$, completing the proof. □

Remark 5.10. Referring to condition B2, we may assume that the B_n are mutually disjoint (by choosing subsets of a subsequence of $\{B_n\}_{n\in\mathbb{N}}$ if necessary). Now take a fixed $R > 0$. Thus, for all sufficiently large n, the disc B_n contains $O(r_n^2)$ disjoint discs of radius R. Since $\int_{B_n} |H|^2 = o(r_n^2)$, it follows that, for all sufficiently large

n, we can find a disc $B_{n,R} \subseteq B_n$ of radius R such that $\int_{B_{n,R}} |H|^2 \leqslant R^{-4}$. Since a similar argument clearly applies to Φ, we can thus replace condition B2 with the following condition.

(B2′) In addition to the requirements imposed by condition A2, we assume there exists a sequence $\{B_n\}_{n\in\mathbb{N}}$ of disjoint discs such that, for each $n \in \mathbb{N}$, the radius of B_n is n and $\|\Phi\|_{L^2(B_n)}, \|H\|_{L^2(B_n)} \leqslant n^{-2}$.

Proof of the second part of theorem 3.1. Choose $\lambda \in (-\infty, -c^2] \cup [c^2, \infty)$ and let $\{B_n\}_{n\in\mathbb{N}}$ be the sequence of balls given by condition B2′. For each $n \in \mathbb{N}$, let B_n' denote the disc with the same centre as B_n, but with a radius of $\frac{1}{2}n$. Let $\boldsymbol{v}_n$ be the function given by proposition 5.7 for the disc B_n' and, using lemma 5.9, choose a $\phi_n \in C_0^\infty$ that satisfies

$$\|A_i - \partial_i \phi_n\|_{L_1^2(B_n')} \leqslant C_1 n^2 \|H\|_{L^2(B_n)} + 1,$$

for $i = 1, 2$. Finally, define a function $\boldsymbol{u}_n \in C_0^\infty \subseteq \operatorname{Dom} \mathcal{A}$ by

$$\boldsymbol{u}_n(x) = \mathrm{e}^{-\mathrm{i}\phi_n(x)} \boldsymbol{v}_n(x).$$

Clearly, $\operatorname{supp}(\boldsymbol{u}_n) \subseteq B_n' \subset B_n$, so the set $\{\boldsymbol{u}_n \mid n \in \mathbb{N}\}$ is linearly independent (since the B_n are mutually disjoint). The result will therefore follow if we can show $\|(\mathcal{A} - \lambda J)\boldsymbol{u}_n\| / \|\boldsymbol{u}_n\| \to 0$ as $n \to \infty$.

Let $n \in \mathbb{N}$. Thus

$$\|\Phi J \boldsymbol{u}_n\| = \|\Phi J \boldsymbol{v}_n\| \leqslant C_2 \|\Phi\|_{L^2(B_n)} \|\boldsymbol{v}_n\|_{L^\infty} \leqslant C_3 \|\Phi\|_{L^2(B_n)} \|\boldsymbol{u}_n\| / n$$

for some constants C_2 and C_3. On the other hand,

$$P_\pm \boldsymbol{u}_n = \mathrm{e}^{-\mathrm{i}\phi_n(x)}(-\mathrm{i}\partial_\pm + (A_\pm - \partial_\pm \phi_n))\boldsymbol{v}_n,$$

so

$$\begin{aligned} \|T^*T\boldsymbol{u}_n - \mathrm{e}^{-\mathrm{i}\phi_n(x)} T_0{}^* T_0 \boldsymbol{v}_n\| &\leqslant C_4 \sum_{i=1,2} \|A_i - \partial_i \phi_n\|_{L_1^2(B_n')} \|\boldsymbol{v}_n\|_{L_1^\infty} \\ &\leqslant C_5 (n^2 \|H\|_{L^2(B_i)} + 1) \|\boldsymbol{u}_n\| / n \end{aligned}$$

for some constants C_4 and C_5. Therefore,

$$\begin{aligned} \|(\mathcal{A} - \lambda J)\boldsymbol{u}_n\| &\leqslant \|(\mathcal{A}_0 - \lambda J)\boldsymbol{v}_n\| + \|T^*T\boldsymbol{u}_n - \mathrm{e}^{-\mathrm{i}\phi_n(x)} T_0^* T_0 \boldsymbol{v}_n\| + \|\Phi J \boldsymbol{u}_n\| \\ &\leqslant C_6 (1 + n^2 \|H\|_{L^2(B_n)} + \|\Phi\|_{L^2(B_n)}) \|\boldsymbol{u}_n\| / n \end{aligned}$$

for some constant C_6. Condition B2′ now completes the proof. □

6. Stability of isolated eigenvalues

The proof of theorem 3.3 will employ the method of asymptotic perturbation theory, as developed in chapter VIII of [7].

