Wedge Product of Submanifold Distributions with Applications to Classical Electromagnetism

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Abstract

In this thesis we propose a new and novel approach to expressing distributions on submanifolds. To this end we introduce a set of objects called submanifold distributions constructed out of the embedding maps and the tools of exterior calculus. We then define a multiplication between these objects called the wedge product. Both of the aforementioned are succinctly defined without reference to a coordinate system. Following with the coordinate free theme we also define the pullback of a submanifold distribution and show that this provides a powerful framework for calculating solutions to linear differential equations via Green’s methods. We then investigate multipoles in terms of submanifold distributions and find an unusual result for quadrupoles. It is remarkable that quadrupoles have been extensively studied for over a century yet there is no mention of quadrupole transformations between general coordinate systems. More so, we show that quadrupoles do not in fact transform as a tensor, as suggested by the literature, but instead possess more complicated transformation rules that involve second order derivatives and an integral. We conclude this thesis with the calculation of the Liénard-Wiechert field for moving dipoles and quadrupoles using the newly developed machinery of wedge products and submanifold distributions.
Declaration

I declare that the original ideas in this thesis are my own work carried out in collaboration with Dr. Jonathan Gratus. No portion of this work has been submitted towards another degree or qualification at this or any other University. An article based on the ideas concerned with quadrupole transformations has been published in Proceedings of the Royal Society A [1].
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Chapter 1

Introduction

Distributions have had a prevailing role in mathematical physics in one form or another for over two centuries. A rigorous theory of distributions was developed by Schwartz in the 1950’s [2]. He defined a distribution as a continuous, linear functional that acts on the space of test functions with compact support. It was not long after that the theory was modified to accommodate manifolds as the underlying base space and this more general theory was developed by de Rham [3]. His approach treats distributions as differential forms which act on test forms; smooth differential forms with compact support. Distributions as a mathematical tool come into their own when modelling extended sources whether that be of mass or charge when concentrated on submanifolds in the ambient space. For instance a shock wave is a lower dimensional surface which propagates through a medium [4, 5]. This is described by a discontinuous function which is continuous on the surface but abruptly falls to zero off it. Another example is a moving point particle in spacetime. This is represented by a worldline which has a constant value of charge/mass on the curve but is zero off the curve. Both of the aforementioned are instances of the Dirac delta function.

In order to perform calculations one needs to set up a coordinate system and then write down the Dirac deltas and its derivatives in these coordinates. The transformation rules going from one coordinate system to another is highly involved. As
we shall see in Chapter 6 the transformations for quadrupoles involve second order derivatives and even an integral. One may argue that a coordinate free definition of the geometric object is more aesthetically pleasing, but a coordinate free approach also leads to useful, geometrically inspired, approximation methods and numerical integration schemes. As already stated the coordinate transformation, especially for distributions, are complicated. Therefore we propose a new approach to defining distributions on submanifolds which avoids coordinates from the outset. The advantage of defining distributions in a coordinate free manner is that one can simply choose a coordinate system to perform calculations in. In the shock wave example one may want to use co-moving coordinates or a coordinate system in the lab frame. The conventional approach would require setting up the problem in one set of coordinates then perform lengthy coordinate transformation to obtain an expression in the other. In almost all case cases this will be a highly involved calculation. However the coordinate free approach bypasses this step and allows the user to choose a coordinate system at will with the knowledge that the system is valid in that set of coordinates.

To achieve this goal we utilize de Rham’s pushforward map [3] which takes distributional forms from manifold $\mathcal{N}$ into another manifold $\mathcal{M}$. Distributions on $\mathcal{M}$, which have support on the submanifold $\mathcal{N}$, are called submanifold distributions (SMDs). These are the Dirac deltas and their derivatives when written in local coordinates.

When two or more submanifolds intersect we define a wedge product, a bilinear map which produces another submanifold distribution on the intersection. The key innovation in this thesis is that we define the wedge product of two submanifold distributions in terms of four axioms without any reference to coordinates. As long as these axioms lead to a well defined wedge product then all the issues of coordinate transformations do not arise. It is therefore necessary to show the wedge product exists and that it is unique. To this end it is necessary to employ adapted coordinates and define another product, which we call the O-wedge.

- For existence we show that the coordinate dependent O-wedge product satisfies
The axioms.

- For uniqueness, we choose a new adapted coordinate system and show that, in this coordinate system, the axioms imply that the wedge product is the same as the O-wedge.

The four axioms needed to define the wedge product are as follows. The first says the wedge product of two pushforwards is another pushforward on the intersection. The second is that the wedge product of two SMDs is antisymmetric whilst the third says it is also associative. Finally the wedge product is Leibniz with respect to internal contractions and Lie derivatives.

We then use the wedge product to investigate solutions to linear differential equations via Green’s functions. The standard approach to Green’s function techniques on manifolds is through the use of bitensors [6, 7] or double forms [3, 8, 9]. The alternative which we employ is to treat it as a distributional form on a product manifold. The advantage of this approach is that mappings between spaces are easily handled by the pushforward and pullback and we can make full use of the tools of exterior calculus.

Having discussed the key properties of Green’s functions we then consider the Laplace-Beltrami operator where the Green’s function is the well known “retarded Green’s function” [10]. One of the key results in this thesis is the definition of the retarded Green’s function in terms of an SMD. Since this has support on the forward light cone one may question whether it can be an SMD, since the forward light cone is not smooth at the vertex, and furthermore the value of the function diverges as one approaches the vertex. We show that we can write the retarded Green’s function as a bona fide SMD over a closed embedding in the product manifold with the diagonal removed.

In Chapter 2 we give an overview of distributions on \( \mathbb{R}^n \), distributions on manifolds, multi-indices, and we construct the set of SMDs using de Rham’s pushforward map. Chapter 3 starts with a discussion of transverse submanifolds then concerns
itself with the wedge product and contains all the proofs necessary for its definition. In order to use Green’s method we need to define the pullback of an SMD, this again is defined in a coordinate free manner in terms of 4 axioms. This forms the content of Chapter 4. In Chapter 5 we investigate solutions to linear differential equations using Green’s functions and the wedge product from the previous chapter. In Chapter 6 we investigate moving multipole sources, specifically dipoles and quadrupoles. Finally in Chapter 7 we calculate the Liénard-Wiechert field for a moving monopole, dipole and quadrupole using the wedge product.
Chapter 2

Preliminaries

We first give an overview of distributions on $\mathbb{R}^n$ then proceed onto manifolds. Let $\mathbb{R}^n$ have coordinates $(x^1, \ldots, x^n)$. Given a function $\rho$ on $\mathbb{R}^n$ the support of $\rho$, $\text{supp}(\rho)$, is the closure on the set of points for which $\rho$ is non-zero. If this set is compact then the function is said to have compact support.

**Definition 2.0.1.** The vector space of all smooth functions on $\mathbb{R}^n$ that have compact support is denoted as $\mathcal{D}(\mathbb{R}^n)$. Elements of this set are called test functions.

$\mathcal{D}(\mathbb{R}^n)$ can be made into a topological vector space by introducing a family of semi-norms. This defines the notion of convergence of sequences of test functions in $\mathcal{D}(\mathbb{R}^n)$ and also provides a means to define continuous maps on this set. The topological dual of $\mathcal{D}(\mathbb{R}^n)$ is another topological vector space denoted as $\mathcal{D}'(\mathbb{R}^n)$. Elements of $\mathcal{D}'(\mathbb{R}^n)$ are called distributions.

**Definition 2.0.2.** A distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is a continuous, linear functional on the space of test functions

$$T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{R},$$

$$(T, \rho) \mapsto \mathbb{R},$$

(2.1)
with vector space structure defined as
\[
(T + S, \rho) = (T, \rho) + (S, \rho), \quad T, S \in \mathcal{D}'(\mathbb{R}^n), \quad \rho \in \mathcal{D}(\mathbb{R}^n),
\]
\[
(\lambda T, \rho) = \lambda (T, \rho), \quad \lambda \in \mathbb{R}.
\]

An important subspace of \( \mathcal{D}'(\mathbb{R}^n) \) is the space of regular distributions. These are locally integrable functions \( f \) (integrable on every compact set of \( \mathbb{R}^n \)) defined as
\[
(f^D, \rho) = \int f \rho \, dx^1 \cdots dx^n, \quad \rho \in \mathcal{D}(\mathbb{R}^n),
\] (2.2)
where \( f^D \in \mathcal{D}'(\mathbb{R}^n) \) is the regular distribution associated with \( f \). Another member of \( \mathcal{D}'(\mathbb{R}^n) \) is the Dirac delta distribution defined by
\[
(\delta, \rho) = \rho(0), \quad \rho \in \mathcal{D}(\mathbb{R}^n). \]
(2.3)

Distributions that do not correspond to a locally integrable function are called singular. The Dirac distribution is an example of a singular distribution. Note that the Dirac delta distribution \( \delta \) in (2.3) acts on \( \rho \) and returns the value of the test function at the origin. Similarly we can define a distribution that evaluates a test function on a hyperplane in \( \mathbb{R}^n \). Let \( N \subset \mathbb{R}^n \) be defined by the equation \( x^1 = 0 \) then the Dirac distribution on \( N, \delta_N \), is defined as
\[
(\delta_N, \rho) = \int_{\mathbb{R}^{n-1}} \rho(0, x^2, \ldots, x^n) \, dx^2 \cdots dx^n.
\] (2.4)

This type of distribution will be cast into the language of submanifold distributions in Section 2.5. Some useful operations on distributions are as follows.

**Definition 2.0.3.** Let \( T \in \mathcal{D}'(\mathbb{R}^n) \).

(i) The product of a distribution with a smooth function \( g \) is another distribution defined by
\[
(gT, \rho) = (T, g\rho), \quad \rho \in \mathcal{D}(\mathbb{R}^n).
\] (2.5)

(ii) The partial derivative of a distribution is again a distribution defined by
\[
\left( \frac{\partial T}{\partial x^i}, \rho \right) = - \left( T, \frac{\partial \rho}{\partial x^i} \right), \quad \rho \in \mathcal{D}(\mathbb{R}^n), \quad i = 1, \ldots, n.
\] (2.6)
A distribution can be differentiated an arbitrary number of times since \( \rho \) is smooth. An important property of distributions is its support. Unlike functions, distributions are not defined by their values on \( \mathbb{R}^n \) but rather by their action on test functions.

**Definition 2.0.4.** The support, \( \text{supp}(T) \subset \mathbb{R}^n \), of \( T \in \mathcal{D}'(\mathbb{R}^n) \) is defined as the set

\[
\text{supp}(T) = \bigcap \left\{ U \subset \mathbb{R}^n \bigg| U \text{ is closed and if} \right. \\
\left. \rho \in \mathcal{D}(\mathbb{R}^n), \text{ supp}(\rho) \in \mathbb{R}^n \setminus U \text{ then } \langle T, \rho \rangle = 0 \right\}.
\] (2.7)

In other words \( \text{supp}(T) \) is the smallest closed subset of \( \mathbb{R}^n \) such that for every test function \( \rho \) with \( \text{supp}(T) \cap \text{supp}(\rho) = \emptyset \) then \( \langle T, \rho \rangle = 0 \). From (2.2), \( \text{supp}(f^D) \) is equal to the support of \( f \) as a function whilst the Dirac delta distribution (2.3) has support at a single point. Closely related to the notion of support is singular support.

**Definition 2.0.5.** The singular support, \( \text{sing supp}(T) \), is the closure on the set of points where \( T \) fails to be a smooth function.

Therefore the singular support is the smallest closed set outside of which the distribution \( T \) is equal to a smooth function. A distribution can be multiplied by a smooth function, but in general one cannot multiply two or more distributions together [11]. However, there is a special case when multiplication is possible. If \( T, S \in \mathcal{D}'(\mathbb{R}^n) \) then the product \( TS \) is defined when

\[
\text{sing supp}(T) \cap \text{sing supp}(S) = \emptyset.
\] (2.8)

### 2.1 Distributions on Manifolds

Now consider a smooth, orientable, Hausdorff and paracompact manifold \( \mathcal{M} \) of dimension \( m \). On a manifold the only objects that can be integrated are \( m \)-forms (densities if the manifold is non-orientable) so in order for regular distributions (2.2) to be defined on manifolds the argument in the integral must be a top form (or a density in the non-orientable case) [12]. Therefore test functions \( \rho \) in (2.2) are to be replaced by test \( m \)-forms.
2.1. Distributions on Manifolds

**Definition 2.1.1.** A test \(m\)-form is a smooth differentiable \(m\)-form on \(\mathcal{M}\) with compact support. The vector space of test \(m\)-forms is denoted as

\[
\Gamma_0 \Lambda^m \mathcal{M} = \{ \phi \in \Gamma \Lambda^m \mathcal{M} \mid \phi \text{ has compact support} \}.
\]

Test forms have smooth components on a compact subset \(U \subset \mathcal{M}\) whereas on the set \(\mathcal{M} \setminus U\) they are all zero.

Continuity on the set of test \(m\)-forms is given by a family of semi-norms. The topological dual of \(\Gamma_0 \Lambda^m \mathcal{M}\) is written as \(\mathcal{D}'(\mathcal{M})\) and elements of this set are called scalar distributions on \(\mathcal{M}\). A more general object that includes scalar distributions as a particular case are currents which were introduced by de Rham [3]. A homogeneous current of degree \(p\) is a dual element to the space of \((m - p)\)-test forms. De Rham’s theory of currents allows for non-homogeneous forms and also applies to non-orientable manifolds. However we only consider orientable manifolds in this thesis and all forms are required to be homogeneous, therefore the objects we want to consider are as follows.

**Definition 2.1.2.** A \(p\)-form distribution \(\Psi\) (or a homogeneous \(p\)-current so called in [3]) is a continuous, linear \(p\)-functional defined as

\[
\Psi : \Gamma_0 \Lambda^{m-p} \mathcal{M} \to \mathbb{R},
\]

\[
\phi \mapsto [\phi|\Psi]_\mathcal{M} \in \mathbb{R}.
\]

The set of \(p\)-form distributions is denoted as \(\Gamma_D \Lambda^p \mathcal{M}\) and has vector space structure

\[
[\phi|\Psi + \Phi]_\mathcal{M} = [\phi|\Psi]_\mathcal{M} + [\phi|\Phi]_\mathcal{M}, \quad \Psi, \Phi \in \Gamma_D \Lambda^p \mathcal{M}, \quad \phi \in \Gamma_0 \Lambda^{m-p} \mathcal{M},
\]

\[
[\phi|\lambda \Psi]_\mathcal{M} = \lambda [\phi|\Psi]_\mathcal{M}, \quad \lambda \in \mathbb{R}.
\]

To distinguish between distributions on \(\mathbb{R}^n\) and distributional forms on \(\mathcal{M}\) we use square brackets \([\cdot|\cdot]\) for the latter and test forms are placed on the left. Note that a 0-form distribution is a scalar distribution on \(\mathcal{M}\). The degree of a \(p\)-form distribution \(\deg(\Psi) = p\) is such that

\[
[\phi|\Psi]_\mathcal{M} \neq 0 \quad \text{if and only if} \quad \deg(\Psi) + \deg(\phi) = \dim(\mathcal{M}).
\]
2.1. Distributions on Manifolds

**Definition 2.1.3.** The subspace of $\Gamma_D \Lambda^p \mathcal{M}$ consisting of piecewise continuous $p$-forms is called the space of regular distributions. A piecewise continuous $p$-form $\alpha$ leads to a distribution $\alpha^D$ as follows:

$$\left[ \phi | \phi^D \right]_\mathcal{M} = \int_\mathcal{M} \phi \wedge \alpha. \quad (2.13)$$

Properties of regular distributions are given in Appendix A. Some useful operations on $p$-form distributions are as follows.

**Definition 2.1.4.** Let $\Psi \in \Gamma_D \Lambda^p \mathcal{M}$.

(i) The exterior derivative of a $p$-form distribution is defined as

$$d: \Gamma_D \Lambda^p \mathcal{M} \rightarrow \Gamma_D \Lambda^{p+1} \mathcal{M},$$

$$\Psi \mapsto d\Psi,$$

such that

$$\left[ \phi | d\Psi \right]_\mathcal{M} = -(-1)^{\deg(\phi)} \left[ d\phi | \Psi \right]_\mathcal{M}, \quad (2.14)$$

where $\phi \in \Gamma_0 \Lambda^{m-p-1} \mathcal{M}$.

