The lattice of closed ideals in the Banach algebra of operators on a certain dual Banach space

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Abstract

We determine the closed operator ideals of the Banach space $(\ell_2^1 \oplus \ell_2^2 \oplus \cdots \oplus \ell_2^n \oplus \cdots)_{\ell_1}$.

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1 Introduction

The aim of this note is to classify the closed ideals in the Banach algebra $\mathscr{B}(F)$ of (bounded, linear) operators on the Banach space

$$F = \left(\bigoplus_{n \in \mathbb{N}} \ell_2^n \right)_{\ell_1}. \tag{1.1}$$

More precisely, we shall show that there are exactly four closed ideals in $\mathscr{B}(F)$, namely $\{0\}$, the compact operators $\mathscr{K}(F)$, the closure $\overline{\mathscr{G}}_{\ell_1}(F)$ of the set of operators factoring through ℓ_1, and $\mathscr{B}(F)$ itself.

The collection of Banach spaces E for which a classification of the closed ideals in $\mathscr{B}(E)$ exists is very sparse. Indeed, the following list appears to be the complete list of such spaces.

(i) For a finite-dimensional Banach space E, $\mathscr{B}(E) \cong M_n$, where n is the dimension of E, and so it is ancient folklore that $\mathscr{B}(E)$ is simple in this case.

(ii) In 1941 Calkin [2] classified all the ideals in $\mathscr{B}(\ell_2)$. In particular he proved that there are only three closed ideals in $\mathscr{B}(\ell_2)$, namely $\{0\}$, $\mathscr{K}(\ell_2)$, and $\mathscr{B}(\ell_2)$.

(iii) In 1960 Gohberg, Markus, and Feldman [5] extended Calkin's theorem to the other classical sequence spaces. More precisely, they showed that $\{0\}$, $\mathscr{K}(E)$, and $\mathscr{B}(E)$ are the only closed ideals in $\mathscr{B}(E)$ for each of the spaces $E = c_0$ and $E = \ell_p$, where $1 \leqslant p < \infty$.

(iv) Later in the 1960's Gramsch [6] and Luft [10] independently extended Calkin's theorem in a different direction by classifying all the closed ideals in $\mathscr{B}(H)$ for each Hilbert space H (not necessarily separable). In particular, they showed that these ideals are well-ordered by inclusion.

(v) In 2003 Laustsen, Loy, and Read [8] proved that, for the Banach space

$$E = \Big(\bigoplus_{n\in\mathbb{N}} \ell_2^n\Big)_{c_0}, \tag{1.2}$$

there are exactly four closed ideals in $\mathscr{B}(E)$, namely $\{0\}$, the compact operators $\mathscr{K}(E)$, the closure $\overline{\mathscr{G}}_{c_0}(E)$ of the set of operators factoring through c_0, and $\mathscr{B}(E)$ itself.

Note that (1.1) is the dual Banach space of (1.2), and so the result of this note can be seen as a 'dualization' of [8]. In fact, our strategy draws heavily on the methods introduced in [8]. However, the present case is more involved because in [8] it was possible to restrict attention to block-diagonal operators of a special kind. In the Banach space (1.1), however, one cannot even reduce to operators with a 'locally finite matrix' (due to the fact that the unit vector basis of ℓ_1 is not shrinking), and so a new trick is required (see Remark 2.13 for details).

(vi) In 2004 Daws [4] extended Gramsch and Luft's result to the Gohberg–Markus–Feldman case by classifying the closed ideals in $\mathscr{B}(E)$ for $E = c_0(\mathbb{I})$ and $E = \ell_p(\mathbb{I})$, where $\mathbb{I}$ is an index set of arbitrary cardinality and $1 \leqslant p < \infty$. Again, these ideals are well-ordered by inclusion.

2 The classification theorem

Throughout, all Banach spaces are assumed to be over the same scalar field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We denote by I_E the identity operator on the Banach space E.

We begin by recalling various definitions and results concerning ℓ_1-direct sums and operators between them.

2.1 ℓ_1-direct sums. Let (E_n) be a sequence of Banach spaces. We denote by $\big(\bigoplus E_n\big)_{\ell_1}$ or $(E_1 \oplus E_2 \oplus \cdots)_{\ell_1}$ the ℓ_1-direct sum of $E_1, E_2, \ldots$, that is, the collection of sequences (x_n) such that $x_n \in E_n$ for each $n \in \mathbb{N}$ and

$$\big\|(x_n)\big\| \overset{\text{defn}}{=} \sum_{n=1}^{\infty} \|x_n\| < \infty. \tag{2.1}$$

This is a Banach space for coordinate-wise defined vector space operations and norm given by (2.1).

