

# THE GROUP OF ENDOTRIVIAL MODULES IN THE NORMAL CASE

NADIA MAZZA

**ABSTRACT.** The group of endotrivial modules has recently been determined for a finite group having a normal Sylow  $p$ -subgroup. In this paper, we give and compare three different presentations of a torsion-free subgroup of maximal rank of the group of endotrivial modules. Finally, we illustrate the constructions in an example.

**Keywords :** Endotrivial modules, finite groups, cohomology of finite groups.

## 1. INTRODUCTION

Endotrivial modules play an important role in the modular representation theory of finite groups, and this may explain why many group theorists have been studying them intensively, since the late seventies. The classification of endotrivial modules of a finite  $p$ -group was recently achieved in [12]. Thereafter, in a joint work, Jon Carlson, Daniel Nakano and the author tackled the question of the classification of endotrivial modules for an arbitrary finite group, and they could give an almost complete classification in the case of a finite group of Lie type in its defining characteristic (cf. [9]). The general case is still an open question, in the sense that no presentation by generators and relations of the group of endotrivial modules is yet known. However, the obstacles have been overcome in the case of a finite group having a normal Sylow  $p$ -subgroup. The results are presented in [9, Theorem 3.4], where the authors show that in this case, the group of endotrivial modules is generated by the classes of the indecomposable endotrivial modules that are extended from the Sylow  $p$ -subgroup. Then, by means of cohomological tools, they construct a minimal set of generators for the group of endotrivial modules.

The primary aim of these notes is to give an alternative construction of a torsion-free subgroup of maximal rank of the group of endotrivial modules of a finite group having a normal Sylow  $p$ -subgroup, that does not appeal to any cohomological knowledge. The method refers to a theorem proven by Dade, and it is presented in Section 3, after that basic facts about endotrivial modules are recapitulated in Section 2. In Section 4, we review the modules and the techniques used in [8] and [9], and we compare with the approach presented in the previous section. Finally, in Section 5, we work out thoroughly an “odd extraspecial” example.

## 2. PRELIMINARIES

Throughout these notes, we let  $k$  be an algebraically closed field of prime characteristic  $p$ . If  $G$  is a finite group, we write  $\mathbf{mod}(kG)$  for the category of finitely generated  $kG$ -modules and  $\mathbf{stmod}(kG)$  for the stable module category. That is, the objects of  $\mathbf{stmod}(kG)$  are the same as those of  $\mathbf{mod}(kG)$ , and the morphisms are equivalence classes of morphisms. Namely, two morphisms are equivalent if their difference factors through a projective module (cf. [7, § 5]). In addition, we write  $k$  for the 1-dimensional trivial  $kG$ -module, and if  $M$  is a finitely generated  $kG$ -module, then  $\mathrm{End}_k M = \mathrm{Hom}_k(M, M)$  denotes the  $kG$ -module that is the  $k$ -algebra of  $k$ -linear endomorphisms of  $M$ . Recall that, for two  $kG$ -modules  $M$  and  $N$ , there is an isomorphism of  $kG$ -modules  $\mathrm{Hom}_k(M, N) \cong M^* \otimes N$ , where  $M^* = \mathrm{Hom}_k(M, k)$  is the  $k$ -linear dual of  $M$ , and the tensor product “ $\otimes$ ” is the tensor product over the field  $k$ .

**Definition 2.1.** Let  $G$  be a finite group. A finitely generated  $kG$ -module  $M$  is endotrivial provided that  $\text{End}_k M \cong k$  in  $\mathbf{stmod}(kG)$ , or equivalently,  $\text{End}_k M \cong k \oplus (\text{proj})$  in  $\mathbf{mod}(kG)$ , for some projective  $kG$ -module ( $\text{proj}$ ).

We say that two endotrivial  $kG$ -modules are equivalent if they are isomorphic in  $\mathbf{stmod}(kG)$ . The set  $T(G)$  of isomorphism classes in  $\mathbf{stmod}(kG)$  of endotrivial  $kG$ -modules is an abelian group, called the group of endotrivial modules. The composition law is defined by  $[M] + [N] = [M \otimes N]$ .

In particular, in  $T(G)$ , we have  $0 = [k]$  and  $-[M] = [M^*]$ .

Endotrivial  $kG$ -modules have been defined by E. Dade (cf. [13]), in 1978, for finite  $p$ -groups, as a particular case of the capped endo-permutation  $kG$ -modules. A *capped endo-permutation  $kG$ -module*, for a finite  $p$ -group  $G$ , is a finitely generated  $kG$ -module whose endomorphism algebra is a permutation module having a trivial direct summand. Modulo a suitable equivalence relation, they form a finitely generated abelian group  $D(G)$ , and the group  $T(G)$  identifies with a subgroup of  $D(G)$ .

Also, for a subgroup  $H$  of  $G$ , the restriction map  $\text{Res}_H^G : \mathbf{mod}(kG) \rightarrow \mathbf{mod}(kH)$  (also denoted by “ $\cdot \downarrow_H^G$ ”) induces a group homomorphism  $\text{Res}_H^G : T(G) \rightarrow T(H)$ .

Non-trivial examples of endotrivial modules are the syzygies of the trivial module, whereas, in the case of a finite  $p$ -group, most of the relative syzygies are capped endo-permutation modules (not endotrivial in general, cf. [1]). Let us recall their definitions.

**Definition 2.2.** If  $X$  is a finite  $G$ -set, then  $\Omega_X^1(k)$  is the relative (to  $X$ ) syzygy of  $k$ , that is, the kernel of the augmentation map  $kX \rightarrow k$ .

If  $X = G$ , then we define the syzygy  $\Omega_G^n(k)$  of  $k$ , for each  $n \in \mathbb{Z}$ , as follows. If  $n \geq 1$ , we let  $\Omega_G^n(k)$  be the kernel of the  $(n-1)$ -st differential in a minimal projective resolution of  $k$ .

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial_{n+1}} & X_n & \xrightarrow{\partial_n} & X_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots \xrightarrow{\partial_2} X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\partial_0} k \\ & & \searrow & & \nearrow & & \searrow & & \nearrow \\ & & & & \Omega_G^n(k) & & & & \Omega_G^1(k) \end{array}$$

If  $n \leq -1$ , we let  $\Omega_G^n(k) = \text{Hom}_k(\Omega_G^{-n}(k), k)$ , and we set  $\Omega_G^0(k) = k$ .

Let  $G$  be a  $p$ -group and suppose that  $\Omega_X^1(k)$  is a capped endo-permutation  $kG$ -module. Then we let  $\Omega_X$  denote its class in  $D(G)$ , or  $T(G)$  in case it is endotrivial. In particular, for any finite group  $G$ , and for any integer  $n$ , the syzygies  $\Omega_G^n(k)$  are indecomposable endotrivial modules and we have  $[\Omega_G^n(k)] = n\Omega_G$  in  $T(G)$ . We refer the reader to Section 4 in [7] for more properties of the syzygies, and to Sections 2 to 5 in [4] for those of the relative syzygies.

Elementary abelian  $p$ -subgroups play an important role in the analysis of  $T(G)$ . In particular, we will need the following group theoretical notions for our purposes.

**Definition 2.3.** Let  $G$  be a finite group and  $p$  be a prime.

- (1) The  $p$ -rank of  $G$  is the largest integer  $r$  such that  $G$  has an elementary abelian  $p$ -subgroup of rank  $r$ .
- (2) We write  $\mathcal{E}_{\geq 2}(G)$  for the poset of  $G$ -conjugacy classes of elementary abelian  $p$ -subgroups of  $G$  of  $p$ -rank at least 2.

Note that the  $p$ -rank of a finite group  $G$  is the  $p$ -rank of a Sylow  $p$ -subgroup of  $G$ . Moreover, if a Sylow  $p$ -subgroup  $P$  is normal in  $G$ , then  $\mathcal{E}_{\geq 2}(G)$  identifies with the quotient of  $\mathcal{E}_{\geq 2}(P)$  by the action by conjugation of  $G$ .

The number of connected components of  $\mathcal{E}_{\geq 2}(G)$  and the torsion-free rank of  $T(G)$  are closely related, as shown by the next two theorems. We state them here as they appear in [12] and in [9] respectively. Let us point out the fact that the first result was first obtained by J. Alperin (cf. [1, Theorem 4]).

