The Dade group of (almost) extraspecial p-groups

Serge Bouc and Nadia Mazza

Abstract: In this paper, we determine a presentation by explicit generators and relations for the Dade group of all (almost) extraspecial p-groups. The proof of the main result uses the cohomological properties of the Tits building corresponding to the natural geometric structure of the lattice of subgroups of such p-groups.

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1. Introduction

Let p be a prime number, and k be a field of characteristic p. If P is a finite p-group, a finitely generated kP-module M is called an *endo-permutation* module if $\mathrm{End}_k(M)$ is a permutation module, i.e. admits a P-invariant k-basis. This notion was introduced at the end of the '70s, as a generalization of the notion of *endo-trivial*, or *invertible* module, and it is now considered as a crucial tool in many aspects of the modular representation theory of finite groups.

In [8], E.C. Dade considered an equivalence relation on the set of all endo-permutation modules whose endomorphism algebra is a permutation kP-module having a trivial direct summand. This allowed him to define a group structure on the set of these equivalence classes, giving rise to the group $D_k(P)$ that we now call the Dade group of P. Hence, the classification of endo-permutation kP-modules comes down to the computation of $D_k(P)$. Moreover, in these papers, Dade determined the structure of $D_k(P)$ when P is abelian.

Shortly afterwards, L. Puig ([14]) proved that for an arbitrary finite p-group P, the group $D_k(P)$ is a finitely generated (abelian) group. This paper only appeared in 1990, and there was no major progress on the structure of the Dade group until J. L. Alperin ([1]), using relative syzygies of the trivial module, determined the torsion-free rank of the subgroup $T_k(P)$ of $D_k(P)$ consisting of the classes of endo-trivial modules. Next (but this was published before Alperin's result), J. Thévenaz and the first author ([4]) determined the torsion free rank of $D_k(P)$, using tensor induction to build endo-permutation modules.

Since that time, J. Carlson and J. Thévenaz ([5], [6], [7]) completed the hard task of finding the structure of $T_k(P)$, using deep cohomological methods. They also determined the structure of $D_k(P)$ when P is a dihedral, semi-dihedral, or generalized quaternion 2-group.

Following that, in her thesis ([13]), the second author determined the structure of $D_k(P)$, when P is a metacyclic p-group for an odd prime p ([12]), or when P is an extraspecial 2-group of a particular type.

In this paper, we will extend this latter result to the case of all extraspecial p-groups and almost extraspecial p-groups, by giving a presentation of the Dade group of these groups by explicit generators and relations. The reason for considering those particular groups is twofold : firstly, our method makes an essential use of the geometric structure associated to (almost) extraspecial p-groups, more precisely the cohomological

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properties of the corresponding Tits building. We proceed by induction, referring to [7] for some of the initial small cases. Secondly, (almost) extraspecial groups appear as a cornerstone in many problems related to p-groups, and our hope is that our results can be used to solve the general problem of determining $D_k(P)$ for an arbitrary p-group, although we have no clear idea at the moment of how this could be achieved.

The paper is organized as follows : first we recall the definition and basic properties of (almost) extraspecial p-groups. In Section 3, we recall some notation and results on the Dade group. In Section 4, we state some combinatorial results on the geometry of (almost) extraspecial p-groups. Section 5 is devoted to the proof of a particular relation between relative syzygies of the trivial module that holds for this class of groups. In section 6, we state a cohomological property of a particular subposet $\mathcal{E}$ of the poset of subgroups of P (and that is where the Tits building mentioned in the abstract appears). In section 7 we describe three coefficient systems related to the poset $\mathcal{E}$, and in the next section, we let them fit into an exact sequence. Sections 9 and 10 give the inductive step of our proof, whereas Section 11 handles the initial steps of induction.

2. (Almost) extraspecial p-groups

In this section, we are going to state some useful facts concerning the structure of (almost) extraspecial p-groups. Let us start by fixing the notation we consider throughout. Let G be a finite group. Denote by $Z(G)$ the center of G, by G' the commutator subgroup, i.e. the subgroup of G generated by the commutators $[x, y]$ for $x, y \in G$, where $[x, y] = xyx^{-1}y^{-1}$, $\forall\, x, y \in G$, and write $\Phi(G)$ for the Frattini subgroup of G. If H, K and L are subgroups of G, we write $H \leq K$, respectively $H \leq_L K$, if H is contained in K, respectively if H is contained in a L-conjugate of K. If $x, y \in G$, we set ${}^xy = xyx^{-1}$ and $y^x = x^{-1}yx$. For our purpose, it is also convenient to recall the definition of the central product of groups (see 2.5.3 [10]).

Definition 2.1 *Let H, K and M be groups such that $M \leq Z(H)$ and such that there exists an injective map $\theta : M \to Z(K)$. If we identify M with $\theta(M)$, then there exists a group G such that $G = HK$ and such that H centralizes K. Such a group is called the* central product *of H and K and we write $G = H * K$. If H is isomorphic to K and $M = Z(H)$, we simply write $G = H^{*2}$. More generally, for all integer $r \geq 0$, we write $G = H^{*r}$ for the central product of r copies of the group H, where $M = Z(H)$, and with the convention that $H^{*0} = Z(H)$, and that $H^{*1} = H$.*

We can now state the main definition of this section.

Definition 2.2 *Let p be a prime number and P be a finite p-group.*

- *We say that P is* extraspecial *if $Z(P) = P' = \Phi(P)$ has order p.*
- *We say that P is* almost extraspecial *if $P' = \Phi(P)$ has order p, and if $Z(P)$ is cyclic of order p^2.*

Hence, if P is an almost extraspecial p-group, then there exists a subgroup Q of index p of P which does not contain the center of P. Then, we can easily verify that Q is extraspecial, and that P is the central product of Q and $Z(P)$. Conversely, if P is an extraspecial p-group, then $P * C_{p^2}$ is almost extraspecial. In particular, if $p = 2$, then the dihedral group D_8 of order 8 and the quaternion group Q_8 are extraspecial 2-groups. While the central product $D_8 * C_4$, which is isomorphic to the group $Q_8 * C_4$, is almost extraspecial of order 16 (where C_n denotes the cyclic group of order n, for all integer n). In order to state the classification theorem of (almost) extraspecial p-groups, it will be convenient to introduce the following notation. For $p \neq 2$, we set

$$M = \langle x, y \mid x^p = y^p = 1,\ {}^y x = x[y,x],\ {}^y[y,x] = [y,x] \rangle \quad \text{and}$$

$$N = \langle x, y \mid x^{p^2} = y^p = 1,\ {}^y x = x^{1+p} \rangle.$$

Theorem 2.3 *Let p be a prime number and P be an (almost) extraspecial p-group. Then, the following holds:*

- *If P is extraspecial, then there exists an integer $r \geq 1$ such that P has order p^{2r+1} and P is isomorphic to one of the following non-isomorphic groups:*
  $$\left.\begin{array}{l} - D_8^{*r} \\ - Q_8 * D_8^{*(r-1)} \end{array}\right\} \text{ if } p = 2;$$
  $$\left.\begin{array}{l} - M^{*r} \\ - N * M^{*(r-1)} \end{array}\right\} \text{ if } p \neq 2.$$
- *If P is almost extraspecial, then there exists an integer $r \geq 1$ such that P has order p^{2r+2}, and P is unique, up to isomorphism. More precisely, P is isomorphic to*
  - $D_8^{*r} * C_4$, *if* $p = 2$;
  - $M^{*r} * C_{p^2}$, *if* $p \neq 2$.

The part concerning extraspecial p-groups is a classical result and a proof of this can be found in section 5.5 of [10], or also in section III.13 of [11]. For almost extraspecial p-groups, one can refer to sections 2 and 4 of [5]. In particular, for each prime number p, we have exactly 3 isomorphism types of (almost) extraspecial p-groups. First of all, notice that we distinguish between almost extraspecial and extraspecial p-groups by looking at the parity of the exponent of p in $|P|$. Then, if p is odd, the exponent of the group characterizes the isomorphism type of extraspecial p-groups. These remarks lead us to introduce the following notation:

Definition 2.4 *Let p be a prime number, P be an (almost) extraspecial p-group and r be the integer defined as in the above theorem. We say that P has* type*:*

- *T*1 *if $P \cong D_8^{*r}$, or if $P \cong M^{*r}$;*
- *T*2 *if $P \cong Q_8 * D_8^{*(r-1)}$, or if $P \cong N * M^{*(r-1)}$;*
- *T*3 *if P is almost extraspecial.*

One of the most remarkable properties of (almost) extraspecial p-groups lies in the geometrical structure that these groups carry. Indeed, let P be an (almost) extraspecial p-group, let Φ be its Frattini subgroup, generated by an element z of order p, and set E for the quotient P/Φ, and π for the canonical surjection $\pi : P \to E$. Then, since $\Phi = \Phi(P)$, the finite p-group E is elementary abelian, and thus, it is also a finite dimensional $\mathbb{F}_p$-vector space. Moreover, this vector space is endowed with the following two forms.

- $$b : E \times E \longrightarrow \mathbb{F}_p \ , \ (u,v) \longmapsto b(u,v) \ ,$$
  where $b(u,v)$ is the element of $\mathbb{F}_p$ satisfying $[\tilde{u}, \tilde{v}] = z^{b(u,v)}$, $\forall\, \tilde{u} \in u$, $\forall\, \tilde{v} \in v$, and $\forall\, u, v \in E$.

- $$q : E \longrightarrow \mathbb{F}_p \ , \ u \longmapsto q(u) \ ,$$

  where $q(u)$ is the element of $\mathbb{F}_p$ satisfying $\tilde{u}^p = z^{q(u)}$, $\forall\, \tilde{u} \in u$, and $\forall\, u \in E$.

Section 20 in [9] is concerned with extraspecial p-groups. However, their results generalize easily to (almost) extraspecial, and we get the following statements (expressed in the terminology used in chapter 7 in [15]).

- If $p = 2$, then q is a quadratic form, and b is the corresponding symmetric bilinear form, i.e. we have $q(u+v) = q(u) + q(v) + b(u,v)$, $\forall\, u, v \in E$. Moreover, the quadratic form q is non-degenerate, since $E^{\perp} \cap q^{-1}(0) = \{0\}$.
- If p is odd, then b is a symplectic form on E, and q is a linear form on E. Moreover, b is non-degenerate if and only if P is extraspecial. If P is almost extraspecial, then the radical $E^{\perp} = \pi(Z(P))$ is a line in E. Here again we have $E^{\perp} \cap q^{-1}(0) = \{0\}$.

Recall that, if $p = 2$, a subspace V of E is said *totally singular* if $q(V) = \{0\}$, and, if p is odd, we say that a subspace V of E is *totally isotropic* if $V \leq V^{\perp}$, i.e. if $b(u,v) = 0$, $\forall\, u, v \in V$.

Notation 2.5 *We set*

$$\mathcal{E} = \{V \leq E \mid \{0\} \neq V \leq V^{\perp} \ , q(V) = \{0\}\} \ , \ \textit{and}\ \underline{\mathcal{E}} = \mathcal{E} \cup \{0\} \ .$$

We denote by $\mathcal{Q}$ be the set of non-trivial subgroups Q of P such that $Q \cap \Phi = \mathbf{1}$, and $\underline{\mathcal{Q}} = \mathcal{Q} \cup \{\mathbf{1}\}$, where $\mathbf{1}$ denotes the trivial group. Set also $\mathcal{Q}_P$ and $\underline{\mathcal{Q}}_P$ for the respective sets of conjugacy classes of subgroups in P.

If $V \in \underline{\mathcal{E}}$, we set

$$N(V) = \pi^{-1}(V^{\perp}) \qquad \textit{and} \qquad m(V) = \{Q \in \underline{\mathcal{Q}} \mid \pi(Q) = V\} \quad .$$

Note that $\underline{\mathcal{E}}$ is a poset for the inclusion of subspaces, and that $\underline{\mathcal{Q}}$ is preordered by the relation $\leq_P$ of inclusion up to conjugacy in P. The associated poset is the set $\underline{\mathcal{Q}}_P$, for the order given by $C \leq D$ for $C, D \in \underline{\mathcal{Q}}_P$ if there exist subgroups $Q \in C$ and $R \in D$, such that $Q \leq R$. The link between the posets $\underline{\mathcal{E}}$ and $\underline{\mathcal{Q}}_P$ is a direct consequence of the following lemma.

Lemma 2.6 *Let P be an (almost) extraspecial p-group, and let Q be a non-trivial subgroup of P. Then,*

1. *$Q \lhd P \iff \Phi \leq Q$.*
2. *$Q \ntriangleleft P \implies N_P(Q) = C_P(Q)$. In particular, it follows that in this case Q is elementary abelian of rank at most r, for the integer r defined as in Theorem 2.3, and we have $|Q|{\cdot}|C_P(Q)| = |P|$. Moreover, $C_P(Q) = Q \times U$, where U is (almost) extraspecial of order $\frac{|P|}{|Q|^2}$, or $U = Z(P)$.*
3. *Let Q, $R \ntriangleleft P$. Then,*

   $$Q <_P R \iff Q <_P R < N_P(R) < N_P(Q) \ , \quad \textit{and} \quad Q =_P R \iff Q\Phi = R\Phi \ .$$

   Moreover, if $Q \leq_P R$, then there exists a unique $u \in [P/N_P(Q)]$ such that ${}^uQ \leq R$.

Proof: Note that $\forall\, x, y \in P$, we have ${}^y x = x$, or $\Phi = <[x,y]>$.