(ii) Given $v \in \Gamma T \mathcal{M}$ the internal contraction is defined as

$$\Gamma T \mathcal{M} \times \Gamma_D \Lambda^p \mathcal{M} \rightarrow \Gamma_D \Lambda^{p-1} \mathcal{M},$$

$$(v, \Psi) \mapsto i_v \Psi,$$

such that

$$\left[ \phi | i_v \Psi \right]_\mathcal{M} = -(-1)^{\deg(\phi)} \left[ i_v \phi | \Psi \right]_\mathcal{M}, \quad \phi \in \Gamma_0 \Lambda^{m-p-1} \mathcal{M}. \quad (2.15)$$

(iii) The wedge product with a smooth form is defined as

$$\Gamma \Lambda^q \mathcal{M} \times \Gamma_D \Lambda^p \mathcal{M} \rightarrow \Gamma_D \Lambda^{q+p} \mathcal{M},$$

$$(\beta, \Psi) \mapsto \beta \wedge \Psi,$$

and satisfies

$$\left[ \phi | \beta \wedge \Psi \right]_\mathcal{M} = (-1)^{\deg(\phi) \deg(\beta)} \left[ \beta \wedge \phi | \Psi \right]_\mathcal{M}, \quad \phi \in \Gamma_0 \Lambda^{m-p-q} \mathcal{M}. \quad (2.16)$$
2.2. Submanifolds and Adapted Coordinate Systems

(iv) If \( \mathcal{M} \) has a metric then the Hodge dual is defined as

\[
* : \Gamma_D^p \mathcal{M} \to \Gamma_D^{m-p} \mathcal{M},
\]

\[\Psi \mapsto \ast \Psi,\]

and satisfies

\[
[\phi \ast \Psi]_\mathcal{M} = (-1)^{p(m-p)} [\ast \phi | \Psi]_\mathcal{M}, \quad \phi \in \Gamma_0^p \mathcal{M}. \tag{2.17}
\]

Using Cartan’s identity, \( L_v = d_i + i_v d \), the Lie derivative follows immediately from (2.14) and (2.15)

\[
[\phi | L_v \Psi]_\mathcal{M} = -[L_v \phi | \Psi]_\mathcal{M}. \tag{2.18}
\]

**Definition 2.1.5.** The support, \( \text{supp}(\Psi) \subset \mathcal{M} \), of \( \Psi \in \Gamma_D^p \mathcal{M} \) is defined as

\[
\text{supp}(\Psi) = \bigcap \left\{ U \subset \mathcal{M} \mid U \text{ is closed and if } \phi \in \Gamma_0^{m-p} \mathcal{M}, \text{ supp}(\phi) \subset \mathcal{M} \setminus U \text{ then } [\phi | \Psi]_\mathcal{M} = 0 \right\}. \tag{2.19}
\]

The space of distributions contains some pathological elements. For instance, there exists distributions that have support on non-trivial sets such as the Cantor set. In this thesis we avoid these types of objects by only considering the class of distributions that have support on submanifolds, and can be constructed out of smooth forms with the operations (2.14)-(2.18).

### 2.2 Submanifolds and Adapted Coordinate Systems

In this section we give a brief overview of submanifolds and embeddings [13]. A submanifold is a subset of \( \mathcal{M} \) such that it is itself a manifold. Suppose \( f : \mathcal{N} \to \mathcal{M} \) is a map between manifolds where \( \dim(\mathcal{N}) = n \) and \( \dim(\mathcal{M}) = m \) such that \( m > n \). We set \( r \) to be the codimension

\[ r = m - n. \tag{2.20} \]
The rank of \( f \) at \( p \in \mathcal{N} \) is the dimension of the image of the pushforward i.e. \( \dim(f_*(T_p\mathcal{N})) \). If \( f_* \) is injective for all \( p \in \mathcal{N} \) (the rank is equal to \( n \)) then \( f \) is said to be an immersion. A stronger condition is when \( f \) is also injective then the map becomes an embedding. In this case \( f \) is homeomorphic onto its image \( f(\mathcal{N}) \) and we can say \( \mathcal{N} \), rather than the technically correct \( f(\mathcal{N}) \), is a submanifold of \( \mathcal{M} \). An embedding is denoted by a hooked arrow

\[ f : \mathcal{N} \hookrightarrow \mathcal{M}. \tag{2.21} \]

An embedding is also said to be closed if \( f(\mathcal{N}) \) is a closed subset of \( \mathcal{M} \). Furthermore the map \( f \) is called proper if the pre-image of a compact set is also compact. A closed embedding is always a proper map [14]. We will only ever consider closed embeddings in this thesis. It is often convenient to express a coordinate system on \( \mathcal{M} \) adapted to the map \( f \). This is defined as follows.

**Definition 2.2.1.** Suppose \( f : \mathcal{N} \hookrightarrow \mathcal{M} \) is an embedding and \( U \subset \mathcal{M} \) is an open subset with coordinate system \( \{U, (y^1, \ldots, y^n, z^1, \ldots, z^r)\} \). Given the coordinates \( (y^1, \ldots, y^n) \) for \( \mathcal{N} \) then

\[ f(y^1, \ldots, y^n) = (y^1, \ldots, y^n, 0, \ldots, 0). \tag{2.22} \]

Locally this means the image of \( \mathcal{N} \) is a \( n \)-dimensional plane in \( \mathbb{R}^m \). We use the notation \( \underline{x} = (x^1, \ldots, x^m) \) to denote a list of coordinates. (2.22) then becomes \( f(\underline{y}) = (\underline{y}, 0) \). One of the advantages of using adapted coordinates is that the pullback of forms satisfies \( f^*(dy^i) = dy^i \) and \( f^*(dz^i) = 0 \). We will frequently deal with intersections of submanifolds and a useful construction is the pullback manifold.

**Definition 2.2.2.** Given the mappings \( f : X \to \mathcal{M} \) and \( g : Y \to \mathcal{M} \) between manifolds the associated pullback manifold \( X \times_\mathcal{M} Y \) is defined as

\[
\begin{array}{ccc}
X \times_\mathcal{M} Y & \xrightarrow{\hat{g}} & Y \\
\downarrow f & & \downarrow g \\
X & \xrightarrow{f} & \mathcal{M}
\end{array}
\]

\[ X \times_\mathcal{M} Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}, \]
where \( \hat{f} \) and \( \hat{g} \) are projection maps given by \( \hat{f}(x, y) = x \) and \( \hat{g}(x, y) = y \). The pullback manifold is also known as the induced manifold and is often written \( f^*Y = g^*X = X \times_\mathcal{M} Y \). Sections of the tangent bundle over \( \mathcal{M} \) are denoted \( \Gamma T\mathcal{M} \) whereas sections of the pullback bundle over \( f \) are denoted \( \Gamma(f, T\mathcal{M}) \).

## 2.3 Multi-indices

We will frequently encounter arbitrary number of Lie derivatives and internal contractions acting on forms. In order to simplify notation we introduce a multi-index which is a list of indices that has a certain ordering. \( I \) represents a tuple of indices \([I_1, I_2, \ldots, I_s]\) of length \( |I| \) with each \( I_i \) lying in the range \( 1 \leq I_i \leq m \) where \( m = \dim(\mathcal{M}) \). In the case of the Lie derivatives the indices can be repeated whereas for internal contractions any two repeated indices reduces the expression to zero. Therefore it is necessary to impose restrictions on the list as follows.

**Definition 2.3.1.** The set of all increasing lists with repetitions of elements is defined as

\[
\text{Rnc}(m) = \{ [I_1, I_2, \ldots, I_s] \mid s, I_1, \ldots, I_s \in \mathbb{Z} : s \geq 0, 1 \leq I_1 \leq I_2 \leq \cdots \leq I_s \leq m \}, \tag{2.23}
\]

whereas the set of all strictly increasing lists is given by

\[
\text{Snc}(m) = \{ [J_1, J_2, \ldots, J_s] \mid s, J_1, \ldots, J_s \in \mathbb{Z} : s \geq 0, 1 \leq J_1 < J_2 < \cdots < J_s \leq m \}. \tag{2.24}
\]

An element \( I \in \text{Rnc}(m) \) is denoted by \( I ||^m \) whereas \( J \in \text{Snc}(m) \) is denoted as \( J^m \).

Lists which have all of the elements of the set present are elements of \( \text{Snc}(m) \).

Thus,

\[
\underline{m} = [1, 2, \ldots, m] \in \text{Snc}(m).
\]

Since \( I \) allows for repeated indices we define the Kronecker delta as

\[
\delta^I_{I'} = \begin{cases} 
1 & \text{if } I \text{ and } I' \text{ have the same elements} \\
0 & \text{otherwise.} 
\end{cases} \tag{2.25}
\]
2.3. Multi-indices

Permutations of the lists may induce a change of sign and this is defined by the symbol

$$\epsilon_{J'} = \begin{cases} 
1 & \text{if } J' \text{ is an even permutation of } J \\
-1 & \text{if } J' \text{ is an odd permutation of } J \\
0 & \text{if } J' \text{ and } J \text{ do not contain the same elements.}
\end{cases} \tag{2.26}$$

Concatenation of strictly increasing lists $J, K \in \text{Snc}(m)$ is defined by

$$J \cdot K = [J_1, \ldots, J_{|J|}, K_1, \ldots, K_{|K|}], \quad a \cdot J = [a, J_1, \ldots, J_{|J|}]. \tag{2.27}$$

Elements can also be removed from lists. $J \setminus K$ denotes the elements $J$ with the elements of $K$ removed whilst preserving the order. If $K \subset J$ then the list $K \cdot (J \setminus K)$ is a permutation of $J$ with elements of $K$ moved to the beginning of the list. The sign of the permutation is given by $\epsilon_{K \cdot (J \setminus K)}$. The Lie derivative, partial derivatives, internal contractions and forms in multi-index notation are

$$L_{iJ}^{(x)} = L_{i_1J_1}^{(x)} L_{i_2J_2}^{(x)} \cdots L_{i_{|J|}J_{|J|}}^{(x)},$$

$$\partial^{(x)}_{i} = \partial^{(x)}_{i_1} \partial^{(x)}_{i_2} \cdots \partial^{(x)}_{i_{|J|}},$$

$$dx^J = dx^{J_1} \wedge dx^{J_2} \wedge \cdots \wedge dx^{J_{|J|}},$$

$$i^J(x) = i^{(x)}_{J_1} \cdots i^{(x)}_{J_{|J|}},$$

$$dx^m = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m,$$

where $L_a^{(x)} = L_{\partial \partial x^a}$, $\partial_a^{(x)} = \partial \partial x^a$ and $i^J_a = i^{(x)}_{J_1} \partial_{J_1} x^a$. Note that the order of $i^J_a$ is reversed. This is to ensure that

$$i^J_a dx^J = i^{(x)}_{J_1} \cdots i^{(x)}_{J_{|J|}} dx^{J_1} \wedge \cdots \wedge dx^{J_{|J|}} = 1.$$

In general for $J, J' \in \text{Snc}(m)$ with $|J'| \geq |J|$, $i^J_{J'} dx^J = \epsilon_{J'}$. However, the case $|J'| < |J|$ may leave some $dx^a$ left over which when pulled back to $\mathcal{N}$ will produce zero, thus

$$f^*(i^J_{J'} dz^J) = \epsilon^J_{J'} \tag{2.33}$$

By using a coordinate basis we have the following commutation rules: $L_a^{(x)} \circ L_b^{(x)} = L_b^{(x)} \circ L_a^{(x)}$ and $i_a^{(x)} \circ L_b^{(x)} = L_b^{(x)} \circ i_a^{(x)}$ such that

$$L^{(x)}_{IJ} \circ L^{(x)}_{I'J'} = L^{(x)}_{I'J'} \circ L^{(x)}_{IJ}, \quad L^{(x)}_I \circ i^{(x)}_J = i^{(x)}_J \circ L^{(x)}_I. \tag{2.34}$$
2.4 Pushforward of Distributions

A smooth map between manifolds induces a corresponding map between distributional form spaces. This is known as de Rham’s pushforward map [3].

**Definition 2.4.1.** Let \( f : N \rightarrow M \) be a proper embedding. The pushforward \( f_\circ \) is a continuous, linear map defined as

\[
f_\circ : \Gamma_D \Lambda^p N \rightarrow \Gamma_D \Lambda^{m-n+p} M,
\]

\[
\Psi \mapsto f_\circ \Psi,
\]

such that

\[
[f \circ \phi]_M = [f^* \phi]_N, \quad \phi \in \Gamma_0 \Lambda^{n-p} M, \quad (2.35)
\]

and \( \text{deg}(f_\circ(\Psi)) = \dim(M) - \dim(N) + \text{deg}(\Psi) \).

Notice that if \( f \) was not a proper map then \( f^* \phi \) would not necessarily have compact support. Linearity follows from the regular pullback (see Appendix A). Below are some properties of the pushforward.

**Lemma 2.4.2.** The exterior derivative commutes with the pushforward

\[
d \circ f_\circ = f_\circ \circ d. \quad (2.36)
\]

**Proof.** Let \( \Psi \in \Gamma_D \Lambda^p M \). Using (2.14) and the fact that the exterior derivative commutes with regular pullbacks i.e. \( f^* \circ d = d \circ f^* \) we have

\[
\begin{align*}
[f \circ d(\phi)]_M &= (-1)^{\text{deg}(\phi)}[d\phi, \phi]_N = (-1)^{\text{deg}(\phi)}[f^*(d\phi)]_N \\
&= (-1)^{\text{deg}(\phi)}[df^*(\phi)]_N = [f^*(\phi)]_N = [\phi, f_\circ(d\Psi)]_M,
\end{align*}
\]

where \( \phi \in \Gamma_0 \Lambda^{n-p-1} M \).

**Definition 2.4.3.** Given the map \( f : N \rightarrow M \) the vector field \( v \in \Gamma TM \) is said to be tangential to \( f \) if there exists a vector field \( u \in \Gamma TN \) such that for each \( x \in N \), \( f_*(u_x) = v|_{f(x)} \).
2.4. Pushforward of Distributions

**Lemma 2.4.4.** Suppose \( v \in \Gamma TM \) is tangential to \( f \). For \( u \in \Gamma TN \) such that \( f_* u = v \) the internal contraction satisfies

\[
i_v \circ f_\varsigma = f_\varsigma \circ i_u.
\] (2.37)

**Proof.** Let \( \Psi \in \Gamma_D \Lambda^p \mathcal{M} \). From (2.15),

\[
[\phi | i_v (f_\varsigma \Psi) ]_\mathcal{M} = - ( -1 )^{\deg(\phi)} [i_v \phi | f_\varsigma \Psi ]_\mathcal{M} = - ( -1 )^{\deg(\phi)} [f^*(i_u \phi) | \Psi ]_\mathcal{N}
\]

\[
= - ( -1 )^{\deg(\phi)} [i_u f^*(\phi) | \Psi ]_\mathcal{N} = [f^*(\phi) | i_u \Psi ]_\mathcal{N}
\]

\[
= [\phi | f_\varsigma (i_u \Psi) ]_\mathcal{M},
\]

where \( \phi \in \Gamma_0 \Lambda^{n-p+1} \mathcal{M} \).

The above Lemma only holds true when vector fields are transverse with respect to \( f \). In adapted coordinates (2.22) we have \( i_{\partial / \partial y^a} \circ f_\varsigma = f_\varsigma \circ i_{\partial / \partial y^a} \) whereas \( i_{\partial / \partial z^a} \) does not commute with the pushforward. It follows immediately from (2.36) and (2.37) using Cartan’s identity that the Lie derivative satisfies

\[
L_v \circ f_\varsigma = f_\varsigma \circ L_u.
\] (2.38)

In the next section we only consider the pushforward of smooth forms so from (2.13), Definition 2.4.1 becomes

\[
[\phi | f_\varsigma (\alpha^D)]_\mathcal{M} = \int_\mathcal{N} f^*(\phi) \wedge \alpha, \quad \alpha \in \Gamma \Lambda^p \mathcal{M}.
\] (2.39)

As \( f \) is a closed embedding it follows that \( \text{supp}(f_\varsigma (\alpha^D)) = f(\text{supp}(\alpha^D)) \). Below are some properties of the pushforward of regular distributions.

**Lemma 2.4.5.** Let \( \alpha \in \Gamma \Lambda^p \mathcal{N} \),

\[
\beta \wedge f_\varsigma (\alpha^D) = f_\varsigma (f^*(\beta) \wedge \alpha^D), \quad \beta \in \Gamma \Lambda^q \mathcal{M}.
\] (2.40)

**Proof.** Acting on the test form \( \phi \in \Gamma_0 \Lambda^{n-p-q} \mathcal{M} \),

\[
[\phi | f_\varsigma (f^*(\beta) \wedge \alpha^D)]_\mathcal{M} = \int_\mathcal{N} f^* \phi \wedge (f^*(\beta) \wedge \alpha) = \int_\mathcal{N} (f^* \phi \wedge f^* \beta) \wedge \alpha
\]

\[
= \int_\mathcal{N} f^* (\phi \wedge \beta) \wedge \alpha = [\phi \wedge \beta | f_\varsigma (\alpha^D)]_\mathcal{M}
\]

\[
= [\phi | \beta \wedge f_\varsigma (\alpha^D)]_\mathcal{M}.
\]
Lemma 2.4.6. If $\gamma : N \to M$ is a diffeomorphism then

$$\gamma_\varsigma = (\gamma^{-1})^*.$$  \hfill (2.41)

Proof. As $\gamma$ is a diffeomorphism this means it has a smooth inverse. Suppose $\alpha \in \Gamma \Lambda^p N$ and $\phi \in \Gamma_0 \Lambda^{m-p} M$,

$$\left[ \phi \right|_M \gamma_\varsigma(\alpha^D) = \int_N \gamma^* \phi \wedge \alpha = \int_M (\gamma^{-1})^* (\gamma^* \phi \wedge \alpha) = \int_M \phi \wedge (\gamma^{-1})^* \alpha$$

$$= \left[ \phi \right| (\gamma^{-1})^* \alpha^D]_M,$$

hence $\gamma_\varsigma = (\gamma^{-1})^*$. \hfill $\square$

Note that for a diffeomorphism the degree is preserved i.e. $\deg(f_\varsigma(\alpha^D)) = \deg(\alpha^D)$. From here on we drop the $D$ for notational convenience.