Before beginning with results, we introduce some new notation that will be used throughout the next two sections. For $c > 0$, we define $\mathcal{A}_c$ to be the shifted operator $\mathcal{A}_c = \mathcal{A} - c^2 J$. Thus $\sigma_J(\mathcal{A}_c) = \sigma_J(\mathcal{A}) - c^2$, and so direct comparison can now be made with the eigenvalues of the Pauli operator H_P^+.

For $z \in \mathbb{C} \setminus \sigma_J(\mathcal{A}_c)$, let $\mathcal{R}_c(z) = (\mathcal{A}_c - zJ)^{-1}$ be the 'resolvent' of $\mathcal{A}_c$. Also, for $z \in \mathbb{C} \setminus \sigma(H_P^+)$, set

$$\mathcal{R}(z) = \begin{pmatrix} R(z) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\mathcal{R}}_c(z) = \begin{pmatrix} R(z) & 0 \\ 0 & \frac{1}{2}c^{-2} \end{pmatrix}, \tag{6.1}$$

where $R(z) = (H_P^+ - z)^{-1}$ is the resolvent of Pauli's operator. We will use $\mathcal{N}_c$ to denote the constant matrix operator given by

$$\mathcal{N}_c = \begin{pmatrix} 1 & 0 \\ 0 & c^{-1} \end{pmatrix}.$$

Operators denoted by some form of the letter Q will be (essentially) projections. For such operators, we will use $\dim Q$ to denote the dimension of the range.

The first lemma allows us to assume extra regularity conditions in the statement and proofs of some subsequent results.

LEMMA 6.1. *The set C_0^∞ is a core of $\mathcal{A}_c$.*

Proof. It is sufficient to prove that T^*T is essentially self-adjoint on C_0^∞ (note that $\mathcal{A}_c = T^*T + \Phi J + c^2(I-J)$ and $\Phi J + c^2(I-J)$ is bounded by condition C). This is equivalent to showing that if $\boldsymbol{v} \in L^2$, then

$$T^*T\boldsymbol{v} = \pm \mathrm{i}\boldsymbol{v} \tag{6.2}$$

implies $\boldsymbol{v} = 0$. Here, we initially define $T\boldsymbol{v}$ and $T^*T\boldsymbol{v}$ as elements of L^2_{-1} and L^2_{-2}, respectively. Setting $w = T\boldsymbol{v} \in L^2_{-1}$, equation (6.2) implies

$$T^*w = -\frac{1}{\sqrt{2}}\begin{pmatrix} \partial_- \\ \mathrm{i}\partial_+ \end{pmatrix} w + \frac{1}{\sqrt{2}}\begin{pmatrix} -\mathrm{i}A_- \\ A_+ \end{pmatrix} w \in L^2. \tag{6.3}$$

However, we have $\begin{pmatrix} -\mathrm{i}A_- & A_+ \end{pmatrix}^{\mathrm{T}} w \in L^2_{-1}$ by condition C, so (6.3) implies $\partial_\pm w \in L^2_{-1}$. It follows that $\partial_i w \in L^2_{-1}$ for $i = 1, 2$, which, coupled with the fact that $w \in L^2_{-1}$, gives $w \in L^2$. Applying the same argument again now gives $w = T\boldsymbol{v} \in L^2_1$. Therefore, $\langle T^*T\boldsymbol{v}, \boldsymbol{v}\rangle = \langle w, w\rangle \in \mathbb{R}$. It follows from (6.2) that $\pm\mathrm{i}\langle \boldsymbol{v}, \boldsymbol{v}\rangle \in \mathbb{R}$ and so $\boldsymbol{v} = 0$. □

The operators $P_\pm$ are first-order elliptic partial-differential operators on $\mathbb{R}^2$ whose coefficients, together with their derivatives of all orders, are bounded in the L^∞ norm (this follows from condition C). Using standard theory (see § 18.1 of [5], for example), we can hence find pseudo-differential operators F_1, F_2, G_1 and G_2 of orders 0, 0, -1 and -1, respectively, such that

$$P_+ = F_1P_- + G_1 \quad \text{and} \quad P_- = P_+F_2 + G_2. \tag{6.4}$$

Furthermore, we can insist that F_1, F_2, $G_1P_\pm$ and $P_\pm G_2$ are bounded operators on L^2_k for any k; this fact will be used in the proof of the next result.

PROPOSITION 6.2. *Let $z \in \mathbb{C}$ and $\varepsilon > 0$. Then there exists a constant $C(\varepsilon, z)$, depending continuously on ε and z, such that, for any $\delta \in [0,1]$, $c > C(\varepsilon, z)$ and $\boldsymbol{u} \in L^2_\infty$ with $\|\mathcal{N}_c^\delta(\mathcal{A}_c - zJ)\boldsymbol{u}\| \leqslant \varepsilon\|\boldsymbol{u}\|$, we have $\|(H_P^+ - z)w\| \leqslant 5\varepsilon\|w\|$, where w is defined by $w = u_1 - \mathrm{i}F_2u_2$. Furthermore, $w = 0$ only if $\boldsymbol{u} = 0$.*

With the help of lemma 6.1, proposition 6.2 (with $\delta = 0$) gives the following.