1. If $\mathbf{1} < Q \leq Z(P)$, then the assertion holds. Assume $\mathbf{1} < Q \not\leq Z(P)$ and $Q \triangleleft P$. Then, there exist $x \in Q$ and $y \in P$ such that $\Phi = <[x,y]>$. Since ${}^y x \in Q$, we have $[x,y] \in Q$ and hence $\Phi \leq Q$. The converse is obvious, since the quotient P/Φ is abelian.
2. If $Q \ntriangleleft P$, then Q is elementary abelian, since $\Phi(Q) \leq Q \cap \Phi = \mathbf{1}$, by the previous assertion. Moreover, the inclusion $[Q, N_P(Q)] \leq Q \cap \Phi = \mathbf{1}$ implies the equality $N_P(Q) = C_P(Q)$. Set $I = \pi(Q)$. Then $C_P(Q) = \pi^{-1}(I^{\perp})$, and $I \cap E^{\perp} = \{0\}$. Hence we have $|E| = |I| \cdot |I^{\perp}|$, and it follows that $|P| = |Q| \cdot |C_P(Q)|$. The subspace $\pi(Q)$ of E is totally isotropic if p is odd, respectively totally singular if $p = 2$. Thus, its dimension is at most equal to the Witt index r of the form b, since $\dim(E) = 2r$ and $\dim(E^{\perp}) \leq 1$ (see section 7 in [15]). Set $G = C_P(Q)/Q$. Then $\Phi(G) \leq Q\Phi/Q$ has order 1 or p. In the first case, $C_P(Q)$ is abelian, since $[G,G] = \mathbf{1}$ implies $[C_P(Q), C_P(Q)] \leq Q \cap \Phi = \mathbf{1}$. Hence, $C_P(Q) \leq C_P C_P(Q) = Q \cdot Z(P)$, and so $C_P(Q) = Q \cdot Z(P)$. In the second case, $\Phi(G)$ has order p. Let H be the subgroup of P containing Q and such that $H/Q = Z(G)$. Since $[H/Q, G] = \mathbf{1}$, we have $[H, C_P(Q)] \leq Q \cap \Phi = \mathbf{1}$, and hence $H \leq C_P C_P(Q) = Q \cdot Z(P)$. Thus $H = Q \cdot Z(P)$, and it follows that H/Q is isomorphic to $Z(P)$. This shows that the p-group G is (almost) extraspecial. Notice now that the p-group $C_P(Q)/\Phi$ is elementary abelian, and contains $Q \cdot \Phi/\Phi$. Therefore, there exists $U \leq P$, such that $\pi(C_P(Q) = \pi(Q) \oplus \pi(U)$. It follows that
$$C_P(Q) = Q \cdot \Phi \cdot U = Q \cdot U \ , \quad \text{and that} \quad Q \cap U \leq Q \cap \Phi = \mathbf{1} \ ,$$
since $Q \cdot \Phi \cap U = \Phi$. In other words, $C_P(Q)$ is the semi-direct product of U by Q, and in fact this product is direct, since Q is a central subgroup of $C_P(Q)$.
3. The previous assertion implies $N_P(Q) = C_P(Q)$, $\forall\, Q \ntriangleleft P$. Hence, if $Q, R \ntriangleleft P$, then the assertion $Q <_P R \iff Q <_P R < N_P(R) < N_P(Q)$ is immediate. Set $V = \pi(Q)$. Then $Q \in m(V)$, and $m(V)$ is the set of complements of Φ in the elementary abelian group $Q \cdot \Phi$. Moreover, the group P acts transitively (by conjugation) on $m(V)$. Indeed, since $Q \cap \Phi = \mathbf{1}$, we have that $|P : N_P(Q)| = |Q|$ is equal to the number of complements of Φ in $Q \cdot \Phi$. In other words, two subgroups Q and R of P are conjugate if, and only if $Q \cdot \Phi = R \cdot \Phi$, as was to be shown. Finally, let $Q \leq_P R$, where $Q, R \ntriangleleft P$. Then $[P/N_P(Q)]$ acts transitively on the set of all conjugates of Q, and so there exists $x \in [P/N_P(Q)]$ such that $Q^x \leq R$. Let $y \in [P/N_P(Q)]$ such that $Q^y \leq R$. Then, $Q^x \cdot Q^y \leq R$, since R is abelian. Now, if $Q^x \neq Q^y$, then $\Phi \leq Q^x \cdot Q^y$. Hence $Q^x = Q^y$, since $\Phi \not\leq R$.

□

Corollary 2.7 *The canonical surjection $\pi : P \to E$ induces an isomorphism of posets $\pi : \underline{\mathcal{Q}}_P \to \underline{\mathcal{E}}$, which restricts to an isomorphism of posets $\pi : \mathcal{Q}_P \to \mathcal{E}$. That is, we have $\forall\, C, D \in \underline{\mathcal{Q}}_P$*
$$C \leq D \text{ in } \underline{\mathcal{Q}}_P \iff Q\Phi \leq R\Phi \ , \ \forall\, Q \in C \ , \ \forall\, R \in D.$$
Moreover, we have $\pi\big(N_P(Q)\big) = \big(\pi(Q)\big)^{\perp}$, $\forall\, Q \in C$, and $\forall\, C \in \underline{\mathcal{Q}}_P$.

This fact has the nice property that it will allow us later to translate the traditional operations of restriction-deflation on the Dade groups of (almost) extraspecial p-groups into a geometrical language.

3. The Dade group

In this section, we consider all notations of the previous section, and we want to recall some essential facts concerning the Dade group of a finite p-group. Then, we will apply them to the particular case of an (almost) extraspecial p-group, in order to get a first reduction of the question we are concerned with in this paper, that is, the classification of capped endo-permutation modules (up to equivalence) in the case of an (almost) extraspecial p-group.

Fix a prime number p and a field k of characteristic p. The Dade group $D(P) = D_k(P)$ of a finite p-group P is the finitely generated abelian group formed by the equivalence classes of capped endo-permutation kP-modules, with composition law induced by tensor product (over k) of the modules. As usual, we let $T(P)$ be the subgroup of $D(P)$ generated by the classes of endo-trivial modules, and we write $D^t(P)$, resp. $T^t(P)$, for the torsion subgroup of $D(P)$, resp. $T(P)$. We will also denote by $D^\Omega(P)$, resp. $T^\Omega(P)$, the subgroups of $D(P)$, resp. $T(P)$, generated by the classes of the relative syzygies of the trivial module k. Recall that if X is a non empty finite P-set, then the relative syzygy $\Omega^1_X(k)$ of the trivial module k, relatively to X, is the kernel of the k-linear map $kX \longrightarrow k$, sending each $x \in X$ to 1. We will then just write Ω_X for the class of $\Omega^1_X(k)$ in $D(P)$. We assume that the reader is familiar with the usual operations of restriction, tensor induction, inflation, and deflation acting on the Dade group.

Notation 3.1 *Let P be a finite p-group. We denote by:*

- $\mathrm{nc}(P)$ *the number of conjugacy classes of non-cyclic subgroups of P, and*
- $\mathrm{c}'(P)$ *the number of conjugacy classes of cyclic subgroups of order at least 3 of P.*

Next lemma collects the known results that we will need, and we let the reader refer to [1], [3], [4], [5], [6], [8] and [14] for the proofs.

Lemma 3.2 *1. ([8]) If P is abelian, then*

$$T(P) \cong \begin{cases} \mathbf{1} & \text{if } |P| \le 2 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } P \text{ is cyclic and } |P| \ge 3 \\ \mathbb{Z} & \text{otherwise .} \end{cases}$$

Moreover, in that case, we have

$$D(P) = D^\Omega(P) \cong \bigoplus_{\mathbf{1} \le Q < P} T(P/Q) \cong \mathbb{Z}^{\mathrm{nc}(P)} \oplus (\mathbb{Z}/2\mathbb{Z})^{\mathrm{c}'(P)}.$$

2. ([14]) The subgroups $T(P)$ and $\bigcap\limits_{\mathbf{1} < Q \le P} \mathrm{Ker}\big(\mathrm{Defres}^P_{N_P(Q)/Q}\big)$ of $D(P)$ are equal.

3. ([6]) If P is not cyclic, quaternion or semi-dihedral, then $T(P)$ is detected on restriction to all elementary abelian subgroups of rank 2. In particular, in that case, the torsion subgroup $T^t(P)$ is trivial.

4. ([1],[4]) If n is the number of conjugacy classes of maximal elementary abelian subgroups of rank 2, then the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} T(P)$ is :

(a) 0 if P is cyclic,

(b) n if P is not cyclic, and if P has no elementary abelian subgroup of rank 3,

(c) $n+1$ *otherwise.*

Moreover, if any maximal elementary abelian subgroup of P has rank at least 3, then $T(P) = <\Omega_P> \cong \mathbb{Z}$.

5. *([4]) Let $\mathcal{S}$ be the set of non-cyclic subgroups of P, and for every $Q \in \mathcal{S}$ choose a normal subgroup $Q_0 \lhd Q$ such that Q/Q_0 is elementary abelian of rank 2. Then, the $\mathbb{Q}$-linear map*

$$\prod_{Q\in[\mathcal{S}/P]} \mathrm{Id}\otimes \mathrm{Defres}^P_{Q/Q_0} : \mathbb{Q}\otimes_{\mathbb{Z}} D(P) \longrightarrow \prod_{Q\in[\mathcal{S}/P]} \mathbb{Q}\otimes_{\mathbb{Z}} D(Q/Q_0)$$

is an isomorphism.

6. *([3]) Let $\mathcal{A}$ denote the set of all elementary abelian sections of P, and set*

$$K(P) = \bigcap_{A\in\mathcal{A}} \mathrm{Ker}\Big(\mathrm{Defres}^P_A\Big).$$

Then $K(P)$ is a subgroup of $D^t(P)$, and the quotient

$$D(P)/\big(D^\Omega(P)+K(P)\big) \text{ is a finite group with exponent dividing } |P|.$$

7. *([5], [6]) If p is odd, let $\mathcal{X}$ denote the set of all subgroups Q of P such that the quotient $N_P(Q)/Q$ is cyclic. If $p=2$, let $\mathcal{X}$ denote the set of all subgroups Q of P such that the quotient $N_P(Q)/Q$ is cyclic of order at least 4, or generalized quaternion, or semi-dihedral. Then the map*

$$\prod_{Q\in\mathcal{X}} \big(\rho_{N_P(Q)/Q}\circ \mathrm{Defres}^P_{N_P(Q)/Q}\big) \; : D^t(P) \longrightarrow \prod_{Q\in\mathcal{X}} T^t(N_P(Q)/Q)$$

is injective, where $\rho_H : D^t(H) \longrightarrow T^t(H)$ is the canonical surjection (that is well defined for all subgroups H that we consider here). More precisely, if p is odd, and $[\mathcal{C}/P]$ is a set of representatives of conjugacy classes of all cyclic non-trivial subgroups of P. Then, the map

$$\prod_{C\in[\mathcal{C}/P]} \mathrm{Defres}^P_{C/\Phi(C)} : D^t(P) \longrightarrow \prod_{C\in[\mathcal{C}/P]} D(C/\Phi(C)) \ ,$$

is an isomorphism, and the set $\{\mathrm{Teninf}^P_{C/\Phi(C)}\Omega_{C/\Phi(C)} \mid C\in[\mathcal{C}/P]\}$ is an $\mathbb{F}_2$-basis of $D^t(P)$. In particular, the torsion subgroup $D^t(P)$ is contained in $D^\Omega(P)$.

Consider now an (almost) extraspecial p-group P, and let $\pi : P \to E$ be the canonical surjection, where $E = P/\Phi(P)$, as defined in the previous section.

Notation 3.3 *We set:*

1. $\partial D(P) = \mathrm{Ker}(\mathrm{Def}^P_E)$.
2. $\partial D^\Omega(P) = \partial D(P)\cap D^\Omega(P)$.

Proposition 3.4 *Let P be an (almost) extraspecial p-group. Then,*

1. $T(P) \le \partial D(P)$.
2. $D(P) = \mathrm{Inf}^P_E\big(D(E)\big)\oplus \partial D(P)$*, and* $D(P) = D^\Omega(P)+\partial D(P)$.

3. *The groups $D(P)/D^{\Omega}(P)$ and $\partial D(P)/\partial D^{\Omega}(P)$ are isomorphic.*

Proof:

1. We have $T(P) = \bigcap_{\mathbf{1}<Q\leq P} \mathrm{Ker}\big(\mathrm{Defres}^P_{N_P(Q)/Q}\big) =$

$$= \mathrm{Ker}(\mathrm{Def}^P_E) \cap \Bigg(\bigcap_{\substack{V\in\mathcal{E}\\ \tilde{V}\in m(V)}} \mathrm{Ker}(\mathrm{Defres}^P_{N(V)/\tilde{V}}) \Bigg) \leq \partial D(P).$$

2. The inflation Inf^P_E is a section of the surjective map $\mathrm{Def}^P_E : D(P) \longrightarrow D(E)$, whose kernel is the subgroup $\partial D(P)$, by definition. Thus $D(P) = \mathrm{Inf}^P_E\big(D(E)\big) \oplus \partial D(P)$. Since E is abelian, we have $D(E) = D^{\Omega}(E)$, and thus it can be identified to a subgroup of $D^{\Omega}(P)$, by inflation.
3. From 2., we have $D(P)/D^{\Omega}(P) \cong \partial D(P)/\big(\partial D(P) \cap D^{\Omega}(P)\big)$.

□

4. Some combinatorial properties

Recall that P is an (almost) extraspecial p-group with Frattini subgroup Φ, and that $\underline{\mathcal{Q}}$ is the set of subgroups Q of P such that $Q \cap \Phi = \mathbf{1}$. The elements of $\underline{\mathcal{Q}}$ are elementary abelian.