Set $E = \big(\bigoplus E_n\big)_{\ell_1}$. For each $m \in \mathbb{N}$, we write J_m^E for the canonical embedding of E_m into E and Q_m^E for the canonical projection of E onto E_m. Both J_m^E and Q_m^E are operators of norm one; in fact, the former is an isometry, and the latter is a quotient map.

We use similar notation for finite direct sums.

2.2 Diagonal operators. For each $n \in \mathbb{N}$, let $T_n \colon E_n \to F_n$ be an operator, where E_n and F_n are Banach spaces. Suppose that $\sup \|T_n\| < \infty$. Then we can define the *diagonal operator*

$$\operatorname{diag}(T_n) \colon \Big(\bigoplus E_n\Big)_{\ell_1} \to \Big(\bigoplus F_n\Big)_{\ell_1}, \quad (x_n) \mapsto (T_n x_n).$$

Clearly, we have $\big\|\operatorname{diag}(T_n)\big\| = \sup \|T_n\|$. In the finite case, we also use the notation $T_1 \oplus \cdots \oplus T_n$ for the diagonal operator from $(E_1 \oplus \cdots \oplus E_n)_{\ell_1}$ to $(F_1 \oplus \cdots \oplus F_n)_{\ell_1}$.

2.3 Definition. Let $T \colon \big(\bigoplus E_n\big)_{\ell_1} \to \big(\bigoplus F_n\big)_{\ell_1}$ be an operator, where (E_n) and (F_n) are sequences of Banach spaces. We associate with T the infinite matrix $(T_{m,n})$, where

$$T_{m,n} = Q_m^F T J_n^E \colon \ E_n \to F_m \qquad (m, n \in \mathbb{N}).$$

The *support of the n^{th} column* of T is

$$\operatorname{colsupp}_n(T) = \{m \in \mathbb{N} : T_{m,n} \neq 0\} \qquad (n \in \mathbb{N}).$$

We say that T has *finite columns* if each column has finite support.

The significance of operators with finite columns lies in the fact that, in the case where each of the spaces E_n ($n \in \mathbb{N}$) is finite-dimensional, given an operator $T \colon \big(\bigoplus E_n\big)_{\ell_1} \to \big(\bigoplus F_n\big)_{\ell_1}$, there is a compact operator $K \colon \big(\bigoplus E_n\big)_{\ell_1} \to \big(\bigoplus F_n\big)_{\ell_1}$ such that $T + K$ has finite columns; in fact K can be picked with arbitrarily small norm (see [8, Lemma 2.7(i)]).

We next introduce a parameter n_ε that is at the heart of our main result (Theorem 2.12). It is the dual version of the parameter m_ε that was introduced in [8].

2.4 Definition. Let G be a closed subspace of a Hilbert space H. We denote by $G^\perp$ the orthogonal complement of G in H, and write proj_G for the orthogonal projection of H onto G (so that proj_G is the idempotent operator on H with image G and kernel $G^\perp$).

Let $k \in \mathbb{N}$, let E be a Banach space, let $H_1, \ldots, H_k$ be Hilbert spaces, and denote by $\mathbb{N}_0$ the set of non-negative integers. For each operator $T \colon E \to (H_1 \oplus \cdots \oplus H_k)_{\ell_1}$ and each $\varepsilon > 0$, set

$$n_\varepsilon(T) = \sup \left\{ n \in \mathbb{N}_0 : \begin{array}{c} \big\|(\operatorname{proj}_{G_1^\perp} \oplus \cdots \oplus \operatorname{proj}_{G_k^\perp})T\big\| > \varepsilon \\ \text{whenever } G_j \subset H_j \text{ are subspaces} \\ \text{with } \dim G_j \leqslant n \text{ for } j = 1, \ldots, k \end{array} \right\} \in \mathbb{N}_0 \cup \{\pm\infty\}.$$

The parameter n_ε gives quantitative information on certain factorizations. This is the content of parts (i) and (ii) of Lemma 2.5, which are dual to the corresponding statements about the parameter m_ε in [8, Lemma 5.3]. We shall indeed prove parts (i) and (ii) via [8, Lemma 5.3], but would like to emphasize that their proofs are fairly elementary (and indeed we could have easily translated them into direct proofs here). The important point in [8] is the definition of m_ε itself. Part (iii) of Lemma 2.5 has no counterpart in [8]; it will be used to deal with the extra difficulty that on ℓ_1-direct sums one has to consider operators whose matrices may have infinite rows.