Corollary 6.3. *Suppose* $z \in \mathbb{C}\backslash\sigma(H_P^+)$. *Then* $z \notin \sigma_J(\mathcal{A}_c)$ *and* $\|\mathcal{R}_c(z)\| \leqslant 5\|R(z)\|$ *for all* $c > C(\frac{1}{5}\|R(z)\|^{-1}, z)$.

Remark 6.4. The spectrum of $\mathcal{A}_c$ thus converges to that of H_P^+ in the following sense: given a compact set $K \subset \mathbb{C} \setminus \sigma(H_P^+)$, we have $K \cap \sigma_J(\mathcal{A}_c) = \emptyset$ for all $c > C$, where C is the maximum value of $C(\frac{1}{5}\|R(z)\|^{-1}, z)$ for $z \in K$.

Proof of proposition 6.2. Let $\boldsymbol{u} \in L^2_\infty$ and set $w = u_1 - \mathrm{i}F_2u_2$. Thus

$$\|\boldsymbol{u}\| \leqslant \|w\| + C_1\|u_2\| \tag{6.5}$$

for some constant C_1. On the other hand, equation (6.4) allows us to write

$$(\mathcal{A}_c - zJ)\boldsymbol{u} = \begin{pmatrix} (H_P^+ - z)w \\ Bw \end{pmatrix} + \begin{pmatrix} 0 \\ 2c^2u_2 \end{pmatrix} + M_zu_2, \tag{6.6}$$

where M_z is the operator

$$M_z = \frac{1}{2}\begin{pmatrix} -\mathrm{i}P_- \\ P_+ \end{pmatrix} G_2 + (z + \varPhi)\begin{pmatrix} -\mathrm{i}F_2 \\ 1 \end{pmatrix}.$$

We can also write $B = \mathrm{i}F_1(H_P^+ - z) + N_z$, with $N_z = \frac{1}{2}\mathrm{i}G_1P_+ + \mathrm{i}F_1(\varPhi + z)$. Now the operators F_1, M_z and N_z are all bounded on L^2. Let $C_2(z)$ denote the maximum of the corresponding operator norms and 1. Setting $C_3(\varepsilon, z) = \min\{1, \varepsilon\}/2C_2(z)$, it follows that $C_3(\varepsilon, z) \in (0, \frac{1}{2}]$. However, on any inner product space we have the inequality $\|a + b\| \geqslant \mu(\|b\| - \|a\|)$ for all $\mu \in [-1, 1]$. It follows that

$$\|Bw + 2c^2u_2\| \geqslant C_3(\varepsilon, z)(2c^2\|u_2\| - \|Bw\|).$$

Choosing any $\delta \in [0, 1]$ and $c > 1$, we then get

$$\begin{aligned} c^{-\delta}\|Bw + 2c^2u_2\| &\geqslant C_3(\varepsilon, z)(2c\|u_2\| - \|\mathrm{i}F_1(H_P^+ - z)w + N_zw\|) \\ &\geqslant 2C_3(\varepsilon, z)c\|u_2\| - \tfrac{1}{2}\|(H_P^+ - z)w\| - \tfrac{1}{2}\varepsilon\|w\|. \end{aligned}$$

Combining this with (6.6), it follows that

$$2\|\mathcal{N}_c^\delta(\mathcal{A}_c - zJ)\boldsymbol{u}\| \geqslant \tfrac{1}{2}\|(H_P^+ - z)w\| - \tfrac{1}{2}\varepsilon\|w\| + 2(C_3(\varepsilon, z)c - C_2(z))\|u_2\|.$$

If we now assume $\|\mathcal{N}_c^\delta(\mathcal{A}_c - zJ)\boldsymbol{u}\| \leqslant \varepsilon\|\boldsymbol{u}\|$ and use (6.5), we get

$$5\varepsilon\|w\| \geqslant \|(H_P^+ - z)w\| + 4(C_3(\varepsilon, z)c - C_2(z) - \varepsilon C_1)\|u_2\|.$$

Therefore, the result follows if we take $C(\varepsilon, z) = (C_2(z) + \varepsilon C_1)/C_3(\varepsilon, z) \geqslant 2$. □

Let $z \in \mathbb{C} \setminus \sigma(H_P^+)$, so $R(z)$, $\mathcal{R}(z)$ and $\tilde{\mathcal{R}}_c(z)$ are all bounded operators on L^2. From corollary 6.3, it follows that $\mathcal{R}_c(z)$ is also a bounded operator on L^2 for all sufficiently large c. The next result deals with the relationship between these operators as $c \to +\infty$.

