On the choice of an auxiliary function in the $M/G/\infty$ estimation

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Abstract

The use of auxiliary functions in nonparametric inference for the $M/G/\infty$ queueinging model is considered. Estimation of the service time distribution G is challenging when only limited information about the busy/idle cycle is available. It is shown, using diagnostic plots of estimators, that a standard auxiliary function aimed at providing numerical stability fails in that regard but that a reliable auxiliary function can be constructed. The improvement made by the alternative auxiliary function is demonstrated.

Key words: inversion, Laplace transform, nonparametric density estimation, $M/G/\infty$ model

1 INTRODUCTION

Interest in internet traffic modeling led to the development of several probability models. Among those, Kulkarni et al. (2001) proposed a product modeling approach to study connection flows. Related to that product modeling framework, the $M/G/\infty$ queueinging model appeared as a limiting process in Resnick and Samorodnitsky (2001). The $M/G/\infty$ model has also been used to model aggregated traffic [Guerin et al.(2003)]. See Chapter 7 of Kulkarni (1995) for an introduction to queueinging theory and this model.

We consider the nonparametric estimation of the service time distribution G, with limited information, in an $M/G/\infty$ queueinging model, where the arrival

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process is Poisson with intensity λ. For the discrete-time $M/G/\infty$ queueing model, see Pickands and Stine (1997). Bingham and Pitts (1999) developed nonparametric methods for the $M/G/\infty$ queueing model based on the Laplace transform and gave an analytic representation in terms of infinite series of convolutions.

The auxiliary function used in Bingham and Pitts (1997) is a simple and natural choice, which forms the basis of our first approach in Section 2. Often the idea of converting a Laplace inversion formulation to that of Fourier inversion simplifies the problem and works equally well. See Dubner and Abate (1968) and Abate et al. (1999). But it turns out that caution is required to prevent numerical problems for certain cases. Particular numerical instability issues encountered by the auxiliary function are investigated in detail in Section 3. In Section 4, we propose a new auxiliary function, which solves the instability problem. The modified auxiliary function provides a basis for the estimators developed by Hall and Park (2004).

2 FORMULATION OF THE PROBLEM

Let Y have the distribution of the busy period in the $M/G/\infty$ queueing model, whose distribution function is denoted by B. The Laplace transform of B is given by

$$\int_0^\infty e^{-st}\,dB(t) = 1 + s/\lambda - \Big(\lambda \int_0^\infty e^{-st} a_0(t)\ dt\Big)^{-1} \qquad \Re\,(s) > 0\,, \tag{1}$$

where

$$a_0(t) = \exp\Big[-\lambda \int_0^t \{1 - G(x)\}\,dx\Big]\,. \tag{2}$$

This fundamental result is due to Shanbhag (1966) and Hall (1988), and is named the Shanbhag-Hall theorem. Note that a_0 leads directly to both the service time distribution G and its density g. In particular, solving (2) for G and differentiating gives

$$G = \frac{a_0'}{\lambda\, a_0} + 1\,, \quad g = \frac{1}{\lambda}\left\{\frac{a_0''}{a_0} - \frac{(a_0')^2}{a_0^2}\right\}. \tag{3}$$

The key to construction of an estimator of G for this situation is to find reliable estimators of a_0, a_0' and a_0''. Define the characteristic function of the busy period distribution, B, to be $\psi(s) = E(e^{isY})$ for real s and denote

$$\alpha(s) = \lambda^{-1}\left\{1 - \psi(s) - is\lambda^{-1}\right\}^{-1}. \tag{4}$$

From (1), it follows that

$$\alpha(s) = \int_0^\infty a_0(t)\, e^{ist}\, dt\,. \tag{5}$$

Thus, the auxiliary function a_0 may be represented using the Fourier inversion formula as

$$a_0(t) = \frac{2}{\pi} \int_0^\infty \{\Re\, \alpha(s)\}\ \cos(st)\, ds\,, \tag{6}$$

which allows simple estimation by replacing α in (4) by its empirical counterpart.

3 NUMERICAL PERFORMANCE

This section provides a numerical example that shows instability of the use of the auxiliary function a_0. Consider an exponential assumption on G with rate parameter, μ, as a special case of an $M/G/\infty$ queueing model. We focus on three cases: $(\lambda = 1, \mu = 1), (\lambda = 2, \mu = .5)$ and $(\lambda = .5, \mu = 2)$, to study the effect of longer or shorter busy periods.