Lemma 4.1 *1. Let Q and R be subgroups of P. Then*

$$C_P\big(Q \cap C_P(R)\big) = C_P(Q){\cdot}R \quad .$$

2. If moreover R is abelian, then

$$C_P\Big(R{\cdot}\big(Q \cap C_P(R)\big)\Big) = R{\cdot}\big(C_P(Q) \cap C_P(R)\big) \quad .$$

3. Let Q and R be subgroups of P such that $Q \cap C_P(R) \leq R \leq C_P(R)$. Then

$$C_P(R)C_P(Q) = C_P\big(Q \cap C_P(R)\big) = C_P(Q \cap R) \quad ,$$

and

$$C_P(R) = \big(C_P(R) \cap C_P(Q)\big){\cdot}R \quad .$$

4. If Q and R are elements of $\underline{\mathcal{Q}}$ such that $Q \cap C_P(R) \leq R$, then $Q{\cdot}\big(C_P(Q) \cap R\big) \in \underline{\mathcal{Q}}$, and

$$|Q{\cdot}\big(C_P(Q) \cap R\big)| = |R| \quad .$$

Proof: For any subgroup R of P, one has that $C_PC_P(R) = R.Z$, where $Z = Z(P)$. Moreover $C_P(R) \geq Z$, hence $C_P(R) \trianglelefteq P$. Now

$$\begin{aligned} C_PC_P\big(Q \cap C_P(R)\big) &= (Q \cap C_P(R){\cdot}Z = Q{\cdot}Z \cap C_P(R) \\ &= C_PC_P(Q) \cap C_P(R) \\ &= C_P\big(C_P(Q){\cdot}R\big) \end{aligned}$$

Taking centralizers once more gives

$$\begin{aligned} C_PC_PC_P\big(Q\cap C_P(R)\big) &= C_PC_P\big(C_P(Q)\cdot R\big) \\ C_P\big(Q\cap C_P(R)\big)\cdot Z &= \big(C_P(Q)\cdot R\big)\cdot Z \end{aligned}$$

and Assertion 1 follows, since $Z\leq C_P\big(Q\cap C_P(R)\big)$ and $Z\leq C_P(Q)\cdot R$. Now

$$\begin{aligned} C_P\Big(R\cdot\big(Q\cap C_P(R)\big)\Big) &= C_P(R)\cap C_P\big(Q\cap C_P(R)\big) \\ &= C_P(R)\cap C_P(Q)\cdot R \\ &= \big(C_P(R)\cap C_P(Q)\big)\cdot R \quad , \end{aligned}$$

since $R\leq C_P(R)$, proving Assertion 2.

For Assertion 3, the inclusion $C_P(Q)\cdot C_P(R)\leq C_P(Q\cap R)$ is obvious. Conversely the assumption implies $Q\cap C_P(R)\leq Q\cap R\leq Q\cap C_P(R)$, hence $Q\cap R=Q\cap C_P(R)$. Thus $C_P(Q\cap R)=C_P(Q)\cdot R\leq C_P(Q)\cdot C_P(R)$ by Assertion 1. In particular $C_P(Q)\cdot C_P(R)=C_P(Q)\cdot R$, thus $R\leq C_P(R)\leq C_P(Q)\cdot R$, hence $C_P(R)=\big(C_P(R)\cap C_P(Q)\big)\cdot R$.

For Assertion 4, let $z\in Q\cdot\big(C_P(Q)\cap R\big)\cap\Phi$. Then $z=qr$, for elements $q\in Q$ and $r\in C_P(Q)\cap R$. Thus $q=zr^{-1}\in Q\cap C_P(R)$, because $R\leq C_P(R)$, and $q\in Q\cap R$. It follows that $z=qr\in\Phi\cap R=\mathbf{1}$, showing that $Q\cdot\big(C_P(Q)\cap R\big)\cap\Phi=\mathbf{1}$. Now

$$\begin{aligned} |Q\cdot\big(C_P(Q)\cap R\big)| &= \frac{|Q||C_P(Q)\cap R|}{|Q\cap R|} \\ &= \frac{|Q|}{|Q\cap R|}\,\frac{|P|}{|C_P\big(C_P(Q)\cap R\big)|} \quad , \end{aligned}$$

since $C_P(Q)\cap R\in\underline{\mathcal{Q}}$, and since $|S||C_P(S)|=|P|$ for any $S\in\underline{\mathcal{Q}}$. Hence

$$\begin{aligned} |Q\cdot\big(C_P(Q)\cap R\big)| &= \frac{|Q|}{|Q\cap R|}\,\frac{|P|}{|Q\cdot C_P(R)|} \\ &= \frac{|Q|}{|Q\cap R|}\,\frac{|P||Q\cap C_P(R)|}{|Q||C_P(R)|} \\ &= \frac{|P||Q\cap C_P(R)|}{|Q\cap R||C_P(R)|} \\ &= \frac{|R||Q\cap C_P(R)|}{|Q\cap R|}=|R| \quad , \end{aligned}$$

since $Q\cap C_P(R)=Q\cap R$. □

Corollary 4.2 *Let $Q\in\underline{\mathcal{Q}}$, and let Q_0 be any maximal element of $\underline{\mathcal{Q}}$. Then :*

1. *There exists a unique double coset $C_P(Q)xC_P(Q_0)$ of elements $y\in P$ such that $Q^y\cap C_P(Q_0)\leq Q_0$.*
2. *The cardinality of the set $Q\backslash C_P(Q)xC_P(Q_0)/C_P(Q_0)$ is equal to $|Q_0|/|Q|$.*
3. *The group $Q^x\cdot\big(C_P(Q^x)\cap Q_0\big)$ is a maximal element of $\underline{\mathcal{Q}}$.*

Proof: Since Q_0 is a maximal element of $\underline{\mathcal{Q}}$, it is contained in a unique maximal elementary abelian subgroup $Q_0{\cdot}\Phi$ of P, and it follows that $\big(Q\cap C_P(Q_0)\big){\cdot}Q_0 \leq Q_0{\cdot}\Phi$. Set $R = Q\cap C_P(Q_0)$. There exists a maximal element Q_1 of $\underline{\mathcal{Q}}$ containing R and contained in $Q_0{\cdot}\Phi$. Since $Q_1{\cdot}\Phi = Q_0{\cdot}\Phi$, it follows $Q_1 = {}^xQ_0$ for some $x\in P$. Hence $R^x = Q^x\cap C_P(Q_0)\leq Q_0$. Clearly if $y\in C_P(Q)xC_P(Q_0)$, then $R^y = Q^y\cap C_P(Q_0)\leq Q_0$.

Conversely, if y is any element of P such that $R^y\leq Q_0$, then R is contained in ${}^xQ_0$ and ${}^yQ_0$. Hence there exists $z\in C_P(R)$ such that ${}^yQ_0 = {}^{zx}Q_0$. Thus $x^{-1}z^{-1}y\in C_P(Q_0)$, and $y\in C_P(R)xC_P(Q_0)$. But

$$C_P(R)x = xC_P(R^x) = xC_P(Q^x)C_P(Q_0)\quad ,$$

since $Q^x\cap C_P(Q_0)\leq Q_0$. Hence $C_P(R)xC_P(Q_0) = C_P(Q)xC_P(Q_0)$, as was to be shown for Assertion 1.

There is a surjective map

$$\begin{array}{rcl}\varphi\ :C_P(Q^x) & \to & Q\backslash C_P(Q)xC_P(Q_0)/C_P(Q_0)\\ z & \mapsto & QxzC_P(Q_0).\end{array}$$

Now for $z,t\in C_P(Q^x)$, one has that $\varphi(z)=\varphi(t)$ if and only if

$$t\in\big(Q^xzC_P(Q_0)\big)\cap C_P(Q^x) = zQ^x{\cdot}(C_P(Q_0)\cap C_P(Q^x))\quad .$$

Hence all the fibers of φ have the same cardinality, and

$$|Q\backslash C_P(Q)xC_P(Q_0)/C_P(Q_0)| = |C_P(Q^x)|/|Q^x{\cdot}\big(C_P(Q_0)\cap C_P(Q^x)\big)|\quad .$$

Now $R^x = Q^x\cap C_P(Q_0)$, and so $C_P(R^x) = C_P(Q^x)C_P(Q_0)$, by Lemma 4.1. Thus

$$|C_P(R^x)| = \frac{|C_P(Q^x)||C_P(Q_0)|}{|C_P(Q^x)\cap C_P(Q_0)|}\quad .$$

It follows that

$$\begin{array}{rcl}|Q\backslash C_P(Q)xC_P(Q_0)/C_P(Q_0)| & = & \dfrac{|C_P(Q^x)||Q^x\cap C_P(Q_0)|}{|Q^x||C_P(Q^x)\cap C_P(Q_0)|}\\ & = & \dfrac{|C_P(Q^x)||R^x||C_P(R^x)|}{|Q^x||C_P(Q^x)||C_P(Q_0)|}\\ & = & \dfrac{|C_P(Q^x)|}{|C_P(Q_0)|}\\ & = & \dfrac{|Q_0|}{|Q^x|}\end{array}$$

since $|S||C_P(S)| = |P|$ for any $S\in\underline{\mathcal{Q}}$. Assertion 2 follows, since $|Q^x| = |Q|$.

Finally the group $Q^x{\cdot}\big(C_P(Q^x)\cap Q_0\big)$ is an element of $\underline{\mathcal{Q}}$ of order equal to $|Q_0|$, by Lemma 4.1. Hence it is maximal in $\underline{\mathcal{Q}}$, and this completes the proof of Corollary 4.2. □

5. Some relations

Theorem 5.1.2 of [3] states a formula for the effect of tensor induction on relative syzygies in the Dade group : if Q is a subgroup of a finite p-group P, if X is a finite

Q-set, then

$$\mathrm{Ten}_Q^P \Omega_X = \sum_{\substack{R,S\in[s_P]\\ R\leq_P S}} \mu_P(R,S)\,|\{a\in S\backslash P/Q \mid X^{S^a\cap Q}\neq\emptyset\}|\,\Omega_{P/R} \quad ,$$

where $[s_P]$ is a set of representatives of the set s_P of conjugacy classes of subgroups of P, and μ_P is the Möbius function of s_P, partially ordered by the relation $\leq_P$. By convention $\Omega_\emptyset = 0$.

We apply this formula for the (almost) extraspecial group P : choose any maximal element Q_0 of $\underline{\mathcal{Q}}$, and consider the subgroup $Q = C_P(Q_0)$ and the finite Q-set $X = Q/Q_0$. The previous formula gives

$$\mathrm{Ten}_Q^P \Omega_X = \sum_{\substack{R,S\in[s_P]\\ R\leq_P S}} \mu_P(R,S)\,|\{a\in S\backslash P/Q \mid S^a\cap Q\leq Q_0\}|\,\Omega_{P/R} \quad ,$$

since $X^{S^a\cap Q}\neq\emptyset$ if and only if $S^a\cap Q\leq Q_0$. This implies in particular that $S^a\cap\Phi\leq Q_0\cap\Phi=\mathbf{1}$, thus $S\in\underline{\mathcal{Q}}$. Conversely, if $S\in\underline{\mathcal{Q}}$, by Corollary 4.2

$$|\{a\in S\backslash P/Q \mid S^a\cap Q\leq Q_0\}| = |Q_0|/|S| \quad .$$

Moreover if $R\leq S$, and if x is an element of P such that $R^x\leq S$, then $R=R^x$. It follows in particular that $\mu_P(R,S)$ is equal to the ordinary Möbius function $\mu(R,S)$ of the poset of subgroups of P for the inclusion relation (recall that $\mu(R,S)=(-1)^k p^{\binom{k}{2}}$ if $|S:R|=p^k$).

This gives finally

$$\mathrm{Ten}_Q^P \Omega_X = \sum_{\substack{R,S\in[s_P]\cap\underline{\mathcal{Q}}\\ R\leq_P S}} \mu(R,S)\,\frac{|Q_0|}{|S|}\,\Omega_{P/R}$$

Notation 5.1 *Let τ denote the order of $\Omega_{C_P(Q_0)/Q_0}$ in the Dade group $D\big(C_P(Q_0)/Q_0\big)$.*

Then τ is equal to 1, 2, or 4 according to $C(Q_0)/Q_0$ being cyclic of order 2, cyclic of order at least 3, or quaternion of order 8. Obviously, the order of $\mathrm{Ten}_Q^P\Omega_X$ in $D(P)$ divides τ.

Moreover, if Y is any finite P-set, and if H is any subgroup of P, one has that $\mathrm{Defres}^P_{N_P(H)/H}\Omega_Y = \Omega_{Y^H}$, where the set Y^H on the right hand side of the equality is considered as a $N_P(H)/H$-set. It follows in particular that

$$\mathrm{Defres}^P_{Q/Q_0}\mathrm{Ten}_Q^P\Omega_X = \Omega_{(P/Q_0)^{Q_0}} = \Omega_X \quad ,$$

since for $R\leq_P S\in\underline{\mathcal{Q}}$, one has that $(P/R)^{Q_0}\neq\emptyset$ if and only if $R=_P S=_P Q_0$, and since moreover $(P/Q_0)^{Q_0}=Q/Q_0=X$. Hence the order of $\mathrm{Ten}_Q^P\Omega_X$ in $D(P)$ is equal to τ. In particular

$$\textbf{(5.2)}\qquad \tau\cdot\left(\sum_{\substack{R,S\in[s_P]\cap\underline{\mathcal{Q}}\\ R\leq_P S}} \mu(R,S)\frac{|Q_0|}{|S|}\Omega_{P/R}\right) = 0$$

in $D(P)$.

Notation 5.3 *If $V \in \underline{\mathcal{E}}$, let $\tilde{V}$ be an element of $m(V)$, and set*

$$r_V = \tau \cdot \left(\sum_{\substack{S \in [s_P] \cap \underline{\mathcal{Q}} \\ S \geq_P \tilde{V}}} \mu(\tilde{V}, S) \frac{|Q_0|}{|S|} \right)$$

5.4. Remark: There is a slight abuse in the notation here : the expression $\mu(\tilde{V}, S)$ actually means $\mu(\tilde{V}, S^x)$, where x is some element of P such that $S^x \geq \tilde{V}$.

Since the elements of $m(V)$ are conjugate in P, the integer r_V, i.e. the coefficient of $\Omega_{P/\tilde{V}}$ in 5.2, does not depend on the choice of $\tilde{V}$ in $m(V)$. Now relation 5.2 can be written as follows :

$$\sum_{V \in \underline{\mathcal{E}}} r_V \Omega_{P/\tilde{V}} = 0 \tag{5.5}$$

6. The cohomology of $\mathcal{E}$

If $p \neq 2$, recall that $\mathcal{E}$ is the set of non zero subspaces V of E which are contained in $H = \mathrm{Ker}(q)$, and such that $V \leq V^\perp$, where $V^\perp$ is the orthogonal of V for the bilinear symplectic form b. In other words $\mathcal{E}$ is the set of subspaces V of H such that $c(V, V) = 0$, where c is the restriction of b to H. Since $H \cap E^\perp = \{0\}$, it follows that the form c is non-degenerate.