2.5 Lemma. *Let $k \in \mathbb{N}$, let $H, K_1, \ldots, K_k$ be Hilbert spaces, let $T\colon H \to (K_1 \oplus \cdots \oplus K_k)_{\ell_1}$ be an operator, and let $0 < \varepsilon < \|T\|$.*

(i) *Suppose that $n_\varepsilon(T)$ is finite. Then there exist a number $d \in \mathbb{N}$ and operators $R\colon H \to \ell_1^d$ and $S\colon \ell_1^d \to (K_1 \oplus \cdots \oplus K_k)_{\ell_1}$ such that $\|T - SR\| \leqslant \varepsilon$, $\|R\| \leqslant \|T\| \sqrt{n_\varepsilon(T) + 1}$, and $\|S\| \leqslant 1$.*

(ii) *For each natural number $n \leqslant \frac{1}{2} n_\varepsilon(T) + 1$, there exist operators $U\colon \ell_2^n \to H$ and $V\colon (K_1 \oplus \cdots \oplus K_k)_{\ell_1} \to \ell_2^n$ such that $I_{\ell_2^n} = VTU$, $\|U\| \leqslant 1/\varepsilon$, and $\|V\| \leqslant 1$.*

(iii) *Let $g \in \mathbb{N}$, let H_0 be a closed subspace of finite codimension in H, and suppose that $n_\varepsilon(T) \geqslant \dim H_0^\perp + g$. Then $n_\varepsilon(T|_{H_0}) \geqslant g$.*

Proof. In [8, Definition 5.2(ii)] the quantity

$$m_\varepsilon(W) = \sup \left\{ m \in \mathbb{N}_0 : \begin{array}{l} \big\|W(\mathrm{proj}_{G_1^\perp} \oplus \cdots \oplus \mathrm{proj}_{G_k^\perp})\big\| > \varepsilon \\ \text{whenever } G_j \subset K_j \text{ are subspaces} \\ \text{with } \dim G_j \leqslant m \text{ for } j = 1, \ldots, k \end{array} \right\} \in \mathbb{N}_0 \cup \{\pm\infty\} \tag{2.2}$$

is introduced for each operator $W\colon (K_1 \oplus \cdots \oplus K_k)_{\ell_\infty} \to H$. Making standard identifications of dual spaces, we may regard the adjoint operator of $T\colon H \to (K_1 \oplus \cdots \oplus K_k)_{\ell_1}$ as an operator $T^*\colon (K_1 \oplus \cdots \oplus K_k)_{\ell_\infty} \to H$, where the subscript ℓ_∞ indicates that we equip the direct sum with the norm

$$\big\|(x_1, \ldots, x_k)\big\| = \max\big\{\|x_1\|, \ldots, \|x_k\|\big\} \qquad (x_1 \in K_1, \ldots, x_k \in K_k).$$

It follows that we may insert $W = T^*$ in (2.2). Standard properties of adjoint operators show that

$$m_\varepsilon(T^*) = n_\varepsilon(T). \tag{2.3}$$

We use this identity and [8, Lemma 5.3] to prove (i) and (ii).

(i). Suppose that $n_\varepsilon(T) < \infty$. By (2.3) and [8, Lemma 5.3(i)], we can find a number $d \in \mathbb{N}$ and operators $A\colon (K_1 \oplus \cdots \oplus K_k)_{\ell_\infty} \to \ell_\infty^d$ and $B\colon \ell_\infty^d \to H$ such that $\|A\| \leqslant 1$, $\|B\| \leqslant \|T\| \sqrt{n_\varepsilon(T) + 1}$, and $\|T^* - BA\| \leqslant \varepsilon$. Dualizing this gives us operators $R = B^*\colon H \to \ell_1^d$ and $S = A^*\colon \ell_1^d \to (K_1 \oplus \cdots \oplus K_k)_{\ell_1}$ such that (i) holds because the adjoint operation is antimultiplicative and an operator has the same norm as its adjoint.

(ii). Suppose that $n \leqslant \frac{1}{2} n_\varepsilon(T) + 1$. Then it follows from (2.3) and [8, Lemma 5.3(ii)] that there are operators $C\colon \ell_2^n \to (K_1 \oplus \cdots \oplus K_k)_{\ell_\infty}$ and $D\colon H \to \ell_2^n$ such that $\|C\| \leqslant 1$, $\|D\| \leqslant 1/\varepsilon$, and $I_{\ell_2^n} = DT^*C$. As before, we dualize this to obtain operators $U = D^*\colon \ell_2^n \to H$ and $V = C^*\colon (K_1 \oplus \cdots \oplus K_k)_{\ell_1} \to \ell_2^n$ such that (ii) is satisfied.