Let  $P$  be a finite  $p$ -group. Assume that  $P$  has  $p$ -rank at least 2 and is not semi-dihedral. Let  $F_1, \dots, F_m$  denote a set of representatives of the components  $\mathcal{D}_1, \dots, \mathcal{D}_m$  of the poset  $\mathcal{E}_{\geq 2}(P)$ . By [12], we know that we can make such a choice satisfying that all  $F_i$ 's have  $p$ -rank 2, all but possibly  $F_m$  are maximal, and  $F_m$  is normal in  $P$ . For each  $1 \leq i \leq m$  let  $S_i$  be a non-central subgroup of  $P$  of order  $p$  that is contained in  $F_i$ . By [12, Lemma 2.2], if  $F_i$  is maximal then  $C_P(S_i) = S_i \times L_i$ , where  $L_i$  has normal  $p$ -rank one. That is,  $L_i$  is cyclic if  $p$  is odd, and can be either cyclic or generalized quaternion in case  $p = 2$ .

**Theorem 2.4** ([12], Theorem 3.1). *The group  $T(P)$  of endotrivial  $kP$ -modules is torsion-free of rank  $m$  and it is generated by  $\Omega_P$  and the classes of the modules  $N_i$  for  $i = 1, \dots, (m-1)$ , where  $N_i$  is the unique indecomposable summand with vertex  $P$  of*

$$\begin{aligned} M_i^{\otimes 2} & \text{ if } C_P(S_i)/S_i \text{ is cyclic of order } \geq 3, \\ M_i & \text{ if } p = 2 \text{ and } |C_P(S_i)/S_i| = 2, \\ M_i^{\otimes 4} & \text{ if } p = 2 \text{ and } C_P(S_i)/S_i \text{ is generalized quaternion,} \end{aligned}$$

and where  $M_i = \Omega_P^{-1}(k) \otimes \Omega_{P/S_i}^1(k)$ . Moreover,

$$\text{Res}_{F_j}^P N_i \cong \begin{cases} \Omega_{F_i}^{a_i}(k) \oplus (\text{proj}) & \text{if } i = j \\ k \oplus (\text{proj}) & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} a_i = -2p & \text{ if } C_P(S_i)/S_i \text{ is cyclic of order } \geq 3, \\ a_i = -2 & \text{ if } p = 2 \text{ and } |C_P(S_i)/S_i| = 2, \\ a_i = -8 & \text{ if } p = 2 \text{ and } C_P(S_i)/S_i \text{ is generalized quaternion.} \end{aligned}$$

In particular, if  $T(P)$  is cyclic, then  $T(P) = \langle \Omega_P \rangle$ .

We refer the reader to [10], [12] and [13] for a detailed description of the group of endotrivial modules in the case that  $m = 1$ , since we will not consider that situation later.

Let  $G$  be a finite group having a normal Sylow  $p$ -subgroup  $P$ , and let  $F_1, \dots, F_m$  be as above. Choose representatives  $E_1, \dots, E_n$  of the components  $\mathcal{C}_1, \dots, \mathcal{C}_n$  of the poset  $\mathcal{E}_{\geq 2}(G)$ .

**Remark 2.5.** *If  $P$  has  $p$ -rank at least 3 and  $m \geq 2$ , then, for any index  $1 \leq i \leq m-1$ , the subgroups  $F_i$  and  $F_m$  are not  $G$ -conjugate. Indeed, since  $P$  is normal in  $G$ , then  $P$  contains all  $p$ -subgroups of  $G$ , and since the  $p$ -rank is at least 3, then  $F_m$  is contained in an elementary abelian  $p$ -subgroup  $F$  of rank 3. Thus, if  ${}^g F_i = F_m < F$ , for some  $g \in G$ , then  $F_i < F^g < P$ . But this means that  $F_i$  is not maximal in  $P$ , which implies  $i = m$ . In other words, there is only one connected component of elementary abelian  $p$ -subgroups of rank at least 3, which is thus invariant by any automorphism of  $P$ , in particular here by  $G$ -conjugation.*

This observation allows us, without loss of generality and only for the convenience of the notation, to choose the indexes of the  $F_i$ 's so that  $E_i = F_i$ , for  $1 \leq i \leq n-1$  and  $E_n = F_m$ . The structure of  $T(G)$  is as follows (cf. [9]).

**Theorem 2.6.** *The group  $T(G)$  is finitely generated, and hence, it splits as a direct sum  $TT(G) \oplus TF(G)$ , where  $TT(G)$  denotes the torsion subgroup and  $TF(G)$  is a torsion-free subgroup of maximal rank (as  $\mathbb{Z}$ -module).*

Moreover,  $T(G)$  is generated by the classes of the indecomposable endotrivial  $kG$ -modules whose restriction to  $P$  is an indecomposable module, and  $TF(G) \cong \mathbb{Z}^n$ .

In particular, if  $TT(P)$  is trivial, then  $TT(G)$  is generated by the classes of the one-dimensional  $kG$ -modules. Also, if  $n = 1$ , then  $TF(G) = \langle \Omega_G \rangle$ .

Besides the purpose of the classification of endotrivial  $kG$ -modules, the fact that the syzygies of  $k$  are indecomposable endotrivial modules has the following consequence.

**Corollary 2.7.** *Let  $P$  be a normal Sylow  $p$ -subgroup of a finite group  $G$ , and let  $(X., \partial.)$  be a  $kG$ -projective resolution of the trivial module  $k$ . Then,  $(X., \partial.)$  is minimal  $\iff (X.\downarrow_P^G, \partial.\downarrow_P^G)$  is a minimal  $kP$ -projective resolution of  $k$ .*

*Proof.* “ $\Leftarrow$ ” is obvious.

“ $\Rightarrow$ ” A projective  $kG$ -resolution of  $k$  is minimal if and only if it is the splice of short exact sequences, one for each integer  $n \geq 0$ ,

$$0 \longrightarrow \Omega_G^{n+1}(k) \xrightarrow{i_n} X_n \xrightarrow{\partial_n} \Omega_G^n(k) \longrightarrow 0$$

By restriction to  $P$ , we get then

$$0 \longrightarrow \Omega_P^{n+1}(k) \oplus (\text{proj}) \xrightarrow{i_n \downarrow_P^G} X_n \downarrow_P^G \xrightarrow{\partial_n \downarrow_P^G} \Omega_P^n(k) \oplus (\text{proj}) \longrightarrow 0$$

But the  $kG$ -modules  $\Omega_G^n(k)$  are indecomposable endotrivial and  $P \triangleleft G$ , and so, by Theorem 2.6,  $\Omega_G^n(k) \downarrow_P^G$  is indecomposable. Therefore the projective factors “(proj)” are all zero. In other words, for each integer  $n \geq 0$ , the above short exact sequence is in fact

$$0 \longrightarrow \Omega_P^{n+1}(k) \xrightarrow{i_n \downarrow_P^G} X_n \downarrow_P^G \xrightarrow{\partial_n \downarrow_P^G} \Omega_P^n(k) \longrightarrow 0$$

Hence,  $(X.\downarrow_P^G, \partial.\downarrow_P^G)$  is a minimal  $kP$ -projective resolution of  $k$ . □

### 3. DADE’S RESULT AND ITS CONSEQUENCES

Throughout this section, we assume that  $G$  is a finite group and that  $P$  is a normal Sylow  $p$ -subgroup of  $G$ . We continue with the notation used in the previous section.

The only cases in which generators for the group  $TF(G)$  are not given by Theorem 2.6 are for  $TF(G)$  not cyclic, i.e. for  $n \geq 2$ , according to our notation. Hence, we will assume that  $n \geq 2$ . In this case, we also have  $m \geq 2$ , and so  $TT(P) = 0$ , by Theorem 2.4. Moreover, this also implies that the center of  $P$  is cyclic non trivial. Thus, it has a unique cyclic subgroup  $Z$  of order  $p$ , which is contained in any maximal elementary abelian  $p$ -subgroup. Furthermore,  $Z \triangleleft G$ , since  $Z$  is characteristic in  $P$ .

A key tool to determine which endotrivial  $kP$ -modules extend to  $G$  is provided by the following theorem, proven by Dade and never published (as far as we are aware of).