If $p = 2$, the set $\mathcal{E}$ is the set of non zero subspaces V of W such that $q(V) = \{0\}$, where q is the quadratic form, with associated bilinear form b. In this case, the form q is non-degenerate.

The set $\mathcal{E}$ is partially ordered by the inclusion relation.

Proposition 6.1 *Let H be a finite dimensional vector space over a field k of characteristic p, and let c be a non-degenerate bilinear form on H if $p \neq 2$, or a non-degenerate quadratic form on H if $p = 2$, with associated bilinear form c. Let $\mathcal{E}$ be the set of non zero totally isotropic subspaces of H for c if $p \neq 2$, or the set of non zero totally singular subspaces of H for q if $p = 2$, ordered by inclusion. Then the cohomology groups $H^n(\mathcal{E}, \mathbb{Z})$ are torsion-free abelian groups, for any integer $n \in \mathbb{N}$.*

Proof: This follows from the Solomon-Tits theorem, since the cohomology of $\mathcal{E}$ is the same as the cohomology of the associated Tits building, which has the homotopy type of a bouquet of spheres of a fixed dimension. Since we are only interested in cohomology groups, and since the full strength of the Solomon-Tits theorem is not required here, we give a more elementary proof of this proposition.

Lemma 6.2 *Let $(X, \leq)$ be a poset. Assume that there exists an acyclic subset Y of X such that $X - Y$ consists of minimal elements of X. Then for any $k \in \mathbb{Z}$, there is an isomorphism*

$$\tilde{H}^k(X) \cong \prod_{x \in X - Y} \tilde{H}^{k-1}(]x, .[_X) \quad ,$$

where $]x, .[_X = \{z \in X \mid x < z\}$, and $\tilde{H}^k(X)$ is the k^{th} integral reduced cohomology group of X.

Proof: Let $\tilde{C}_*(X)$ denote the reduced chain complex of X over $\mathbb{Z}$. Then the reduced integral cohomology of X is the cohomology of the reduced cochain complex

$$\tilde{C}^*(X) = \mathrm{Hom}_{\mathbb{Z}}\big(\tilde{C}_*(X), \mathbb{Z}\big) \quad .$$

Let $i \in \mathbb{Z}$. Then $\tilde{C}_i(X)$ is equal to $\{0\}$ if $i < -1$, to $\mathbb{Z}$ if $i = -1$, and if $i > 0$, it is the free abelian group with basis the set of strictly increasing sequences $x_0 < x_1 < \cdots < x_i$ of elements of X. Such a sequence has at most one element outside Y, namely x_0, since $X - Y$ consists of minimal elements of X. It follows that

$$\tilde{C}_i(X) = \tilde{C}_i(Y) \oplus \left(\underset{x \in X-Y}{\oplus} \tilde{C}_{i-1}(]x, \cdot[_X) \right) \quad ,$$

and it is easy to check that this decomposition leads to an exact sequence of complexes

$$0 \to \tilde{C}_*(Y) \to \tilde{C}_*(X) \to \underset{x \in X-Y}{\oplus} \tilde{C}_{*-1}(]x, \cdot[_X) \to 0 \quad .$$

Since these are complexes of free $\mathbb{Z}$-modules, taking $\mathbb{Z}$-duals gives an exact sequence of cochain complexes

$$\textbf{(6.3)} \qquad 0 \to \prod_{x \in X-Y} \tilde{C}^{*-1}(]x, \cdot[_X) \to \tilde{C}^*(X) \to \tilde{C}^*(Y) \to 0 \quad .$$

But the complex $\tilde{C}^*(Y)$ is acyclic if Y is, and the long exact sequence of cohomology groups associated to the exact sequence 6.3 gives the required isomorphisms. □

The following lemma (see for instance [2] Corollary 4.1.3) is a weak form of a theorem of Quillen. Recall that two maps of posets $h, h' : X \to Y$ are said to be *comparable* if $h(x) \leq h'(x)$ for any $x \in X$, or if $h(x) \geq h'(x)$ for any $x \in X$.

Lemma 6.4 *1. Let X and Y be posets, and let $f : X \to Y$ and $g : Y \to X$ be maps of posets. If $g \circ f$ is comparable to Id_X, and if $f \circ g$ is comparable to Id_Y, then the maps of complexes $\tilde{C}_*(f)$ and $\tilde{C}_*(g)$ are mutual inverse homotopy equivalences between the complexes $\tilde{C}_*(X)$ and $\tilde{C}_*(Y)$. In particular f and g induce inverse group isomorphisms $\tilde{H}^k(Y) \cong \tilde{H}^k(X)$, for any $k \in \mathbb{Z}$.*

2. If the poset X has a largest element, or a smallest element, then the chain complex $\tilde{C}_*(X)$ is contractible. In particular $\tilde{H}^k(X) = \{0\}$ for any $k \in \mathbb{Z}$.

Corollary 6.5 *Let $f : X \to Y$ be a map of posets. Suppose that for any $y \in Y$, the set $f^y = \{x \in X \mid f(x) \leq y\}$ has a largest element, or that for any $y \in Y$, the set $f_y = \{x \in X \mid f(x) \geq y\}$ has a smallest element. Then f induces an isomorphism $\tilde{H}^k(Y) \cong \tilde{H}^k(X)$, for any $k \in \mathbb{Z}$.*

Proof: If the first assertion holds, let $y \in Y$, and call $g(y)$ the largest element of f^y. Then $g : Y \to X$ is a map of posets. Moreover $f \circ g \leq \mathrm{Id}_Y$, and $g \circ f \geq \mathrm{Id}_X$. Similarly, if the second assertion holds, call $g(y)$ the smallest element of f_y. Then $g : Y \to X$ is a map of posets. Moreover $g \circ f \leq \mathrm{Id}_X$ and $f \circ g \geq \mathrm{Id}_Y$. □

Proof of Proposition 6.1: By induction on the dimension of H. If $H = 0$, there is nothing to prove : in that case indeed $\mathcal{E} = \emptyset$, and the reduced cohomology group of $\emptyset$ are all zero except $\tilde{H}^{-1}(\emptyset) \cong \mathbb{Z}$. Suppose that the result holds for spaces of dimension lower than $\dim_k H$. If $\mathcal{E} = \emptyset$, again there is nothing to prove. And if $\mathcal{E} \neq \emptyset$, let d be a minimal element of $\mathcal{E}$. Then d is a line. Consider the following subposets of $\mathcal{E}$:

$$\begin{aligned}[] [d, .[_{\mathcal{E}} &= \{V \in \mathcal{E} \mid V \geq d\} \\ ]., d^\perp] \cap \mathcal{E} &= \{V \in \mathcal{E} \mid V \leq d^\perp\} \\ \mathcal{E}_d &= \{V \in \mathcal{E} \mid V \cap d^\perp \neq \{0\}\} \quad , \end{aligned}$$

where $d^\perp$ is the orthogonal of d for the bilinear form c. There are obvious inclusion maps $i : [d, .[_{\mathcal{E}} \hookrightarrow]., d^\perp] \cap \mathcal{E}$ and $j :]., d^\perp] \cap \mathcal{E} \hookrightarrow \mathcal{E}_d$. Let $V \in]., d^\perp] \cap \mathcal{E}$. Then the set

$$i_V = \{W \in [d, .[_{\mathcal{E}} \mid W \geq V\}$$

has a smallest element $d + V$: indeed $d + V$ is a non-zero totally isotropic/singular subspace of H if V is a totally isotropic/singular subspace of H contained in $d^\perp$. By Corollary 6.5, the map i induces an isomorphism of reduced cohomology groups $\tilde{H}^k([d, .[_{\mathcal{E}}) \cong \tilde{H}^k(]., d^\perp] \cap \mathcal{E})$, for any $k \in \mathbb{Z}$. But $[d, .[_{\mathcal{E}}$ is an acyclic poset by Lemma 6.4, since it has a smallest element. Thus $]., d^\perp] \cap \mathcal{E}$ is also acyclic.

Now let $V \in \mathcal{E}_d$. The set

$$j^V = \{W \in]., d^\perp] \cap \mathcal{E} \mid W \leq V\}$$

has a largest element $V \cap d^\perp$. Again by Corollary 6.5 and Lemma 6.4, this shows that $\mathcal{E}_d$ is an acyclic poset. The set $\mathcal{E} - \mathcal{E}_d$ is the set of elements V of $\mathcal{E}$ such that $V \cap d^\perp = \{0\}$. Since $d^\perp$ is an hyperplane of H (because $d \cap H^\perp = \{0\}$ if $d \in \mathcal{E}$), it follows that $\mathcal{E} - \mathcal{E}_d$ consists of lines, i.e. of minimal elements of $\mathcal{E}$. By Lemma 6.2, there are group isomorphisms

$$\tilde{H}^k(\mathcal{E}) \cong \prod_{\substack{V \in \mathcal{E} \\ V \cap d^\perp = \{0\}}} \tilde{H}^{k-1}(]V, .[_{\mathcal{E}}) \quad ,$$

for any $k \in \mathbb{Z}$. But for $V \in \mathcal{E} - \mathcal{E}_d$, the poset $]V, .[_{\mathcal{E}}$ is isomorphic to the poset of non zero totally isotropic/singular subspaces for the non-degenerate bilinear (and quadratic) form induced by c (and q) on the space $V^\perp/V$. Since $\dim_k(V^\perp/V) < \dim H$, by induction hypothesis, the cohomology groups $\tilde{H}^{k-1}(]V, .[_{\mathcal{E}})$ are torsion-free. Hence the groups $\tilde{H}^k(\mathcal{E})$ are torsion-free, as was to be shown. □

6.6. Remark: The previous proof gives actually a more precise result : the cohomology groups $\tilde{H}^k(\mathcal{E})$ are products of copies of $\mathbb{Z}$.

7. The coefficient system

Recall from Notation 2.5 that if $\pi : P \to P/\Phi$ denotes the canonical projection, and if $V \in \underline{\mathcal{E}}$, we set $N(V) = \pi^{-1}(V^\perp)$ and $m(V) = \{Q \in \underline{\mathcal{Q}} \mid \pi(Q) = V\}$. Recall that if $Q \in m(V)$, then $N_P(Q) = C_P(Q) = N(V)$, by Lemma 2.6, and that the group P acts transitively on $m(V)$ by conjugation.

It follows that P acts on the group $\Sigma_V = \underset{Q \in m(V)}{\oplus} \partial D\Big(N(V)/Q\Big)$, and we denote by

$$F(V) = \Big(\underset{Q \in m(V)}{\oplus} \partial D\Big(N(V)/Q\Big)\Big)_P \quad ,$$

the co-invariants of P on this group, i.e. the largest quotient of Σ_V on which P acts trivially. Since the action of P on $m(V)$ is transitive, and since the stabilizer $N_P(Q) = N(V)$ of $Q \in m(V)$ acts trivially on $\partial D\Big(N(V)/Q\Big)$, the group $F(V)$ is isomorphic to $\partial D\Big(N(V)/Q\Big)$, for any $Q \in m(V)$. The consideration of $F(V)$ avoids the choice of a particular element in $m(V)$. Also note that $F(\{0\}) = \partial D(P)$.

Now if $V \leq W$ in $\underline{\mathcal{E}}$, then $W^\perp \leq V^\perp$, thus $N(W) \leq N(V)$. Moreover $\pi^{-1}(V) \leq \pi^{-1}(W)$. If $Q \in m(V)$, then there is an $R \in m(W)$ such that $R \geq Q$, and any two R's are conjugate by $N_P(Q)$. In this case $Q \leq R \leq N(W) \leq N(V)$, and the map

$$\mathrm{Defres}^{N(V)/Q}_{N(W)/R} : \partial D\Big(N(V)/Q\Big) \to \partial D\Big(N(W)/R\Big)$$

gives a map

$$\partial D\Big(N(V)/Q\Big) \to F(W)$$

which does not depend on the choice of $R \geq Q$. This leads to a well defined map

$$F_V^W : F(V) \to F(W)$$

whose restriction to $\partial D\Big(N(V)/Q\Big)$ is the previous map.

8. A sequence of coefficient systems

We will consider the constant coefficient system $\mathbf{Z}$ on $\underline{\mathcal{E}}$, defined by $\mathbf{Z}(V) = \mathbb{Z}$ for any $V \in \underline{\mathcal{E}}$, and $\mathbf{Z}_V^W = Id_{\mathbb{Z}}$ for any $V \leq W \in \underline{\mathcal{E}}$.

If $V \in \underline{\mathcal{E}}$, set

$$\underline{\mathcal{E}}(V) = \{V' \in \underline{\mathcal{E}} \mid V \leq V'\} \quad .$$

Define another coefficient system $\mathbf{Z}\underline{\mathcal{E}}$ on $\underline{\mathcal{E}}$ by

$$\mathbf{Z}\underline{\mathcal{E}}(V) = \mathbb{Z}^{\underline{\mathcal{E}}(V)} \quad ,$$

for $V \in \underline{\mathcal{E}}$. It is the free abelian group on the set $\underline{\mathcal{E}}(V)$: an element of $\mathbf{Z}\underline{\mathcal{E}}(V)$ is a linear combination with integer coefficients of elements of $\underline{\mathcal{E}}(V)$. If $V \leq W \in \underline{\mathcal{E}}$, the transition map $\mathbf{Z}\underline{\mathcal{E}}_V^W : \mathbf{Z}\underline{\mathcal{E}}(V) \to \mathbf{Z}\underline{\mathcal{E}}(W)$ is the canonical projection map corresponding to the inclusion $\underline{\mathcal{E}}(W) \subseteq \underline{\mathcal{E}}(V)$.

Fix $V \in \underline{\mathcal{E}}$. Let $i_V : \mathbb{Z} \to \mathbb{Z}^{\underline{\mathcal{E}}(V)}$ be the linear map sending $1 \in \mathbb{Z}$ to the element $\sum\limits_{U \in \underline{\mathcal{E}}(V)} r_U U$. Let $s_V : \mathbb{Z}^{\underline{\mathcal{E}}(V)} \to F(V)$ be the map sending $U \in \underline{\mathcal{E}}(V)$ to the image in $F(V)$ of the element $\Omega_{\big(N(V)/\tilde{V}\big)/(\tilde{U}/\tilde{V})}$, where $\tilde{U}$ is any element of $m(U)$ containing $\tilde{V}$.