(iii). For each $j = 1, \ldots, k$, let G_j be a subspace of K_j with $\dim G_j \leqslant g$. Set $F_j = G_j + Q_j T(H_0^\perp) \subset K_j$. Then F_j is finite-dimensional with $\dim F_j \leqslant n_\varepsilon(T)$, and so we can find a unit vector $x \in H$ such that $\big\|(\mathrm{proj}_{F_1^\perp} \oplus \cdots \oplus \mathrm{proj}_{F_k^\perp})Tx\big\| > \varepsilon$. It follows that

$$\begin{aligned} \big\|(\mathrm{proj}_{G_1^\perp} \oplus \cdots \oplus \mathrm{proj}_{G_k^\perp})T|_{H_0}\big\| &\geqslant \big\|(\mathrm{proj}_{G_1^\perp} \oplus \cdots \oplus \mathrm{proj}_{G_k^\perp})T(\mathrm{proj}_{H_0} x)\big\| \\ &\geqslant \big\|(\mathrm{proj}_{F_1^\perp} \oplus \cdots \oplus \mathrm{proj}_{F_k^\perp})T(\mathrm{proj}_{H_0} x)\big\| \\ &= \big\|(\mathrm{proj}_{F_1^\perp} \oplus \cdots \oplus \mathrm{proj}_{F_k^\perp})Tx\big\| > \varepsilon, \end{aligned}$$

and so $n_\varepsilon(T|_{H_0}) \geqslant g$. □

2.6 Remark. Let T be an operator on $\big(\bigoplus K_n\big)_{\ell_1}$ with finite columns, where (K_n) is an (infinite) sequence of Hilbert spaces. As in [8, Remark 5.4], there is a natural way to define $n_\varepsilon(TJ_m)$ for each $\varepsilon \geqslant 0$ and each $m \in \mathbb{N}$, namely by ignoring the cofinite number of Hilbert spaces K_k such that $Q_kTJ_m = 0$.

The proof of our classification result (Theorem 2.12) has two non-trivial parts. The first part is done in Proposition 2.8 relying on older results. The second part is dealt with in Proposition 2.10 using the parameter n_ε and a small trick to take care of matrices with infinite rows. Before proceeding we prove a little lemma which will be useful at a number of places.

2.7 Lemma. *Let $\mathscr{J}$ be an ideal in a Banach algebra $\mathscr{A}$. If $P \in \overline{\mathscr{J}}$ is idempotent, then in fact $P \in \mathscr{J}$.*

Proof. Let (T_n) be a sequence in $\mathscr{J}$ converging to P. Replacing T_n with PT_nP, we may assume that $T_n \in P\mathscr{A}P$ for each $n \in \mathbb{N}$. Note that $P\mathscr{A}P$ is a Banach algebra with identity P, and so there exists $n \in \mathbb{N}$ such that T_n is invertible in $P\mathscr{A}P$. Thus there is $S \in \mathscr{A}$ with $P = (PSP)T_n$, which implies that $P \in \mathscr{J}$. □

For each pair (E, F) of Banach spaces, set

$$\mathscr{G}_{\ell_1}(E,F) = \big\{TS : S \in \mathscr{B}(E,\ell_1),\ T \in \mathscr{B}(\ell_1,F)\big\}.$$

The fact that ℓ_1 is isomorphic to $\ell_1 \oplus \ell_1$ implies that $\mathscr{G}_{\ell_1}$ is an operator ideal, and so its closure $\overline{\mathscr{G}}_{\ell_1}$ is a closed operator ideal. As usual, we write $\overline{\mathscr{G}}_{\ell_1}(E)$ instead of $\overline{\mathscr{G}}_{\ell_1}(E,E)$.

2.8 Proposition. *Set $F = \big(\bigoplus \ell_2^n\big)_{\ell_1}$. Then $\overline{\mathscr{G}}_{\ell_1}(F)$ is a proper ideal in $\mathscr{B}(F)$.*

Proof. Assume towards a contradiction that $I_F \in \overline{\mathscr{G}}_{\ell_1}(F)$. Then $I_F \in \mathscr{G}_{\ell_1}(F)$ by Lemma 2.7, and so F is isomorphic to ℓ_1, which is false. (It is well-known that F is not isomorphic to ℓ_1, but this is by no means obvious. One may for example use the fact that ℓ_1 has a unique unconditional basis up to equivalence (see [9, §2.b], or [7, §5] for a simpler proof relying only on Khintchine's inequality), whereas it is easy to see that F does not have this property.) □

The following construction is a dual version of [8, Construction 4.2].