**Theorem 3.1** (Theorem 7.1, [14]). *Let  $G$  be a finite group having a normal Sylow  $p$ -subgroup  $P$ , let  $k$  be an algebraically closed field of characteristic  $p$ , and let  $M$  be an endo-permutation  $kP$ -module. Then  $M$  extends to a  $kG$ -module if and only if  $M$  is  $G$ -stable.*

By definition, if  $H$  is a normal subgroup of  $G$  and  $M$  is a  $kH$ -module, then we define the conjugate  $kH$ -module  ${}^gM$  of  $M$  by setting  $\underbrace{h \cdot m}_{\text{in } {}^gM} = \underbrace{{}^g(h^g \cdot m)}_{\text{in } M}$ ,  $\forall h \in H, \forall m \in M, \forall g \in G$ . We

say then that  $M$  is  $G$ -stable if we have an isomorphism of  $kH$ -modules  $M \cong {}^gM, \forall g \in G$ . Equivalently, a  $kH$ -module  $M$  is  $G$ -stable if and only if  ${}^gM \cong M, \forall g \in G/H$ .

Since an endotrivial module is an endo-permutation module, and since a module is endotrivial if and only if its restriction to a Sylow  $p$ -subgroup is, we have the following.

**Corollary 3.2.** *The map  $\text{Res}_P^G : T(G) \rightarrow T(P)$  induces an isomorphism of abelian groups  $TF(G) \cong TF(P)^{G/P}$ , where  $TF(P)^{G/P}$  is the subset of the  $G/P$ -fixed points. That is,  $TF(P)^{G/P}$  is the subgroup of  $TF(P)$  generated by the classes of the  $G$ -stable endotrivial  $kP$ -modules.*

*Proof.* By Theorem 2.6, we have  $\ker(\text{Res}_P^G) = TT(G)$ , and the image of  $\text{Res}_P^G$  is necessarily a subgroup of  $T(P)^{G/P}$ . Hence,  $\text{Res}_P^G$  induces an injection  $TF(G) \rightarrow TF(P)^{G/P}$ . Now, by Theorem 3.1, an endotrivial  $kP$ -module  $M$  extends to a  $kG$ -module (necessarily endotrivial) if and only if  $M$  is  $G$ -stable. That is, the above map is also surjective.  $\square$

At this stage, determining a set of generators for  $TF(G)$  comes down to a question of linear algebra. Indeed, if  $x \in T(P)$  then  $x$  is uniquely expressible as a  $\mathbb{Z}$ -linear combination  $x = \sum_{i=1}^m x_i e_i$ , where  $e_i = [N_i]$ , for the module  $N_i$  defined in Theorem 2.4, for  $1 \leq i \leq m-1$ , and  $e_m = \Omega_P$ .

**Lemma 3.3.** *For all  $g \in G$ , and for all  $1 \leq i \leq m$ , we have  ${}^g e_i = e_{\sigma_g(i)}$ , for the permutation  $\sigma_g$ , acting on the set  $\{1, \dots, m\}$ , defined by  ${}^g S_j =_P S_{\sigma_g(j)}$ ,  $\forall 1 \leq j \leq m-1$  and  $\sigma_g(m) = m$ .*

Here, the notation “ $=_P$ ” means “is  $P$ -conjugate to”.

*Proof.* Fix an index  $1 \leq i \leq m-1$ , and write  $S = S_i$  and  $N = N_i$ . Let  $g \in G$  and consider the map  $\varphi : {}^g k[P/S] \rightarrow k[P/{}^g S]$ , defined by  $\varphi({}^g(uS)) = {}^g u {}^g S$  on a  $k$ -basis  $\{{}^g(uS) \mid uS \in P/S\}$  of the permutation  $kP$ -module  ${}^g k[P/S]$ . Let us verify that  $\varphi$  is an isomorphism of  $kP$ -modules. It is immediate that  $\varphi$  is an isomorphism of  $k$ -vector spaces, and so we only need to check that  $\varphi$  commutes with the action of  $P$ :

$$v \cdot \varphi({}^g(uS)) = v \cdot {}^g u {}^g S = {}^g(v \cdot u) {}^g S = \varphi({}^g(v \cdot u) S) = \varphi(v \cdot {}^g(uS)), \quad \forall u \in P, \forall uS \in P/S, \forall g \in G.$$

It follows that we have exact sequences forming the commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & {}^g \Omega_{P/S}^1(k) & \longrightarrow & {}^g k[P/S] & \xrightarrow{{}^g \varepsilon} & {}^g k & \longrightarrow & 0 \\ & & \downarrow \varphi|_{\ker({}^g \varepsilon)} & & \downarrow \varphi & & \downarrow \text{Id} & & \\ 0 & \longrightarrow & \Omega_{P/{}^g S}^1(k) & \longrightarrow & k[P/{}^g S] & \xrightarrow{\varepsilon'} & k & \longrightarrow & 0 \end{array}$$

Since  $\varphi$  is an isomorphism, the left hand  $kP$ -modules  ${}^g \Omega_{P/S}^1(k)$  and  $\Omega_{P/{}^g S}^1(k)$  are isomorphic. Similarly, we have  ${}^g \Omega_P^1(k) \cong \Omega_P^1(k)$  and the result follows.  $\square$

This lemma has an immediate consequence.

**Proposition 3.4.**  $x = \sum_{i=1}^m x_i e_i \in T(P)^G \iff x_i = x_j$  whenever  $S_i =_G S_j$ ,  $\forall 1 \leq i \leq m-1$ .

We will need a group theoretical result, complementary to Lemma 2.2 in [12]. For this, let us fix an index  $1 \leq i \leq n-1$  and write  $S = S_i$ ,  $E = E_i$  and  $N = N_i$ . Recall that our choice of the  $E_i$ 's forces  $E_i = F_j$  for some  $j$ .

**Lemma 3.5.** *The stabilizer of the  $P$ -conjugacy class of  $S$  is the subgroup  $PN_G(E)$  of  $G$ .*

*Proof.* By [12, Lemma 2.2], we know that  $C_P(S) = N_P(S) = S \times L$ , where  $L$  has normal  $p$ -rank 1 and  $L$  contains the unique central subgroup  $Z$  of  $P$  of order  $p$ . Moreover, the index of  $N_P(S)$  in  $N_P(E)$  is  $p$ . In particular, we deduce that  $E = S \times Z$ , and that  $N_P(E)/N_P(S)$  acts transitively (by conjugation) on the  $p$  non central subgroups of order  $p$  of  $E$ .

Since  $Z \triangleleft G$ , we have  $N_G(S) \leq N_G(E)$ , and so  $PN_G(S) \leq PN_G(E)$ .

Conversely, for  $g \in N_G(E)$ , we have  ${}^gS \leq {}^gE = E$ . Thus, there exists  $v \in P$  (actually, we can take  $v \in N_P(E) - N_P(S)$ ) such that  ${}^gS = {}^vS$ , and so,  $v^{-1}g \in N_G(S)$ . It follows then that  $g \in PN_G(S)$ , as was to be shown.  $\square$

Since  $PN_G(E)$  is the stabilizer of the  $P$ -conjugacy class of  $S$ , Theorem 3.1 implies that  $N$  extends to a  $k[PN_G(E)]$ -module  $\tilde{N}$ , which is necessarily endotrivial.

Let  $C$  be a set of representatives of the left cosets  $G/PN_G(E)$ . Then the correspondence  $c \mapsto {}^cN$  is a bijection from  $C$  to the  $G$ -conjugacy class of  $N$ .

**Definition 3.6.** *Let  $H$  be a subgroup of  $G$  and let  $M$  be a  $kH$ -module. We define the tensor induced module  $\text{Ten}_H^G M$  as follows (cf. [15, § 5.1]). It is the  $k$ -vector space*

$$\bigotimes_{s \in [G/H]} (s \otimes M), \quad \text{where } [G/H] \text{ is a set of representatives of the left cosets } G/H$$

endowed with the structure of  $kG$ -module given by

$$g \cdot \otimes_s (s \otimes m_s) = \otimes_s (\tau_g(s) \otimes h_s m_s) = \otimes_s (s \otimes h_{\tau_g^{-1}(s)} m_{\tau_g^{-1}(s)}),$$

where  $gs = \tau_g(s)h_s$ , with  $h_s \in H$  and a permutation  $\tau_g$  of  $[G/H]$ .