Lemma 8.1 *The maps i_V and s_V, for $V \in \underline{\mathcal{E}}$, define maps of coefficient systems*

$$i : \mathbf{Z} \to \mathbf{Z}\underline{\mathcal{E}} \qquad \text{and} \qquad s : \mathbf{Z}\underline{\mathcal{E}} \to F \quad ,$$

such that $s \circ i = 0$.

Proof: Saying that i is a map of coefficient systems is equivalent to saying that for any $V \le W$ in $\underline{\mathcal{E}}$, the square

$$\begin{tikzcd} \mathbf{Z}(V) \arrow[r, "i_V"] \arrow[d, "\mathbf{Z}_V^W"'] & \mathbf{Z}\underline{\mathcal{E}}(V) \arrow[d, "\mathbf{Z}\underline{\mathcal{E}}_V^W"] \\ \mathbf{Z}(W) \arrow[r, "i_W"'] & \mathbf{Z}\underline{\mathcal{E}}(W) \end{tikzcd}$$

is commutative. Equivalently, this amounts to checking that

$$\mathbf{Z}\underline{\mathcal{E}}_V^W\Big(\sum_{U\in\underline{\mathcal{E}}(V)} r_U U\Big) = \sum_{U\in\underline{\mathcal{E}}(W)} r_U U$$

which is obvious. Thus i is a map of coefficient systems.

Similarly, for the map s, we must check that the square

$$\begin{tikzcd} \mathbf{Z}\underline{\mathcal{E}}(V) \arrow[r, "s_V"] \arrow[d, "\mathbf{Z}\underline{\mathcal{E}}_V^W"'] & F(V) \arrow[d, "F_V^W"] \\ \mathbf{Z}\underline{\mathcal{E}}(W) \arrow[r, "s_W"'] & F(W) \end{tikzcd}$$

is commutative. Equivalently, we must check that

(8.2) $$F_V^W \circ s_V(U) = s_W \circ \mathbf{Z}\underline{\mathcal{E}}_V^W(U) \quad ,$$

for any $U \in \underline{\mathcal{E}}(V)$. There are two cases : first if $U \not\geq W$, then $\mathbf{Z}\underline{\mathcal{E}}_V^W(U) = 0$. But $s_V(U) = \Omega_{\big(N(V)/\tilde{V}\big)/(\tilde{U}/\tilde{V})}$, where $\tilde{U}$ is any element of $m(U)$ containing $\tilde{V}$. Thus $F_V^W \circ s_V(U)$ is equal to the image in $F(W)$ of

$$\mathrm{Defres}_{N(W)/\tilde{W}}^{N(V)/\tilde{V}} \Omega_{\big(N(V)/\tilde{V}\big)/(\tilde{U}/\tilde{V})} \quad ,$$

where $\tilde{W}$ is an element of $m(W)$ containing $\tilde{V}$. This is equal to Ω_Y, where Y is the set of fixed points of the subgroup $\tilde{W}/\tilde{V}$ of $N(V)/\tilde{V}$ on the set $\big(N(V)/\tilde{V}\big)/(\tilde{U}/\tilde{V})$. This is zero unless $\tilde{W}/\tilde{V}$ is contained in $\tilde{U}/\tilde{V}$ up to conjugation by $N(V)/\tilde{V}$. In this case $\tilde{W} \le \tilde{U}^n$, for some $n \in N(V)$, hence $W = \pi(\tilde{W}) \le \pi(\tilde{U}^n) = U$. Hence both sides of equation 8.2 are zero if $W \not\le U$.

Suppose now that $W \le U$. Choose $\tilde{W} \in m(W)$ such that $\tilde{W} \ge \tilde{V}$, and $\tilde{U} \in m(U)$ such that $\tilde{U} \ge \tilde{W}$. Then $\mathbf{Z}\underline{\mathcal{E}}_V^W(U) = U$, and $s_W \circ \mathbf{Z}\underline{\mathcal{E}}_V^W(U)$ is equal to the image of $\Omega_{\big(N(W)/\tilde{W}\big)/(\tilde{U}/\tilde{W})}$ in $F(W)$. On the other hand $F_V^W \circ s_V(U)$ is equal to the image in $F(W)$ of Ω_Y, where Y is the set of fixed points of the subgroup $\tilde{W}/\tilde{V}$ of $N(V)/\tilde{V}$ on the set $\big(N(V)/\tilde{V}\big)/(\tilde{U}/\tilde{V})$. But if $\tilde{W}^n \le \tilde{U}$ for some $n \in N(V)$, then $\tilde{W}^n = \tilde{W}$. Hence $Y = \big(N(W)/\tilde{W}\big)/(\tilde{U}/\tilde{V})$, and equation 8.2 also holds in this case.

It remains to check that $s \circ i = 0$, or equivalently that

$$\sum_{U\in\underline{\mathcal{E}}(V)} r_U \Omega_{\big(N(V)/\tilde{V}\big)/(\tilde{U}/\tilde{V})} = 0 \quad .$$

But this is relation 5.5 for the group $N(V)/\tilde{V}$. □

In the proof of our main theorem, we will have to consider inverse limits of the above coefficient systems on the poset $\mathcal{E}$, and we summarize the results we need in the following lemma :

Lemma 8.3 *1. The group $\varprojlim_{V\in\mathcal{E}} \mathbf{Z}(V)$ is isomorphic to $\mathbb{Z}^{\Gamma}$, where Γ is the set of connected components of the poset $\mathcal{E}$.*

2. The group $\varprojlim_{V\in\mathcal{E}} \mathbf{Z}\underline{\mathcal{E}}(V)$ is isomorphic to $\mathbb{Z}^{\mathcal{E}}$.

Proof: Assertion 1 is well known (and straightforward). Now for assertion 2 : an element of $\varprojlim_{V\in\mathcal{E}} \mathbf{Z}\underline{\mathcal{E}}(V)$ is a sequence of elements $l_V \in \mathbf{Z}\underline{\mathcal{E}}(V)$, for $V \in \mathcal{E}$, such that $\mathbf{Z}\underline{\mathcal{E}}_V^W(l_V) = l_W$ whenever $V \leq W$ in $\mathcal{E}$. If $l_V = \sum_{U\in\underline{\mathcal{E}}(V)} s_{V,U}U$, for coefficients $s_{V,U} \in \mathbb{Z}$, this gives

$$s_{V,U} = s_{W,U}$$

whenever $V \leq W \leq U$ in $\mathcal{E}$. Set $t_V = s_{V,V}$, for $V \in \mathcal{E}$. Then $l_V = \sum_{U\in\underline{\mathcal{E}}(V)} t_U U$, and the sequence $(l_V)_{V\in\mathcal{E}}$ is determined by the element $l = \sum_{U\in\mathcal{E}} t_U U$. Conversely, if such an element $l = \sum_{U\in\mathcal{E}} t_U U$ is given, then setting $l_V = \sum_{U\in\underline{\mathcal{E}}(V)} t_U U$ for any $V \in \mathcal{E}$ gives an element of the inverse limit. □

Notation 8.4 *Let $a : F(\{0\}) \to \varprojlim_{V\in\mathcal{E}} F(V)$ be the map sending $u \in F(\{0\})$ to the sequence $(F_{\{0\}}^V(u))_{V\in\mathcal{E}}$. Let $b : \mathbb{Z}^{\underline{\mathcal{E}}} = \mathbf{Z}\underline{\mathcal{E}}(\{0\}) \to \varprojlim_{V\in\mathcal{E}} \mathbf{Z}\underline{\mathcal{E}}(V) \cong \mathbb{Z}^{\mathcal{E}}$ denote the corresponding map for the coefficient system $\mathbf{Z}\underline{\mathcal{E}}$, and let $c : \mathbb{Z} = \mathbf{Z}(\{0\}) \to \varprojlim_{V\in\mathcal{E}} \mathbf{Z}(V) \cong \mathbb{Z}^{\Gamma}$ the corresponding map for the constant coefficient system $\mathbf{Z}$. Let $t = \varprojlim_{V\in\mathcal{E}} s_V : \mathbb{Z}^{\underline{\mathcal{E}}} \to \varprojlim_{V\in\mathcal{E}} F(V)$, and let $j = \varprojlim_{V\in\mathcal{E}} i_V : \mathbb{Z}^{\Gamma} \cong \varprojlim_{V\in\mathcal{E}} \mathbf{Z} \to \varprojlim_{V\in\mathcal{E}} \mathbf{Z}\underline{\mathcal{E}}(V) \cong \mathbb{Z}^{\mathcal{E}}$.*

It is easy to see that $b : \mathbb{Z}^{\underline{\mathcal{E}}} \to \mathbb{Z}^{\mathcal{E}}$ is the canonical projection map corresponding to the inclusion $\mathcal{E} \subseteq \underline{\mathcal{E}}$, that $c : \mathbb{Z} \to \mathbb{Z}^{\Gamma}$ is the "diagonal" inclusion, and that $j : \mathbb{Z}^{\Gamma} \to \mathbb{Z}^{\mathcal{E}}$ is the linear map sending the basis vector $\gamma \in \Gamma$ to the sum $\sum_{U\in\gamma} r_U U$.

Proposition 8.5 *The diagram*

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{i} & \mathbb{Z}^{\underline{\mathcal{E}}} & \xrightarrow{s} & F(\{0\}) \\
 & & \Big\downarrow{\scriptstyle c} & & \Big\downarrow{\scriptstyle b} & & \Big\downarrow{\scriptstyle a} \\
0 & \longrightarrow & \mathbb{Z}^{\Gamma} & \xrightarrow{j} & \mathbb{Z}^{\mathcal{E}} & \xrightarrow{t} & \varprojlim_{V\in\mathcal{E}} F(V)
\end{array}$$

is commutative. Moreover $s \circ i = 0$ and $t \circ j = 0$.

Proof: This is a straightforward consequence of Lemma 8.1, since taking inverse limit is functorial with respect to coefficient systems. □

9. The main theorem in the general case

Theorem 9.1 *Let P be an (almost) extraspecial p-group. If $p = 2$ and if the ground field contains cubic roots of unity, assume moreover that P is not isomorphic to $D_8^{*n} * Q_8$. Then :*

1. *The sequence*
$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z}^{\underline{\mathcal{E}}} \xrightarrow{s} \partial D(P) \to 0$$
is exact.
In other words the group $\partial D(P)$ is generated by the elements $\Omega_{P/\tilde{V}}$, for $V \in \underline{\mathcal{E}}$, subject to the single relation
$$\sum_{V \in \underline{\mathcal{E}}} r_V \Omega_{P/\tilde{V}} = 0 \quad . \tag{9.2}$$
In particular, the group $\partial D(P)$ is isomorphic to $\mathbb{Z}^{|\mathcal{E}|} \oplus \mathbb{Z}/\tau\mathbb{Z}$.
2. *The group $D(P)$ admits the following presentation : it is generated by the elements $\Omega_{P/Q}$, where $Q \supseteq \Phi$ with $|P:Q| \geq 3$, and the elements $\Omega_{P/\tilde{V}}$, for $V \in \underline{\mathcal{E}}$, subject to the above relation 9.2, together with the relations $2\Omega_{P/Q} = 0$ if $|P:Q| = p$.*

Proof: Since $D(P) = \partial D(P) \oplus D(E)$, the second assertion follows from the first one, and from Dade's Theorem giving generators and relations for the Dade group of the elementary abelian group E.

To prove Assertion 1, we proceed by induction on the order of P. The initial case of induction will be checked in section 11. Now let P be an (almost) extraspecial p-group, and assume that the theorem holds for all (almost) extraspecial p-groups of order strictly smaller than $|P|$. Assume further that $\mathcal{E} = \underline{\mathcal{E}} - \{\{0\}\}$ is a connected poset.

With these assumptions, the restriction to $\mathcal{E}$ of the coefficient systems $\mathbf{Z}$, $\mathbf{Z}\underline{\mathcal{E}}$, and F fit together in an exact sequence of coefficient systems
$$0 \to \mathbf{Z} \xrightarrow{i} \mathbf{Z}\underline{\mathcal{E}} \xrightarrow{s} F \to 0 \quad ,$$
because every evaluation on $\mathcal{E}$ of this sequence involves (almost) extraspecial p-groups of order (strictly) less than $|P|$. Since taking inverse limits of coefficients systems on $\mathcal{E}$ is a covariant left exact functor, we get a long exact sequence starting with
$$0 \to \varprojlim_{V \in \mathcal{E}} \mathbf{Z}(V) \to \varprojlim_{V \in \mathcal{E}} \mathbf{Z}\underline{\mathcal{E}}(V) \to \varprojlim_{V \in \mathcal{E}} F(V) \xrightarrow{\delta} H^1(\mathcal{E}, \mathbf{Z}) \to \dots \quad ,$$
where δ is the connecting homomorphism of the long exact sequence of cohomology groups. Now $\varprojlim_{V \in \mathcal{E}} \mathbf{Z}(V) \cong \mathbb{Z}$ since $\mathcal{E}$ is a connected poset. Moreover $\varprojlim_{V \in \mathcal{E}} \mathbf{Z}\underline{\mathcal{E}}(V) \cong \mathbb{Z}^{\mathcal{E}}$ by Lemma 8.3. By Proposition 8.5, the above exact sequence fits into the commutative diagram
$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "i"] \arrow[d, "Id"'] & \mathbb{Z}^{\underline{\mathcal{E}}} \arrow[r, "s"] \arrow[d, "b"'] & F(\{0\}) \arrow[d, "a"'] & \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "j"] & \mathbb{Z}^{\mathcal{E}} \arrow[r, "t"] & \varprojlim_{V \in \mathcal{E}} F(V) \arrow[r, "\delta"] & H^1(\mathcal{E}, \mathbf{Z})
\end{tikzcd}$$
In particular the images of $a \circ s$ and $t \circ b$ are equal. Now the image of s is clearly equal to $\partial D^{\Omega}(P)$, and since b is surjective, it follows that
$$a\big(\partial D^{\Omega}(P)\big) = \mathrm{Im}(t) = \mathrm{Ker}(\delta) \quad .$$

This shows that the group $L = a\big(\partial D(P)\big)/a\big(\partial D^{\Omega}(P)\big)$ maps injectively via δ into $H^1(\mathcal{E},\mathbf{Z})$. But this group is a torsion-free abelian group by Proposition 6.1. On the other hand L is a factor group of $\partial D(P)/\partial D^{\Omega}(P)$, which is a torsion group by Lemma 3.2. Hence $L=\{0\}$, and $a\big(\partial D(P)\big) = a\big(\partial D^{\Omega}(P)\big)$. Note that $\mathrm{Ker}(a) = T(P) \leq \partial D^{\Omega}(P)$. Indeed, we have $x \in \mathrm{Ker}(a)$ if and only if $x \in \partial D(P)$ and $0 = a(x) = \big(F^V_{\{0\}}(x)\big)_{V\in\mathcal{E}}$. But, for any $\tilde{V} \in m(V)$, the restriction of $F^V_{\{0\}}(x)$ to the component $\partial D(N(V)/\tilde{V})$ is the element $\mathrm{Defres}^P_{N(V)/\tilde{V}}(x)$. Hence

$$x \in \mathrm{Ker}(a) \text{ if and only if } x \in \partial D(P) \cap \bigcap_{V\in\mathcal{E}} \mathrm{Ker}\big(\mathrm{Defres}^P_{N(V)/\tilde{V}}\big),$$

for any choice of subgroups $\tilde{V} \in m(V)$. That is

$$\mathrm{Ker}(a) = \bigcap_{\mathbf{1}<Q\leq P} \mathrm{Ker}\big(\mathrm{Defres}^P_{N_P(Q)/Q}\big) = T(P),$$

by Lemma 3.2. Moreover, our assumption on the set $\mathcal{E}$ implies that any maximal elementary abelian subgroup of P has rank at least 3, and so $T(P) = <\Omega_P>$, by Lemma 3.2. Hence $\mathrm{Ker}(a) = T(P) \leq \partial D^{\Omega}(P)$, as asserted.