2.9 Construction. Let $E_1, E_2, E_3, \dots$ and F be Banach spaces. Set $E = \big(\bigoplus E_n\big)_{\ell_1}$ and $\widetilde{F} = (F \oplus F \oplus \cdots)_{\ell_1}$, and let $T\colon E \to F$ be an operator. Since $\|TJ_n^E\| \leqslant \|T\|$ for each $n \in \mathbb{N}$, we have a diagonal operator $\operatorname{diag}(TJ_n^E)\colon E \to \widetilde{F}$. For each $y \in \widetilde{F}$ the series $\sum_{n=1}^\infty Q_n^{\widetilde{F}} y$ converges absolutely in F, and it is easy to check that

$$W\colon\ \widetilde{F} \to F, \quad y \mapsto \sum_{n=1}^\infty Q_n^{\widetilde{F}} y,$$

defines an operator of norm 1 satisfying

$$T = W\,\mathrm{diag}(TJ_n^E). \tag{2.4}$$

2.10 Proposition. *Set $F = \big(\bigoplus \ell_2^n\big)_{\ell_1}$. For each operator T on F with finite columns, the following three conditions are equivalent:*

(i) $T \notin \overline{\mathscr{G}}_{\ell_1}(F)$,

(ii) $\sup\big\{n_\varepsilon(TJ_k^F) : k \in \mathbb{N}\big\} = \infty$ *for some* $\varepsilon > 0$,

(iii) *there are operators U and V on F such that $VTU = I_F$.*

Proof. We begin by proving the implication "not (ii) $\Rightarrow$ not (i)". We may suppose that $T \neq 0$. Let $0 < \varepsilon < \|T\|$, and suppose that $n' = \sup\big\{n_\varepsilon(TJ_k^F) : k \in \mathbb{N}\big\} < \infty$. Then Lemma 2.5(i) implies that, for each $k \in \mathbb{N}$, we can find a number $d_k \in \mathbb{N}$ and operators $R_k\colon \ell_2^k \to \ell_1^{d_k}$ and $S_k\colon \ell_1^{d_k} \to F$ such that $\|TJ_k^F - S_kR_k\| \leqslant \varepsilon$, $\|R_k\| \leqslant \|T\|\sqrt{n'+1}$, and $\|S_k\| \leqslant 1$. Put $\widetilde{F} = (F \oplus F \oplus \cdots)_{\ell_1}$ as in Construction 2.9. Then the diagonal operators $\mathrm{diag}(R_k)\colon F \to \big(\bigoplus \ell_1^{d_k}\big)_{\ell_1} = \ell_1$ and $\mathrm{diag}(S_k)\colon \ell_1 = \big(\bigoplus \ell_1^{d_k}\big)_{\ell_1} \to \widetilde{F}$ exist and satisfy

$$\big\|\mathrm{diag}(TJ_k^F) - \mathrm{diag}(S_k)\,\mathrm{diag}(R_k)\big\| = \sup \|TJ_k^F - S_kR_k\| \leqslant \varepsilon.$$

It follows that $\mathrm{diag}(TJ_k^F) \in \overline{\mathscr{G}}_{\ell_1}(F, \widetilde{F})$, and so $T \in \overline{\mathscr{G}}_{\ell_1}(F)$ by (2.4), as required.

To show "(ii)$\Rightarrow$(iii)", suppose that $\sup\big\{n_\varepsilon(TJ_k^F) : k \in \mathbb{N}\big\} = \infty$ for some $\varepsilon > 0$. We construct inductively a strictly increasing sequence (k_j) in $\mathbb{N}$ such that the following three conditions are satisfied:

(a) $\mathrm{colsupp}_{k_j}(T) \neq \emptyset$ for each $j \in \mathbb{N}$.

(b) Set $m_j = \max\big(\mathrm{colsupp}_{k_j}(T)\big) \in \mathbb{N}$. Then $m_{j+1} > m_j$ for each $j \in \mathbb{N}$.

(c) Set $E_j = \big(\bigoplus_{i=m_{j-1}+1}^{m_j} \ell_2^i\big)_{\ell_1}$, where $m_0 = 0$ and m_j is defined as in (b) for $j \in \mathbb{N}$, and let $P_j = \sum_{i=m_{j-1}+1}^{m_j} J_i^{E_j} Q_i^F : F \to E_j$ be the canonical projection. Then there are operators $U_j\colon \ell_2^j \to \ell_2^{k_j}$ and $V_j\colon E_j \to \ell_2^j$ with $\|U_j\| \leqslant 1/\varepsilon$ and $\|V_j\| \leqslant 1$ such that the diagram

$$\begin{tikzcd}[column sep=large]
\ell_2^j \arrow[rrr, "I_{\ell_2^j}"] \arrow[d, "U_j"'] & & & \ell_2^j \\
\ell_2^{k_j} \arrow[r, "J_{k_j}^F"] & F \arrow[r, "T"] & F \arrow[r, "P_j"] & E_j \arrow[u, "V_j"']
\end{tikzcd} \tag{2.5}$$

is commutative, and $U_j(\ell_2^j) \subset \bigcap_{i=1}^{m_{j-1}} \ker T_{i,k_j}$ for each $j \in \mathbb{N}$.