Theorem 6.5. *For any* $z \in \mathbb{C} \setminus \sigma(H_P^+)$, *we have* $\mathcal{R}_c(z) \xrightarrow{\mathrm{s}} \tilde{\mathcal{R}}_c(z)$ *(where '*$\xrightarrow{\mathrm{s}}$*' is used to denote strong operator convergence as* $c \to +\infty$*). Since* $\tilde{\mathcal{R}}_c(z) \to \mathcal{R}(z)$ *in operator norm, it follows that* $\mathcal{R}_c(z) \xrightarrow{\mathrm{s}} \mathcal{R}(z)$.

Proof. For $\boldsymbol{u} \in C_0^\infty$, we have

$$(\mathcal{R}_c(z) - \tilde{\mathcal{R}}_c(z))\boldsymbol{u} = -\mathcal{R}_c(z)\mathcal{N}_c^{-1}\left(\mathcal{N}_c(\mathcal{A}_c - zJ)\mathcal{N}_c - \begin{pmatrix} H_P^+ - z & 0 \\ 0 & 2 \end{pmatrix}\right)\mathcal{N}_c^{-1}\tilde{\mathcal{R}}_c(z)\boldsymbol{u}.$$

Now

$$\mathcal{N}_c^{-1}\tilde{\mathcal{R}}_c(z)\boldsymbol{u} = \begin{pmatrix} R(z)u_1 \\ 0 \end{pmatrix} + \frac{1}{2c}\begin{pmatrix} 0 \\ u_2 \end{pmatrix},$$

$R(z)u_1 \in L^2_\infty$ by condition C (see proposition 7.2 for more details) and

$$\mathcal{N}_c(\mathcal{A}_c - zJ)\mathcal{N}_c - \begin{pmatrix} H_P^+ - z & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & c^{-1}B^* \\ c^{-1}B & c^{-2}(H_P^- + z) \end{pmatrix}.$$

It follows that

$$\left(\mathcal{N}_c(\mathcal{A}_c - zJ)\mathcal{N}_c - \begin{pmatrix} H_P^+ - z & 0 \\ 0 & 2 \end{pmatrix}\right)\mathcal{N}_c^{-1}\tilde{\mathcal{R}}_c(z)\boldsymbol{u} \to 0$$

as $c \to +\infty$. On the other hand, $\mathcal{R}_c(z)\mathcal{N}_c^{-1} = (\mathcal{N}_c(\mathcal{A}_c - zJ))^{-1}$. By proposition 6.2 (with $\delta = 1$), this is uniformly bounded in operator norm for $c > C(\frac{1}{5}\|R(z)\|^{-1}, z)$. Thus we have $(\mathcal{R}_c(z) - \tilde{\mathcal{R}}_c(z))\boldsymbol{u} \to 0$ as $c \to +\infty$ for any $\boldsymbol{u} \in C_0^\infty$. However, C_0^∞ is dense in L^2, while corollary 6.3 shows that $\mathcal{R}_c(z) - \tilde{\mathcal{R}}_c(z)$ is uniformly bounded in operator norm for $c > C(\frac{1}{5}\|R(z)\|^{-1}, z)$. It follows that $\mathcal{R}_c(z) - \tilde{\mathcal{R}}_c(z) \xrightarrow{\mathrm{s}} 0$. □

Let $\lambda < 0$ be an isolated eigenvalue of H_P^+ and let Γ be a simple closed contour in $\mathbb{C} \setminus \sigma(H_P^+)$ enclosing λ but no other part of $\sigma(H_P^+)$. Set

$$Q = -\frac{1}{2\pi\mathrm{i}}\int_\Gamma R(z)\,\mathrm{d}z \quad \text{and} \quad \mathcal{Q} = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} = -\frac{1}{2\pi\mathrm{i}}\int_\Gamma \mathcal{R}(z)\,\mathrm{d}z.$$

Thus Q is just the projection onto the eigenspace of H_P^+ associated with λ. Now, by remark 6.4, we know that the contour Γ does not intersect $\sigma_J(\mathcal{A}_c)$ for all sufficiently large c. Hence we can define an operator $\mathcal{Q}_c$ by

$$\mathcal{Q}_c = -\frac{1}{2\pi\mathrm{i}}\int_\Gamma \mathcal{R}_c(z)\,\mathrm{d}z. \tag{6.7}$$

Remark 6.6. Owing to the fact that we are dealing with a slightly non-standard spectral problem, the operator $\mathcal{Q}_c$ is not a projection. However, both of the operators $J\mathcal{Q}_c$ and $\mathcal{Q}_cJ$ are projections (see remark 2.2 for more details).