2.5 Submanifold Distributions

We now consider the set of distributions which are constructed using the pushforward map (2.35) of smooth forms $\alpha$ with operations (2.14)-(2.18).

Definition 2.5.1. Given the closed embedding $f : N \hookrightarrow M$ the real vector space of SMDs over $f$ is denoted as $\Upsilon^{k,p}(f)$ where $k \in \mathbb{Z}^+$ is the order and $p$ is the degree.

We say that $f_\varsigma(\alpha)$ has order 0 so that

$$f_\varsigma(\alpha) \in \Upsilon^{0,p}(f) \quad \text{where} \quad p = r + \deg(\alpha).$$  \hfill (2.42)

A SMD of order $k$ has no more than $k$ derivatives. Higher order SMDs are constructed by application of the exterior and Lie derivatives on $f_\varsigma(\alpha)$. Given $\Psi, \Phi \in \Upsilon^{k,p}(f)$ and $v \in \Gamma T M$

$$(\Psi + \Phi) \in \Upsilon^{k,p}(f), \quad \left[ \phi \right| \Psi + \Phi]_M = \left[ \phi \right| \Psi]_M + \left[ \phi \right| \Phi]_M.$$  \hfill (2.43)
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\[ d\Psi \in \mathcal{Y}^{k+1,p+1}(f), \quad [\phi | d\Psi]_M = -(-1)^{\deg(\phi)}[d\phi | \Psi]_M, \quad (2.44) \]
\[ i_v\Psi \in \mathcal{Y}^{k,p-1}(f), \quad [\phi | i_v\Psi]_M = -(-1)^{\deg(\phi)}[i_v\phi | \Psi]_M, \quad (2.45) \]
\[ L_v\Psi \in \mathcal{Y}^{k+1,p}(f), \quad [\phi | L_v\Psi]_M = -[L_v\phi | \Psi]_M. \quad (2.46) \]

If \( \mathcal{M} \) has a metric then the Hodge dual is defined as

\[ *\Psi \in \mathcal{Y}^{k,m-p}(f), \quad [\phi | *\Psi]_M = (-1)^{p(m-p)}[*\phi | \Psi]_M. \quad (2.47) \]

Furthermore given a smooth form \( \beta \in \Gamma\Lambda^q\mathcal{M} \) the regular wedge product with an SMD is

\[ \beta \wedge \Psi \in \mathcal{Y}^{k,p+q}(f), \quad [\phi | \beta \wedge \Psi]_M = (-1)^{\deg(\phi)\deg(\beta)}[\beta \wedge \phi | \Psi]_M. \quad (2.48) \]

It is possible to add together \( k > 0 \) order SMDs and get a lower order result. For example if \( u, v \in \Gamma T\mathcal{M} \) a SMD of order 2 can be written as \( L_uL_vf_\varsigma(1) \in \mathcal{Y}^{2,p}(f) \) and \( L_uLvf_\varsigma(1) \in \mathcal{Y}^{2,p}(f) \) then the sum is

\[ L_uL_vf_\varsigma(1) - L_vL_uf_\varsigma(1) = L_{[u,v]}f_\varsigma(1) \in \mathcal{Y}^{1,p}(f). \]

This means each \( \mathcal{Y}^{j,p}(f) \) is a subspace of higher order SMDs,

\[ \mathcal{Y}^{j,p}(f) \subset \mathcal{Y}^{k,p}(f), \quad j \leq k. \quad (2.49) \]

As one would expect SMDs have support on the image of \( f \).

Lemma 2.5.2.

\[ \Psi \in \mathcal{Y}^{k,p}(f) \quad \text{implies} \quad \text{supp}(\Psi) \subset f(\mathcal{N}). \quad (2.50) \]

Proof. Since \( f(\mathcal{N}) \) is closed its complement \( U = \mathcal{M}\setminus f(\mathcal{N}) \) is open. Let \( \Psi \in \mathcal{Y}^{k,p}(f) \) and \( \phi \in \Gamma_0\Lambda^{m-p}\mathcal{M} \) with \( \text{supp}(\phi) \subset U \) and consider \([\phi | \Psi]_M\). The application of (2.44)-(2.48) on \( \phi \) gives a new test form \( \hat{\phi} \in \Gamma_0\Lambda^q\mathcal{M} \) for some \( q \) such that \( \text{supp}(\hat{\phi}) \subset \text{supp}(\phi) \). Thus \( \text{supp}(\hat{\phi}) \subset \mathcal{M}\setminus f(\mathcal{N}) \). We now have

\[ [\phi | \Psi]_M = [\hat{\phi} | f_\varsigma(\alpha)]_M = \int_\mathcal{N} f^*(\hat{\phi}) \wedge \alpha = 0, \]

since \( f^*(\hat{\phi}) = 0 \). However \( \text{supp}(f(\alpha)) \subset f(\mathcal{N}) \) hence \( \text{supp}(\Psi) \subset f(\mathcal{N}) \). \( \square \)
2.5. Submanifold Distributions

We define the internal degree $\text{ideg}(\Psi)$ as

$$\text{ideg}(\Psi) = \deg(\Psi) - r,$$  \hspace{1cm} (2.51)

where $r = \dim(M) - \dim(N)$. Observe that $\text{ideg}(\varsigma(\alpha)) = \deg(\alpha)$. To make the connection with standard Dirac notation we may write $(\Psi)_{\text{Dirac}}$ which is placed under an integral and is constructed out of Dirac delta functions

$$[\phi \mid \Psi]_{\mathcal{M}} = \int_{\mathcal{M}} \phi \wedge (\Psi)_{\text{Dirac}}.$$  \hspace{1cm} (2.52)

All SMDs can be written locally in this way. As an example consider the map $f : N \to M$ with $\dim(M) = 3$ and $\dim(N) = 2$ such that $N = g^{-1}\{0\} = \{ x \in M \mid g(x) = 0 \}$ where $g \in \Gamma^0\mathcal{M}$. In other words the submanifold $N$ of $M$ is defined by the points that are mapped to zero by $g$. Then using (2.39)

$$[\phi \mid f^\varsigma(1)]_{\mathcal{M}} = \int_{N} f^* \phi,$$  \hspace{1cm} (2.53)

where $\phi \in \Gamma_0\Lambda^2\mathcal{M}$. Introduce adapted coordinates $(y^1, y^2, z^1)$ such that $g = z^1$ and with orientation $dy^1 \wedge dy^2 \wedge dz^1$. The test form $\phi$ has local representation

$$\phi = \phi_{12,6}(y^1, y^2, z^1)dy^1 \wedge dy^2 + \phi_{11,1}(y^1, y^2, z^1)dy^1 \wedge dz^1 + \phi_{2,1}(y^1, y^2, z^1)dy^2 \wedge dz^1.$$  \hspace{1cm} (2.54)

Now using the fact that $f^*(dz^1) = 0$, $f^*(dy^1) = dy^1$ and $f^*(dy^2) = dy^2$ we have

$$f^*(\phi) = (\phi_{12,6} \circ f(y^1, y^2))dy^1 \wedge dy^2 = \phi_{12,6}(y^1, y^2, 0)dy^1 \wedge dy^2,$$  \hspace{1cm} (2.55)

thus

$$\int_{N} f^* \phi = \int_{\mathbb{R}^2} \phi_{12,6}(y^1, y^2, 0)dy^1 \wedge dy^2.$$  \hspace{1cm} (2.56)

However

$$\int_{N} f^* \phi = \int_{\mathbb{R}^2} \phi_{12,6}(y^1, y^2, 0)dy^1 \wedge dy^2$$

$$= \int_{\mathbb{R}^2} \int_{\mathbb{R}} \delta(z^1)(\phi_{12,6}(y^1, y^2, z^1)dy^1 \wedge dy^2) \wedge dz^1$$

$$= \int_{\mathbb{R}^3} \phi_{12,6}(y^1, y^2, z^1)dy^1 \wedge dy^2 \wedge (\delta(z^1)dz^1)$$

$$= \int_{\mathcal{M}} \phi \wedge (\delta(z^1)dz^1).$$
2.5. Submanifold Distributions

Comparing with (2.52)

\[(f, (1))_{\text{Dirac}} = \delta(z^1)dz^1. \tag{2.57}\]

We see that \((f, (1))_{\text{Dirac}}\) is a one form with a Dirac delta distribution as its argument. Compare (2.56) with (2.4) and we see the action of the pullback is equivalent to evaluating the test form components on the submanifold. Taking the internal contraction and Lie derivative in the \(z^1\) direction gives familiar expressions

\[(i_1^{(z)} f, (1))_{\text{Dirac}} = \delta(z^1), \quad (L_1^{(z)} f, (1))_{\text{Dirac}} = \delta'(z^1)dz^1. \tag{2.58}\]

We can carry out the same procedure with \(f, (\alpha)\) where \(\alpha \in \Gamma\Lambda^1N\),

\[
\left[ \phi | f, (\alpha) \right]_M = \int_N f^\ast (\phi) \wedge (\alpha dy^1 + \alpha_2 dy^2)
= \int_{\mathbb{R}^2} \left( \phi_{1,0}(y^1, y^2, 0) dy^1 + \phi_{2,0}(y^1, y^2, 0) dy^2 \right) \wedge (\alpha dy^1 + \alpha_2 dy^2)
= \int_{\mathbb{R}^2} \left( \alpha_2 \phi_{1,0}(y^1, y^2, 0) - \alpha_1 \phi_{2,0}(y^1, y^2, 0) \right) dy^1 \wedge dy^2
= \int_{\mathbb{R}^2} \int_{\mathbb{R}} \phi \wedge \left( \delta(z^1) \alpha_1 dy^1 \wedge dz^1 + \delta(z^1) \alpha_2 dy^2 \wedge dz^1 \right)
= \int_M \phi \wedge \left( \alpha \wedge \delta(z^1)dz^1 \right),
\]

hence

\[(f, (\alpha))_{\text{Dirac}} = \alpha \wedge \delta(z^1)dz^1. \tag{2.59}\]

In the following we use the notation \(\delta^{(r)}(z) = \delta(z^1)\delta(z^2)\cdots\delta(z^r)\).

**Lemma 2.5.3.** In the adapted coordinate system \((y, z)\) the Dirac delta function can be written

\[(f, (\alpha))_{\text{Dirac}} = \sum_{K^1, \ldots, K^r \in p} \alpha_K(y) \delta^{(r)}(z)dy^K \wedge dz^r, \tag{2.60}\]

where \(\alpha_K = i_{(y)}^{(z)} \alpha\).

**Proof.** Let \(\phi \in \Gamma_0\Lambda^\alpha\nu M\). In adapted coordinates \(\phi = \sum_{K^{\alpha}, J^{\nu}} \phi_{K, J}y \delta^{(z)}(z)dy^K \wedge dz^J\)
such that $|K'| + |J| = n - p$ and also $|K| = p$,

$$\left[ \phi \left| f_{\varsigma}(\alpha) \right. \right]_{M} = \int_{\mathcal{N}} f^*(\phi) \wedge \alpha$$

$$= \int_{\mathcal{N}} f^* \left( \sum_{K'^{\prime},J'} \phi_{K',J}(y, z) dy^{K'} \wedge dz^{J} \right) \wedge \left( \sum_{K} \alpha_{K}(y) dy^{K} \right)$$

$$= \sum_{K'^{\prime},J',K^{\prime}} \int_{\mathcal{N}} f^* \left( \phi_{K',J}(y, z) dy^{K'} \wedge dz^{J} \right) \wedge \left( \alpha_{K}(y) dy^{K} \right)$$

$$= \sum_{K'^{\prime},J',K^{\prime}} \int_{\mathbb{R}^{n}} \phi_{K',\varnothing}(y, \emptyset) \alpha_{K}(y) dy^{K'} \wedge dy^{K}$$

$$= \sum_{K'^{\prime}} \epsilon_{n}^{K'^{\prime}(\emptyset|K')} \int_{\mathbb{R}^{n}} \phi_{K',\varnothing}(y, \emptyset) \alpha_{K}(y) dy^{n},$$

whereas

$$\int_{\mathcal{M}} \phi \wedge \left( f_{\varsigma}(\alpha) \right)_{\text{Dirac}}$$

$$= \int_{\mathcal{M}} \phi \wedge \left( \sum_{K'^{\prime},|K'|=p} \alpha_{K}(y) \delta^{(r)}(\bar{z}) dy^{K'} \wedge dz^{K} \right)$$

$$= \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{r}} \left( \sum_{K'^{\prime},J'} \phi_{K',J}(y, \bar{z}) dy^{K'} \wedge dz^{J} \right) \wedge \sum_{K'^{\prime},|K'|=p} \alpha_{K}(y) \delta^{(r)}(\bar{z}) dy^{K'} \wedge dz^{K}$$

$$= \sum_{K'^{\prime},J',K^{\prime}} \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{r}} \left( \phi_{K',\varnothing}(y, \bar{z}) \alpha_{K}(y) \delta^{(r)}(\bar{z}) dy^{K'} \wedge dz^{K} \right)$$

$$= \sum_{K'^{\prime},J',K^{\prime}} \int_{\mathbb{R}^{n}} \phi_{K',\varnothing}(y, \emptyset) \alpha_{K}(y) dy^{n}. $$

Lemma 2.5.4. The Dirac delta distribution can be written

$$\delta^{(r)}(\bar{z}) = \left( \tilde{i}^{(z)}_{\xi} f_{\varsigma}(1) \right)_{\text{Dirac}}.$$  \hspace{1cm} (2.61)

Proof. Let $\phi \in \Gamma_{0} \Lambda^{m} \mathcal{M}$ and again $\phi = \sum_{K'^{\prime},J'} \phi_{K',J}(y, \bar{z}) dy^{K'} \wedge dz^{J}$ such that $|K| + |J| = m$. 


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From (A.15) in Appendix A we have

\[
\left[ \frac{\phi}{i^i_z f_i(1)} \right]_M = (-1)^{r(m+1)} \left[ \frac{i^i_z \phi}{f_i(1)} \right]_M
\]

\[
= (-1)^{r(|J|+|K|+1)} \int_N f^* (i^i_z \phi)
\]

\[
= \sum_{\mathcal{Y}^r, K \uparrow^n} (-1)^{r(|J|+|K|+1)} \int_N f^* (i^i_z (\phi_{K,L}(y, z) dy^K \wedge dz^J))
\]

\[
= \sum_{\mathcal{Y}^r, K \uparrow^n} (-1)^{r(|J|+|K|+1)+|K|} \int_N f^* (\phi_{K,L}(y, z) dy^K \wedge f^* (i^i_z dz^J))
\]

\[
= \sum_{\mathcal{Y}^r, K \uparrow^n} (-1)^{r(|J|+1)} \int_N f^* (\phi_{K,L}(y, z) dy^K \wedge f^* (i^i_z dz^J))
\]

\[
= \sum_{\mathcal{Y}^r, K \uparrow^n} (-1)^{r(r+1)} \int_{\mathbb{R}^n} \phi_{K,L}(y, 0) dy^K
\]

\[
= \int_{\mathbb{R}^n} \phi_{K,L}(y, 0) dy^K.
\]

However

\[
\int_M \phi \wedge (i^i_z f_i(1))_{\text{Dirac}} = \sum_{\mathcal{Y}^r, K \uparrow^n} \int_M (\phi_{K,L}(y, z) dy^K \wedge dz^J) \wedge (\delta^r(\zeta))
\]

\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi_{K,L}(y, z) dy^K \wedge (\delta^r(\zeta) dz^L)
\]

\[
= \int_{\mathbb{R}^n} \phi_{K,L}(y, 0) dy^K.
\]

\[\square\]

**Theorem 2.5.5.** Locally the elements of \(\Psi \in \Upsilon^{k,p}(f)\) can be written in adapted coordinates as

\[
\left[ \frac{\phi}{\Psi} \right]_M = \sum_{\text{Rng}(I,J,K)} (-1)^{|I|+|J|+|K|} \int_N \Psi^I_J dy^K \wedge f^* (i^i_z L^i_I(\phi)), \tag{2.62}
\]

where the support of \(\phi\) lies inside a coordinate patch \(\mathcal{N}_C\). The components of \(\Psi\) are

\[
\Psi^I_J = \Gamma_{\Lambda^0} \mathcal{N}_C \text{ and }
\]

\[
\text{Rng}(I,J,K) = \{ I \uparrow^n, J \uparrow^n, K \uparrow^n \mid |K|-|J|=p-r, |I| \leq k \}. \tag{2.63}
\]

All the \(\Psi\) defined in (2.62) belong to \(\Upsilon^{k,p}(f)\).
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Proof. See Appendix A. \hfill \square

Lemma 2.5.6. In adapted coordinates $\Psi \in \Upsilon^k \psi(f)$ can be written

$$\Psi = \sum_{\text{Rng}(I,J,K)} i_{j}^{(z)} L_{I}^{(z)} f_{x}(\Psi_{K}^{I,J} dy^{K}). \tag{2.64}$$

Proof. Let $\phi \in \Gamma_{0} \Lambda^{m-p} \mathcal{M}$. Using (2.62) and (A.15),

$$[\phi | \Psi]_{\mathcal{M}} = \sum_{\text{Rng}(I,J,K)} \int_{\mathcal{N}} \Psi_{K}^{I,J} dy^{K} \wedge f^{*}(i_{j}^{(z)} L_{I}^{(z)} \phi)$$

$$= \sum_{\text{Rng}(I,J,K)} (-1)^{|K|(m-p-|J|)} \int_{\mathcal{N}} f^{*}(i_{j}^{(z)} L_{I}^{(z)} \phi) \wedge \Psi_{K}^{I,J} dy^{K}$$

$$= \sum_{\text{Rng}(I,J,K)} (-1)^{|K|(m-p-|J|)-|J|} [i_{j}^{(z)} L_{I}^{(z)} \phi f_{x}(\Psi_{K}^{I,J} dy^{K})]_{\mathcal{M}}$$

$$= \sum_{\text{Rng}(I,J,K)} (-1)^{|K|(m-p-|J|)+|J|} [L_{I}^{(z)} i_{j}^{(z)} f_{x}(\Psi_{K}^{I,J} dy^{K})]_{\mathcal{M}}$$

$$= \sum_{\text{Rng}(I,J,K)} (-1)^{|I|+(p-r)(m-p-|J|)} [\phi L_{I}^{(z)} i_{j}^{(z)} f_{x}(\Psi_{K}^{I,J} dy^{K})]_{\mathcal{M}}. \hfill \square$$

Lemma 2.5.7. The internal degree of (2.64) is

$$\text{ideg}(\Psi) = |K| - |J|. \tag{2.65}$$

Proof. From (2.51)

$$\text{ideg}(\Psi) = \text{deg}(\Psi) - r = \text{deg}(f_{x}(\Psi_{K}^{I,J} dy^{K})) - |J| - r = |K| + r - |J| - r = |K| - |J|. \hfill \square$$

Using (2.48), (2.64) becomes

$$\Psi = \sum_{\text{Rng}(I,J,K)} i_{j}^{(z)} L_{I}^{(z)} (\Psi_{K}^{I,J} |_{y} dy^{K} \wedge f_{x}(1)). \tag{2.66}$$

Note that in (2.62) the components $\Psi_{K}^{I,J}$ are defined on $\mathcal{N}$ whereas $\Psi_{K}^{I,J} |_{y}$ are defined on the whole of $\mathcal{M}$. 