We start the induction by choosing $k_1 \in \mathbb{N}$ such that $n_\varepsilon(TJ_{k_1}^F) \geqslant 1$. Then $\mathrm{colsupp}_{k_1}(T)$ is non-empty and $\|TJ_{k_1}^F\| > \varepsilon$. Take a unit vector $x \in \ell_2^{k_1}$ such that $\|TJ_{k_1}^F x\| > \varepsilon$, and define

$$U_1\colon\ \ell_2^1 = \mathbb{K} \to \ell_2^{k_1}, \quad \alpha \mapsto \frac{\alpha}{\|TJ_{k_1}^F x\|}\,x.$$

Further, take a functional $V_1\colon E_1 \to \mathbb{K} = \ell_2^1$ of norm 1 such that

$$V_1(P_1 T J_{k_1}^F x) = \big\| P_1 T J_{k_1}^F(x) \big\|.$$

Then the diagram (2.5) is commutative because $\big\| P_1 T J_{k_1}^F(x) \big\| = \big\| T J_{k_1}^F(x) \big\|$, and the inclusion $U_1(\ell_2^1) \subset \bigcap_{i=1}^{m_0} \ker T_{i,k_1}$ is trivially satisfied because $\bigcap_{i\in\emptyset} \ker T_{i,k_1} = \ell_2^{k_1}$ by convention.

Now let $j \geqslant 2$, and suppose that $k_1 < k_2 < \cdots < k_{j-1}$ have been chosen. Set $h = \sum_{i=1}^{m_{j-1}} i$, take $k_j > k_{j-1}$ such that $n_\varepsilon(T J_{k_j}^F) \geqslant h + 2(j-1)$, and set

$$H_0 = \bigcap_{i=1}^{m_{j-1}} \ker T_{i,k_j} = \ker\big((Q_1^F \oplus \cdots \oplus Q_{m_{j-1}}^F) T J_{k_j}^F\big) \subset \ell_2^{k_j}.$$

Since $\dim H_0 \geqslant k_j - h$, it follows that $\dim H_0^\perp \leqslant h$. Hence Lemma 2.5(iii) implies that $n_\varepsilon(T J_{k_j}^F|_{H_0}) \geqslant 2(j-1)$. In particular $T J_{k_j}^F|_{H_0} \neq 0$, so that $\operatorname{colsupp}_{k_j}(T) \neq \emptyset$, and $m_j > m_{j-1}$ by the choice of H_0. Further, we note that $n_\varepsilon(P_j T J_{k_j}^F|_{H_0}) = n_\varepsilon(T J_{k_j}^F|_{H_0})$ because $Q_i^F T J_{k_j}^F|_{H_0} = 0$ whenever $i \leqslant m_{j-1}$ or $i > m_j$. Lemma 2.5(ii) then shows that there are operators $U_j\colon \ell_2^j \to H_0 \subset \ell_2^{k_j}$ and $V_j\colon E_j \to \ell_2^j$ with $\|U_j\| \leqslant 1/\varepsilon$ and $\|V_j\| \leqslant 1$ making the diagram (2.5) commutative, and the induction continues.

Next we 'glue' the sequences of operators (U_j) and (V_j) together to obtain operators U and V on F. Specifically, given $x \in F$, we define $y_i \in \ell_2^i$ by

$$y_i = \begin{cases} U_j Q_j^F x & \text{if } i = k_j \text{ for some } j \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \qquad (i \in \mathbb{N}).$$

Then

$$\sum_{i=1}^\infty \|y_i\| = \sum_{j=1}^\infty \|U_j Q_j^F x\| \leqslant \frac{\|x\|}{\varepsilon},$$

and so $Ux = (y_i)_{i=1}^\infty$ defines an operator U on F. Further, since

$$\sum_{j=1}^\infty \|V_j P_j x\| \leqslant \sum_{j=1}^\infty \|P_j x\| = \|x\|,$$

we can define an operator V on F by $Vx = (V_j P_j x)_{j=1}^\infty$.