**Proposition 3.7.** *The  $kG$ -module  $\text{Ten}_{PN_G(E)}^G \tilde{N}$  is endotrivial.*

*Proof.* It is enough to check that  $\text{Res}_P^G \text{Ten}_{PN_G(E)}^G \tilde{N}$  is endotrivial. By the tensor version of Mackey formula (cf. [15, Prop. 5.2.1]), and since  $P \triangleleft G$ , we have isomorphisms of  $kP$ -modules

$$\begin{aligned} \text{Res}_P^G \text{Ten}_{PN_G(E)}^G \tilde{N} &\cong \bigotimes_{c \in C} \text{Ten}_{P \cap {}^c(PN_G(E))}^P \text{Res}_{P \cap {}^c(PN_G(E))}^{({}^c(PN_G(E)))} {}^c\tilde{N} = \\ &= \bigotimes_{c \in C} \text{Res}_P^{({}^c(PN_G(E)))} {}^c\tilde{N} \cong \bigotimes_{c \in C} {}^c(\text{Res}_P^{PN_G(E)} \tilde{N}) \cong \bigotimes_{c \in C} {}^cN \end{aligned}$$

which is a tensor product of endotrivial  $kP$ -modules and hence is endotrivial.  $\square$

For any elementary abelian  $p$ -group  $E$  of rank at least 2, we may identify the group  $T(E) = \langle \Omega_E \rangle$  with  $\mathbb{Z}$ , via  $n\Omega_E \mapsto n$ .

**Definition 3.8.** *Let  $H$  be a finite group and let  $E_1, \dots, E_l$  be representatives of the connected components of  $\mathcal{E}_{\geq 2}(H)$ . The product of all restriction maps  $\text{Res}_{E_i}^H : T(H) \rightarrow T(E_i)$ , composed with the isomorphism  $T(E_i) \cong \mathbb{Z}$ ,  $1 \leq i \leq l$ , yields a well-defined homomorphism*

$$\mathbf{res}_{\mathcal{E}(H)} : T(H) \longrightarrow \prod_{i=1}^l T(E_i) \longrightarrow \mathbb{Z}^l$$

For any endotrivial  $kH$ -module  $M$ , the element  $\mathbf{res}_{\mathcal{E}(H)}([M]) \in \mathbb{Z}^l$  is called the type of  $M$ .

Throughout these notes,  $\delta_{i,j}$  denotes the Kronecker symbol.

**Proposition 3.9.** *We have  $\ker(\mathbf{res}_{\mathcal{E}(G)}) = TT(G)$ . Moreover,*

$$\mathbf{res}_{\mathcal{E}(G)}([\text{Ten}_{PN_G(E_j)}^G \tilde{N}_j]) = (\delta_{i,j} a_i)_{i=1}^n,$$

where  $a_i$  is the integer defined in Theorem 2.4.

*Proof.* The first statement has been proved in [9, Proposition 2.3], so we only need to show the second one.

By transitivity of the restriction and Proposition 3.7, we have

$$\mathrm{Res}_{E_i}^G \mathrm{Ten}_{PN_G(E_j)}^G \tilde{N}_j \cong \bigotimes_{c \in C_j} \mathrm{Res}_{E_i}^P {}^c N_j$$

where  $C_j$  is a set of representatives of the left cosets  $G/PN_G(E_j)$ . We may assume that  $1 \in C_j$  for all  $j$ . Now, for any  $c \in C_j$ , the  $kP$ -module  ${}^c N_j$  is endotrivial and defined in Theorem 2.4 as the “unique” indecomposable direct summand with vertex  $P$  of a  $r$ -fold tensor product of the module

$${}^c(\Omega_P^{-1}(k) \otimes \Omega_{P/S_j}^1(k)) \cong \Omega_P^{-1}(k) \otimes \Omega_{P/{}^c S_j}^1(k)$$

where the integer  $r$  is either 1, 2 or 4. By “unique”, we mean unique up to isomorphism.

Even though it is not an endotrivial module,  $\mathrm{Res}_{E_i}^P \Omega_{P/{}^c S_j}^1(k)$  has a “unique” indecomposable summand  $V$  with vertex  $E_i$ , and  $V$  is isomorphic to  $\Omega_{(P/{}^c S_j) \downarrow E_i}^1(k)$ . By Mackey formula (for  $E_i$ -sets) we have that

$$(P/{}^c S_j) \downarrow_{E_i}^P \cong \coprod_{x \in [E_i \backslash P/{}^c S_j]} E_i / (E_i \cap {}^x c S_j)$$

If  $i \neq j$  then  $E_j \neq_G E_i$  and no  $G$ -conjugate of  $S_j$  is contained in  $E_i$ , since  $E_i = S_i \times Z$  and  $Z \triangleleft G$ . It follows that

$$(P/{}^c S_j) \downarrow_{E_i}^P \cong \coprod_{x \in [E_i \backslash P/{}^c S_j]} E_i. \quad \text{Thus, } V \cong \Omega_{E_i}^1(k) \text{ if } i \neq j, \text{ by [4, Lemma 3.2.7],}$$

and therefore  ${}^c N_j$  is trivial.

Otherwise, we have  $i = j$ . If  ${}^x c S_i < E_i$ , for  $x \in P$ , then there is  $y \in N_P(E_i) - C_P(S_i)$  such that  ${}^x c S_i = {}^y S_i$ , since  $N_P(E_i)/C_P(S_i)$  acts transitively on the  $p$  non central subgroups of order  $p$  of  $E_i$ . It follows that  $y^{-1} x c \in N_G(S_i)$ , and so  $c \in PN_G(S_i) = PN_G(E_i)$ . By the choice of representatives in  $C_i$ , this implies  $c = 1$ . Then, Theorem 2.4 shows that  $\mathrm{Res}_{E_i}^P N_i \cong \Omega_{E_i}^{a_i} \oplus (\mathrm{proj})$ .

Finally, if  $i = j$  and  $c \notin PN_G(E_i)$ , then the previous argument shows that  ${}^x c S_i$  is not contained in  $E_i$ , for any  $x \in P$ . Therefore,  $\mathrm{Res}_{E_i}^P {}^c N_i \cong k \oplus (\mathrm{proj})$ , as in the case  $i \neq j$ .

In conclusion, by tensoring all the pieces together, we deduce that

$$\mathrm{Res}_{E_i}^G \mathrm{Ten}_{PN_G(E_j)}^G \tilde{N}_j \cong \bigotimes_{c \in C_j} \mathrm{Res}_{E_i}^P {}^c N_j \cong \begin{cases} \Omega_{E_i}^{a_i}(k) \oplus (\mathrm{proj}) & \text{if } i = j \\ k \oplus (\mathrm{proj}) & \text{if } i \neq j \end{cases}$$

□

**Theorem 3.10.** *Let  $G$  be a finite group having a normal Sylow  $p$ -subgroup  $P$ . Consider the same notation as above, and write the group  $T(G)$  of endotrivial  $kG$ -modules as a direct sum  $T(G) = TT(G) \oplus TF(G)$ , where  $TT(G)$  is the torsion subgroup and  $TF(G)$  is torsion-free. Then, we may choose the set*

$$\left\{ \Omega_G, x_i = [\mathrm{Ten}_{PN_G(E_i)}^G \tilde{N}_i] \mid 1 \leq i \leq n-1 \right\} \quad \text{as a basis for the free } \mathbb{Z}\text{-module } TF(G).$$

Moreover, we have  $\mathbf{res}_{\mathcal{E}(G)}(\Omega_G) = (1)_{i=1}^n$ , and  $\mathbf{res}_{\mathcal{E}(G)}(x_j) = (\delta_{i,j} a_i)_{i=1}^n$ .

*Proof.* The  $kG$ -modules  $\mathrm{Ten}_{PN_G(E_i)}^G \tilde{N}_i$  are endotrivial and their classes are linearly independent, by Theorem 2.4 and the proof of Proposition 3.7. Hence they generate a torsion-free submodule of  $T(G)$  of rank equal to the rank of  $TF(G)$ .