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Lemma 2.5.8. (2.64) in Dirac notation is

\[
(\Psi)_{\text{Dirac}} = \sum_{\text{Rng}(I,J,K)} \epsilon^{r}_{J(\zeta\backslash\zeta')}(\zeta)\frac{\partial}{\partial z^{(r)}}(\zeta) (\Psi)^{I,J}_{K}(y) dy^{K} \wedge dz^{\zeta \backslash \zeta'}. \quad (2.67)
\]

Proof. Using (2.66)

\[
\Psi = \sum_{\text{Rng}(I,J,K)} i^{(z)}_{J} L^{(z)}_{I}(\Psi^{I,J}_{K}|_{y} dy^{K} \wedge f_{\zeta}(1))
\]

\[
= \sum_{\text{Rng}(I,J,K)} i^{(z)}_{J} L^{(z)}_{I}(\Psi^{I,J}_{K}|_{y} dy^{K} \wedge dz^{\zeta} (i^{(z)}_{\zeta} f_{\zeta}(1)))
\]

\[
= \sum_{\text{Rng}(I,J,K)} i^{(z)}_{J} (\Psi^{I,J}_{K}|_{y} dy^{K} \wedge dz^{\zeta} (L^{(z)}_{I} i^{(z)}_{\zeta} f_{\zeta}(1)))
\]

\[
= \sum_{\text{Rng}(I,J,K)} (-1)^{|J||K|} (\Psi^{I,J}_{K}|_{y} dy^{K} \wedge dz^{\zeta} (\partial^{(z)}_{I} i^{(z)}_{\zeta} f_{\zeta}(1)))
\]

\[
= \sum_{\text{Rng}(I,J,K)} (-1)^{|J||K|} \epsilon^{r}_{J(\zeta\backslash\zeta')} (\partial^{(z)}_{I} i^{(z)}_{\zeta} f_{\zeta}(1)) (\Psi^{I,J}_{K}|_{y} dy^{K} \wedge dz^{\zeta \backslash \zeta'}). \quad \square
\]

In the definition of SMDs there is a significant amount of redundant information. For example the internal contraction requires defining a vector field on the whole of \(\mathcal{M}\) or at the very least in a neighbourhood of \(f(\mathcal{N})\). However, as shown in Theorem 2.5.5 only the value of the fields and its derivatives on the submanifold are required. We now have the necessary properties of SMDs to define the wedge product.

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Chapter 3

Wedge Product of SMDs

3.1 Introduction

In this chapter we use the results of the previous section to define a wedge product of SMDs. It is a well known result by Schwartz that multiplication between distributions is not well defined [15]. However, there are instances where one can define a multiplication (not necessarily an algebra) between certain elements: multiplication of a distribution with a smooth function and multiplication of distributions whose singular supports are disjoint [11, 16, 17]. An important criteria for when distributions can be multiplied together is the determined by the wavefront set. Singular support specifies where the function is singular while the wavefront set gives the directions of the singularities. If the directions are transverse to one another then multiplication is defined [16]. Motivated by this we consider the wedge product of SMDs that have singular support on submanifolds that intersect transversely. This type of product excludes objects of the type $(\delta(x))^2$. The aim of this chapter is to prove that such a wedge product exists and is unique. We then use this new formalism to express solutions to linear differential equations via Green’s function techniques. However first we define what it means for two submanifolds to be transverse.
3.2 Transverse Embeddings

Consider the closed embeddings $A : A \hookrightarrow \mathcal{M}$ and $B : B \hookrightarrow \mathcal{M}$. From Definition 2.2.2 the associated pullback manifold is

$$A \times_\mathcal{M} B = \{(a, b) \in A \times B | A(a) = B(b)\}, \quad (3.1)$$

with commutative diagram

\[ \begin{array}{ccc}
A \times_\mathcal{M} B & \xrightarrow{\hat{A}} & A \\
\downarrow \hat{B} & & \downarrow A \\
B & \xrightarrow{\hat{B}} & \mathcal{M}
\end{array} \]

(3.2)

We require that the intersection of closed embeddings be also a closed embedding, that is we want a map $C$ from $C = A \times_\mathcal{M} B$ to $\mathcal{M}$. This is given by the transversality condition [18].

**Definition 3.2.1.** The closed embeddings $A : A \hookrightarrow \mathcal{M}$ and $B : B \hookrightarrow \mathcal{M}$ are said to be transverse if for all points $p \in \mathcal{M}$ on the intersection $A \cap B$, the tangent spaces decompose as $T_p\mathcal{M} = T_pA + T_pB$ such that $T_pA \cap T_pB = T_p(A \cap B)$. Consequently the dimensions satisfy $\dim(A) + \dim(B) - \dim(C) = \dim(\mathcal{M})$.

If $A$ and $B$ are transverse let $C = A \times_\mathcal{M} B$ with embedding map $C : C \hookrightarrow \mathcal{M}$. (3.2) now becomes

\[ \begin{array}{ccc}
C & \xrightarrow{\hat{A}} & A \\
\downarrow \hat{B} & & \downarrow A \\
B & \xrightarrow{\hat{B}} & \mathcal{M}
\end{array} \]

(3.3)
3.2. Transverse Embeddings

We can construct an adapted coordinate system on $U$ using (2.22) as follows

$$\{ U, (x^1, \ldots, x^{\dim(A) - \dim(C)}, y^1, \ldots, y^{\dim(B) - \dim(C)}, z^1, \ldots, z^{\dim(C)}) \}, \quad (3.4)$$

such that

$$A(A) \cap U : (x^1, \ldots, x^n, 0, \ldots, 0, z^1, \ldots, z^n), \quad (3.5)$$

$$B(B) \cap U : (0, \ldots, 0, y^1, \ldots, y^t, z^1, \ldots, z^n), \quad (3.6)$$

$$C(C) \cap U : (0, \ldots, 0, 0, \ldots, 0, z^1, \ldots, z^n), \quad (3.7)$$

where $s = \dim(A) - \dim(C)$, $t = \dim(B) - \dim(C)$ and $n = \dim(C)$. If we have the forms $\alpha \in \Gamma^pA$ and $\beta \in \Gamma^qB$ then the corresponding forms on $C$ are $\hat{A}^*(\alpha)$ and $\hat{B}^*(\beta)$. Now if we take the usual wedge product $\hat{A}^*(\alpha) \wedge \hat{B}^*(\beta) \in \Gamma^{p+q}C$ and pushforward with respect to the map $C$ we have a means to construct SMDs over the map $C$ i.e. on the intersection. See Figure 3.2. From (2.37), (2.38) and (2.40) we have the following commutation relations with respect to the pushforward of $A$,

$$dx^a \wedge A(\alpha) = A(\alpha) \wedge dx^a, \quad dz^c \wedge A(\alpha) = A(\alpha) \wedge dz^c, \quad (3.8)$$

$$L^{(x)}_{a} \circ A = A \circ L^{(x)}_{a}, \quad \iota^{(x)}_{a} \circ A = A \circ \iota^{(x)}_{a}, \quad L^{(z)}_{c} \circ A = A \circ L^{(z)}_{c}, \quad \iota^{(z)}_{c} \circ A = A \circ \iota^{(z)}_{c}. \quad (3.9)$$

Similarly for $B$,

$$dy^b \wedge B(\beta) = B(\beta) \wedge dy^b, \quad dz^c \wedge B(\beta) = B(\beta) \wedge dz^c, \quad (3.10)$$

$$L^{(y)}_{b} \circ B = B \circ L^{(y)}_{b}, \quad \iota^{(y)}_{b} \circ B = B \circ \iota^{(y)}_{b}, \quad L^{(z)}_{c} \circ B = B \circ L^{(z)}_{c}, \quad \iota^{(z)}_{c} \circ B = B \circ \iota^{(z)}_{c}, \quad (3.11)$$

and finally $C$,

$$dz^c \wedge C(\gamma) = C(\gamma) \wedge dz^c, \quad (3.12)$$

$$L^{(z)}_{c} \circ C = C \circ L^{(z)}_{c}, \quad \iota^{(z)}_{c} \circ C = C \circ \iota^{(z)}_{c}, \quad (3.13)$$

where $\gamma \in \Gamma^pC$. These relations will be used extensively in this chapter. The above definition of transversality can be extended to include more than two submanifolds. We will need this when proving associativity of the wedge product. Let $I_{AB}$ be the
3.2. Transverse Embeddings

Figure 3.1: The intersection of two transverse submanifolds $A(A)$ and $B(B)$ embedded in $\mathcal{M}$ is also a submanifold $C(C)$. At each point on $\mathcal{M}$ it is possible to construct a locally adapted coordinate system $(x, y, z)$. 
3.2. Transverse Embeddings

intersection of $A$ and $B$, $I_{BD}$ of $B$ and $D$, $I_{AD}$ of $A$ and $D$ and $I_{ABD}$ the intersection of three submanifolds $A, B$ and $D$ such that $A \cap B \cap D \neq \emptyset$. Now $\dim(A) + \dim(B) + \dim(D) - 2\dim(I_{ABD}) = \dim(M)$. The commutative diagrams below along with Figure 3.2, show the intersections $I_{AB}, I_{BD}$ and $I_{AD}$.

Figure 3.2: The intersection of three transverse submanifolds in three dimensional space.
3.2. Transverse Embeddings

\[3.8\]

The intersections are also embeddings which intersect transversely, so we have additional maps

\[3.9\]

and

\[3.10\]

However when we prove associativity of the wedge product we will not require all the maps above. The essential maps are

\[3.11\]
3.3. Definition of the Wedge Product

Let $V \subset M$ be an open set such that $I_M(\mathcal{I}_{ABD}) \cap V \neq \emptyset$ then an adapted coordinate system on $V$ is

$$\{V, (x^1, \ldots, x^{s'}, y^1, \ldots, y^{t'}, z^1, \ldots, z^{n'}, w^1, \ldots, w^{m'})\},$$  \hspace{1cm} (3.12)

such that

$$A(A) \cap V : (x^1, \ldots, x^{s'}, y^1, \ldots, y^{t'}, 0, \ldots, 0, w^1, \ldots, w^{m'}) ,$$  \hspace{1cm} (3.13)

$$B(B) \cap V : (0, \ldots, 0, y^1, \ldots, y^{t'}, z^1, \ldots, z^{n'}, w^1, \ldots, w^{m'}) ,$$  \hspace{1cm} (3.14)

$$D(D) \cap V : (x^1, \ldots, x^{s'}, 0, \ldots, 0, z^1, \ldots, z^{n'}, w^1, \ldots, w^{m'}) ,$$  \hspace{1cm} (3.15)

$$I_M(\mathcal{I}_{ABD}) \cap V : (0, \ldots, 0, 0, 0, 0, w^1, \ldots, w^{m'}) ,$$  \hspace{1cm} (3.16)

where $s' = \dim(A) - \dim(\mathcal{I}_{ABD})$, $t' = \dim(B) - \dim(\mathcal{I}_{ABD})$, $n' = \dim(D) - \dim(\mathcal{I}_{ABD})$ and $m' = \dim(\mathcal{I}_{ABD})$.

### 3.3 Definition of the Wedge Product

**Definition 3.3.1. Wedge product of SMDs**

Given two submanifold distributions $\Psi \in \mathcal{V}^{k,p}(A)$ and $\Phi \in \mathcal{V}^{k',q}(B)$ then we say the product

$$\bullet \wedge \bullet : \mathcal{V}^{k,p}(A) \times \mathcal{V}^{k',q}(B) \to \mathcal{V}^{k+k',p+q}(C)$$  \hspace{1cm} (3.18)

is a **wedge product of transverse SMDs** if the following conditions hold:

(W1).  

$$A_\ast(\alpha) \wedge B_\ast(\beta) = C_\ast(\hat{A}^\ast(\alpha) \wedge \hat{B}^\ast(\beta))$$  \hspace{1cm} (3.19)

with $\alpha \in \Lambda^p A$, $\beta \in \Lambda^q B$ and maps given by (3.3).

(W2). **Associativity**

$$(\Psi \wedge \Phi) \wedge \Omega = \Psi \wedge (\Phi \wedge \Omega) \quad \Omega \in \mathcal{V}^{k',l}(D).$$  \hspace{1cm} (3.20)
3.3. Definition of the Wedge Product

(W3). Graded commutativity

\[ \Psi \wedge \Phi = (-1)^{\text{ideg}(\Psi) \text{ideg}(\Phi)} \Phi \wedge \Psi. \]  \hspace{1cm} (3.21)

(W4). Graded Leibniz

\[ i_v(\Psi \wedge \Phi) = i_v \Psi \wedge \Phi + (-1)^{\text{ideg}(\Psi)} \Psi \wedge i_v \Phi, \]  \hspace{1cm} (3.22)

\[ L_v(\Psi \wedge \Phi) = L_v \Psi \wedge \Phi + \Psi \wedge L_v \Phi, \]  \hspace{1cm} (3.23)

for \( v \in \Gamma T\mathcal{M} \).

We have defined a wedge product \( \Psi \wedge \Phi \) of two transverse SMDs \( \Psi, \Phi \) by giving a set of coordinate free axioms (3.18)-(3.23) we wish it to obey. At this point we make no assumption as to whether such a wedge product exists or is unique. However the goal of this section is to demonstrate that the axioms do indeed give a well defined product. In order to do this we define another product \( \Psi \circledast (x,y,z) \wedge \Phi \) by giving an explicit form (3.26) in terms of the adapted coordinate system \((x,y,z)\). The subscript indicates that \( \Psi \circledast (x,y,z) \wedge \Phi \) has a dependence on the adapted coordinate system. Our goal is to show that a wedge product of two transverse SMDs exists, is unique and has a coordinate representation which will be given below. For an adapted coordinate system \((\x,\y,\z)\), which respects the transverse embeddings, we define a new wedge product \( \circledast (x,y,z) \wedge \Phi \) called the “O-wedge” product. We will usually drop the subscript if it is clear which coordinate system is being used. As stated in the introduction, this involves two steps.

- We prove that the coordinate dependent product \( \Psi \circledast (x,y,z) \wedge \Phi \) satisfies the axioms (3.18)-(3.23). This guarantees that a wedge product of two SMDs exists.

- For uniqueness, we choose a new adapted coordinate system \((\x,\y,\z)\) and show that, in this coordinate system, the axioms (3.18)-(3.23) imply that the wedge product is the same as the O-wedge \( \Psi \circledast (x,y,z) \wedge \Phi \).
3.4. Properties of the O-Wedge Product

Thus a wedge product of two transverse SMDs defined axiomatically exists, is unique and has a coordinate representation given by (3.26). Note that the wedge product of distributions does not form an algebra as the product $\Psi \wedge \Phi$ is not defined.