Figure 1(a) shows the estimates of $\lambda\, a_0(\cdot)$ along with the true value for the three different cases. They were computed based on a sample size of 1000. The top and bottom cases in the left shows very poor estimation, while the second appears better.

To examine the possibility of too small a sample size, the estimators were calculated for much larger sample sizes. Figure 2 shows a zoomed in view for $\lambda = 2, \mu = .5$ (the best case shown in Figure 1(a)), which suggests rapid convergence to a curve different from the true curve.

A Q-Q diagnostic plot can be used to explore the cause of this poor estimation. We examined the tail behavior of the samples of the busy period distribution with the relatively longer $(\lambda = 2, \mu = .5)$ and shorter $(\lambda = .5, \mu = 2)$ ones. The tail index was estimated by distribution matching at two quantiles (.95, .995), assuming both Pareto-like tails and Weibull-like tails, which are presented in

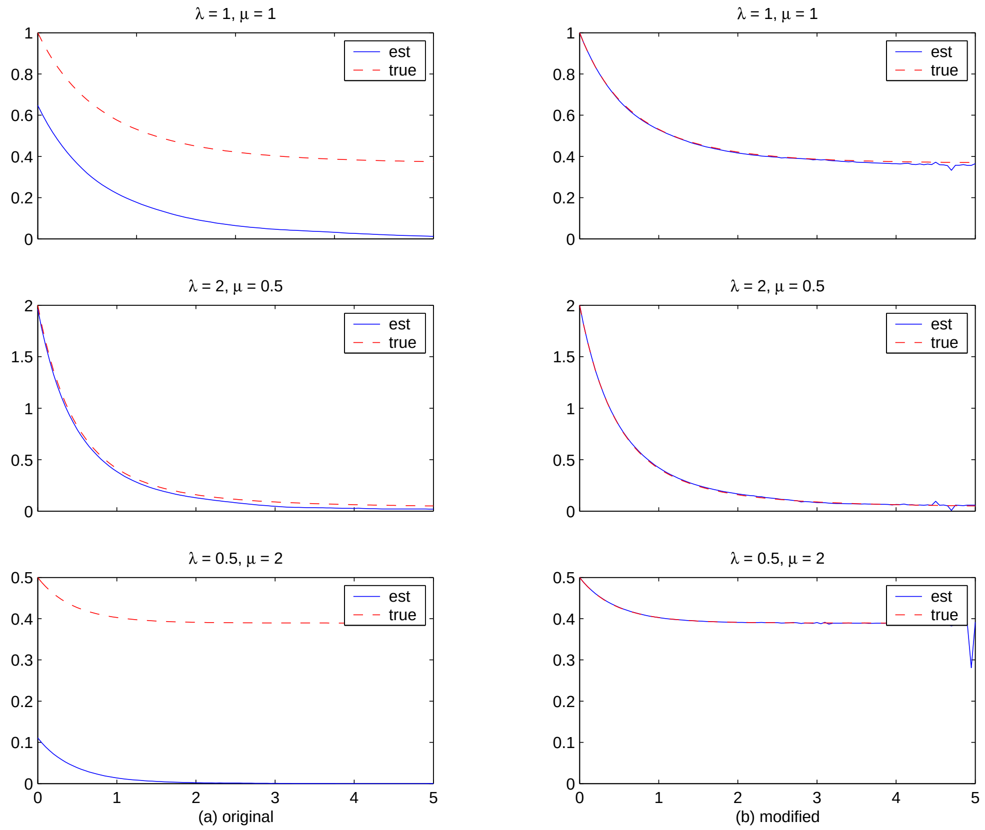

Fig. 1. Estimates $\lambda\, a_0(t)$ from simulated data for different rate parameter. While plots in the left ((a) original) suggest large estimation error for $\mu \geq \lambda$, plots in the right ((b) modified) show serious improvements.