Moreover, the proof of Proposition 3.7 shows that in  $T(P)$  we have

$$\mathrm{Res}_P^G [\mathrm{Ten}_{PN_G(E_i)}^G \tilde{N}_i] = \sum_{c \in C_i} [{}^c N_i]$$

where  $C_i$  denotes a set of the representatives of the left cosets  $G/PN_G(E_i)$ . Since  $\{{}^cN_i \mid c \in C_i\}$  is a sequence without repetition that contains all the  $G$ -conjugates of  $N_i$ , it also makes the element  $\sum_{c \in C_i} [{}^cN_i]$  of  $T(P)$  satisfy the minimal necessary condition for being  $G$ -stable, by Proposition 3.4.

This proves the first statement. The second claim has been proved in Proposition 3.9.  $\square$

#### 4. ALTERNATIVE CONSTRUCTIONS

Throughout this section, we let  $G$  be a finite group having a normal Sylow  $p$ -subgroup  $P$ . We assume  $n \geq 2$ , and we choose the representatives  $E_1, \dots, E_n$  and  $F_1, \dots, F_m$  of the connected components of  $\mathcal{E}_{\geq 2}(G)$  and  $\mathcal{E}_{\geq 2}(P)$ , respectively, such that  $E_i = F_i, \forall 1 \leq i \leq n-1$  and  $E_n = F_m$ , as before. Write also  $Z = \langle z \rangle$  for the unique central subgroup of  $P$  of order  $p$ . Finally, let  $a_i$  be the integer defined in Theorem 2.4.

The purpose of this section is to review the two constructions of endotrivial modules providing free sets of generators for  $TF(G)$ , as given in [9]. Then, we compare the modules obtained for each construction. We assume that the reader is familiar with the notions of cohomology of finite groups and (cohomological) varieties of modules given in [7] or [15]. The notation we use in this section is as follows. For an inclusion of finite groups  $H \rightarrow K$ , we denote the induced restriction map in cohomology  $H^*(K, k) \rightarrow H^*(H, k)$  either  $\text{Res}_H^K$  or  $\downarrow_H^K$  indifferently. Meanwhile, we denote by  $\text{res}_{K,H}^* : V_H(k) \rightarrow V_K(k)$  the induced map on the varieties.

Let us now recall the presentation of  $TF(G)$  given in [9].

**Theorem 4.1** ([9, Theorem 3.4]). *For each  $1 \leq i \leq n-1$  there exists an endotrivial  $kG$ -module  $U_i$  such that*

$$U_i \downarrow_{E_i}^G \cong \Omega_{E_i}^{a_i} \oplus (\text{proj}) \quad \text{and} \quad U_i \downarrow_{E_j}^G \cong k \oplus (\text{proj}), \quad \forall j \neq i,$$

where the  $a_i$ 's are the integers defined in Theorem 2.4.

The classes  $\Omega_G, [U_i], 1 \leq i \leq n-1$  form a free set of generators for  $TF(G)$ .

The structure of the modules  $U_i$  is described in the proof, which uses the construction originating from [11] and quoted by the authors as the ‘‘deconstruction method’’. Briefly, it consists in analysing the variety  $V = V_{G/Z}(\overline{\Omega_G^{a_i}(k)})$ , where  $\overline{\Omega_G^{a_i}(k)}$  is the  $kG/Z$ -module that is the quotient of  $\Omega_G^{a_i}(k)$  by the submodule  $(z-1)^{p-1}\Omega_G^{a_i}(k)$ . The arguments of [8, Theorem 7.2] show that for each index  $j$ , the variety  $V$  decomposes as a union of varieties  $V_j \cup V_j'$  such that

$$V_j \cap V_j' = \{0\}, \quad V_j = \text{res}_{G/Z, E_j/Z}^*(V_{E_j/Z}(k)) \quad \text{and} \quad \text{res}_{G/Z, E_l/Z}^*(V_{E_l/Z}(k)) \subseteq V_j', \quad \forall l \neq j$$

Then, by [8, Corollary 4.3], there exists an endotrivial module of the desired type, whose construction appeals to previous results (and ultimately from a theorem of [2], applied to the Lyndon-Hochschild-Serre spectral sequence of the group extension  $1 \rightarrow Z \rightarrow G \rightarrow G/Z \rightarrow 1$ ). Indeed, it turns out that each  $V_j$  is the variety of the  $kG/Z$ -module that is the quotient  $U_j/(z-1)^{p-1}U_j$  of an indecomposable endotrivial  $kG$ -module  $U_j$ .

We now turn to the third construction. Note that it does not require  $P \triangleleft G$ , and that, instead, it necessitates an additional assumption on the cohomology group  $H^*(G, k)$ .

**Proposition 4.2** ([8, Corollary 4.6]). *Suppose that  $Z$  is a subgroup of order  $p$  in the center of a Sylow  $p$ -subgroup  $P$  of  $G$ . Suppose that for  $d > 0$  the group  $H^{2d}(G, k)$  has an element  $\zeta$  such that  $\text{Res}_Z^G(\zeta) \neq 0$ . Then  $G$  has an endotrivial module of type  $(2d\delta_{i,j})_{i=1}^n$  for each  $1 \leq j \leq n$ .*

Let us recall how these endotrivial modules are obtained, starting from a generalization of the definition of Carlson's  $L_\zeta$ 's modules (cf. [7, § 6]).



**Definition 4.3.** Let  $\zeta \in H^s(G, S)$  be non zero, where  $S$  is a 1-dimensional  $kG$ -module and  $s \geq 1$ . Let  $\tilde{\zeta} \in \text{Hom}_G(\Omega_G^s(k), S)$  represent  $\zeta$ . We define the  $kG$ -module  $L_{\zeta, S} = \ker \tilde{\zeta}$ .

Among many properties of these modules, all that we need to know for our concern is that the modules  $L_{\zeta, S}$  are defined up to isomorphism by the class  $\zeta \in H^s(G, S)$ . In particular, for  $s = 2d$ , as in Proposition 4.2, we have a short exact sequence of  $kG$ -modules

$$0 \longrightarrow L_{\zeta, S} \longrightarrow \Omega_G^{2d}(k) \xrightarrow{\tilde{\zeta}} S \longrightarrow 0$$

which does not depend on the choice of  $\tilde{\zeta}$ , and which does not involve any projective summand. Since  $\Omega_G^{2d}(k) \downarrow_Z^G \cong k \oplus (\text{proj})$ . Thus, the assumption that  $\zeta \downarrow_Z^G \neq 0$  implies that  $L_{\zeta, S} \downarrow_Z^G$  is projective. By Quillen Dimension Theorem (cf. [15, Corollary 8.3.3]), it follows that the variety  $V_G(L_{\zeta, S})$  of  $L_{\zeta, S}$  is disconnected, since the number  $n$  of connected components of  $\mathcal{E}_{\geq 2}(G)$  is at least 2. More precisely,  $V_G(L_{\zeta, S})$  is a union of  $n$  subspaces  $V_1, \dots, V_n$ , each  $V_i$  being the variety of a  $kG$ -submodule  $L_i$  of  $L_{\zeta, S}$ , such that  $L_{\zeta, S}$  decomposes as direct sum  $L_1 \oplus \dots \oplus L_n$ . Note that  $V_i$  is a line for each  $1 \leq i \leq n-1$ . The similar reasoning applies to the group  $P$  instead of  $G$ , and since  $\Omega_G(k) \downarrow_P^G \cong \Omega_P(k)$ , we conclude that  $L_{\zeta, S} \downarrow_P^G \cong L_{\zeta|_P} \downarrow_P^G$  splits as a direct sum  $X_1 \oplus \dots \oplus X_m$ , where  $m \geq 2$  is the number of connected components of  $\mathcal{E}_{\geq 2}(P)$ . Moreover, if  $V_P(L_{\zeta, S} \downarrow_P^G) = W_1 \cup \dots \cup W_m$ , with  $W_i = V_P(X_i)$ , then  $G$  acts by permutation on the  $W_i$ 's and hence on the  $X_i$ 's. It follows that  $L_i \downarrow_P^G \cong \bigoplus_{g \in [G/G_i]} g \cdot X_i$ , for the stabilizer  $G_i = PN_G(F_i)$  of the  $P$ -conjugacy class of  $F_i$ , as in the previous section.

For each index  $1 \leq i \leq n$ , set  $L'_i = \bigoplus_{j \in J_i} L_j$ , where  $J_i = \{j \mid 1 \leq j \leq n, i \neq j\}$ .