It follows that $\partial D(P) = \partial D^{\Omega}(P)$, showing that the map s is surjective.

Now let $u = \sum_{U\in\underline{\mathcal{E}}} n_U U \in \mathrm{Ker}(s)$, for coefficients $n_U \in \mathbb{Z}$. Then $a\circ s(u) = t\circ b(u) = 0$, thus $b(u) \in \mathrm{Ker}(t) = \mathrm{Im}(j)$. It follows that there is an integer m such that $n_U = mr_U$, for any $U \neq \{0\}$. Then

$$\begin{aligned} s(u) &= n_{\{0\}}\Omega_{P/\mathbf{1}} + m\sum_{U\in\mathcal{E}} r_U\Omega_{P/\tilde{U}} \\ &= n_{\{0\}}\Omega_{P/\mathbf{1}} - mr_{\{0\}}\Omega_{P/\mathbf{1}} \\ &= (n_{\{0\}} - mr_{\{0\}})\Omega_{P/\mathbf{1}} = 0 \end{aligned}$$

It follows that $n_{\{0\}} = mr_{\{0\}}$, since $\Omega_{P/\mathbf{1}}$ has infinite order in $D(P)$. Hence $u = i(m)$, showing that $\mathrm{Ker}(s) = \mathrm{Im}(i)$, and completing the proof of the first assertion of the theorem.

The second assertion is just a reformulation of the first one. The last assertion follows from the fact that the g.c.d. of the coefficients r_V, for $V \in \underline{\mathcal{E}}$, is equal to τ, since actually $r_{V_0} = \tau$, for any maximal element V_0 of $\underline{\mathcal{E}}$. □

10. The main theorem in the exceptional case

In this section we suppose that $p=2$ and that P is isomorphic to $D_8^n * Q_8$, for $n \geq 2$, and that the ground field contains cubic roots of unity. The remaining "small" cases $n \leq 1$ will be handled in section 11. The case where the ground field does not contain cubic roots of unity is part of Theorem 9.1.

In this situation, let Q_0 be a maximal element of $\mathcal{Q}$. Then $C_P(Q_0)/Q_0$ is isomorphic to the quaternion group Q_8. Now by Theorem 6.3 of [5], the group $D(Q_8)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. There are exactly three elements of order 2 in $D(Q_8)$, one of which is equal to $2\Omega_{Q_8/\mathbf{1}}$. The two others are equal to M and $M' = M + 2\Omega_{Q_8/\mathbf{1}}$, where M and M' denote the images in $D(Q_8)$ of exceptional endo-trivial modules of dimension 5, with the same names. Note that M and M' depend on the choice of cubic roots in the field of coefficients : they are permuted by any endomorphism of the ground field exchanging these two non trivial roots, so there is no canonical way to specify which of them should be called M and which should be called M'.

Notation 10.1 *Choose any maximal element Q_0 in $\mathcal{Q}$, and choose one element M of order 2 in $D\big(C_P(Q_0)/Q_0\big) - D^{\Omega}\big(C_P(Q_0)/Q_0\big)$. Set*

$$u_0 = \mathrm{Teninf}_{C_P(Q_0)/Q_0}^{P} M \quad .$$

This is an element of $\partial D(P) = F(\{0\})$. An element of $\partial D(P)$ which can be obtained by this construction will be called an exotic element of $\partial D(P)$.

If $V \in \underline{\mathcal{E}}$, set

$$u_V = F_{\{0\}}^{V}(u_0) \quad ,$$

so that with this notation $u_{\{0\}} = u_0$.

Lemma 10.2 *Let Q be any element of $\mathcal{Q}$. Then* $\mathrm{Defres}_{C_P(Q)/Q}^{P} u_0$ *is an exotic element in $D\big(C_P(Q)/Q\big)$.*

Proof: Set $R = C_P(Q_0)/Q_0 \cong Q_8$. Recall that $M \in D(R)$ is the class of a *critical* endo-trivial module in the sense of [6] Definition 5.1 : this means that $\mathrm{Res}_H^R M = 0$ for all proper subgroups H of R, and that $\mathrm{Def}_{R/N}^R M = 0$ for any non-trivial normal subgroup N of R.

Now setting $C = C_P(Q)$ and $C_0 = C_P(Q_0)$, we have that

$$\begin{aligned}
\mathrm{Res}_{C_P(Q)}^{P} u_0 &= \mathrm{Res}_C^P \mathrm{Ten}_{C_0}^P \mathrm{Inf}_{C_0/Q_0}^{C_0} M \\
&= \sum_{x \in C\backslash P/C_0} \mathrm{Ten}_{C\cap {}^xC_0}^{C}\, {}^x\Big(\mathrm{Res}_{C^x\cap C_0}^{C_0} \mathrm{Inf}_{C_0/Q_0}^{C_0} M\Big) \quad ,
\end{aligned}$$

by the Mackey formula. Now

$$\mathrm{Res}_{C^x\cap C_0}^{C_0} \mathrm{Inf}_{C_0/Q_0}^{C_0} M = \mathrm{Inf}_{(C^x\cap C_0)/(C^x\cap Q_0)}^{C^x\cap C_0} \mathrm{Iso}_{(C^x\cap C_0)\cdot Q_0/Q_0}^{(C^x\cap C_0)/(C^x\cap Q_0)} \mathrm{Res}_{(C^x\cap C_0)\cdot Q_0/Q_0}^{C_0/Q_0} M \quad ,$$

where $\mathrm{Iso}_{(C^x\cap C_0)\cdot Q_0/Q_0}^{(C^x\cap C_0)/(C^x\cap Q_0)}$ denotes the transfer by the canonical isomorphism

$$(C^x \cap C_0)\cdot Q_0/Q_0 \to (C^x \cap C_0)/(C^x \cap Q_0) \quad .$$

But $\mathrm{Res}_{(C^x\cap C_0)\cdot Q_0/Q_0}^{C_0/Q_0} M = 0$, unless $(C^x \cap C_0)\cdot Q_0 = C_0$. Hence

$$\mathrm{Res}_{C_P(Q)}^{P} u_0 = \sum_{\substack{x \in C\backslash P/C_0 \\ (C^x\cap C_0)\cdot Q_0 = C_0}} \mathrm{Ten}_{C\cap {}^xC_0}^{C}\, {}^x\Big(\mathrm{Inf}_{(C^x\cap C_0)/(C^x\cap Q_0)}^{C^x\cap C_0} \mathrm{Iso}_{C_0/Q_0}^{(C^x\cap C_0)/(C^x\cap Q_0)} M\Big) \quad .$$

Set $M_x = \mathrm{Iso}_{C_0/Q_0}^{(C^x\cap C_0)/(C^x\cap Q_0)} M$. This is a critical element in $D\big((C^x \cap C_0)/(C^x \cap Q_0)\big)$. Set moreover $N_x = {}^x\Big(\mathrm{Inf}_{(C^x\cap C_0)/(C^x\cap Q_0)}^{C^x\cap C_0} M_x\Big)$. We get

$$\mathrm{Res}_{C_P(Q)}^{P} u_0 = \sum_{\substack{x \in C\backslash P/C_0 \\ (C^x\cap C_0)\cdot Q_0 = C_0}} \mathrm{Ten}_{C\cap {}^xC_0}^{C}\, {}^xN_x \quad .$$

Now by Proposition 3.10 of [4], we have that

$$\mathrm{Def}^C_{C/Q}\mathrm{Res}^P_C u_0 = \sum_{\substack{x\in C\backslash P/C_0\\ (C^x\cap C_0)\cdot Q_0=C_0}} L_x \quad ,$$

where

$$L_x = \gamma_{|Q:Q\cap {}^xC_0|}\mathrm{Ten}^{C/Q}_{(C\cap {}^xC_0)\cdot Q/Q}\mathrm{Iso}^{(C\cap {}^xC_0)\cdot Q/Q}_{(C\cap {}^xC_0)/(Q\cap {}^xC_0)}\mathrm{Def}^{C\cap {}^xC_0}_{(C\cap {}^xC_0)/(Q\cap {}^xC_0)}N_x \quad ,$$

and $\gamma_{|Q:Q\cap {}^xC_0|}$ denotes the effect on the Dade group of raising the scalars of the ground field to the power $|Q:Q\cap {}^xC_0|$. Now

$$\begin{aligned}\mathrm{Def}^{C\cap {}^xC_0}_{(C\cap {}^xC_0)/(Q\cap {}^xC_0)}N_x &= {}^x\Big(\mathrm{Def}^{C^x\cap C_0}_{(C^x\cap C_0)/(Q^x\cap C_0)}\mathrm{Inf}^{C^x\cap C_0}_{(C^x\cap C_0)/(C^x\cap Q_0)}M_x\Big)\\ &= {}^x\Big(\mathrm{Inf}^{(C^x\cap C_0)/(Q^x\cap C_0)}_{S_x}\mathrm{Def}^{(C^x\cap C_0)/(C^x\cap Q_0)}_{S_x}M_x\Big) \quad ,\end{aligned}$$

where $S_x=(C^x\cap C_0)/(Q^x\cap C_0)(C^x\cap Q_0)$. Since M_x is critical, this is zero unless $Q^x\cap C_0\leq C^x\cap Q_0$. By Corollary 4.2, there is a unique double coset CxC_0 in P such that $Q^x\cap C_0\leq Q_0$, and for such a double coset, by Lemma 4.1, we have $(C^x\cap C_0)\cdot Q_0=C_0$. For such an element x, we have that

$$\begin{aligned}L_x &= \gamma_{|Q:Q\cap {}^xC_0|}\mathrm{Ten}^{C/Q}_{(C\cap {}^xC_0)\cdot Q/Q}\mathrm{Iso}^{(C\cap {}^xC_0)\cdot Q/Q}_{(C\cap {}^xC_0)/(Q\cap {}^xC_0)}\mathrm{Inf}^{(C^x\cap C_0)/(Q^x\cap C_0)}_{(C^x\cap C_0)/(C^x\cap Q_0)}\,{}^xM_x\\ &= \gamma_{|Q:Q\cap {}^xC_0|}\mathrm{Ten}^{C/Q}_{(C\cap {}^xC_0)\cdot Q/Q}\mathrm{Inf}^{(C\cap {}^xC_0)\cdot Q/Q}_{(C\cap {}^xC_0)\cdot Q/(C\cap {}^xQ_0)\cdot Q}\mathrm{Iso}^{(C\cap {}^xC_0)\cdot Q/(C\cap {}^xQ_0)\cdot Q}_{(C^x\cap C_0)/(C^x\cap Q_0)}\,{}^xM_x\end{aligned}$$

Finally, we get

$$\begin{aligned}\mathrm{Def}^C_{C/Q}\mathrm{Res}^P_C u_0 &= \gamma_{|Q:Q\cap {}^xC_0|}\mathrm{Ten}^{C/Q}_{(C\cap {}^xC_0)\cdot Q/Q}\mathrm{Iso}^{(C\cap {}^xC_0)\cdot Q/Q}_{(C\cap {}^xC_0)/(Q\cap {}^xC_0)}\,{}^xM_x\\ &= \gamma_{|Q:Q\cap {}^xC_0|}\mathrm{Ten}^{C/Q}_{(C\cap {}^xC_0)\cdot Q/Q}\mathrm{Iso}^{(C\cap {}^xC_0)\cdot Q/Q}_{(C\cap {}^xC_0)/(Q\cap {}^xC_0)}\mathrm{Iso}^{(C\cap {}^xC_0)/(C\cap {}^xQ_0)}_{{}^xC_0/{}^xQ_0}\,{}^xM\end{aligned}$$

Now the group $Q_1=(C\cap {}^xQ_0)\cdot Q$ is a maximal element of $\underline{\mathcal{Q}}$ by Corollary 4.2, thus $\overline{Q}_1=Q_1/Q$ is a maximal element of $\underline{\mathcal{Q}}(\overline{C})$, where $\underline{\mathcal{Q}}(\overline{C})$ is the set of subgroups of $\overline{C}=C_P(Q)/Q$ which intersect trivially the Frattini subgroup of this (almost) extra-special group. Moreover

$$C_{\overline{C}}(\overline{Q}_1)=N_{\overline{C}}(\overline{Q}_1)=N_{C_P(Q)/Q}(Q_1/Q)=N_{C_P(Q)}(Q_1)/Q=C_P(Q_1)/Q \quad .$$

Moreover $C_P(Q_1)=(C\cap {}^xC_0)\cdot Q$, again by Lemma 4.1. Since $\gamma_{|Q:Q\cap {}^xC_0|}$ commutes with tensor induction and inflation (by Lemma 3.2 of [4]), it follows that

$$\mathrm{Def}^C_{C/Q}\mathrm{Res}^P_C u_0=\mathrm{Ten}^{\overline{C}}_{C_{\overline{C}}(\overline{Q}_1)}\mathrm{Inf}^{C_{\overline{C}}(\overline{Q}_1)}_{C_{\overline{C}}(\overline{Q}_1)/\overline{Q}_1}\overline{M} \quad ,$$

where $\overline{M}=\gamma_{|Q:Q\cap {}^xC_0|}\mathrm{Iso}^{C_{\overline{C}}(\overline{Q}_1)/\overline{Q}_1}_{C_0^x/Q_0^x}M^x$.