Figure 3. The Q-Q plot shows how closely the data correspond to the assumed distribution. They are closer to the true distribution when the curve is closer to the 45 degree line. To account for sampling variability, 100 samples from the true distribution, also of the same size of 10000, are overlayed to give an envelope indicating the natural sampling variation. The thick curve is the actual data Q-Q plot and α is an estimate of the respective shape parameter. Although possibly longer, the tail index is far from that of heavy tails, which rules out heaviness of the tails as a cause of the problem.

Another potential cause of the instability seen in Figure 1(a) is the Fourier inversion of a_0. Consider $s = -iy \, (y \neq 0)$. Then we have

$$\int_0^\infty e^{iyt} a_0(t)\, dt = \lambda^{-1}\{1 - iy\lambda^{-1} - \phi(y)\}^{-1} .$$

Observe that the right hand side is well defined so that the left hand side

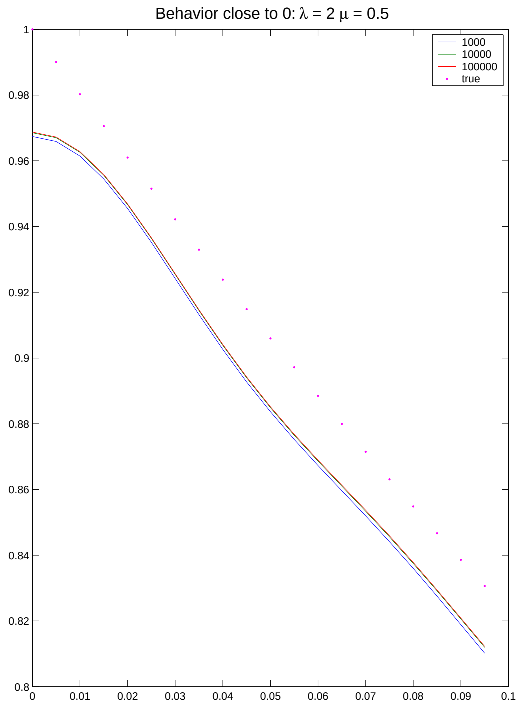

Fig. 2. Estimating $a_0(t)$ with increasing sample size: zoomed in (i.e. $t \in (0, .1)$) view for the second case ($\lambda = 2, \mu = .5$) shown in Figure 1(a).

converges for $y \neq 0$. But it does not converge absolutely because

$$\int_0^\infty |e^{ist} a_0(t)|\, dt = \int_0^\infty a_0(t) \geq \int_0^\infty e^{-\lambda\tau} = \infty\,,$$

where τ is the mean of the service time distribution satisfying $\tau = \int_0^\infty \{1 - G(x)\}\, dx$. Similarly, it can be shown that a_0 is not square integrable. In either case, Fourier inversion formula cannot be verified. While such integrability conditions often seem like a technicality, this turns out to be critical. Alternatively we may consider $\Re\alpha(s)$ for real $s > 0$. Write $\phi_1(s) = \Re\phi(s)$ and $\phi_2(s) = \Im\phi(s)$. Then, we have

$$\lambda\Re\alpha(s) = \frac{1 - \phi(s)}{\{\phi_1(s) - 1\}^2 + \{\phi_2(s) + (s/\lambda)\}^2}\,.$$

As $s \to 0$, it can be shown that

$$\lim_{s\to 0} \lambda\Re\alpha(s) = \frac{EY^2}{2(EY + \lambda^{-1})^2}\,,$$

while it is not defined at $s = 0$. Again, one would need to consider modifications in the transform as in Beylkin (1995) to deal with such cases. Some