It remains to prove that $VTU = I_F$. For this, it suffices to check that

$$Q_i^F V T U J_j^F(x) = \begin{cases} x & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad (i, j \in \mathbb{N},\ x \in \ell_2^j).$$

By definition, we have $Q_i^F V T U J_j^F(x) = V_i P_i T J_{k_j}^F U_j(x)$. For $i = j$, the latter equals x by (2.5). For $i < j$, we have

$$P_i T J_{k_j}^F U_j(x) = \sum_{h=m_{i-1}+1}^{m_i} J_h^{E_i} T_{h,k_j} U_j(x) = 0$$

because $U_j x \in \ker T_{h,k_j}$ for each $h \leqslant m_{j-1}$. For $i > j$,

$$P_i T J_{k_j}^F = \sum_{h=m_{i-1}+1}^{m_i} J_h^{E_i} T_{h,k_j} = 0$$

because $T_{h,k_j} = 0$ for each $h > m_j$. This completes the proof of the implication "(ii)$\Rightarrow$(iii)".

Finally, the implication "(iii)$\Rightarrow$(i)" follows from Proposition 2.8. $\square$

In fact conditions (i) and (iii), above, are equivalent also for operators that do not have finite columns.

2.11 Corollary. *Let T be an operator on the Banach space $F = \left(\bigoplus \ell_2^n\right)_{\ell_1}$. Then $T \notin \overline{\mathscr{G}}_{\ell_1}(F)$ if and only if there exist operators R and S on F such that $I_F = STR$.*

Proof. As before, the implication "$\Leftarrow$" follows from Proposition 2.8.

Conversely, suppose that $T \notin \overline{\mathscr{G}}_{\ell_1}(F)$, and let K be a compact operator on F such that $T - K$ has finite columns (*cf.* [8, Lemma 2.7(i)]). By the ideal property we have $T - K \notin \overline{\mathscr{G}}_{\ell_1}(F)$. Proposition 2.10 implies that there are operators U and V on F such that $I_F = V(T-K)U$. Thus VTU is a compact perturbation of the identity, and hence it is a Fredholm operator. It follows that, for some $W \in \mathscr{B}(F)$, the operator $WVTU$ is a cofinite-rank projection. This completes the proof because F is isomorphic to its closed subspaces of finite codimension. (This latter fact is a consequence of the existence of a left and a right shift operator on the basis of F obtained by stringing together the natural bases of $\ell_2^1, \ell_2^2, \ldots, \ell_2^n, \ldots$). $\square$

Our main result classifying the closed ideals in $\mathscr{B}(F)$ is now easy to deduce.

2.12 Theorem. *The lattice of closed ideals in $\mathscr{B}(F)$, where $F = \left(\bigoplus \ell_2^n\right)_{\ell_1}$, is given by*

$$\{0\} \subsetneq \mathscr{K}(F) \subsetneq \overline{\mathscr{G}}_{\ell_1}(F) \subsetneq \mathscr{B}(F). \tag{2.6}$$

Proof. It is clear that $\mathscr{B}(F)$ contains the chain of closed ideals (2.6). The right-hand inclusion is proper by Proposition 2.8. The middle inclusion is proper because F contains ℓ_1 as a complemented subspace, the projection onto which is an example of a non-compact operator in $\overline{\mathscr{G}}_{\ell_1}(F)$.

It remains to show that the ideals in (2.6) are the *only* closed ideals in $\mathscr{B}(F)$. Standard basis arguments show that the identity on ℓ_1 factors through any non-compact operator in $\mathscr{B}(F)$ (see [8, §3] for details). It follows that, for each non-zero, closed ideal $\mathscr{J}$ in $\mathscr{B}(F)$, either $\mathscr{J} = \mathscr{K}(F)$ or $\overline{\mathscr{G}}_{\ell_1}(F) \subset \mathscr{J}$. However, Corollary 2.11 implies that $\overline{\mathscr{G}}_{\ell_1}(F)$ is a maximal ideal in $\mathscr{B}(F)$, and so there are no other closed ideals in $\mathscr{B}(F)$ than the four listed in (2.6). $\square$

2.13 Remark. We can now explain where the present proof differs in an essential way from the proof for the Banach space $E = \big(\bigoplus \ell_2^n\big)_{c_0}$ given in [8]. Indeed, each operator on E has a compact perturbation which has a 'locally finite matrix' in the sense that its associated matrix (*cf.* Definition 2.3) has only finitely many non-zero entries in each row and in each column. This is not true for all operators on $F = \big(\bigoplus \ell_2^n\big)_{\ell_1}$ (an example of this is given below). We circumvent this difficulty by arranging that the operators U_j map into $\bigcap_{i=1}^{m_{j-1}} \ker T_{i,k_j}$ in the proof of Proposition 2.10.