**Proposition 4.4.** Each  $L_i$  gives rise to an indecomposable endotrivial  $kG$ -module  $M_i$ , of type  $(2d\delta_{i,j})_{j=1}^n$ , built as the push-out in the following commutative diagram.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & L'_i & \xlongequal{\quad} & L'_i & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & L_{\zeta, S} & \longrightarrow & \Omega_G^{2d}(k) & \xrightarrow{\tilde{\zeta}} & S \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & L_i & \longrightarrow & M_i & \longrightarrow & S \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

*Proof.* The proof is a paraphrase of an argument used in the demonstration of [9, Theorem 3.1], and goes as follows.

Let  $1 \leq i, j \leq n$ . We need to show that

$$M_i \downarrow_{E_j}^G \cong \begin{cases} \Omega_{E_j}^{2d}(k) \oplus (\text{proj}) & \text{if } i = j \\ k \oplus (\text{proj}) & \text{otherwise} \end{cases}$$

We have  $L_i \downarrow_P^G = \bigoplus_{g \in [G/G_i]} g \cdot X_i$ , with  $X_i \downarrow_{E_j}^P$  projective (and hence injective) if and only if  $j \neq i$ , i.e.  $E_j \neq F_i$ , by our choice of the representatives of the connected components of  $\mathcal{E}_{\geq 2}(P)$  and  $\mathcal{E}_{\geq 2}(G)$ . This implies that exactly one of the restrictions to  $E_j$  of the exact sequences consisting

in the second column or the bottom row is split, depending on whether  $X_j$  is a summand in  $L_i \downarrow_P^G$  or not. Namely, the restriction to  $E_j$  of the second column splits if and only if  $X_j \mid L_i \downarrow_P^G$ , that is, if and only if  $X_j = g \cdot X_i$ , for some  $g \in G$ . This happens if and only if  $i = j$ , and in this case we have

$$M_j \downarrow_{E_j}^G \oplus (\text{proj}) \cong \Omega_P^{2d}(k) \downarrow_{E_j}^P \cong \Omega_{E_j}^{2d}(k) \oplus (\text{proj}).$$

Hence  $M_j \downarrow_{E_j}^G \cong \Omega_{E_j}^{2d}(k) \oplus (\text{proj})$ .

Otherwise,  $i \neq j$  and the restriction to  $E_j$  of the bottom row splits, since the left term is injective. In this case, we have  $M_i \downarrow_{E_j}^G \cong k \oplus (\text{proj})$ .

Thus  $M_i \downarrow_{E_j}^G$  is endotrivial for all  $1 \leq i, j \leq n$ , and the type of  $M_i$  is  $(2d\delta_{i,j})_{i=1}^n$ , as asserted.  $\square$

We discuss the possible additional assumptions needed for this construction to be carried out. That is, we want to give a sufficient and necessary condition on the cohomology group  $H^{2d}(P, k)$ , that would detect when there exists a one-dimensional  $kG$ -module  $S$  such that  $H^{2d}(G, S)$  contains an element that restrict non trivially to  $Z$ .

Let  $H$  be a normal subgroup of  $G$ . Then  $\text{Hom}_H(M \downarrow_H^G, k)$  is a  $kG$ -module, for any  $kG$ -module  $M$ , and since the restriction map commutes with the differentials in complexes of  $kG$ - and  $kH$ -modules, it induces a map of  $kG$ -modules  $\text{Res}_H^G : H^s(G, k) \rightarrow H^s(H, k)$  in cohomology (cf. [15, § 4.1]). In fact,  $H$  acts trivially on  $\text{Hom}_H(M \downarrow_H^G, k)$ , and so, we can consider  $\text{Res}_H^G$  as a map of  $kG/H$ -modules. Similarly, for  $G, P, Z$  and  $2d$  as in Proposition 4.2, the map  $\text{Res}_Z^P : H^{2d}(P, k) \rightarrow H^{2d}(Z, k)$  is a map of  $kG/P$ -modules. Since  $p$  does not divide the order of  $G/P$ , the map  $\text{Res}_Z^P$  splits. Now, the fact that  $H^{2d}(Z, k)$  is one dimensional implies that, if  $\text{Res}_Z^P$  is non zero, then any non zero element  $\beta \in H^{2d}(Z, k)$  lifts to an element  $\zeta \in H^{2d}(P, k)$  that restricts non trivially to  $Z$  and such that  $(g \cdot \zeta) \downarrow_Z^P = g \cdot \zeta \downarrow_Z^P = \mu_g \beta$ , for some  $|G : P|$ -root of unity  $\mu_g \in k$ , for any  $g \in G$ .

On the other hand, we also have that the action of  $G$  on the 1-dimensional  $k$ -vector space  $H^s(Z, k)$ , for any integer  $s \geq 0$ , defines a representation  $\rho : G \rightarrow \text{Aut}(H^s(Z, k))$  of  $G$ . Set  $H$  for the kernel of  $\rho$ , that is,  $H = \{g \in G \mid g \cdot \zeta = \zeta, \forall \zeta \in H^s(Z, k)\}$  is the stabilizer of  $H^s(Z, k)$  in  $G$ . In particular,  $H$  is a normal subgroup of  $G$  containing  $P$ , and the quotient group  $G/H$  is cyclic of order prime to  $p$ , since it is isomorphic to a finite subgroup of the group of units in  $k$ . Hence,  $H^s(Z, k)$  is a  $kG/H$ -module.

**Lemma 4.5.** *Let  $s \geq 0$  be an integer and assume there exists  $\zeta \in H^s(P, k)$ , such that  $\zeta \downarrow_Z^P \neq 0$ . Then  $\text{Tr}_P^H \zeta \in H^s(H, k)$  and we have  $(\text{Tr}_P^H \zeta) \downarrow_Z^H \neq 0$ , where  $\text{Tr}_P^H$  denotes the transfer in cohomology (cf. [15, § 4.2]).*

*Proof.* Let  $\zeta \in H^s(P, k)$ , for some  $s \geq 0$ , and assume that  $\zeta \downarrow_Z^P \neq 0$ . By definition of  $H$ , we have

$$\text{Res}_Z^H \text{Tr}_P^H \zeta = \sum_{g \in [H/P]} g \cdot \zeta \downarrow_Z^P = |H : P| \cdot \zeta \downarrow_Z^P.$$

Since  $|H : P|$  is invertible in  $k$ , we have that  $\text{Res}_Z^H \text{Tr}_P^H \zeta \neq 0$  if  $\zeta \downarrow_Z^P \neq 0$ .  $\square$

This observation leads us to a sufficient and necessary condition for determining when the last construction of endotrivial modules can be applied, provided that it applies for  $T(P)$ .

**Proposition 4.6.** *There exist a 1-dimensional  $kG$ -module  $S$  and  $\eta \in H^s(G, S)$  such that  $\eta \downarrow_Z^G \neq 0$  if and only if there exists  $\zeta \in H^s(P, k)$  such that  $\zeta \downarrow_Z^P \neq 0$ .*

*In particular, for  $s = 2d$ , there exists then an indecomposable endotrivial module of type  $(2d\delta_{i,j})_{i=1}^n$ , for each index  $1 \leq j \leq n$ .*

*Proof.* The “if” part is clear.

Assume there exists  $\zeta \in \mathbb{H}^s(P, k)$  such that  $\zeta \downarrow_Z^P \neq 0$ , and let  $H$  be the stabilizer of  $\mathbb{H}^s(Z, k)$  in  $G$ , as above. By Lemma 4.5, we have that  $\text{Tr}_P^H \zeta \in \mathbb{H}^s(H, k)$  and that  $(\text{Tr}_P^H \zeta) \downarrow_Z^H \neq 0$ . By the Eckmann-Shapiro Lemma (cf. [15, Proposition 4.1.3]), we have  $\mathbb{H}^s(H, k) \cong \mathbb{H}^s(G, k \uparrow_H^G)$ . Thus, the image of  $\text{Tr}_P^H \zeta$  under this isomorphism is an element of  $\mathbb{H}^s(G, k \uparrow_H^G)$  that restricts non trivially to the subgroup  $Z$ . Now,  $k \uparrow_H^G$  is a direct sum of 1-dimensional  $kG$ -modules, since  $G/H$  is a cyclic group of order prime to  $p$ , as noted above, and since  $k$  is algebraically closed. Therefore, there is a 1-dimensional direct summand  $S$  of  $k \uparrow_H^G$  and  $\eta \in \mathbb{H}^s(G, S)$  such that  $\eta \downarrow_Z^G \neq 0$ , as was to be shown.