**Definition 3.3.2.** In the local adapted coordinates (3.4) and using Theorem 2.5.5 the SMDs $\Psi \in \mathcal{Y}^{k,p}(A)$ and $\Phi \in \mathcal{Y}^{k',q}(B)$ can be expressed as

$$\Psi = \sum_{R_{\alpha k,p}(I,J,K,L)} \sum_{R_{\beta k,p}(J')} i^{(y)}_J L^{(y)}_I A_i \left( \Psi_{K,L} \right) dx^K \wedge dz^L, \quad (3.24)$$

and

$$\Phi = \sum_{R_{\alpha k,q}(I,J,K')} \sum_{R_{\beta k,q}(K',L')} i^{(x)}_J L^{(x)}_I B_i \left( \Phi_{K',L'} \right) dy^{K'} \wedge dz^{L'}, \quad (3.25)$$

where the multi-indices obey the following constraints

$$R_{\alpha k,p}(I,J,K,L) = \left\{ I \uparrow^s, J \uparrow^t, K \uparrow^s, L \uparrow^n \mid |K| + |L| - |J| = p - s, |I| \leq k \right\},$$

and

$$R_{\alpha k,q}(I',J',K',L') = \left\{ I' \uparrow^s, J' \uparrow^s, K' \uparrow^t, L' \uparrow^n \mid |K'| + |L'| - |J'| = q - t, |I'| \leq k' \right\}.$$ 

The O-wedge product $\Psi \wedge \Phi \in \mathcal{Y}^{k+k',p+q}(C)$ is defined to be

$$\Psi \wedge \Phi = \sum_{R_{\alpha k,p}(I,J,K,L)} \sum_{R_{\beta k,p}(J')} \sum_{R_{\gamma k,q}(I',J',K',L')} (-1)^{|R|+|R'|} \omega_{K,K'}^{I,J,K,L} \omega_{J',J',K'}^{I,J,K,L} \left( \hat{A}^i \left( \partial_{R'}^{(y)} \Psi_{K,L} \right) \hat{B}^i \left( \partial_{R}^{(x)} \Phi_{K',L'} \right) \right) dx^K \wedge dz^L \wedge dy^{K'} \wedge dz^{L'}, \quad (3.26)$$

with sign

$$\omega_{J',J',K'}^{I,J,K,L} = (-1)^{|K'|+|J'|+|L'|+|K|+|J|} \epsilon_{I,J,K,L}^{(J,K,K')} \epsilon_{J',J',K'}^{(J,K,K')} \epsilon_{I,J,K,L}^{(J,K,K')}, \quad (3.27)$$

### 3.4 Properties of the O-Wedge Product

As it stands we have a coordinate free definition of a wedge of SMDs and also a coordinate dependent definition given by (3.26). In our chosen adapted coordinate system we show (3.26) satisfies the following.
Proposition 3.4.1. **SMD coordinate basis O-wedge**

In the adapted coordinate system \((x, y, z)\) the O-wedge product satisfies

\[
A_\varsigma(\alpha) \hat{\wedge} B_\varsigma(\beta) = C_\varsigma(\hat{A}(\alpha) \wedge \hat{B}(\beta)),
\]

(3.28a)

\[
(\lambda \Psi) \hat{\wedge} \Phi = \Psi \hat{\wedge} (\lambda \Phi), \quad \lambda \in \Gamma\Lambda^0\mathcal{M},
\]

(3.28b)

\[
i_a^{(x)}(\Psi \hat{\wedge} \Phi) = i_a^{(x)}\Psi \hat{\wedge} \Phi + (-1)^{\text{ideg}(\Psi)}\Psi \hat{\wedge} i_a^{(x)}\Phi,
\]

(3.28c)

\[
L_a^{(x)}(\Psi \hat{\wedge} \Phi) = L_a^{(x)}\Psi \hat{\wedge} \Phi + \Psi \hat{\wedge} L_a^{(x)}\Phi,
\]

(3.28d)

\[
(dx^a \hat{\wedge} \Psi) \hat{\wedge} \Phi = (-1)^{\text{ideg}(\Psi)}\Psi \hat{\wedge} (dx^a \hat{\wedge} \Phi),
\]

(3.28e)

\[
(\Psi \hat{\wedge} \Phi) \hat{\wedge} \Omega = \Psi \hat{\wedge} (\Phi \hat{\wedge} \Omega),
\]

(3.28f)

\[
\Psi \hat{\wedge} \Phi = (-1)^{\text{ideg}(\Psi)\text{ideg}(\Phi)}\Phi \hat{\wedge} \Psi,
\]

(3.28g)

with identical expressions (3.28c)-(3.28e) for \(y\) and \(z\) components.

We prove Proposition 3.4.1 by breaking it down into a series of lemmas. To simplify matters we only need to consider \(\Psi\) and \(\Phi\) where the lists \(I, J, K, L\) and \(I', J', K', L'\) all have fixed lengths. Clearly all the properties in Proposition 3.4.1 are linear so we can neglect the \(\text{Rng}_{k,p}(I, J, K, L)\) and \(\text{Rng}_{k'q}(I', J', K', L')\) summations in (3.26). Thus we can just consider a single term in the summations

\[
\Psi = i_j^{(y)}L_i^{(y)}A_\varsigma\left(\Psi_{K,L}^{I,J}dx^K \wedge dz^L\right),
\]

(3.29)

\[
\Phi = i_{J'}^{(x)}L_{I'}^{(x)}B_\varsigma\left(\Phi_{K',L'}^{I',J'}dy^{K'} \wedge dz^{L'}\right).
\]

(3.30)
3.4. Properties of the O-Wedge Product

For $\Psi \hat{\wedge} \Phi$ to be non-zero $J = K'$ and $J' = K$ since any remaining $dx$'s and $dy$'s pullback to zero. The subsets $R$ and $R'$ respectively are not subject to the same constraints as the internal contractions so we must retain summations over sublists of $I$ and $I'$. $\Psi \hat{\wedge} \Phi$ with the aforementioned constraints on indices is given by

$$\Psi \hat{\wedge} \Phi = \sum_{R, R' \subseteq I' \atop \forall, R \subseteq I} \omega^{K,K'}_R \cdot \iota^{(x)}_j \cdot \iota^{(y)}_j \cdot s'(y) \cdot L^{(y)}_{A,R} C_s(\bar{A}^*(\partial^{(x)}_R \Psi)^\circ_{\circ,L}) \bar{B}^*(\partial^{(y)}_R \Phi)^\circ_{\circ,L'} dz^{L-L'}.$$  

(3.31)

**Lemma 3.4.2.** The product (3.26) satisfies (3.28a), i.e.

$$A_s(\alpha) \hat{\wedge} B_s(\beta) = C_s(\bar{A}^*(\alpha) \wedge \bar{B}^*(\beta)).$$

**Proof.** Let $\Psi = A_s(\alpha)$ and $\Phi = B_s(\beta)$ with $\alpha = \Psi^{\circ,\circ}_{K,L} dx^K \wedge dz^L$ and $\beta = \Phi^{\circ,\circ}_{K',L'} dy^{K'} \wedge dz^{L'}$ then

$$\Psi \hat{\wedge} \Phi = A_s(\Psi^{\circ,\circ}_{K,L} dx^K \wedge dz^L) \hat{\wedge} B_s(\Phi^{\circ,\circ}_{K',L'} dy^{K'} \wedge dz^{L'}) = A_s(\alpha) \hat{\wedge} B_s(\beta).$$

However from (3.26) we also have

$$\Psi \hat{\wedge} \Phi = C_s(\bar{A}^*(\Psi^{\circ,\circ}_{\circ,L}) \bar{B}^*(\Phi^{\circ,\circ}_{\circ,L'}) dz^L \wedge dz^{L'})$$

$$= C_s(\bar{A}^*(\Psi^{\circ,\circ}_{\circ,L} dz^L) \wedge \bar{B}^*(\Phi^{\circ,\circ}_{\circ,L'} dz^{L'}))$$

$$= C_s(\bar{A}^*(\alpha) \wedge \bar{B}^*(\beta)), $$

hence the result follows. \hfill \Box

**Lemma 3.4.3.** The product (3.31) satisfies (3.28b) i.e.

$$(\lambda \Psi) \hat{\wedge} \Phi = \Psi \hat{\wedge} (\lambda \Phi), \quad \lambda \in \Gamma \Lambda^0 \mathcal{M}.$$ 

**Proof.** The internal contractions commute with scalar fields, so in light of the discussion above, we only need to consider the Lie derivatives. Multiplying $\Psi$ by $\lambda$ and commuting the Lie derivatives with the pushforward we have

$$\lambda \Psi = \lambda L^{(y)}_{I} A_s(\Psi^{\circ,\circ}_{K,L} dx^K \wedge dz^L) = \sum_{R, R' \subseteq I} (-1)^{|R|} L^{(y)}_{I',R} A_s(\Psi^{\circ,\circ}_{K,L} dx^K \wedge dz^L).$$ 

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and likewise
\[ \lambda \Phi = \sum_{R', R'' \in I'} (-1)^{|R'|} L_{I' \setminus R'}^{(x)} B_{\lambda} \left( B^{*} \left[ \partial_{R'}^{(y)} \lambda \right] \Phi_{K', L'}^{I', J'} \, dy^{K'} \land dz^{L'} \right). \]

Taking the appropriate wedge product and then the difference we get
\[ (\lambda \Psi) \circ \Phi - \Psi \circ (\lambda \Phi) \]

\[ = \sum_{R''_h, R'' \subset I'} (-1)^{|R''_h|} L_{I' \setminus R''}^{(y)} \left( A_{\lambda} \left[ \partial_{R''}^{(y)} \lambda \right] \Psi_{K, L}^{I, J} \, dx^{K} \land dz^{L} \right) \circ L_{I'}^{(x)} B_{\lambda} \left( B^{*} \left[ \partial_{R''}^{(x)} \lambda \right] \Phi_{K', L'}^{I', J'} \, dy^{K'} \land dz^{L'} \right) \]

\[ - \sum_{R''_h, R'' \subset I'} (-1)^{|R''_h|} L_{I' \setminus R''}^{(y)} \left( A_{\lambda} \left[ \partial_{R''}^{(y)} \lambda \right] \Psi_{K, L}^{I, J} \, dx^{K} \land dz^{L} \right) \circ L_{I'}^{(x)} B_{\lambda} \left( B^{*} \left[ \partial_{R''}^{(x)} \lambda \right] \Phi_{K', L'}^{I', J'} \, dy^{K'} \land dz^{L'} \right) \]

The argument of the pushforward with respect to \( C \) can be simplified by using Leibniz and (3.3) such that
\[ A_{\lambda} \left[ \partial_{R''}^{(y)} \lambda \right] \Psi_{K, L}^{I, J} \, A^{*} \left[ \partial_{R''}^{(y)} \lambda \right] \Psi_{K', L'}^{I', J'} = \sum_{S, S' \subset R''} C^{*} \left[ \partial_{S'}^{(y)} \partial_{S}^{(y)} \lambda \right] A_{\lambda} \left[ \partial_{R'' \setminus S'}^{(x)} \partial_{R'' \setminus S}^{(x)} \lambda \right] \Psi_{K, L}^{I, J}, \Psi_{K', L'}^{I', J'}. \]

Similarly for the pullback \( \hat{B}^{*} \):
\[ \hat{B}^{*} \left[ \partial_{R''}^{(y)} \left( B^{*} \left[ \partial_{R''}^{(x)} \lambda \right] \Phi_{K', L'}^{I', J'} \right) \right] = \sum_{S, S' \subset R''} C^{*} \left[ \partial_{S'}^{(y)} \partial_{S}^{(y)} \lambda \right] \hat{B}^{*} \left[ \partial_{R'' \setminus S'}^{(y)} \partial_{R'' \setminus S}^{(y)} \lambda \right] \Phi_{K', L'}^{I', J'}. \]

Substituting in the above we have two summations of the form
\[ (\lambda \Psi) \circ \Phi - \Psi \circ (\lambda \Phi) = \sum_{R, S \subset I} \cdots - \sum_{R', S' \subset I'} \cdots. \]
3.4. Properties of the O-Wedge Product

In order to make apparent the symmetry between the two terms let $R \setminus S = T$ and $R \setminus S' = T'$. Using this substitution the condition $\{R', S' | S' \subset R' \subset I'\}$ is equivalent to $\{T', S' \subset I | T' \cap S' = \emptyset\}$, likewise with $T$ so

\[
(\lambda \Psi) \hat{\wedge} \check{\Phi} - \Psi \hat{\wedge} (\lambda \check{\Phi})
= \sum_{R, S \subset I, R \cap S = \emptyset, T', S' \subset I', T' \cap S' = \emptyset} (-1)^{|R| + |T'| + |S| + |S'|}
L_{I \setminus T}^{(x)} L_{I' \setminus (R, S)}^{(y)} C \left[ C^* \left[ \partial_{S'}^{(x)} \partial_R^{(y)} \lambda \right] \hat{A} \left[ \partial_{T'}^{(x)} \bar{\Psi}_{I', J}^{L, K} \right] \hat{B} \left[ \partial_S^{(y)} \bar{\Phi}_{K', L'}^{J', I'} \right] dz^{L, L'} \right]
- \sum_{R', S' \subset I', R' \cap S' = \emptyset, T, S \subset I, T \cap S = \emptyset} (-1)^{|R'| + |T| + |S'| + |S|}
L_{I \setminus (R', S')}^{(x)} L_{I' \setminus I}^{(y)} C \left[ C^* \left[ \partial_{R'}^{(x)} \partial_S^{(y)} \lambda \right] \hat{A} \left[ \partial_{S'}^{(x)} \bar{\Psi}_{I', J}^{L, K} \right] \hat{B} \left[ \partial_T^{(y)} \bar{\Phi}_{K', L'}^{J', I'} \right] \right]
= 0.
\]

\[\square\]

**Lemma 3.4.4.** The product (3.31) satisfies (3.28c)

\[i_a^{(x)} (\Psi \hat{\wedge} \check{\Phi}) = i_a^{(x)} \Psi \hat{\wedge} \check{\Phi} + (-1)^{\deg(\Psi)} \Psi \hat{\wedge} i_a^{(x)} \check{\Phi}.\]

**Proof.** The Lie derivatives commute with internal contractions. Without any loss in generality we can also set $\Psi_{I', J}^{L, K} = \Phi_{K', L'}^{J, I} = 1$. (3.27) incurs an additional change in sign whenever an extra element $a$ is added or taken away from the lists $J'$ and $K$ and also if the ordering of the list $K$ is changed. This is summarized in the following

\[
\omega^a_{J', a, J, L} = (-1)^{|J'| - |J| - |L|} \omega^a_{J', a, J, L} \quad a \in K \quad \text{and} \quad a \notin J',
\]

\[
\omega^a_{J', a, J, L} = (-1)^{|J'| - |J| - |L|} \omega^a_{J', a, J, L} \quad a \notin K \quad \text{and} \quad a \notin J',
\]

\[
\omega^a_{J', a, J, L} = (-1)^{|J'| - |J| - |L| - 1} \omega^a_{J', a, J, L} \quad a \in K \quad \text{and} \quad a \notin J'.
\]

\[
\omega^a_{J', a, J, L} = (-1)^{|J'| - |J| - |L| - 1} \omega^a_{J', a, J, L} \quad a \in K \quad \text{and} \quad a \notin J'.
\]
If we first consider the action of \( i_a^{(x)} \) then we get

\[
i_a^{(x)}(\Psi \wedge \Phi) = \begin{cases} 0 & a \notin K, a \in J' \\ 0 & a \in K, a \notin J \\ \omega_{j',j;L}^K \epsilon_a(K\backslash a) i_{j'(K\backslash a)}^{(x)} i_{j(K\backslash a)}^{(y)} C_\varsigma(dz^{L-L'}) & a \in K, a \in J, \end{cases}
\]

\[
i_a^{(x)}\Psi \circ \wedge \Phi = \begin{cases} 0 & a \notin K, a \in J' \\ \omega_{j',j;L}^K \epsilon_a(K\backslash a) i_{j'(K\backslash a)}^{(x)} i_{j(K\backslash a)}^{(y)} C_\varsigma(dz^{L-L'}) & a \in K, a \notin J \\ 0 & a \in K, a \in J', \end{cases}
\]

\[
\Psi \circ i_a^{(x)} \Phi = \begin{cases} 0 & a \notin K, a \in J' \\ (-1)^{|K|+|L|-|J|-1} \omega_{j',j;L}^K \epsilon_a(K\backslash a) i_{j'(K\backslash a)}^{(x)} i_{j(K\backslash a)}^{(y)} C_\varsigma(dz^{L-L'}) & a \in K, a \notin J \\ (-1)^{|K|+|L|-|J|} \omega_{j',j;L}^K \epsilon_a(K\backslash a) i_{j'(K\backslash a)}^{(x)} i_{j(K\backslash a)}^{(y)} C_\varsigma(dz^{L-L'}) & a \in K, a \notin J \\ 0 & a \in K, a \in J'. \end{cases}
\]

Therefore summing over all the above cases we get

\[
i_a^{(x)}\Psi \circ \wedge \Phi = (-1)^{\deg(\Psi)} \Psi \circ i_a^{(x)} \Phi,
\]

and

\[
i_a^{(x)}(\Psi \circ \wedge \Phi) = i_a^{(x)}\Psi \circ \wedge \Phi + (-1)^{\deg(\Psi)} \Psi \circ i_a^{(x)} \Phi.
\]

\[
\text{Lemma 3.4.5. The product (3.31) satisfies (3.28c) for the y-coordinates}
\]

\[
i_a^{(y)}(\Psi \circ \wedge \Phi) = i_a^{(y)}\Psi \circ \wedge \Phi + (-1)^{\deg(\Psi)} \Psi \circ i_a^{(y)} \Phi.
\]

\[
\text{Proof. Same reasoning as for } i_a^{(x)}.
\]
Lemma 3.4.6. The product (3.31) satisfies (3.28c) for the z-coordinates

\[ i^{(z)}_a (\Psi \circ \Phi) = i^{(z)}_a \Psi \circ \Phi + (-1)^{\deg(\Psi)} \Psi \circ i^{(z)}_a \Phi. \]