The next result plays a key role in making the asymptotic perturbation theory 'work'. The proof is somewhat technical but essentially centres on showing that $\|(\mathcal{A}_c - \lambda J)\mathcal{Q}_c\| \to 0$ as $c \to +\infty$.

Proposition 6.7. *For all sufficiently large c, we have* $\dim \mathcal{Q}_c \leqslant \dim \mathcal{Q}$.

Proof. Let $d_\lambda = \operatorname{dist}(\lambda, \sigma(H_P^+)\setminus\{\lambda\}) > 0$ denote the separation of λ from the rest of $\sigma(H_P^+)$. Set $\varepsilon = \frac{1}{10}d_\lambda$, $r = \frac{1}{10}\varepsilon$ and define Γ_r to be the circular contour centred at λ of radius r. Since H_P^+ is self-adjoint and λ is the closest point of $\sigma(H_P^+)$ to any point on Γ_r, we immediately get $\|R(z)\| = 1/r$ for all $z \in \Gamma_r$. Now let K be the closure of the region between Γ and Γ_r. Therefore, K is compact and $K \cap \sigma(H_P^+) = \emptyset$, so we can define a bounded constant C_1 by

$$C_1 = \sup\{C(\varepsilon, \lambda)\} \cup \{C(\tfrac{1}{5}\|R(z)\|^{-1}, z) \mid z \in K\}$$

(where $C(\cdot,\cdot)$ is given by proposition 6.2). Now suppose $c > C_1$. Corollary 6.3 then gives $\|\mathcal{R}_c(z)\| \leqslant 5/r$ for any $z \in \Gamma_r$. Corollary 6.3 also implies $K \cap \sigma_J(\mathcal{A}_c) = \emptyset$. Using standard properties of resolvents and Cauchy's theorem to deform the contour, we then get

$$(\mathcal{A}_c - \lambda J)\mathcal{Q}_c = -\frac{1}{2\pi\mathrm{i}} \int_{\Gamma_r} (z - \lambda) J \mathcal{R}_c(z) \, \mathrm{d}z.$$

Since $\|J\| = 1$, it follows that

$$\|(\mathcal{A}_c - \lambda J)\mathcal{Q}_c\| \leqslant \frac{1}{2\pi} \int_{\Gamma_r} |z - \lambda| \frac{5}{r} \, \mathrm{d}z \leqslant \frac{5}{2\pi}|\Gamma_r| = \frac{\varepsilon}{2}.$$

Therefore, $\|(\mathcal{A}_c - \lambda J)\boldsymbol{u}\| \leqslant \frac{1}{2}\varepsilon\|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \operatorname{Ran} \mathcal{Q}_c$. By lemma 6.1, we can thus choose a $\dim \mathcal{Q}_c$ subspace L of C_0^∞ such that $\|(\mathcal{A}_c - \lambda J)\boldsymbol{u}\| \leqslant \varepsilon\|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in L$. Now let

$$L' = \{u_1 - \mathrm{i}F_2 u_2 \mid \boldsymbol{u} \in L\} \subseteq L^2_\infty \subset \operatorname{Dom} H_P^+.$$

Since $c > C(\varepsilon, \lambda)$, proposition 6.2 (with $\delta = 0$) gives us $\|(H_P^+ - \lambda)w\| \leqslant 5\varepsilon\|w\|$ for all $w \in L'$. Since $5\varepsilon < d_\lambda$, the minimax principle (see theorem 10.2.3 in [1], for example) applied to the non-negative self-adjoint operator $(H_P^+ - \lambda)^2$ immediately gives us $\dim L' \leqslant \dim Q$. However, the last part of proposition 6.2 also implies $\dim L' = \dim L = \dim \mathcal{Q}_c$, completing the result. □

Theorem 6.5, combined with the definitions of $\mathcal{Q}_c$ and $\mathcal{Q}$ and the fact that $\|\mathcal{R}_c(z) - \mathcal{R}(z)\|$ depends continuously on $z \in \Gamma$ for all sufficiently large c, gives us $\mathcal{Q}_c \xrightarrow{\mathrm{s}} \mathcal{Q}$. On the other hand, the fact that $\mathcal{A}_c$ is self-adjoint can be used to show $\mathcal{Q}_c = \mathcal{Q}_c^*$. Since $J\mathcal{Q} = \mathcal{Q} = \mathcal{Q}J$, it follows that the projections $J\mathcal{Q}_c$ and $(J\mathcal{Q}_c)^*$ both converge strongly to $\mathcal{Q}$. By combining this observation with proposition 6.7 and lemmas VIII.1.23 and VIII.1.24 from [7], we get the following result.