The last assertion is then the application of Proposition 4.2.

□

We end this section with the comparison of the modules obtained in each construction, and, for convenience, we use the same notation. First, observe that a module  $\text{Ten}_{G_i}^G \tilde{N}_i$  of Section 3 is not indecomposable and it has a large dimension, in general. Moreover the corresponding  $kP$ -module  $N_i$  is the *cap* of a (larger, in general) endo-permutation module. That is,  $N_i$  is the unique direct summand with vertex  $P$  of a capped endo-permutation  $kP$ -module. Consequently, if one wishes to compute  $T(G)$  with a computer support, the generators for  $TF(G)$  provided by the  $\text{Ten}_{G_i}^G \tilde{N}_i$  are likely to involve time-consuming algorithms. Instead, the modules of Section 4 are indecomposable and their construction is likely to be more handy by an algebra software.

In order to compare the three presentations of  $TF(G)$ , we use the injectivity of the restriction map  $\text{res}_{\mathcal{E}(G)} : TF(G) \rightarrow \prod_{i=1}^n T(E_i)$ . Then, it is the simple matter of comparing the computations in Theorems 3.10, 4.1, and Proposition 4.2.

**Remark 4.7.** *In  $\text{stmod}(kG)$ , we have isomorphisms  $\text{Ten}_{G_i}^G \tilde{N}_i \cong U_i^* \cong M_i^*$ ,  $\forall 1 \leq i \leq n$ . In particular, since  $U_i$  and  $M_i$  are indecomposable, we have  $U_i \cong M_i$  in  $\text{mod}(kG)$ , for all  $1 \leq i \leq n$ .*

It is remarkable, and remains yet unexplained, that the modules  $U_i$  and  $M_i$  are isomorphic, since their construction is apparently different. Moreover, the modules  $M_i$  cannot be built in general, whereas the modules  $U_i$  always exist.

## 5. ODD EXTRASPECIAL EXAMPLE

Let  $p$  be an odd prime and let  $P$  be a Sylow  $p$ -subgroup of  $\text{PSL}_3(p)$ . That is,  $P$  is an extraspecial  $p$ -group of order  $p^3$  and exponent  $p$ . Let  $G$  be its normalizer in  $\text{PSL}_3(p)$ .

There are  $p+1$  conjugacy classes in  $P$  of maximal elementary abelian  $p$ -subgroups of rank 2, each consisting of a single maximal normal subgroup of  $P$ . Let us call these subgroups  $F_1, \dots, F_{p+1}$ . Choose also non central subgroups  $S_i < F_i$ ,  $1 \leq i \leq (p+1)$  of order  $p$ .

By Theorem 2.4, the group  $T(P)$  of endotrivial modules is free abelian of rank  $p+1$  and admits a presentation  $T(P) = \langle \Omega_P, [N_i], 1 \leq i \leq p \rangle$ , where  $[N_i]$  denotes the class of the indecomposable endotrivial summand of the  $kP$ -module  $(\Omega_P^{-1}(k) \otimes \Omega_{P/S_i}^1(k))^{\otimes 2}$ , for each index  $i$ . The elements  $[N_i]$ , for  $1 \leq i \leq p+1$ , verify a non trivial relation in  $T(P)$  (cf. also [5, § 11])

$$\sum_{i=1}^{p+1} [N_i] = 2p \Omega_P, \quad \text{since both have the same type, namely } (2p, \dots, 2p) \in \mathbb{Z}^{p+1}.$$

Let us also point out that, for each index  $i$ , the module  $(\Omega_P^{-1}(k) \otimes \Omega_{P/S_i}^1(k))^{\otimes 2}$  is not endotrivial (and hence it is decomposable). Indeed, it has dimension  $(p^3 - 1)^2 (p^2 - 1)^2$ , which is not congruent to  $\pm 1 \pmod{p^3}$ ,  $\forall p > 2$ .

Let us now consider  $T(G)$ . By Theorem 2.6, the torsion subgroup  $TT(G)$  is generated by the one-dimensional  $kG$ -modules, since  $TT(P)$  is trivial. That is,  $TT(G)$  is isomorphic to the character

group of the abelian  $p'$ -group  $G/P \cong C_{p-1} \times C_{p-1}$ . On the other hand, for  $TF(G)$ , we note that there are exactly two elementary abelian  $p$ -subgroups which are normal in  $G$ , and the  $p-1$  others form a unique  $G$ -conjugacy class. Say  $F_1, F_2 \triangleleft G$ , and  $F_3 =_G \cdots =_G F_{p+1}$ . Then, we may choose  $E_i = F_i$ ,  $1 \leq i \leq 3$ , and so we get  $G_1 = G_2 = G$  and  $G_3$  has index  $p-1$  in  $G$ , where  $G_i = PN_G(E_i) = N_G(E_i)$  for  $1 \leq i \leq 3$ . Indeed, by the Class Formula, the cardinality of the  $G$ -orbit of  $E_3$  is  $p-1$ , which is hence the index in  $G$  of the stabilizer  $PN_G(E_3)$  of the  $G$ -conjugacy class of  $E_3$ .

From Proposition 3.4 and Lemma 3.5, we get  $T(P)^{G/P} = \langle \Omega_P, [N_1], [N_2] \rangle \cong \mathbb{Z}^3$ . Note also that  $\sum_{i=3}^{p+1} [N_i] = -[N_1] - [N_2] \in T(P)^{G/P}$ .

Theorem 3.1 implies that the modules  $N_1$  and  $N_2$  extend to  $G$ , into indecomposable endotrivial modules  $\tilde{N}_1$  and  $\tilde{N}_2$ , respectively. Therefore,  $TF(G) = \langle \Omega_P, [\tilde{N}_1], [\tilde{N}_2] \rangle$ , by Theorem 3.10.

By Proposition 3.7, the module  $N_3$  extends to an indecomposable endotrivial  $kG_3$ -module  $\tilde{N}_3$ , yielding then an endotrivial  $kG$ -module  $\text{Ten}_{G_3}^G \tilde{N}_3$ . Moreover,  $[\text{Ten}_{G_3}^G \tilde{N}_3] + [\tilde{N}_1] + [\tilde{N}_2] \in TT(G)$ .

Lastly, we remark that in the presentation of  $TF(G)$  above, the underlying modules are indecomposable, which is not the case in general. But it needs to be pointed out that the modules  $N_i$  are proper direct summands of (endo-permutation) modules of dimension  $(p^3-1)^2(p-1)^2$ , and that at this point we do not know what  $\text{Dim}(N_i)$  is. The second half of this section will give us the answer.

We turn now to the presentation of  $T(P)$  and  $TF(G)$  described in Section 4, and for convenience, we adopt the same notation. In particular, since  $p$  is odd, we have  $a_i = 2p$ , for the integer  $a_i$  of Theorem 2.4. Hence, we can apply Proposition 4.2 to find generators for  $T(P)$ . Indeed, by [3, Theorem 10.1] (or [16, Theorems 6 and 7]), there exists  $\zeta \in H^{2p}(P, k)$  such that  $\zeta \downarrow_Z^P \neq 0$ . In fact,  $\zeta \in H^{2p}(P, \mathbb{F}_p)$  and  $\zeta$  is obtained as the norm from any  $F \in \mathcal{E}_{>2}(P)$  to  $P$  of the inflation of a non zero element  $\eta \in H^2(Z, \mathbb{F}_p)$ , and where  $Z$  is identified with a quotient of  $F$ .

Let  $\tilde{\zeta} : \Omega_P^{2p}(k) \rightarrow k$  represent  $\zeta$  and set  $L_\zeta = \ker \tilde{\zeta}$ . Then  $L_\zeta$  is the direct sum  $L_1 \oplus \cdots \oplus L_{p+1}$  of indecomposable  $kP$ -modules. Moreover, all the modules  $L_i$  have same dimension, since the automorphism group  $\text{Aut}(P)$  of  $P$  permutes transitively the subgroups  $F_1, \dots, F_{p+1}$  and hence  $\text{Aut}(P)$  also acts by permuting transitively the  $L_i$ 's.