**Proof.** If we reduce the length of the list \( L \) by one then the resulting change in sign is

\[ \omega^{K,K'}_{J',L} = (-1)^{|K'|-|J'|} \omega^{K,K'}_{J',L}. \]

Thus

\[ i^{(z)}_c (\Psi \circ \Phi) = (-1)^{|J'\setminus K|+|J\setminus K'|} \omega^{K,K'}_{J',L} \]

\[ \cdot i^{(x)}_{J'\setminus K} i^{(y)}_{J\setminus K'} C_c \left( e^L_{c(L\setminus z \setminus c)} dz^L \wedge dz^L' \right) + (-1)^{|L|} (-1)^{|J'\setminus K|+|J\setminus K'|} \omega^{K,K'}_{J',L} i^{(x)}_{J'\setminus K} i^{(y)}_{J\setminus K'} C_c \left( e^{L'}_{c(L\setminus z \setminus c)} dz^L \wedge dz^L' \right) \]

\[ = (-1)^{|J'\setminus K|+|J\setminus K'|} \omega^{K,K'}_{J',L} \]

\[ \cdot i^{(x)}_{J'\setminus K} i^{(y)}_{J\setminus K'} C_c \left( e^L_{c(L\setminus z \setminus c)} dz^L \wedge dz^L' \right) + (-1)^{|L|} (-1)^{|J'\setminus K|+|J\setminus K'|} \omega^{K,K'}_{J',L} i^{(x)}_{J'\setminus K} i^{(y)}_{J\setminus K'} C_c \left( e^{L'}_{c(L\setminus z \setminus c)} dz^L \wedge dz^L' \right) \]

\[ = (-1)^{|J'\setminus K|} i^{(y)}_J A_c \left( dx^K \wedge e^L_{c(L\setminus c)} dz^L \right) \circ i^{(x)}_{J'} B_c \left( dy^{K'} \wedge dz^{L'} \right) \]

\[ + (-1)^{|L|} (-1)^{|J'\setminus K|} \]

\[ \cdot i^{(y)}_J A_c \left( dx^K \wedge dz^L \right) \circ i^{(x)}_{J'} B_c \left( dy^{K'} \wedge e^{L'}_{c(L\setminus c)} dz^L \wedge dz^{L'} \right) \]

\[ = i^{(z)}_c \cdot i^{(y)}_J A_c \left( dx^K \wedge dz^L \right) \circ i^{(x)}_{J'} B_c \left( dy^K \wedge dz^{L'} \right) \]

\[ + (-1)^{|L|} (-1)^{|J'\setminus K|} \]

\[ \cdot i^{(y)}_J A_c \left( dx^K \wedge dz^L \right) \circ i^{(x)}_{J'} B_c \left( dy^{K'} \wedge e^{L'}_{c(L\setminus c)} dz^L \wedge dz^{L'} \right) \]

\[ = i^{(z)}_c \Psi \circ \Phi + (-1)^{\deg(\Psi)} \Psi \circ i^{(z)}_c \Phi. \]
Lemma 3.4.7. The product (3.31) satisfies (3.28d)

\[ L^{(x)}_{a}(\Psi \overset{\circ}{\wedge} \Phi) = L^{(x)}_{a}\Psi \overset{\circ}{\wedge} \Phi + \Psi \overset{\circ}{\wedge} L^{(x)}_{a}\Phi. \]

Proof. The Lie derivatives follow a similar pattern to the previous proof but this time we use (3.31) with the internal contractions dropped, so

\[ L^{(x)}_{a}\Psi \overset{\circ}{\wedge} \Phi + \Psi \overset{\circ}{\wedge} L^{(x)}_{a}\Phi \]

\[ \begin{align*}
&= L^{(y)}_{I} A_{c} \left( \partial_{a}^{(x)} \Psi^{I,\otimes} \delta_{L} d\zeta^{L} \right) \overset{\circ}{\wedge} L^{(x)}_{I'} B_{c} \left( \Phi^{I',\otimes} \delta_{L} d\zeta^{L'} \right) \\
&\quad + L^{(y)}_{I} A_{c} \left( \Psi^{I,\otimes} \delta_{L} d\zeta^{L} \right) \overset{\circ}{\wedge} L^{(x)}_{I'} B_{c} \left( \Phi^{I',\otimes} \delta_{L} d\zeta^{L'} \right) \\
&= \sum_{R, R' \subset I} (-1)^{|R|+|R'|} L^{(y)}_{I \cap R} L^{(x)}_{I' \cap R'} C \left[ \hat{A}^{\ast} \left( \partial_{a}^{(x)} \Psi^{I,\otimes} \delta_{L} \right) \hat{B}^{\ast} \left( \partial_{a}^{(y)} \Phi^{I,\otimes} \delta_{L} \right) d\zeta^{L-L'} \right] \\
&\quad + \sum_{R, R' \subset I} (-1)^{|R|+|R'|} L^{(y)}_{I \cap R} L^{(x)}_{I' \cap R'} C \left[ \hat{A}^{\ast} \left( \partial_{a}^{(x)} \Psi^{I,\otimes} \delta_{L} \right) \hat{B}^{\ast} \left( \partial_{a}^{(y)} \Phi^{I,\otimes} \delta_{L} \right) d\zeta^{L-L'} \right] \\
&\quad - \sum_{R, R' \subset I} (-1)^{|R|+|R'|} L^{(y)}_{I \cap R} L^{(x)}_{I' \cap R'} C \left[ \hat{A}^{\ast} \left( \partial_{a}^{(x)} \Psi^{I,\otimes} \delta_{L} \right) \hat{B}^{\ast} \left( \partial_{a}^{(y)} \Phi^{I,\otimes} \delta_{L} \right) d\zeta^{L-L'} \right] \\
&= \sum_{R, R' \subset I} (-1)^{|R|+|R'|} L^{(y)}_{I \cap R} L^{(x)}_{I' \cap R'} C \left[ \hat{A}^{\ast} \left( \partial_{a}^{(x)} \Psi^{I,\otimes} \delta_{L} \right) \hat{B}^{\ast} \left( \partial_{a}^{(y)} \Phi^{I,\otimes} \delta_{L} \right) d\zeta^{L-L'} \right] \\
&= L^{(x)}_{a} \left( \Psi \overset{\circ}{\wedge} \Phi \right).
\]

Lemma 3.4.8. The product (3.31) satisfies (3.28d) for the y and z coordinates

\[ L^{(y)}_{a}(\Psi \overset{\circ}{\wedge} \Phi) = L^{(y)}_{a}\Psi \overset{\circ}{\wedge} \Phi + \Psi \overset{\circ}{\wedge} L^{(y)}_{a}\Phi, \]

\[ L^{(z)}_{a}(\Psi \overset{\circ}{\wedge} \Phi) = L^{(z)}_{a}\Psi \overset{\circ}{\wedge} \Phi + \Psi \overset{\circ}{\wedge} L^{(z)}_{a}\Phi. \]
3.4. Properties of the O-Wedge Product

Proof. The $y$ component follows the same pattern as the above. The $z$ component is similar to the $i^a_{(z)}$ case. \hfill $\square$

Lemma 3.4.9. The product (3.31) satisfies (3.28e)

$$(dx^a \overset{\circ}{\wedge} \Psi) \overset{\circ}{\wedge} \Phi = (-1)^{\text{deg}(\Psi)} \overset{\circ}{\Psi} \overset{\circ}{\wedge} (dx^a \overset{\circ}{\wedge} \Phi).$$

Proof. The Lie derivatives in (3.31) can be dropped since they commute with $dx^a$ thus

$$dx^a \overset{\circ}{\wedge} \Psi = \begin{cases} 0 & a \in K, \\ \left((-1)^{|j|} \varepsilon_{j}^{(y)} A_{j} (\Psi_{K,L} \overset{\circ}{dx}^{a-K} \wedge dz^{L})\right) & a \notin K. \end{cases}$$

Taking the wedge product with $\Phi$ we have

$$(dx^a \overset{\circ}{\wedge} \Psi) \overset{\circ}{\wedge} i_{J'}^{(x)} L_{I'}^{(x)} B_c (\Phi_{K',L'} dy^{K'} \wedge dz^{L'}) \neq 0 \quad \text{only if} \quad a \in J', a \notin K.$$ 

The list $J'$ is re-ordered such that the element $a$ is last i.e. $(J' \setminus a) \cdot a$. This is to ensure that it respects the ordering of the list $a \cdot K$. Therefore we have

$$\begin{align*}
(dx^a \overset{\circ}{\wedge} \Psi) \overset{\circ}{\wedge} \Phi &= \begin{cases} 0 & a \in K, a \in J' \\ 0 & a \in K, a \notin J' \\ (-1)^{|j|} \varepsilon_{J'}^{(x)} i_{J'}^{(y)} \varepsilon_{K'(a-K') \cdot J'(a), J} C_c \left( \hat{A} \left( \Psi_{K,L} \overset{\circ}{dx}^{a-K} \wedge dz^{L} \right) \right) & a \notin K, a \in J' \\ 0 & a \notin K, a \notin J'. \end{cases} \\
\end{align*}$$

(3.35)

Now consider $dx^a \overset{\circ}{\wedge} \Phi$:

$$dx^a \overset{\circ}{\wedge} \Phi = \begin{cases} 0 & a \notin J', \\ \varepsilon_{J'}^{(x)} i_{J'}^{(y)} \varepsilon_{J' \setminus (a-K) \cdot J, J} B_c (\Phi_{K',L'} dy^{K'} \wedge dz^{L'}) & a \in J'. \end{cases}$$

Thus

$$\Psi \overset{\circ}{\wedge} (dx^a \overset{\circ}{\wedge} \Phi) = \varepsilon_{J'}^{(x)} \varepsilon_{J' \setminus (a-K') \cdot J' \setminus (a-K) \cdot J, J} C_c \left( \hat{A} \left( \Psi_{K,L} \overset{\circ}{dx}^{a-K} \wedge dz^{L} \right) \right).$$

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3.4. Properties of the O-Wedge Product

However we have the relation

\[
(1 - 1) \omega_{J \setminus a, J, L} = (-1)^{\deg(\Psi)} \omega_{J \setminus a, J, L} \quad a \notin K, \ a \in J',
\]

which gives the final result

\[
(dx^a \wedge \Psi) \wedge \Phi = (-1)^{\deg(\Psi)} \Psi \wedge (dx^a \wedge \Phi).
\]

\[
(3.37)
\]

Lemma 3.4.10. The product (3.31) satisfies (3.28e) for the $y$ coordinates

\[
(dy^b \wedge \Psi) \wedge \Phi = (-1)^{\deg(\Psi)} \Psi \wedge (dy^b \wedge \Phi).
\]

Proof. The proof is the same as for $dx^a$.

\[
(3.31)
\]

Lemma 3.4.11. The product (3.31) satisfies (3.28e) for the $z$ coordinates

\[
(dz^c \wedge \Psi) \wedge \Phi = (-1)^{\deg(\Psi)} \Psi \wedge (dz^c \wedge \Phi).
\]

Proof. $dz^c$ passes through both pushfowards $A_\varsigma$ and $B_\varsigma$ thus

\[
(dz^c \wedge \Psi) \wedge \Phi = (-1)^{|J| + |K|} i_J^y A_\varsigma(\Psi_{K, L}^y d\pi K \wedge dz^c) \wedge i_J^x B_\varsigma(\Phi_{K', L'}^y d\pi K' \wedge dz^c)
\]

\[
= (-1)^{|J| - |K|} \omega_{J, K, L} i_J^{(x)} i_J^{(y)} C_\varsigma (dz^c - L')
\]

whereas

\[
\Psi \wedge (dz^c \wedge \Phi) = (-1)^{|J| + |K'|} i_J^y A_\varsigma(\Psi_{K, L}^y d\pi K \wedge dz^c) \wedge i_J^x B_\varsigma(\Phi_{K', L'}^y d\pi K' \wedge dz^c)
\]

\[
= (-1)^{|J| + |K'|} \omega_{J, K, L} i_J^{(x)} i_J^{(y)} C_\varsigma (dz^c - L')
\]

\[
= (-1)^{|J| + |K|} \omega_{J, K, L} i_J^{(x)} i_J^{(y)} C_\varsigma (dz^c - L')
\]

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3.4. Properties of the O-Wedge Product

However from (3.27)

\[ \omega^{K,K'}_{J',J,L} = (-1)^{|J'|-|K'|} \omega^{K,K'}_{J',J,L}. \]

So

\[ (dz^c \wedge \Psi) \circ \Lambda (d_\varphi) = (-1)^{|J'|+|K'|-|\varphi|} \omega^{K,K'}_{J',J,L} \]

\[ \psi \circ \Lambda (d_\varphi) = (-1)^{|J'|+|K'|} \psi \circ \Lambda (d_\varphi). \]

In order to prove associativity we require the following proposition.

**Proposition 3.4.12.** The wedge product of three transverse embeddings satisfies the following associativity rule

\[ (A_\varphi(\alpha) \wedge B_\varphi(\beta)) \wedge D_\varphi(\gamma) = A_\varphi(\alpha) \wedge (B_\varphi(\beta) \wedge D_\varphi(\gamma)) \] (3.38)

where the maps are given by (3.11).

**Proof.** From (3.8) the intersection of \( A \) and \( B \) is

\[ I_{AB} \xrightarrow{\hat{A}} A \]

\[ B \]

\[ \hat{B} \]

\[ I_{AB} \]

\[ B \]

\[ A \]

\[ M \]

\[ A_\varphi(\alpha) \wedge B_\varphi(\beta) = I_{AB}(\hat{A}_\varphi(\alpha) \wedge \hat{B}_\varphi(\beta)), \] (3.39)

whereas for \( B \) and \( D \)

\[ B \]

\[ \hat{B} \]

\[ I_{BD} \]

\[ D \]

\[ \hat{D} \]

\[ M \]

\[ B_\varphi(\beta) \wedge D_\varphi(\gamma) = I_{BD}(\hat{B}_\varphi(\beta) \wedge \hat{D}_\varphi(\gamma)). \] (3.40)
3.4. Properties of the O-Wedge Product

The intersection of $I_{AB}$ and $D$ is

$$I_{ABD} \xrightarrow{I_3} I_{AB} \quad I_{AB},(\delta) \wedge D,(\gamma) = I_{MB} \left( I^{*}_3(\delta) \wedge I^{*}_D(\gamma) \right). \quad (3.41)$$

Thus we have

$$(A,(\alpha) \wedge B,(\beta)) \wedge D,(\gamma) = I_{AB}, \left( \hat{A}^{*}(\alpha) \wedge \hat{B}^{*}(\beta) \right) \wedge D,(\gamma)$$

$$= I_{MB} \left( I^{*}_3(\hat{A}^{*}(\alpha) \wedge \hat{B}^{*}(\beta)) \wedge I^{*}_D(\gamma) \right)$$

$$= I_{MB} \left( (I^{*}_A(\alpha) \wedge I^{*}_B(\beta)) \wedge I^{*}_D(\gamma) \right)$$

$$= I_{MB} \left( I^{*}_A(\alpha) \wedge I^{*}_B(\beta) \wedge \hat{D}^{*}(\gamma) \right)$$

$$= A,(\alpha) \wedge I_{BD}, \left( I^{*}_B(\beta) \wedge \hat{D}^{*}(\gamma) \right)$$

$$= A,(\alpha) \wedge \left( B,(\beta) \wedge D,(\gamma) \right). \quad (3.42)$$

**Lemma 3.4.13.** The product (3.26) satisfies the associativity law (3.28f), i.e.

$$\Psi \circ (\Phi \circ \Omega) = (\Psi \circ \Phi) \circ \Omega.$$ 

**Proof.** By induction. In the local adapted coordinate system given by (3.12) the SMDs $\Psi$, $\Phi$, $\Omega$ have the form

$$\Psi = \sum i^{(z)}_J L^{(z)}_I A,(\Psi^{I,J}_{K,M,L} dx^K \wedge dy^M \wedge dw^L), \quad (3.43a)$$

$$\Phi = \sum j^{(x)}_J L^{(x)}_I B,(\Phi^{I,J'}_{K',M',L'} dy^{K'} \wedge dz^{M'} \wedge dw^{L'}), \quad (3.43b)$$

$$\Omega = \sum m^{(y)}_J L^{(y)}_I D,(\Omega^{I,J''}_{K'',M'',L''} dx^{K''} \wedge dz^{M''} \wedge dw^{L''}). \quad (3.43c)$$

Associativity holds for zero order SMDs as shown in Proposition 3.4.12. Now assume true for $\Psi \in \Upsilon^{k,p}(A)$, $\Phi \in \Upsilon^{k',p'}(B)$ and $\Omega \in \Upsilon^{k'',p''}(D)$. The Lie derivative with respect
3.4. Properties of the O-Wedge Product

to the $z$ components has the following effect

$$L_a^{(z)} \Psi \in \mathcal{Y}^{k+1,p}(A), \quad L_a^{(z)} \Phi = \tilde{\Phi} \in \mathcal{Y}^{k',p'}(B), \quad L_a^{(z)} \Omega = \tilde{\Omega} \in \mathcal{Y}^{k'',p''}(D). \quad (3.44)$$