An operator T on F such that no compact perturbation of T has a locally finite matrix can be constructed as follows. Let $(N_m)_{m=1}^\infty$ be a partition of $\mathbb{N}$ such that N_m is infinite for each $m \in \mathbb{N}$, and define an operator of norm 1 by

$$T\colon\ F \to F, \quad (y_n) \mapsto \bigg(\sum_{n\in N_m} \langle y_n, x_n\rangle\, x_m\bigg)_{m=1}^{\infty},$$

where $\langle\cdot\,,\cdot\rangle$ denotes the inner product in ℓ_2^n and $x_n = (1,0,\ldots,0) \in \ell_2^n$ for each $n \in \mathbb{N}$.

Suppose that $S \in \mathscr{B}(F)$ has a locally finite matrix. Inductively we choose a strictly increasing sequence (n_m) in $\mathbb{N}$ such that $n_m \in N_m$ and $S_{m,j} = 0$ for each $j \geqslant n_m$ and $m \in \mathbb{N}$. We note that no subsequence of $\big((T-S)J_{n_m}^F x_{n_m}\big)$ is convergent because

$$\begin{aligned}\big\|(T-S)(J_{n_k}^F x_{n_k} - J_{n_m}^F x_{n_m})\big\| &\geqslant \big\|Q_m^F(T-S)(J_{n_k}^F x_{n_k} - J_{n_m}^F x_{n_m})\big\| \\ &= \|T_{m,n_k}x_{n_k} - S_{m,n_k}x_{n_k} - T_{m,n_m}x_{n_m} + S_{m,n_m}x_{n_m}\| \\ &= \|0 - 0 - x_m + 0\| = 1\end{aligned}$$

whenever $k > m$. Since the sequence $(J_{n_m}^F x_{n_m})$ is bounded, we conclude that the operator $T - S$ is not compact. In other words, no compact perturbation of T has a locally finite matrix, as claimed.

3 An application

In [1, §8] Bourgain, Casazza, Lindenstrauss, and Tzafriri prove that every infinite-dimensional, complemented subspace of the Banach space $F = \big(\bigoplus \ell_2^n\big)_{\ell_1}$ is isomorphic to either F or ℓ_1. Here we present a new proof of this fact using only the ideal structure of $\mathscr{B}(F)$. More precisely, we shall deduce it from Corollary 2.11.

3.1 Theorem. (Bourgain, Casazza, Lindenstrauss, and Tzafriri [1]) *Each infinite-dimensional, complemented subspace of $F = \big(\bigoplus \ell_2^n\big)_{\ell_1}$ is isomorphic to either F or ℓ_1.*

Proof. Let G be an infinite-dimensional, complemented subspace of F, and let $P \in \mathscr{B}(F)$ be an idempotent operator with image G. If $P \in \overline{\mathscr{G}_{\ell_1}(F)}$, then by Lemma 2.7 we have $P \in \mathscr{G}_{\ell_1}(F)$, and hence G is isomorphic to ℓ_1. If $P \notin \overline{\mathscr{G}_{\ell_1}(F)}$, then by Corollary 2.11 the identity on F factors through P, *i.e.*, F is isomorphic to a complemented subspace of G. We can thus write $F \sim G \oplus X$ and $G \sim F \oplus Y$ for suitable Banach spaces X

and Y. We now use Pełczyński's decomposition method and the fact that F is isomorphic to $(F \oplus F \oplus \cdots)_{\ell_1}$ to show that G is isomorphic to F:

$$\begin{aligned} F \sim G \oplus X \sim F \oplus Y \oplus X \sim (F \oplus F \oplus \cdots)_{\ell_1} \oplus Y \oplus X \\ \sim (G \oplus X \oplus G \oplus X \oplus \cdots)_{\ell_1} \oplus Y \oplus X \\ \sim (G \oplus X \oplus G \oplus X \oplus \cdots)_{\ell_1} \oplus Y \sim F \oplus Y \sim G. \end{aligned} \qquad \square$$

3.2 Remark. In [8, §6] a new proof is presented for the corresponding result of Bourgain, Casazza, Lindenstrauss, and Tzafriri for the Banach space $E = \left(\bigoplus \ell_2^n\right)_{c_0}$, which says that every infinite-dimensional, complemented subspace of E is isomorphic to either E or c_0. The proof in [8] relies on a theorem of Casazza, Kottman, and Lin [3] that implies that E is primary. The results of [3], however, do not show that our space $F = \left(\bigoplus \ell_2^n\right)_{\ell_1}$ is primary, and so the argument in [8] cannot be used here. We note in passing that F is in fact primary — this follows easily from Theorem 3.1. Further, we note that the proof presented above works also for the space $E = \left(\bigoplus \ell_2^n\right)_{c_0}$.

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