PROPOSITION 6.8. *We have* $\dim \mathcal{Q}_c = \dim \mathcal{Q} = \dim Q$ *for all sufficiently large* c*. Furthermore,* $\mathcal{Q}_c \to \mathcal{Q}$ *in operator norm as* $c \to +\infty$*.*

7. Asymptotic expansions of isolated eigenvalues

The next theorem is based on theorems VIII.2.1 and VIII.2.2 in [7]. It gives an asymptotic expansion for the resolvent $\mathcal{R}_c(z)$.

THEOREM 7.1. *For* $\boldsymbol{u} \in L^2_\infty$*, we have*

$$\mathcal{R}_c(z)\boldsymbol{u} = \mathcal{R}(z)\boldsymbol{u} + \tfrac{1}{2}c^{-2}\begin{pmatrix} R(z)B^*BR(z) & -R(z)B^* \\ -BR(z) & I \end{pmatrix}\boldsymbol{u} + O(c^{-4}),$$

where $O(c^{-4})$ *denotes an element of* L^2 *with norm of order* $O(c^{-4})$ *as* $c \to +\infty$*.*

Proof. We have

$$\mathcal{A}_c - zJ = \left[I + \begin{pmatrix} 0 & B^* \\ B & H_P^- + z \end{pmatrix} \begin{pmatrix} R(z) & 0 \\ 0 & c^{-2}/2 \end{pmatrix} \right] \begin{pmatrix} H_P^+ - z & 0 \\ 0 & 2c^2 \end{pmatrix},$$

so $\mathcal{R}_c(z) = \tilde{\mathcal{R}}_c(z)(I + \mathcal{G})^{-1}$, where

$$\mathcal{G} = \begin{pmatrix} 0 & B^* \\ B & H_P^- + z \end{pmatrix} \tilde{\mathcal{R}}_c(z). \tag{7.1}$$

Now let $N \geqslant 0$ and suppose $\boldsymbol{u} \in L^2_\infty$. Therefore, $\boldsymbol{u} \in \operatorname{Dom} \mathcal{G}^k$ for $k = 0, \dots, N$ and

$$\boldsymbol{u} = (I + \mathcal{G}) \sum_{k=0}^{N-1} (-\mathcal{G})^k \boldsymbol{u} + (-\mathcal{G})^N \boldsymbol{u}.$$

Hence

$$\mathcal{R}_c(z)\boldsymbol{u} = \sum_{k=0}^{N} \tilde{\mathcal{R}}_c(z)(-\mathcal{G})^k \boldsymbol{u} + (\mathcal{R}_c(z) - \tilde{\mathcal{R}}_c(z))(-\mathcal{G})^N \boldsymbol{u}.$$

Now $\mathcal{G}^N$ is a polynomial in c^{-2} whose (operator-valued) coefficients are non-zero only for powers between $[\frac{1}{2}N]$ and N (where $[\frac{1}{2}N]$ is the greatest integer not exceeding $\frac{1}{2}N$). On the other hand, $\mathcal{R}_c(z) \xrightarrow{\mathrm{s}} \tilde{\mathcal{R}}_c(z)$ by theorem 6.5. Thus $(\mathcal{R}_c(z) - \tilde{\mathcal{R}}_c(z))(-\mathcal{G})^N \boldsymbol{u} = o(c^{-2[N/2]})$ and so

$$\mathcal{R}_c(z)\boldsymbol{u} = \sum_{k=0}^{N} \tilde{\mathcal{R}}_c(z)(-\mathcal{G})^k \boldsymbol{u} + o(c^{-2[N/2]}). \tag{7.2}$$

The result now follows from a direct computation using (6.1), (7.1) and (7.2) with $N = 4$. □

The operator H_P^+ is a first-order perturbation of the Laplacian Δ on $\mathbb{R}^2$. Furthermore, the coefficients of this perturbation have bounded derivatives of all orders (by condition C on Φ and $\boldsymbol{A}$). Induction and the fact that $u \in L^2_k$, $\Delta u \in L^2_{k-1}$ implies $u \in L^2_{k+1}$ for any $k \in \mathbb{Z}$ now leads to the following result.

Proposition 7.2. *Let $u \in L^2$, $z \in \mathbb{C}$ and suppose $(H_P^+ - z)u \in L^2_\infty$. Then $u \in L^2_\infty$.*

Since elements of $\operatorname{Ran} Q$ are eigenvectors of H_P^+ it follows that $\operatorname{Ran} \mathcal{Q} \subset L^2_\infty$. In turn, this means that we can apply the asymptotic expansion given by theorem 7.1 to elements of $\operatorname{Ran} \mathcal{Q}$. This fact underlies the next result (which is based on part of theorem VIII.2.6 in [7]).