Since the $z$ component is transverse to $\Psi$ the Lie derivative increases the order by one, however for $\Phi$ and $\Omega$ it passes through the pushforward, thus not affecting the order. Therefore using (3.28d) we get

$$\left( L_a^{(z)} \Psi \overset{\circ}{\wedge} \Phi \right) \overset{\circ}{\wedge} \Omega = L_a^{(z)} \left( \Phi \overset{\circ}{\wedge} \Phi \right) \overset{\circ}{\wedge} \Omega - \left( \Phi \overset{\circ}{\wedge} L_a^{(z)} \Phi \right) \overset{\circ}{\wedge} \Omega$$

$$= L_a^{(z)} \left( \left( \Psi \overset{\circ}{\wedge} \Phi \right) \overset{\circ}{\wedge} \Omega \right) - \left( \Psi \overset{\circ}{\wedge} \Phi \right) \overset{\circ}{\wedge} L_a^{(z)} \Omega - \left( \Psi \overset{\circ}{\wedge} L_a^{(z)} \Phi \right) \overset{\circ}{\wedge} \Omega$$

$$= L_a^{(z)} \Psi \overset{\circ}{\wedge} \left( \Phi \overset{\circ}{\wedge} \Omega \right) - \Psi \overset{\circ}{\wedge} \left( \Phi \overset{\circ}{\wedge} \Omega \right) - \Psi \overset{\circ}{\wedge} \left( \Phi \overset{\circ}{\wedge} L_a^{(z)} \Omega \right) - \Psi \overset{\circ}{\wedge} \left( L_a^{(z)} \Phi \overset{\circ}{\wedge} \Omega \right)$$

$$= L_a^{(z)} \Psi \overset{\circ}{\wedge} \left( \Phi \overset{\circ}{\wedge} \Omega \right).$$

Similar reasoning applies for the Lie derivatives $L_a^{(x)}$ and $L_b^{(y)}$ also the internal contractions $i_a^{(x)}$, $i_b^{(y)}$ and $i_c^{(z)}$. Hence it is now straightforward to build up a general $\Psi$, $\Phi$ and $\Omega$ from repeated applications of Lie derivatives and internal contractions on (3.38). From (3.43a)

$$\left( A_c \left( \Psi_{K,M,L}^{I,J} dx^K \wedge dy^M \wedge dw^L \right) \overset{\circ}{\wedge} B_c(\beta) \right) \overset{\circ}{\wedge} D_c(\gamma)$$

$$= A_c \left( \Psi_{K,M,L}^{I,J} dx^K \wedge dy^M \wedge dw^L \right) \overset{\circ}{\wedge} \left( B_c(\beta) \overset{\circ}{\wedge} D_c(\gamma) \right).$$

By induction on the list lengths $I$ and $J$ we have

$$\left( i_{I,J}^{(z)} L_I^{(z)} A_c \left( \Psi_{K,M,L}^{I,J} dx^K \wedge dy^M \wedge dw^L \right) \overset{\circ}{\wedge} B_c(\beta) \right) \overset{\circ}{\wedge} D_c(\gamma)$$

$$= i_{I,J}^{(z)} L_I^{(z)} A_c \left( \Psi_{K,M,L}^{I,J} dx^K \wedge dy^M \wedge dw^L \right) \overset{\circ}{\wedge} \left( B_c(\beta) \overset{\circ}{\wedge} D_c(\gamma) \right).$$

Using ‘+’ linearity

$$\left( \Psi \overset{\circ}{\wedge} B_c(\beta) \right) \overset{\circ}{\wedge} D_c(\gamma) = \Psi \overset{\circ}{\wedge} \left( B_c(\beta) \overset{\circ}{\wedge} D_c(\gamma) \right).$$
3.5. O-wedge to SMD Wedge

Repeating the same procedure with \( i^{(x)}_{j'} \) and \( L^{(x)}_{j''} \) we get

\[
(\Psi \wedge \Phi) \wedge D,(\gamma) = \Psi \wedge (\Phi \wedge D,(\gamma)).
\]

Finally we do the same with \( i^{(y)}_{j''} \) and \( L^{(y)}_{j''} \) in order to construct \( \Omega \).

Lemma 3.4.14. The product (3.26) satisfies the commutativity law (3.28f), i.e.

\[
\Psi \wedge \Phi = (-1)^{\text{deg}(\Psi)\text{deg}(\Phi)}(\Phi \wedge \Psi).
\]

Proof. The procedure is the same as for associativity. For zeroth order we have

\[
A_{\dot{\zeta}}(\alpha) \wedge B_{\dot{\zeta}}(\beta) = C_{\dot{\xi}}(\dot{A} \wedge \dot{B})(\beta) = (-1)^{\text{deg}(\alpha)\text{deg}(\beta)}C_{\dot{\xi}}(\dot{B} \wedge \dot{A})(\beta)
\]

\[
= (-1)^{\text{deg}(\alpha)\text{deg}(\beta)}B_{\dot{\xi}}(\beta) \wedge A_{\dot{\xi}}(\alpha)
\]

\[
= (-1)^{\text{deg}(A_{\dot{\xi}}(\alpha))\text{deg}(B_{\dot{\xi}}(\beta))}B_{\dot{\xi}}(\beta) \wedge A_{\dot{\xi}}(\alpha).
\]

Higher orders are constructed from repeated application of (3.28d) and (3.28c).

3.5 O-wedge to SMD Wedge

Lemma 3.5.1. \( \wedge \) is a wedge product of SMDs, therefore locally a wedge product of SMDs exists.

Proof. (3.28a), (3.28f) and (3.28g) clearly satisfy (W1), (W2) and (W3) respectively so all that remains is (W4). Let \( v \in \Gamma \mathcal{T} \mathcal{M} \). With respect to the adapted coordinate system (3.4), \( v \) has the decomposition

\[
v = v^a_x \partial^x_a + v^b_y \partial^y_b + v^c_z \partial^z_c.
\]

As the the internal contractions and Lie derivatives are ‘f’-linear in \( v \) we only show the
3.5. O-wedge to SMD Wedge

$x$ components, the procedure being the same for the remaining two basis coordinates.

\[ i_x(\Psi \wedge \Phi) = v_x^a i_x^a (\Psi \wedge \Phi) \]
\[ = v_x^a (i_x^a \Psi \wedge \Phi + (-1)^{i_x^a} \Psi \wedge (i_x^a \Phi)) \]
\[ = (v_x^a i_x^a) \wedge \Phi + (-1)^{i_x^a} \Psi \wedge (v_x^a i_x^a) \Phi \]
\[ = i_x \Psi \wedge \Phi + (-1)^{i_x^a} \Psi \wedge i_x \Phi. \]

\[ L_v(\Psi \wedge \Phi) \]
\[ = v_x^a L_v^a (\Psi \wedge \Phi) + dv_x^a \wedge i_x^a (\Psi \wedge \Phi) \]
\[ = v_x^a (L_v^a \wedge \Phi + \Psi \wedge (v_x^a L_v^a \Phi) + d v_x^a \wedge (i_x^a \wedge (v_x^a L_v^a \Phi) + (-1)^{i_x^a} \Psi \wedge (i_x^a L_v^a \Phi)) \]
\[ = (v_x^a L_v^a) \wedge \Phi + \Psi \wedge (v_x^a L_v^a \Phi) + d v_x^a \wedge (i_x^a \wedge (v_x^a L_v^a \Phi) + (-1)^{i_x^a} \Psi \wedge (i_x^a L_v^a \Phi)) \]
\[ = (v_x^a L_v^a + d v_x^a \wedge i_x^a) \wedge \Phi + \Psi \wedge (v_x^a L_v^a \Phi + d v_x^a \wedge i_x^a) \Phi \]
\[ = L_v \Psi \wedge \Phi + \Psi \wedge L_v \Phi. \]

Thus the O-wedge product satisfies (W1-W4) and hence is a wedge product of SMDs.

Lemma 3.5.2. Given another adapted coordinate system \( (\hat{x}, \hat{y}, \hat{z}) \)

\[ \Psi \wedge (\hat{x}, \hat{y}, \hat{z}) = \Psi \wedge \Phi. \] (3.45)
3.5. O-wedge to SMD Wedge

Proof. In the coordinates system \((\hat{x}, \hat{y}, \hat{z})\), \(\Psi\) has the same form as (2.64). Therefore since we have shown (3.26) satisfies Definition 3.3.1, which is defined independently of any coordinate system, the O-wedge product must hold in the new adapted coordinate system. Hence the O-wedge is a wedge product of two transverse SMDs.

Before we give the proof of the next theorem we first give a specific lower dimensional example without multi-indices. Let \(\dim(M) = 3\) and consider two transverse maps \(A\) and \(B\) such that \(\dim(A) = 2\) and \(\dim(B) = 2\). The intersection has dimension 1. See for example Figure 3.2. Local adapted coordinates for \(M\) are \((x_1, y_1, z_1)\).

Consider \(\Psi \in \Upsilon_{1,3}(A)\) and \(\Phi \in \Upsilon^{0,0}(B)\) and let is find \(\Psi \wedge \Phi\). In adapted coordinates we have

\[
\Psi = L_1^{(y)} A_\varsigma \left( \Psi_{1,1}^{1,\partial} dx^1 \wedge dz^1 \right),
\]

\[
\Phi = i_1^{(x)} B_\varsigma \left( \Phi_{\partial,\partial}^{1,1} \right).
\]

First we apply the Leibniz rule, (3.22), with respect to \(i_1^{(x)}\) to get

\[
\Psi \wedge \Phi = L_1^{(y)} A_\varsigma \left( \Psi_{1,1}^{1,\partial} dx^1 \wedge dz^1 \right) \wedge i_1^{(x)} B_\varsigma \left( \Phi_{\partial,\partial}^{1,1} \right)
\]

\[
= (-1)^{\text{ideg}(\Psi)} i_1^{(x)} \left( L_1^{(y)} A_\varsigma \left( \Psi_{1,1}^{1,\partial} dx^1 \wedge dz^1 \right) \wedge B_\varsigma \left( \Phi_{\partial,\partial}^{1,1} \right) \right)
\]

\[
- (-1)^{\text{ideg}(\Psi)} i_1^{(x)} \left( L_1^{(y)} A_\varsigma \left( \Psi_{1,1}^{1,\partial} dx^1 \wedge dz^1 \right) \wedge B_\varsigma \left( \Phi_{\partial,\partial}^{1,1} \right) \right).
\]

From (2.51)

\[
\text{ideg}(\Psi) = \text{ideg} \left( L_1^{(y)} A_\varsigma \left( \Psi_{1,1}^{1,\partial} dx^1 \wedge dz^1 \right) \right) = 1.
\]

Now since \(i_1^{(x)} \circ L_1^{(y)} = L_1^{(y)} \circ i_1^{(x)}\) and \(i_1^{(x)} \circ A_\varsigma = A_\varsigma \circ i_1^{(x)}\) by virtue of (2.37) the second term in the above simplifies to

\[
L_1^{(y)} A_\varsigma \left( \Psi_{1,1}^{1,\partial} dz^1 \right) \wedge B_\varsigma \left( \Phi_{\partial,\partial}^{1,1} \right).
\]

The same applies for \(L_1^{(y)}\): we apply Leibniz with respect to \(L_1^{(y)}\) in (3.48) and obtain
the four terms

\[ \Psi \wedge \Phi = -i_{1}^{(x)}L_{1}^{(y)}\left(A_{\varsigma}(\Psi_{1,1}^{1,\partial}dx^{1} \wedge dz^{1}) \wedge B_{\varsigma}\left(\Phi_{\partial,\partial}^{\partial,1}\right)\right) + i_{1}^{(x)}\left(A_{\varsigma}(\Psi_{1,1}^{1,\partial}dx^{1} \wedge dz^{1}) \wedge L_{1}^{(y)}B_{\varsigma}\left(\Phi_{\partial,\partial}^{\partial,1}\right)\right) + L_{1}^{(y)}\left(A_{\varsigma}(\Psi_{1,1}^{1,\partial}dz^{1}) \wedge B_{\varsigma}\left(\Phi_{\partial,\partial}^{\partial,1}\right)\right) - A_{\varsigma}(\Psi_{1,1}^{1,\partial}dz^{1}) \wedge L_{1}^{(y)}B_{\varsigma}\left(\Phi_{\partial,\partial}^{\partial,1}\right). \]

From (2.38) we have \( L_{1}^{(y)} \circ B_{\varsigma} = B_{\varsigma} \circ L_{1}^{(y)} \) so

\[ \Psi \wedge \Phi = -i_{1}^{(x)}L_{1}^{(y)}\left(A_{\varsigma}(\Psi_{1,1}^{1,\partial}dx^{1} \wedge dz^{1}) \wedge B_{\varsigma}\left(\Phi_{\partial,\partial}^{\partial,1}\right)\right) + i_{1}^{(x)}\left(A_{\varsigma}(\Psi_{1,1}^{1,\partial}dx^{1} \wedge dz^{1}) \wedge B_{\varsigma}\left(\partial_{1}^{(y)}\phi_{\partial,\partial}^{\partial,1}\right)\right) + L_{1}^{(y)}\left(A_{\varsigma}(\Psi_{1,1}^{1,\partial}dz^{1}) \wedge B_{\varsigma}\left(\Phi_{\partial,\partial}^{\partial,1}\right)\right) - A_{\varsigma}(\Psi_{1,1}^{1,\partial}dz^{1}) \wedge B_{\varsigma}\left(\partial_{1}^{(y)}\phi_{\partial,\partial}^{\partial,1}\right). \]

Now we apply (3.19). The first term in the above simplifies to

\[ A_{\varsigma}(\Psi_{1,1}^{1,\partial}dx^{1} \wedge dz^{1}) \wedge B_{\varsigma}\left(\Phi_{\partial,\partial}^{\partial,1}\right) = C_{\varsigma}\left(\hat{A}^{\ast}(\Psi_{1,1}^{1,\partial}dx^{1} \wedge dz^{1}) \wedge \hat{B}^{\ast}(\Phi_{\partial,\partial}^{\partial,1})\right) = 0, \]

since \( \hat{A}^{\ast}(dx^{1}) = 0 \). Likewise the second term is also zero. Only the third and fourth terms are non-vanishing resulting in

\[ \Psi \wedge \Phi = -L_{1}^{(y)}C_{\varsigma}\left(\hat{A}^{\ast}(\Psi_{1,1}^{1,\partial}dz^{1}) \wedge \hat{B}^{\ast}(\Phi_{\partial,\partial}^{\partial,1}) + C_{\varsigma}\left(\hat{A}^{\ast}(\Psi_{1,1}^{1,\partial}dz^{1}) \wedge \hat{B}^{\ast}(\partial_{1}^{(y)}\phi_{\partial,\partial}^{\partial,1})\right)\right) = C_{\varsigma}\left(\hat{A}^{\ast}(\Psi_{1,1}^{1,\partial})\hat{B}^{\ast}(\partial_{1}^{(y)}\phi_{\partial,\partial}^{\partial,1})dx^{1}\right) - L_{1}^{(y)}C_{\varsigma}\left(\hat{A}^{\ast}(\Psi_{1,1}^{1,\partial})\hat{B}^{\ast}(\phi_{\partial,\partial}^{\partial,1})dz^{1}\right). \]

That is \( \Psi \wedge \Phi \in \mathcal{Y}^{1,3}(C) \). Note that in \( \Phi = i_{1}^{(x)}B_{\varsigma}\left(\Phi_{\partial,\partial}^{\partial,1}\right) \) the internal contraction \( i_{1}^{(x)} \) removes the \( dx^{1} \) in \( \Psi = i_{1}^{(y)}L_{1}^{(y)}A_{\varsigma}(\Psi_{1,1}^{1,\partial}dx^{1} \wedge dz^{1}) \). In order for the wedge product to be non-zero only \( dz \) terms must remain when applying (3.19). In general for lists of internal contractions and Lie derivatives we apply the Leibniz rule repeatedly until we obtain terms where we can use (3.19). Appendix B contains expressions of how to deal with lists of operators.