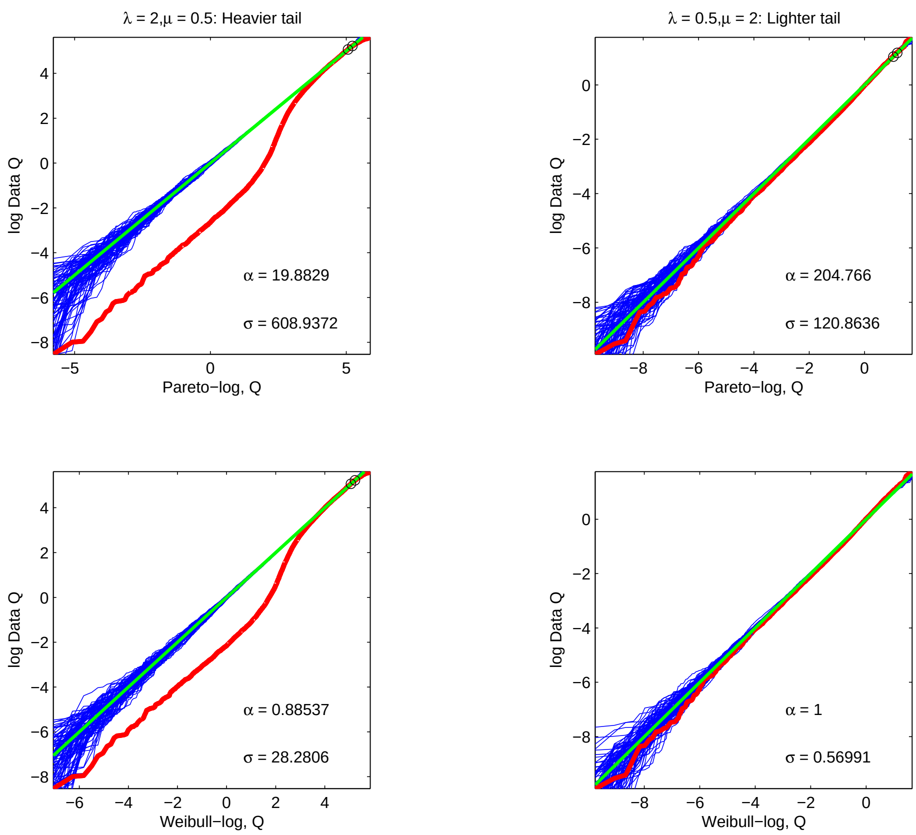

Fig. 3. Q-Q plots for testing Weibull-like tail and Pareto-like tail for the busy period distribution. Top plots are for Weibull-like tail and bottom ones are for Pareto-like tail. Left plots assume $(\lambda = 2, \mu = .5)$ and right ones assume $(\lambda = .5, \mu = 2)$. The estimates suggest to exclude the possibility of heaviness of tails.

other numerical stability issues related to Fourier analysis can be found e.g. in Deraedt (1996) and Korn (2005). In the next section, we propose a simple solution by reformulating the problem.

4 MODIFIED AUXILIARY FUNCTION

A means of overcoming this integrability problem is to consider complex values of s in (1). In particular, define an analytic characteristic function $\psi(s) = E(e^{isY})$ for a complex value $s = u + iv$, where u and v are real numbers. Equivalently, we have

$$\int_0^\infty e^{ist}\, a_0(t)\, dt = \int_0^\infty e^{iut}\{e^{-vt} a_0(t)\}\ dt$$

$$= \lambda^{-1}\Big\{1 + (v - iu)\lambda^{-1} - \int_0^\infty e^{iut}e^{-vt}\,dB(t)\Big\}^{-1}.$$

We shall always take $v > 0$. Observe that

$$\int_0^\infty e^{-vt}a_0(t)\,dt \le \int_0^\infty e^{-vt}\,dt < \infty\,.$$

So, a more tractable auxiliary function is $a(t) = e^{-vt}a_0(t)$. The Fourier transforms in the complex plane has been introduced and developed in Paley and Wiener (1934). For the transform of a function vanishing exponentially, the complex Fourier transform is well defined and converges absolutely and uniformly. Here we have only borrowed the complex formulation to obtain an exponential decaying function and the modified problem allows us to apply the ordinary real Fourier transform. Therefore, we may still be able to express the service time distribution function G and its density g in terms of a and its derivatives as

$$G = \frac{a'}{\lambda\, a} + \frac{v}{\lambda} + 1\,, \qquad g = \frac{1}{\lambda}\left\{\frac{a''}{a} - \frac{(a')^2}{a^2}\right\}.$$

Thus, an estimator of G and its density g may be constructed based on estimators of a, a' and a''. Figure 1(b) in the right shows serious improvement of the estimation using the same setup in the left. Now the large gaps between the estimated and the true curves have all disappeared, although there is some numerical instability towards the right end.

The choice of v is related to finite approximation error. Some simulation studies in Hall and Park (2004) suggest robustness of the choice of v in the performance of the estimator on a reasonable range of values. However numerical error analysis is beyond our scope and for further details we refer the reader to Dubner and Abate (1968).

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