Theorem 7.3. *For all sufficiently large c, we have*

$$\mathcal{Q}_c \mathcal{Q} = \mathcal{Q} + c^{-2} \mathcal{T} \mathcal{Q} + O(c^{-4}),$$

where $O(c^{-4})$ denotes an operator with norm of order $O(c^{-4})$ as $c \to +\infty$ and $\mathcal{T}$ is the operator defined by

$$\mathcal{T} = \frac{1}{2} \begin{pmatrix} SB^*B & 0 \\ -B & 0 \end{pmatrix}, \qquad S = -\frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{R(z)}{\lambda - z} \, \mathrm{d}z.$$

Proof. The operator H_P^+ is self-adjoint and so can only have semi-simple eigenvalues. Therefore,

$$R(z)Q = QR(z) = (\lambda - z)^{-1}Q \quad \Rightarrow \quad \mathcal{R}(z)\mathcal{Q} = (\lambda - z)^{-1}\mathcal{Q}. \tag{7.3}$$

Now let $\boldsymbol{u} \in L^2$. Thus $\mathcal{Q}\boldsymbol{u} \in L^2_\infty$, and so theorem 7.1 gives

$$\begin{aligned} &\mathcal{R}_c(z)\mathcal{Q}\boldsymbol{u} \\ &\quad = \mathcal{R}(z)\mathcal{Q}\boldsymbol{u} + \tfrac{1}{2}c^{-2}\begin{pmatrix} R(z)B^*BR(z) & -R(z)B^* \\ -BR(z) & I \end{pmatrix}\begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}\boldsymbol{u} + O(c^{-4})_{z\boldsymbol{u}} \\ &\quad = (\lambda - z)^{-1}\mathcal{Q}\boldsymbol{u} + \tfrac{1}{2}c^{-2}(\lambda - z)^{-1}\begin{pmatrix} R(z)B^*B & 0 \\ -B & 0 \end{pmatrix}\mathcal{Q}\boldsymbol{u} + O(c^{-4})_{z\boldsymbol{u}} \end{aligned} \tag{7.4}$$

for any $z \in \Gamma$ (note that because $\mathcal{Q}$ contains a non-zero entry only in its upper-left corner, the second column of any matrix operator appearing immediately to its left can be chosen arbitrarily). In (7.4), $O(c^{-4})_{z\boldsymbol{u}}$ denotes an element of L^2 such that $c^4O(c^{-4})_{z\boldsymbol{u}}$ is bounded as $c \to +\infty$; this bound depends continuously on z and $\boldsymbol{u}$. Now, $\mathcal{Q}$ has finite rank and Γ is compact so the strong convergence given by (7.4) implies convergence in operator norm, while the error term can be bounded uniformly on Γ; that is,

$$\mathcal{R}_c(z)\mathcal{Q} = (\lambda - z)^{-1}\mathcal{Q} + \tfrac{1}{2}c^{-2}(\lambda - z)^{-1}\begin{pmatrix} R(z)B^*B & 0 \\ -B & 0 \end{pmatrix}\mathcal{Q} + O(c^{-4}), \tag{7.5}$$

where $O(c^{-4})$ denotes an operator with norm of order $O(c^{-4})$ as $c \to +\infty$. Now

$$-\frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{\mathrm{d}z}{\lambda - z} = 1,$$

so the proof can be completed by integrating (7.5) around Γ and using the definitions of $\mathcal{Q}_c$ (see (6.7)) and $\mathcal{T}$. □

Using theorem 7.3, the proof of theorem 3.3 now follows from an argument quite similar to that used in the second part of the proof of theorem VIII.2.6 in [7]. It will be included here for completeness.

Proof of theorem 3.3. Define (finite-dimensional) subspaces of L^2 by

$$L = \operatorname{Ran}\mathcal{Q} \quad \text{and} \quad L_c = \operatorname{Ran}\mathcal{Q}_c = \operatorname{Ran}\mathcal{Q}_cJ,$$

and define an operator $\mathcal{V}_c$ by

$$\mathcal{V}_c = I - \mathcal{Q} + \mathcal{Q}_c\mathcal{Q} = I + c^{-2}\mathcal{T}\mathcal{Q} + O(c^{-4}),$$

where the second equality follows from theorem 7.3. Since $\mathcal{Q}$ is a projection, we have $\mathcal{V}_c\mathcal{Q} = \mathcal{Q}_c\mathcal{Q}$ and $\mathcal{V}_c(I - \mathcal{Q}) = (I - \mathcal{Q})$. Combining these observations with proposition 6.8, it follows that $\mathcal{V}_c$ maps L onto L_c and leaves every element of the complementary space $\operatorname{Ran}(I - \mathcal{Q})$ unchanged (for all sufficiently large c). We also have $\dim L = \dim L_c$, so $\mathcal{V}_c$ is invertible,

$$\mathcal{V}_c^{-1} = I - c^{-2}\mathcal{T}\mathcal{Q} + O(c^{-4})$$

and ${\mathcal{V}_c}^{-1}$ maps L_c onto L and leaves every element of $\mathrm{Ran}(I - \mathcal{Q})$ unchanged.