Now, the endotrivial modules  $M_i$ , for  $1 \leq i \leq p+1$ , are obtained as the push-out

$$\begin{array}{ccc} L_\zeta & \longrightarrow & \Omega_P^{2p}(k) \\ \downarrow & & \downarrow \\ L_i & \dashrightarrow & M_i \end{array}$$

In addition, by [11, Corollary 4.4], we have  $\text{Dim}(\Omega_P^{2p}(k)) = p^3(p+1) + 1$ . Therefore, we get  $\text{Dim}(L_i) = p^3$ , and so  $\text{Dim}(M_i) = p^3 + 1$ . Note that this is the minimal possible dimension that we could expect for  $M_i$ . Indeed, it has to be greater than 1 and congruent to 1 (mod  $|P|$ ).

Last but not least, we handle  $TF(G)$ . Note that since  $H^{2p}(P, k)$  contains an element that restricts non trivially to  $Z$ , we can apply any of the two constructions presented in Section 4. We choose the ‘‘deconstruction method’’ (cf. Theorem 4.1).

We consider the  $kG/Z$ -module

$$U = \Omega_G^{2p}(k) / (z-1)^{p-1} \Omega_G^{2p}(k).$$

Note that the action of  $(z-1)^{p-1} = \sum_{i=0}^{p-1} z^i$  on a  $kZ$ -module  $M$  annihilates any non-free summand of  $M$ , and sends a free module onto its socle. By a dimension argument (cf. [11, Corollary 4.4]),

we deduce that  $\Omega_G^{2p}(k) \downarrow_Z^G \cong k \oplus (kZ)^{p^2(p+1)}$ , and so  $(z-1)^{p-1} \Omega_G^{2p}(k)$  has dimension  $p^2(p+1)$ . It follows that  $\text{Dim}(U) = p^2(p+1)(p-1) + 1$ .

In addition, in the notation of the discussion after Theorem 4.1, the variety  $V = V_{G/Z}(U)$  decomposes as the union  $V_1 \cup V_2 \cup V_3 \cup W$ , where  $W = V_1' \cap V_2' \cap V_3'$ . This is because  $\mathcal{E}_{\geq 2}(G)$  consists in three isolated vertices  $E_1, E_2, E_3$ . Then, each  $V_i$  is the variety of the  $kG/Z$ -module  $\bar{U}_i = U_i/(z-1)^{p-1}U_i$ , for an indecomposable endotrivial  $kG$ -module  $U_i$  satisfying

$$U_i \downarrow_{E_j}^G = \begin{cases} \Omega_{E_i}^{2p}(k) \oplus (\text{proj}) & \text{if } i = j \\ k \oplus (\text{proj}) & \text{if } i \neq j \end{cases}$$

Since the group  $\text{Aut}(G)$  of automorphisms of  $G$  permutes transitively  $E_1, E_2$  and  $E_3$ , the induced action on  $V$  permutes transitively  $V_1, V_2$  and  $V_3$ . Consequently, the modules  $\bar{U}_1, \bar{U}_2$  and  $\bar{U}_3$  have same dimension. Since the  $U_i$ 's are endotrivial, we have  $U_i \downarrow_Z^G = k \oplus (\text{proj})$ . It follows that  $\text{Dim}(U_i) = 1 + p \text{Dim}(\bar{U}_i)$ , for  $1 \leq i \leq 3$ , and thus the modules  $U_i$  have same dimension.

Now,  $E_i = F_i$ , for  $1 \leq i \leq 3$ , and  $E_3 =_G F_i$ ,  $\forall 3 \leq i \leq p+1$ . That is, the  $G$ -conjugacy classes of  $F_1$  and  $F_2$  coincide with their respective  $P$ -conjugacy classes. Therefore,  $U_1 \downarrow_P^G$  and  $U_2 \downarrow_P^G$  are isomorphic to the indecomposable endotrivial modules obtained using the techniques of Theorem 4.1 applied to  $P$  instead of  $G$ , and corresponding to  $F_1$  and  $F_2$  respectively. By Remark 4.7, we have  $U_i \downarrow_P^G \cong M_i$  for  $i = 1, 2$ . Thus,  $\text{Dim}(U_i) = p^3 + 1$  and  $\text{Dim}(\bar{U}_i) = p^2$ , for  $i = 1, 2$  and 3. Note also that the isomorphism  $U_3 \downarrow_P^G \oplus (\text{proj}) \cong \otimes_{i=3}^{p+1} M_i$  tells us that  $\text{Dim}((\text{proj})) = (p^3 + 1)^{p-1} - (p^3 + 1)$ . On the other hand, the variety  $W$  is the variety of a  $kG/Z$ -module of dimension  $\text{Dim}(U) - 3 \text{Dim}(\bar{U}_1) = p^2((p+1)(p-1) - 3) + 1$ .

We can now answer the question concerning the dimension of the modules  $\tilde{N}_1, \tilde{N}_2$  and  $\text{Ten}_{G_3}^G \tilde{N}_3$ . Indeed, by duality and indecomposability, we have  $\text{Dim}(\tilde{N}_i) = p^3 + 1$  for  $i = 1, 2$ , whereas  $\text{Dim}(\text{Ten}_{G_3}^G \tilde{N}_3) = (p^3 + 1)^{p-1}$ .

In particular,  $\text{Ten}_{G_3}^G \tilde{N}_3$  is not indecomposable, since

$$\text{Dim}(M_3) = (p-1)p^3 + 1 < (p^3 + 1)^{p-1} = \text{Dim}(\text{Ten}_{G_3}^G \tilde{N}_3), \forall p > 2.$$

More precisely, we have  $\text{Ten}_{G_3}^G \tilde{N}_3 \cong M_3^* \oplus (\text{proj})$ , where

$$\text{Dim}((\text{proj})) = \text{Dim}(\text{Ten}_{G_3}^G \tilde{N}_3) - \text{Dim}(M_3) = \sum_{i=2}^{p-1} \binom{p-1}{i} p^3 i.$$

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## REFERENCES

- [1] J. L. Alperin, *A construction of endo-permutation modules*, J. Group Theory **4** (2001), 3–10.
- [2] D. Benson, J. Carlson, *Periodic modules with large period*, Quart. J. Math. **43** (1992), 283–296.
- [3] D. Benson, J. Carlson, *The cohomology of extraspecial groups*, Bull. London Math. Soc. **24** (1992), no. 3, 209–235.
- [4] S. Bouc, *Tensor induction of relative syzygies*, J. Reine Angew. Math., **523** (2003), 113–171.
- [5] S. Bouc, N. Mazza, *The Dade group of (almost) extraspecial  $p$ -groups*, Journal of Pure and Applied Algebra, **192** (2004), 21–51.

- [6] S. Bouc, J. Thévenaz, *The group of endo-permutation modules*, Invent. math. **139** (2000), 275–349.
- [7] J. Carlson, *Modules and Group Algebras*, Birkhäuser Verlag, Basel, 1996.
- [8] J. Carlson, *Constructing endotrivial modules*, Journal of Pure and Applied Algebra, to appear.
- [9] J. Carlson, N. Mazza, D. Nakano, *Endotrivial modules for finite groups of Lie type*, J. Reine Angew. Math., to appear.
- [10] J. Carlson, J. Thévenaz, *Torsion endo-trivial modules*, Algebras and representation Theory, **3** (2000), 303–335.
- [11] J. Carlson, J. Thévenaz, *The classification of torsion endo-trivial modules*, Ann. Math **162** (2005), no. 2, 823–883.
- [12] J. Carlson, J. Thévenaz, *The classification of endo-trivial modules*, Invent. Math. **158** (2004), no. 2, 389–411.
- [13] E. C. Dade, *Endo-permutation modules over  $p$ -groups, I, II*, Ann. Math. **107** (1978), 459–494, **108** (1978), 317–346.
- [14] E. C. Dade, *Extending endo-permutation modules*, preprint (unpublished), 1982.
- [15] L. Evens, *The Cohomology of Groups*, Oxford Science Publications, New York, 1991.
- [16] I. J. Leary, *The mod- $p$  cohomology rings of some  $p$ -groups*, Math. Proc. Cambridge Philos. Soc. **112** (1992), no. 1, 63–75.

NADIA MAZZA, DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF ABERDEEN, ABERDEEN, AB24 3UE, SCOTLAND UK