But $\overline{M}$ is an element of order 2 in $D\big(C_{\overline{C}}(\overline{Q}_1)/\overline{Q}_1\big)-D^{\Omega}\big(C_{\overline{C}}(\overline{Q}_1)/\overline{Q}_1\big)$, since it is the image of M by a group isomorphism followed by a Galois twist. Thus $\mathrm{Def}^C_{C/Q}\mathrm{Res}^P_C u_0$ is an exotic element in $D\big(C_{\overline{C}}(\overline{Q}_1)/\overline{Q}_1\big)$, as was to be shown. □

Let s'_{u_0} be the linear map from $\mathbb{Z}$ to $\partial D(P)$ defined by $s'_{u_0}(1)=u_0$, and let $i':\mathbb{Z}\to\mathbb{Z}$ be multiplication by 2. Since $2u_0=0$ in $D(P)$, it follows that $s'_{u_0}\circ i'=0$.

Theorem 10.3 *Let k be a field of characteristic 2, containing all cubic roots of unity, and let $P \cong D_8^n * Q_8$. Let u_0 be any exotic element in $D(P)$. Then*

1. *The sequence*

$$0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{i \oplus i'} \mathbb{Z}^{\underline{\mathcal{E}}} \oplus \mathbb{Z} \xrightarrow{s+s'_{u_0}} \partial D(P) \longrightarrow 0$$

   is exact.

   In other words the group $\partial D(P)$ is generated by u_0 and by the elements $\Omega_{P/\tilde{V}}$, for $V \in \underline{\mathcal{E}}$, subject to the relations

$$(\mathbf{10.4}) \qquad 2u_0 = 0 \qquad \sum_{V \in \underline{\mathcal{E}}} r_V \Omega_{P/\tilde{V}} = 0 \quad .$$

   In particular the group $\partial D(P)$ is isomorphic to $\mathbb{Z}^{|\mathcal{E}|} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

2. *The group $D(P)$ admits the following presentation : it is generated by the elements $\Omega_{P/Q}$, where $Q \supseteq \Phi$ with $|P:Q| \geq 4$, and the elements u_0 and $\Omega_{P/\tilde{V}}$, for $V \in \underline{\mathcal{E}}$, subject to the above relations 10.4.*

Proof: Here again, Assertion 2 follows from Assertion 1 and Dade's Theorem. We prove Assertion 1 by induction on $n \geq 2$. We will check that the result holds for $n \leq 1$ in section 11.

By induction hypothesis, and by Lemma 10.2, for any $V \in \mathcal{E}$, we have an exact sequence

$$0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{i_V \oplus i'_V} \mathbb{Z}^{\underline{\mathcal{E}}(V)} \oplus \mathbb{Z} \xrightarrow{s_V + s'_{u_V}} F(V) \longrightarrow 0 \quad ,$$

where i_V and s_V are defined in section 8, where i'_V is multiplication by 2, and $s'_{u_V} : \mathbb{Z} \to F(V)$ is obtained from the exotic element u_V, after identification $F(V) \cong \partial D\big(N(V)/\tilde{V}\big)$.

By construction, these maps actually define an exact sequence of coefficient systems on $\mathcal{E}$

$$0 \longrightarrow \mathbf{Z} \oplus \mathbf{Z} \xrightarrow{\mathbf{i} \oplus \mathbf{i}'} \mathbf{Z}\underline{\mathcal{E}} \oplus \mathbf{Z} \xrightarrow{\mathbf{s}+\mathbf{s'}_{u_0}} F \longrightarrow 0 \quad ,$$

and we can take inverse limits on $\mathcal{E}$. Since $n \geq 2$, the poset $\mathcal{E}$ is connected, and we get a long exact sequence of cohomology groups starting with

$$0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j \oplus j'} \mathbb{Z}^{\mathcal{E}} \oplus \mathbb{Z} \xrightarrow{t+t'_{u_0}} \varprojlim_{V \in \mathcal{E}} F(V) \xrightarrow{\delta} H^1(\mathcal{E}, \mathbf{Z}) \quad ,$$

where δ is the connecting homomorphism. As in the general case, we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{i \oplus i'} & \mathbb{Z}^{\underline{\mathcal{E}}} \oplus \mathbb{Z} & \xrightarrow{s+s'_{u_0}} & F(\{0\}) & & \\
 & & {\scriptstyle Id \oplus Id}\big\downarrow & & {\scriptstyle b \oplus Id}\big\downarrow & & {\scriptstyle a}\big\downarrow & & \\
0 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{j \oplus i'} & \mathbb{Z}^{\mathcal{E}} \oplus \mathbb{Z} & \xrightarrow{t+t'_{u_0}} & \varprojlim_{V \in \mathcal{E}} F(V) & \xrightarrow{\delta} & H^1(\mathcal{E}, \mathbf{Z})
\end{array} \quad ,$$

where the top line is a complex of abelian groups, and the bottom line is an exact sequence. Now setting $S = s + s'_{u_0}$, $T = t + t'_{u_0}$, and $B = b \oplus Id$, we have $a \circ S = T \circ B$, thus $\mathrm{Im}(a \circ S) = \mathrm{Im}(T \circ B)$. But obviously $\mathrm{Im}(S) \supseteq \partial D^{\Omega}(P)$, which has finite index in

$\partial D(P) = F(\{0\})$. Hence the group $a\big(F(\{0\})\big)/\mathrm{Im}(a \circ S)$ is finite. On the other hand $\mathrm{Im}(a \circ S) = \mathrm{Im}(T \circ B) = \mathrm{Im}(T) = \mathrm{Ker}(\delta)$ since B is surjective.

Hence $a\big(F(\{0\})\big)/\mathrm{Im}(a \circ S)$ maps injectively by δ into $H^1(\mathcal{E}, \mathbb{Z})$, which is a torsion-free group by Proposition 6.1. It follows that $a\big(F(\{0\})\big) = \mathrm{Im}(a \circ S)$, hence $F(\{0\}) = \mathrm{Im}(S) + \mathrm{Ker}(a)$. But $\mathrm{Ker}(a) = T(P) \subseteq \partial D^{\Omega}(P) \subseteq \mathrm{Im}(S)$. Thus $F(\{0\}) = \mathrm{Im}(S)$, and S is surjective.

Now let $(u, m) \in \mathrm{Ker}(S)$, for $u = \sum_{U \in \underline{\mathcal{E}}} n_U U \in \mathbb{Z}^{\underline{\mathcal{E}}}$ and $m \in \mathbb{Z}$. Thus

$$\sum_{U \in \underline{\mathcal{E}}} n_U \Omega_{P/\tilde{U}} + m u_0 = 0 \quad .$$

Applying $\mathrm{Defres}^P_{C(Q_0)/Q_0}$ to both sides gives

$$n_{U_0} \Omega_{C(Q_0)/Q_0} + mM = 0 \quad ,$$

for any maximal element U_0 of $\mathcal{E}$. It follows that m is even, i.e. $m \in \mathrm{Im}(i')$, and that n_{U_0} is a multiple of 4. Note that $4 = \tau = r_{U_0}$ in this case. Now since $2u_0 = 0$, we have

$$\sum_{V \in \underline{\mathcal{E}}} n_U \Omega_{P/\tilde{U}} = 0 \quad .$$

It follows that $b(u) \in \mathrm{Ker}(t) = \mathrm{Im}(i)$, and there exists an integer n such that $n_U = n r_U$, for any $U \in \mathcal{E}$. Hence

$$0 = \sum_{V \in \underline{\mathcal{E}}} n_U \Omega_{P/\tilde{U}} = n_{\{0\}} \Omega_{P/\mathbf{1}} + \sum_{V \in \mathcal{E}} n r_U \Omega_{P/\tilde{U}} = (n_{\{0\}} - n r_{\{0\}}) \Omega_{P/\mathbf{1}} \quad ,$$

hence $n_{\{0\}} = n r_{\{0\}}$ since $\Omega_{P/\mathbf{1}}$ has infinite order in $D(P)$. But now $u = i(n)$, and $(u, n) \in \mathrm{Im}(i \oplus i')$, as was to be shown. This completes the proof of the theorem. □

11. The main theorem in the initial cases

The aim of this paper consists in computing the Dade group of all (almost) extraspecial p-groups P for any prime number p. Since the proof of the main result of the previous section proceeds on induction on the order of P, we treat the anchoring points of the induction in a section apart, for the sake of clarity. Let us start by saying which groups constitute the "initial cases". In the hypothesis of the main theorem, we include the condition that the set $\mathcal{E}$ is connected. The reason for this lies in the two following facts. The first one concerns the part of the proof which is based on the structure of $T(P)$. Indeed, for all (almost) extraspecial p-groups, except those of type $T2$ when $p = 2$, the group $T(P)$ turns out to be an infinite cyclic group, generated by Ω_P, in case the set $\mathcal{E}$ is connected, and hence, it is contained in $\partial D^{\Omega}(P)$. Secondly, this assumption on $\mathcal{E}$ allows us to apply the cohomological argument that we recalled in section 2, and that we used in order to finish the proof. Recall that the group $T(P)$ has no torsion for an (almost) extraspecial p-group P of order at least 16, by Lemma 3.2. In this case, $T(P)$ is a free abelian group whose rank depends only on the number of maximal elementary abelian subgroups of P of rank 2, that is, on the number of maximal elements of $\mathcal{E}$ of dimension 1. It is an easy exercise to calculate this number for each (almost) extraspecial p-group, and so we come to the conclusion that $\mathcal{E}$ is not connected if

$$P \in \{M, M * C_{p^2}, D_8, Q_8, D_8 * Q_8, D_8 * C_4\} \quad ,$$

in the notation of section 2, that is M is the extraspecial p-group of order p^3 and exponent p, for an odd prime number p. Indeed, in these cases, the set $\mathcal{E}$ is formed respectively by $p+1$, $p+1$, 2, 0, 5, and 3 lines exactly. So, for each of these 6 groups we have to check that the theorem of previous section holds, that is, we must prove that the group

$$\partial D(P) = \partial D^{\Omega}(P) \cong \mathbb{Z}^{|\mathcal{E}|} \oplus \mathbb{Z}/\tau\mathbb{Z}$$

is generated by the set $\{\Omega_{P/\tilde{V}} \mid \tilde{V} \in m(V),\ V \in \underline{\mathcal{E}}\}$, subject to the single relation $\sum_{V \in \underline{\mathcal{E}}} r_V \Omega_{P/\tilde{V}} = 0$, except maybe in the case $p = 2$ and the ground field contains a cubic root of unity, and P has type $T2$. Let us also recall that the integer τ is defined as being the order of $\Omega_{C_P(Q)/Q}$ in the group $D(C_P(Q)/Q)$ for a subgroup Q of P such that $\pi(Q)$ is a maximal element in $\underline{\mathcal{E}}$. In other words, we have

$$\tau = \begin{cases} 1 & , \quad \text{if } P = D_8 \\ 2 & , \quad \text{if } P = D_8 * C_4, \text{ or if } p \text{ is odd} \\ 4 & , \quad \text{if } P = Q_8, \text{ or if } P = D_8 * Q_8 \end{cases} \quad .$$

Moreover, by definition, we have $r_V = \tau \cdot \left(\sum_{\substack{S \in [s_P] \cap \underline{Q} \\ S \geq_P \tilde{V}}} \mu(\tilde{V}, S) \frac{|Q|}{|S|} \right)$. Hence,

$$r_{\{0\}} = (p - |\mathcal{E}|) \cdot \tau \quad \text{and} \quad r_V = \tau \quad , \forall\, V \in \mathcal{E} \quad .$$

Consider now each "initial case" separately :

• If P has order 8, then the Dade group of P has been computed in [5]. Comparing Proposition 3.4 with Lemma 10.2 in [5], we deduce that the subgroup $\partial D(P)$ coincides with $T(P)$. Then, Theorem 5.4 and Theorem 6.4 in [5] prove the results we want to show, for the dihedral group of order 8, and, respectively, for the quaternion group.