Now set $\mathcal{R}'_c(z) = {\mathcal{V}_c}^{-1}\mathcal{R}_c(z)J\mathcal{V}_c\mathcal{Q}$. By standard properties of resolvents, $\mathcal{R}_c(z)J$ commutes with $\mathcal{Q}_cJ$, from which it follows that $\mathrm{Ran}\,\mathcal{R}'_c(z) \subseteq L$. Therefore,

$$\begin{aligned} \mathcal{R}'_c(z) &= \mathcal{Q}\mathcal{R}'_c(z) \\ &= \mathcal{Q}{\mathcal{V}_c}^{-1}\mathcal{R}_c(z)J\mathcal{V}_c\mathcal{Q} \\ &= (\mathcal{Q} + O(c^{-4}))\mathcal{R}_c(z)(\mathcal{Q} + c^{-2}J\mathcal{T}\mathcal{Q} + O(c^{-4})), \end{aligned} \tag{7.6}$$

where we have used the facts that $J\mathcal{Q} = \mathcal{Q}$ and $\mathcal{Q}\mathcal{T} = 0$ (the later being a consequence of the identity $QS = 0$). Now B, B^* and S all map L^2_∞ into itself (this follows from condition C for B and B^*, and from proposition 7.2 and the identity $(H_P^+ - \lambda)S = I - Q$ for S). Coupled with the fact that $\mathrm{Ran}\,\mathcal{Q}$ is a finite-dimensional subspace of L^2_∞ and the definition of $\mathcal{T}$, it follows that $\mathrm{Ran}(J\mathcal{T}\mathcal{Q})$ is also a finite-dimensional subspace of L^2_∞. Since $\mathcal{Q}\mathcal{R}(z)J\mathcal{T} = \mathcal{R}(z)\mathcal{Q}\mathcal{T} = 0$, theorem 7.1 now gives

$$\mathcal{Q}\mathcal{R}_c(z)J\mathcal{T}\mathcal{Q} = \mathcal{Q}\mathcal{R}(z)J\mathcal{T}\mathcal{Q} + O(c^{-2}) = O(c^{-2}), \tag{7.7}$$

where $O(c^{-2})$ denotes an operator with norm of order $O(c^{-2})$ as $c \to +\infty$. On the other hand, equations (7.3) and (7.5) give

$$\mathcal{Q}\mathcal{R}_c(z)\mathcal{Q} = (\lambda - z)^{-1}\mathcal{Q} + c^{-2}(\lambda - z)^{-2}\mathcal{Q}\mathcal{B}\mathcal{Q} + O(c^{-4}), \tag{7.8}$$

where

$$\mathcal{B} = \frac{1}{2}\begin{pmatrix} B^*B & 0 \\ 0 & 0 \end{pmatrix}$$

(note that because $\mathcal{Q}$ contains a non-zero entry only in its upper-left corner, the second row of any matrix operator appearing immediately to its right can be chosen arbitrarily). Combining the definition of $\mathcal{R}'_c(z)$ with (7.6), (7.7) and (7.8), we now obtain

$${\mathcal{V}_c}^{-1}\mathcal{R}_c(z)J\mathcal{V}_c\mathcal{Q} = (\lambda - z)^{-1}\mathcal{Q} + c^{-2}(\lambda - z)^{-2}\mathcal{Q}\mathcal{B}\mathcal{Q} + O(c^{-4})_z. \tag{7.9}$$

The remainder term $O(c^{-4})_z$ can be estimated uniformly for $z \in \Gamma$. Furthermore, by standard properties of resolvents,

$$J\mathcal{A}_c\mathcal{Q}_cJ = -\frac{1}{2\pi\mathrm{i}}\int_\Gamma z\mathcal{R}_c(z)J\,\mathrm{d}z.$$

Integration of (7.9) along Γ after multiplication by $-z/2\pi\mathrm{i}$ thus gives

$$\mathcal{V}_c^{-1}J\mathcal{A}_c\mathcal{Q}_cJ\mathcal{V}_c\mathcal{Q} = \lambda\mathcal{Q} - c^{-2}\mathcal{Q}\mathcal{B}\mathcal{Q} + O(c^{-4}). \tag{7.10}$$

The earlier remarks about ranges, etc., of $\mathcal{V}_c$, etc., means that the left-hand side of (7.10) maps L into L. Furthermore, the eigenvalues of this restricted map can be seen to be $\lambda_1, \dots, \lambda_k$. Standard results about the perturbation of eigenvalues of finite-dimensional matrices (see, for example, theorem II.5.4 in [7]) now complete the proof of the theorem. □

Acknowledgments

The author is grateful to Dmitri Vassiliev for several fruitful discussions of some of the ideas central to this paper. The author also wishes to thank Alex Sobolev and Christiane Tretter for useful comments.

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(*Issued 19 October 2001*)