• Let P be one of the groups $M, M * C_{p^2}$, or $D_8 * C_4$. Then we know by [7] that $T(P) = T^{\Omega}(P)$, and hence $T(P) \leq \partial D^{\Omega}(P)$. In this case, the set $\mathcal{E}$ is formed by lines, i.e. it is in bijection with the elementary abelian subgroups of P of rank 2, which are then all maximal in P. In other words, the torsion-free rank of $\partial D(P)$ is equal to $|\mathcal{E}|$, and it is also equal to the rank of the free abelian subgroup $T(P)$ of $\partial D(P)$. Hence the quotient $\partial D(P)/T(P)$ is finite. Consider the map

$$\alpha = \prod_{V \in \mathcal{E}} \mathrm{Defres}^P_{N(V)/\tilde{V}} : \partial D(P) \longrightarrow \prod_{V \in \mathcal{E}} D\big(N(V)/\tilde{V}\big) \quad .$$

The quotients $N(V)/\tilde{V}$ are all cyclic groups of order at least 3, isomorphic to the center Z of P, and thus $D\big(N(V)/\tilde{V}\big) = <\Omega_{N(V)/\tilde{V}}> \cong \mathbb{Z}/2\mathbb{Z} \quad , \forall\, V \in \mathcal{E}$. Moreover,

$$\mathrm{Ker}(\alpha) = \partial D(P) \cap \left(\bigcap_{V \in \mathcal{E}} \mathrm{Ker}\Big(\mathrm{Defres}^P_{N(V)/\tilde{V}}\Big) \right) = T(P) \quad .$$

On the other side, since

$$\mathrm{Defres}^P_{N(V)/\tilde V}\Omega_{P/\tilde U} = \delta_{U,V}\cdot\Omega_{N(V)/\tilde V} \quad ,\forall U,V\in\mathcal{E} \quad ,$$

we have $\alpha\big(\partial D^\Omega(P)\big) = \prod_{V\in\mathcal{E}} D\big(N(V)/\tilde V\big)$. Therefore, we finally get the following commutative diagram with exact rows (where i denotes the inclusion), proving the equality $\partial D(P) = \partial D^\Omega(P)$, as required.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T(P) & \longrightarrow & \partial D^\Omega(P) & \xrightarrow{\alpha} & \prod_{V\in\mathcal{E}} D(N(V)/\tilde V) & \longrightarrow 0 \\
 & & {\scriptstyle \mathrm{Id}}\big\downarrow & & {\scriptstyle i}\big\downarrow & & {\scriptstyle \mathrm{Id}}\big\downarrow & \\
0 & \longrightarrow & T(P) & \longrightarrow & \partial D(P) & \xrightarrow{\alpha} & \prod_{V\in\mathcal{E}} D(N(V)/\tilde V) & \longrightarrow 0
\end{array}$$

Notice also that the relation 5.2 gives in this case

$$2\cdot\Omega_P = 2\cdot\sum_{V\in\mathcal{E}}\Omega_{P/\tilde V} \quad ,$$

but $\Omega_P - \sum_{V\in\mathcal{E}}\Omega_{P/\tilde V} \neq 0$, since $\Omega_P - \sum_{V\in\mathcal{E}}\Omega_{P/\tilde V} \notin T(P)$. Hence, $D^t(P)$ is a cyclic group of order 2, generated by $\Omega_P - \sum_{V\in\mathcal{E}}\Omega_{P/\tilde V}$. Thus, the statement of the main theorem also holds in these three "initial" cases.

• Finally, consider the case where $P = D_8 * Q_8$, and the ground field k contains a cubic root of unity. Then $\mathcal{E}$ consists in 5 lines, and the group $N(V)/\tilde V$ is a quaternion group of order 8 for each subspace $V\in\mathcal{E}$. Hence, we have an exact sequence

$$0\longrightarrow T_k(P)\xrightarrow{\ i\ }\partial D_k(P)\xrightarrow{\alpha_k}\prod_{V\in\mathcal{E}} T_k\big(N(V)/\tilde V\big),$$

where i denotes the inclusion and where α_k is the map

$$\prod_{V\in\mathcal{E}}\mathrm{Defres}^P_{N(V)/\tilde V} : \partial D_k(P)\longrightarrow \prod_{V\in\mathcal{E}} T_k\big(N(V)/\tilde V\big) \quad ,$$

which is well defined, since we have $\partial D_k\big(N(V)/\tilde V\big) = T_k\big(N(V)/\tilde V\big)$, $\forall\, V\in\mathcal{E}$. Since the results depend *a priori* on the ground field k, we have specified which Dade groups we are dealing with by subscripts.

The group $\partial D_k\big(N(V)/\tilde V\big)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. It is generated by $\Omega_{N(V)/\tilde V}$, of order 4, and by an exotic element Λ_V, of order 2 (see sections 6 and 10 in [5]). There are exactly two exotic elements in $D_k\big(N(V)/\tilde V\big)$, and the other one is equal to $\Lambda_V + 2\Omega_{N(V)/\tilde V}$.

The group $T_k(P) = \mathrm{Ker}(\alpha_k)$ is a free abelian group of rank 5, and it is generated by Ω_P and the five elements $4\Omega_{P/\tilde V}$, $V\in\mathcal{E}$. Indeed $4x\in T_k(P)$, $\forall\, x\in\partial D_k(P)$, since $\prod_{V\in\mathcal{E}} T_k\big(N(V)/\tilde V\big)$ has exponent 4. In order to show that these elements generate the whole group of endo-trivial modules, we need the less obvious results proved in [7]. Consider now $V\in\mathcal{E}$ and an elementary abelian subgroup A of P of rank 2. That is, there exists $U_A\in\mathcal{E}$ such that $A = \pi^{-1}(U_A)$. Let $\tilde V\in m(V)$, and let us determine the element $\mathrm{Res}_A^P \Omega_{P/\tilde{V}}$ of $D_k(A) = T_k(A)$. We have $\mathrm{Res}_A^P \Omega_{P/\tilde{V}} = \Omega_X$, where X is the A-set $\mathrm{Res}_A^P(P/\tilde{V})$. Since A is normal in P, the Mackey formula gives us

$$X = \coprod_{g \in [P/\tilde{V}\cdot A]} A/A \cap {}^g\tilde{V} = \begin{cases} 4{\cdot}A\,, & \text{if } V \neq U_A \\ 2{\cdot}(A/\tilde{V}) \coprod 2{\cdot}(A/{}^g\tilde{V})\,, & \text{if } V = U_A \end{cases} ,$$

for any $g \in P$ such that ${}^g\tilde{V} \neq \tilde{V}$. It follows that

$$\mathrm{Res}_A^P \Omega_{P/\tilde{V}} = (-1)^{\delta_{V,U_A}} \Omega_A .$$

From this computation, and using some elementary linear algebra, we deduce the relationship between the six elements of $T_k(P)$ described above, unique up to multiplication by a scalar. Namely, we get:

$$-12\Omega_P + \sum_{V \in \mathcal{E}} 4\Omega_{P/\tilde{V}} = 0 \quad .$$

Notice that this equality corresponds to the one given in section 5.2. Indeed, in the present case $\tau = 4$ and the maximal subgroups in $\mathcal{Q}$ have order 2. Thus, $r_V = \tau = 4$ for any $V \in \mathcal{E}$ and

$$r_{\{0\}} = 4\big(\mu(\mathbf{1},\mathbf{1}){\cdot} \mid Q_0 \mid + \sum_{V \in \mathcal{E}} \mu(\mathbf{1},\tilde{V}){\cdot}1\big) = 4\big(2 + 5{\cdot}(-1)\big) = -12 \quad .$$

We now want to show that the image of the map α_k is equal to the inverse image Δ_k under the canonical surjection

$$\prod_{V \in \mathcal{E}} T_k\big(N(V)/\tilde{V}\big) \longrightarrow \prod_{V \in \mathcal{E}} T_k\big(N(V)/\tilde{V}\big)/T_k^{\Omega}\big(N(V)/\tilde{V}\big)$$

of the diagonal subgroup of order 2 of the quotient group (which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^5$). To prove this, we can make the following assumption on k :

Assumption 11.1 *There exists and $\mathbb{F}_2$-endomorphism γ of k exchanging the non-trivial cubic roots of unity.*

Indeed there exists a subfield l of k such that the extension $l/\mathbb{F}_2$ is purely transcendental and the extension k/l is algebraic (such an l can be obtained by considering the subfield of k generated by a transcendence basis of k over $\mathbb{F}_2$). Let L denote an algebraic closure of l containing k, and let f denote the subfield of k generated by l and the cubic roots of unity. Then there is an l-isomorphism γ of f exchanging the cubic roots of unity, and such a γ can be extended to an l-isomorphism of L. Hence the assumption holds for the fields f and L. Now we have a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & T_L(P) & \longrightarrow & \partial D_L(P) & \xrightarrow{\alpha_L} & \prod_{V \in \mathcal{E}} T_L\big(N(V)/\tilde{V}\big) \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & T_k(P) & \longrightarrow & \partial D_k(P) & \xrightarrow{\alpha_k} & \prod_{V \in \mathcal{E}} T_k\big(N(V)/\tilde{V}\big) \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & T_f(P) & \longrightarrow & \partial D_f(P) & \xrightarrow{\alpha_f} & \prod_{V \in \mathcal{E}} T_f\big(N(V)/\tilde{V}\big)
\end{array}$$

where the vertical arrows are given by scalar extension from f to k and from k to L. The left hand side vertical arrows are isomorphisms, since the three groups of endotrivial modules have the same generators by the above computation of $T_k(P)$. The right hand side vertical arrows are also isomorphisms, by the above description of the group $T_K(Q_8)$, for a field K containing cubic roots of unity. If we know that $\mathrm{Im}(\alpha_L) = \Delta_L$ and $\mathrm{Im}(\alpha_f) = \Delta_f$, then it follows that $\mathrm{Im}(\alpha_k) = \Delta_k$.

Thus we can suppose that Assumption 11.1 holds for k. Then the endomorphism γ acts on the Dade group of any 2-group P over k. In particular, it maps the element Λ_V of $D\big(N(V)/\tilde{V}\big)$ to the element $\Lambda_V + 2\Omega_{N(V)/\tilde{V}}$.

Now, the equalities $\mathrm{Defres}^P_{N(V)/\tilde{V}}(\Omega_{P/\tilde{U}}) = \delta_{V,U}\Omega_{N(V)/\tilde{V}}$, $\forall U, V \in \mathcal{E}$ imply that

$$\alpha\big(\partial D^\Omega(P)\big) = \prod_{V\in\mathcal{E}} T^\Omega\big(N(V)/\tilde{V}\big) \quad .$$

Set $\Lambda = \mathrm{Teninf}^P_{N(U)/\tilde{U}}\Lambda_U$, for an arbitrary $U \in \mathcal{E}$ and an arbitrary exotic element Λ_U in $D\big(N(U)/\tilde{U}\big)$. Then, $\Lambda \in \partial D(P)$ and $\mathrm{Defres}^P_{N(V)/\tilde{V}}\Lambda$ is an exotic element of $D\big(N(V)/\tilde{V}\big)$, $\forall V \in \mathcal{E}$, by Lemma 10.2. It follows that the image of $\alpha_k(\Lambda)$ by the surjection

$$\prod_{V\in\mathcal{E}} T_k\big(N(V)/\tilde{V}\big) \longrightarrow \prod_{V\in\mathcal{E}} T\big(N(V)/\tilde{V}\big)/T_k^\Omega\big(N(V)/\tilde{V}\big)$$

generates a diagonal subgroup of order 2 in the quotient group. Hence $\mathrm{Im}(\alpha_k) \supseteq \Delta_k$.

Conversely, suppose that u is an element of $\partial D_k(P)$. For each $V \in \mathcal{E}$, fix an exotic element λ_V of $D\big(N(V)/\tilde{V}\big)$, and set $\omega_V = \Omega_{N(V)/\tilde{V}}$ for short. Then

$$\alpha_k(u) = \Big(a_V\omega_V + x_V\lambda_V\Big)_{V\in\mathcal{E}} \quad ,$$

for uniquely defined coefficients $a_V \in \mathbb{Z}/4\mathbb{Z}$ and $x_V \in \mathbb{Z}/2\mathbb{Z}$. Since γ and α_k commute, it follows that

$$\alpha_k\big(\gamma(u)\big) = \Big((a_V + 2x_V)\omega_V + x_V\lambda_V\Big)_{V\in\mathcal{E}} \quad ,$$

Set $v = \gamma(u) - u - \sum_{V\in\mathcal{E}} 2x_V\Omega_{P/\tilde{V}}$. Then $\alpha_k(v) = 0$, hence $v \in T(P)$, and so there are (non unique) integers m_V, for $V \in \mathcal{E}$, and a (non unique) integer n such that

$$v = n\Omega_{P/1} + \sum_{V\in\mathcal{E}} 4m_V\Omega_{P/\tilde{V}} \quad . \tag{11.2}$$

Let U be any element of $\mathcal{E}$. By restriction to $A = \pi^{-1}(U)$, which is an elementary abelian subgroup of rank 2 of P, equality 11.2 gives

$$\mathrm{Res}^P_A v = (n - 4m_U + 4\sum_{V\neq U} m_V)\Omega_{A/1} \quad .$$

On the other hand, since γ commutes with restriction, and since γ acts trivially on $D_k(A)$, it follows that $\mathrm{Res}^P_A(\gamma(u) - u) = 0$, and the definition of v gives

$$\mathrm{Res}^P_A v = (2x_U - \sum_{V\neq U} 2x_V)\Omega_{A/1} \quad .$$

Since $\Omega_{A/1}$ has infinite order in $D_k(A)$, it follows that

$$2x_U - \sum_{V\neq U} 2x_V = n - 4m_U + 4\sum_{V\neq U} m_V \quad .$$

Setting $x = \sum_{V \in \mathcal{E}} x_V$ and $m = \sum_{V \in \mathcal{E}} m_V$, this can also be written as follows

$$4x_U - 2x = n - 8m_U + 4m \quad ,$$

which gives $4x_U + 8m_U = n + 2x + 4m$, which is independent of U. This shows in particular that for any U and U' in $\mathcal{E}$, we have $4x_U \equiv 4x_{U'}$ modulo 8. Thus $x_U \equiv x_{U'}$ modulo 2, showing that $\alpha_k(u) \in \Delta_k$.

Finally, we conclude that the group $\partial D_k(P)$ is isomorphic to $\mathbb{Z}^5 \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and that it is generated by the seven elements Ω_P, $\Omega_{P/\tilde{V}}$, $V \in \mathcal{E}$ and Λ, submitted to the two following relations

$$2\Lambda = 0 \quad \text{and} \quad -12\Omega_P + \sum_{V \in \mathcal{E}} 4\Omega_{P/\tilde{V}} = 0 \quad .$$

This ends the proof of the main theorem in the initial cases. □

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Serge Bouc, Institut de Mathématiques, Université Paris 7-Denis Diderot, 2 Place Jussieu, 75251 - Paris Cedex 05. FRANCE.
`E-mail : bouc@math.jussieu.fr`

Nadia Mazza, Department of Mathematics, University of Georgia, Athens, Georgia 30602, USA.
`E-mail : nmazza@math.uga.edu`