**Theorem 3.5.3.** The wedge product of two transverse SMDs exists and is unique and can be written locally by (3.26).
Proof. Existence has been proved above. All that is now required is uniqueness. From (3.24) and (3.25) we apply the Leibniz rule with respect to internal contractions and Lie derivatives repeatedly. First using (B.4),

\[
\Phi \wedge \Psi = \sum_{\text{Rng}_{k,a}(I,J,K,L)} i_j^{(y)} L_I^{(y)} A_\varsigma \left( \Psi_{K,L}^{I,J} dx^K \wedge dz^L \right)
\]

\[
\wedge \sum_{\text{Rng}_{k',a'}(I',J',K',L')} i_{j'}^{(x)} L_{I'}^{(x)} B_\varsigma \left( \Phi_{K',L'}^{I',J'} dy^{K'} \wedge dz^{L'} \right)
\]

\[
= \sum_{\text{Rng}_{k,b}(I,J,K,L)} \sum_{\text{Rng}_{k',a'}(I',J',K',L')} \epsilon^{(j'|S')} (-1)^{|J|(p-|S'|)}
\]

\[
i_{j'|S'}^{(x)} \left( i^{(x)}_{S'} i_j^{(y)} L_I^{(y)} A_\varsigma \left( \Psi_{K,L}^{I,J} dx^K \wedge dz^L \right) \right)
\]

\[
\wedge L_{I'}^{(x)} B_\varsigma \left( \Phi_{K',L'}^{I',J'} dy^{K'} \wedge dz^{L'} \right)
\]

Now we use the fact \( i_{S'}^{(x)} \circ i_j^{(y)} = (-1)^{|S'||J|} i_j^{(y)} \circ i_{S'}^{(x)} \) and \( i_{S'}^{(x)} \circ L_I^{(y)} = L_I^{(y)} \circ i_{S'}^{(x)} \) along with \( i_{S'}^{(x)} \circ A_\varsigma = A_\varsigma \circ i_{S'}^{(x)} \) to get,

\[
= \sum_{\text{Rng}_{k,b}(I,J,K,L)} \sum_{\text{Rng}_{k',a'}(I',J',K',L')} \epsilon^{(j'|S')} (-1)^{|J|(p-|S'|)+|J||S'|}
\]

\[
i_{j'|S'}^{(x)} \left( i_j^{(y)} L_I^{(y)} A_\varsigma \left( \Psi_{K,L}^{I,J} (i_{S'}^{(x)} dx^K) \wedge dz^L \right) \right)
\]

\[
\wedge L_{I'}^{(x)} B_\varsigma \left( \Phi_{K',L'}^{I',J'} dy^{K'} \wedge dz^{L'} \right)
\]
3.5. O-wedge to SMD Wedge

Using (B.3) we get,

\[
\sum_{\text{Rng}_{k,p}(I',J,K,L)} \text{Rng}_{k,q}(I'',J',K',L') \epsilon_{J'}^{(J''\backslash S')} \epsilon_{J}^{(J')\backslash S} (-1)^{|J''|(|p-|S'|)+|J'||S'|+|S||(|L|-|J|)}
\]

\[
i_{J''\backslash S'}^{(y)} i_{J\backslash S}^{(y)}(L_{I}^{(y)} A_{\xi}(\Psi_{K,L}^{I,J} i_{S'}^{(x)} d\overline{x}^{K}) \wedge d\overline{z}^{L})
\]

\[
\wedge i_{S}^{(y)} (L_{I'}^{(x)} B_{\varsigma}(\Phi_{K',L}^{I',J'} d\overline{y}^{K'} \wedge d\overline{z}^{L'})).
\]

Again using the fact \(i_{S}^{(y)} \circ L_{I'}^{(x)} = L_{I'}^{(x)} \circ i_{S}^{(y)}\) and \(i_{S}^{(y)} \circ B_{\varsigma} = B_{\varsigma} \circ i_{S}^{(y)}\),

\[
\sum_{\text{Rng}_{k,p}(I',J,K,L)} \text{Rng}_{k,q}(I'',J',K',L') \epsilon_{J'}^{(J''\backslash S')} \epsilon_{J}^{(J')\backslash S} (-1)^{|J''|(|p-|S'|)+|J'||S'|+|S||(|L|-|J|)}
\]

\[
i_{J''\backslash S'}^{(x)} i_{J\backslash S}^{(y)}(L_{I}^{(y)} A_{\xi}(\Psi_{K,L}^{I,J} i_{S'}^{(x)} d\overline{x}^{K}) \wedge d\overline{z}^{L})
\]

\[
\wedge L_{I'}^{(x)} B_{\varsigma}(\Phi_{K',L}^{I',J'} d\overline{y}^{K'} \wedge d\overline{z}^{L'}).\]

Now we do the same with the Lie derivatives. Using (B.1)

\[
\sum_{\text{Rng}_{k,p}(I',J,K,L)} \text{Rng}_{k,q}(I'',J',K',L') \epsilon_{J'}^{(J''\backslash S')} \epsilon_{J}^{(J')\backslash S} \delta_{J''}^{(I'\backslash R')} (-1)^{|J''|(|p-|S'|)+|J'||S'|+|S||(|L|-|J|)} (-1)^{|R'|}
\]

\[
i_{J''\backslash S'}^{(x)} i_{J\backslash S}^{(y)}(L_{I}^{(y)} A_{\xi}(\partial_{R'}^{(x)} \Psi_{K,L}^{I,J} i_{S'}^{(x)} d\overline{x}^{K}) \wedge d\overline{z}^{L})
\]

\[
\wedge B_{\varsigma}(\Phi_{K',L}^{I',J'} (i_{S}^{(y)} d\overline{y}^{K'} \wedge d\overline{z}^{L'})).
\]
3.6. Summary

Here we have used $L_I^{(y)} \circ L_R^{(x)} = L_R^{(x)} \circ L_I^{(y)}$ and $L_R^{(x)} \circ A_c = A_c \circ L_R^{(x)}$. Note that $L_R^{(x)} \varPsi_{K,L}^{I,J} \equiv \partial_R^{(x)} \varPsi_{K,L}^{I,J}$. Finally from (B.2),

\[
\begin{align*}
&\sum_{\text{Rng}_{k,p}(I,J,K,L)} \sum_{\text{Rng}_{k',q}(I',J',K',L')} \epsilon_J^{(J')^{S'}} \epsilon_J^{(J^{S'})} \left( \delta_I^{(I')^{R'}} \delta_I^{(I^{R'})} \delta^{(I^{R'})} \delta^{(I^{R'})} \epsilon^{(I')^{R'}} \epsilon^{(I^{R'})} \right) \left( -1 \right)^{|I'|(|I'|+|J'|+|K'|)(|K|+|J|)}
\end{align*}
\]

Now we are in a position to apply (3.19). The expression in the large braces directly above becomes

\[
C_s \left( \hat{A}^* \left( \partial_R^{(x)} \varPsi_{K,L}^{I,J} (\mathcal{I}_S^{(x)} dx^K) \wedge dz^L \right) \wedge \hat{B}^* \left( \partial_R^{(y)} \varPhi_{K',L'}^{I',J'} (\mathcal{I}_S^{(y)} dy^K) \wedge dz^{L'} \right) \right).
\]

However $\hat{A}^* (\mathcal{I}_S^{(x)} dx^K) = 0$ and $\hat{B}^* (\mathcal{I}_S^{(y)} dy^K) = 0$ unless $K = S'$ and $K' = S$. Therefore we have

\[
\varPhi \wedge \varPsi = \sum_{\text{Rng}_{k,p}(I,J,K,L)} \sum_{\text{Rng}_{k',q}(I',J',K',L')} \epsilon_J^{(J')^{K'}} \epsilon_J^{(J^{K'})} \left( \delta_I^{(I')^{R'}} \delta_I^{(I^{R'})} \right) \left( -1 \right)^{|J'|(|I'|+|J'|+|K'|)(|K|+|J|)}
\]

\[
\mathcal{I}_S^{(x)} L_I^{(x)} \mathcal{I}_J^{(y)} L_J^{(y)} C_s \left( \hat{A}^* \left( \partial_R^{(x)} \varPsi_{K,L}^{I,J} \right) \hat{B}^* \left( \partial_R^{(y)} \varPhi_{K',L'}^{I',J'} \right) dz^{L'L'} \right),
\]

which is (3.26).

3.6 Summary

We have shown that the wedge product exists and is unique. We reiterate that the wedge product was defined without coordinates and having established that it is well defined and we are free to use it.
Chapter 4

Pullback of SMDs

In order to use Green’s functions in our new notation we need the pullback of a distribution. Given the map \( a : A \to M \) which is transverse to \( f \) the pullback diagram is

\[
\begin{array}{ccc}
\mathcal{N} & \xrightarrow{\hat{f}} & \mathcal{N} \\
\downarrow \hat{a} & & \downarrow f \\
A & \xrightarrow{a} & M
\end{array}
\]

(4.1)

with pullback manifold

\[
a^*\mathcal{N} = \{(y, z) \in \mathcal{N} \times A \mid f(y) = a(z)\}.
\]

(4.2)

The map \( a \) is required to be regular and the image closed. In a similar manner to the wedge product we define the pullback in a coordinate free way in terms of four axioms.

**Definition 4.0.1.** The pullback of a SMD is the map

\[
a^\varsigma : \Upsilon^{k,p}(f) \to \Upsilon^{k,p}(\hat{a}),
\]

(4.3)

which has the properties

(i)

\[
a^\varsigma f_\varsigma(\alpha) = \hat{a}_\varsigma \hat{f}^*(\alpha),
\]

(4.4)
(ii) \[ a^\Phi(\Psi + \Phi) = a^\Phi(\Psi) + a^\Phi(\Phi), \] (4.5) 

(iii) \[ da^\Phi(\Psi) = a^\Phi(d\Psi), \] (4.6) 

(iv) \[ a^\Phi(i_u \Psi) = i_u a^\Phi(\Psi), \] (4.7) 

for \( v \in \Gamma T A \) and \( u \in \Gamma T M \) which are tangential to \( a \).

Clearly (4.6) and (4.7) imply

\[ L_v a^\Phi(\Psi) = a^\Phi(L_u \Psi). \] (4.8)

The proof of the existence and uniqueness of the pullback is similar but simpler to that of the wedge product and can be found in Appendix C. Relevant here is the coordinate expression in adapted coordinates. We can choose an adapted coordinate system in the same manner as (3.4)

\[ M : (y^1, \ldots, y^t, x^1, \ldots, x^{n-t}, z^1, \ldots, z^r), \]
\[ N : (y^1, \ldots, y^t, x^1, \ldots, x^{n-t}), \]
\[ A : (w^1, \ldots, w^s, y^1, \ldots, y^t, z^1, \ldots, z^r), \]
\[ a^* N : (w^1, \ldots, w^s, y^1, \ldots, y^t). \]

From (2.64) the general element \( \Psi \in \mathcal{Y}^{k,p}(f) \) may be written

\[ \Psi = \sum_{I \mid J^r, K^t, H^{n-t}} i^{(z)}_J L^{(z)}_I f_c(\Psi^{I,J}_{K,H} dy^K \wedge dx^H), \] (4.9)

then the pullback with respect to \( a^\Phi \) is

\[ a^\Phi(\Psi) = \sum_{I \mid J^r, K^t} i^{(z)}_J L^{(z)}_I \hat{a}_c(\hat{f}^*(\Psi^{I,J}_{K,H}) dy^K). \] (4.10)
It is instructive to consider a low dimension example with \( t = 1, n = 2, r = 1, s = 1 \) and \( \Psi \in \mathcal{T}^{1,1}(f) \). In adapted coordinates
\[
\Psi = i_1^{(z)} L_1^{(z)} f_\varsigma (\Psi_{1,\varnothing}^{1,1} dy^1 + \Psi_{\varnothing,1}^{1,1} dx^1).
\] (4.11)

Taking the pullback with respect to \( a \) first note that \( a_*(\partial_1^{(z)}) = \partial_{i_1}^{(z)} \), so using (4.7) then (4.8) we have
\[
a^\varsigma(\Psi) = a^\varsigma(i_1^{(z)} L_1^{(z)} f_\varsigma (\Psi_{1,\varnothing}^{1,1} dy^1 + \Psi_{\varnothing,1}^{1,1} dx^1)) = i_1^{(z)} a^\varsigma(L_1^{(z)} f_\varsigma (\Psi_{1,\varnothing}^{1,1} dy^1 + \Psi_{\varnothing,1}^{1,1} dx^1))
\]
\[
= i_1^{(z)} L_1^{(z)} a^\varsigma(f_\varsigma (\Psi_{1,\varnothing}^{1,1} dy^1 + \Psi_{\varnothing,1}^{1,1} dx^1)).
\]

Now using (4.4) and the fact \( \hat{f}^*(dx^1) = 0 \),
\[
a^\varsigma(\Psi) = i_1^{(z)} L_1^{(z)} \hat{a}_c(\hat{f}^*(\Psi_{1,\varnothing}^{1,1} dy^1 + \Psi_{\varnothing,1}^{1,1} dx^1)) = i_1^{(z)} L_1^{(z)} \hat{a}_c(\Psi_{1,\varnothing}^{1,1} dy^1).
\]

**Lemma 4.0.2.** The pullback, defined by (4.4)-(4.7), of \( \Psi \) given by (4.9) is uniquely given by (4.10).

**Proof.** The vector field \( \partial_a^{(z)} \in \Gamma \mathcal{T} \mathcal{M} \) and \( \partial_{a}^{(z)} \in \Gamma \mathcal{T} \mathcal{A} \) satisfy \( a_*(\partial_a^{(z)}) = \partial_{a}^{(z)} \) due to choice of adapted coordinates. Thus
\[
\begin{align*}
a^\varsigma(\Psi) &= \sum_{I^\gamma, J^\gamma, K^\gamma, H^\gamma} \mathcal{I}_{n-t} \ a^\varsigma(i_1^{(z)} L_1^{(z)} f_\varsigma (\Psi_{1,\varnothing}^{I,J,H} dy^K \wedge dx^H)) \\
&= \sum_{I^\gamma, J^\gamma, K^\gamma, H^\gamma} \mathcal{I}_{n-t} \ i_1^{(z)} L_1^{(z)} \hat{a}_c(\hat{f}^*(\Psi_{1,\varnothing}^{I,J,H} dy^K \wedge dx^H)) \\
&= \sum_{I^\gamma, J^\gamma, K^\gamma, H^\gamma} \mathcal{I}_{n-t} \ i_1^{(z)} L_1^{(z)} \hat{a}_c(\hat{f}^*(\Psi_{1,\varnothing}^{I,J,H} dy^K)).
\end{align*}
\]
Chapter 5

Green’s Distributions

The standard approach to solving linear, inhomogeneous differential equations is via Green’s functions (more accurately Green’s distributions). In the scalar case this says the linear differential operator $\mathcal{L}$ acting on the distribution $G$, called the Green’s distribution, is equal to a Dirac delta distribution. The solution to the linear partial differential equation is thus obtained by integrating the Green’s function with the inhomogeneous term. Therefore, the key to solving the differential equation reduces to finding the Green’s function.

In this chapter we restrict ourselves to linear differential operators $\mathcal{L}$ on $\mathcal{M}$ that can be constructed out of exterior derivatives and internal contractions, and we only consider flat manifolds. We wish to solve the linear differential equation of the type

$$\mathcal{L}[\Psi] = \Phi,$$  \hspace{1cm} (5.1)

where $\Phi$ and $\Psi$ are submanifold distributions. Let $\mathcal{M}_X$ and $\mathcal{M}_Y$ be two copies of $\mathcal{M}$ with natural projections $\pi_X : \mathcal{M}_X \times \mathcal{M}_Y \rightarrow \mathcal{M}_X$ and $\pi_Y : \mathcal{M}_X \times \mathcal{M}_Y \rightarrow \mathcal{M}_Y$. The diagonal map $\Delta : \mathcal{M} \rightarrow \mathcal{M}_X \times \mathcal{M}_Y$ is defined as $\Delta(x) = (x,x)$ and $f : \mathcal{N} \rightarrow \mathcal{M}_Y$ is a closed embedding. Note that the diagonal map is also a closed embedding. The identity maps are $\mathbb{1}_{XY} : \mathcal{M}_X \rightarrow \mathcal{M}_Y$ and $\mathbb{1}_{YX} : \mathcal{M}_Y \rightarrow \mathcal{M}_X$. In order to solve (5.1) we introduce the Green’s distribution $G$ which is a SMD over an embedding in
\( M_X \times M_Y \). The maps are quite subtle but in the next section, when we deal with a specific example, we will be able to define the maps and the embedding explicitly. We will now show that a solution to \( \mathcal{L}[\Psi] = \Phi \) with \( \Phi \in \Upsilon^{k,p}(f) \) can be expressed as 
\[
\Psi = \pi_X(G \wedge \pi_Y^\dagger \Phi)
\]
where 
\[
\mathcal{L}_X[G] = \Delta(1),
\]
and \( \mathcal{L}_X \) is the operator acting only on the \( M_X \) part of \( M_X \times M_Y \). Observe that on the right hand side we have a Dirac delta distribution over the diagonal map cf. (2.60). The equation above is the geometric version of the differential operator \( \mathcal{L}_X \) acting on the Green’s distribution to give a Dirac delta.

**Definition 5.0.1.** We say the linear operator \( \mathcal{L} \) is **extendible to the product manifold** if there exists a linear operator \( \mathcal{L}_X \) on \( M_X \times M_Y \) such that
\[
\mathcal{L} \circ \pi_X = \pi_X \circ \mathcal{L}_X,
\]
and
\[
\mathcal{L}_X(G \wedge \pi_Y^\dagger \Phi) = \mathcal{L}_X[G] \wedge \pi_Y^\dagger \Phi, \quad \Phi \in \Upsilon^{k,p}(f).
\]

In order to solve (5.1) we first show the following.

**Lemma 5.0.2.**
\[
\pi_X(\Delta \wedge \pi_Y^\dagger \Phi) = \mathbf{1}_{Y X} \Phi.
\]

**Proof.** A generic SMD can be written as a list of exterior derivatives and internal contractions acting on \( f_\varsigma(\alpha) \). Therefore, it suffices to show the above by induction, starting with \( \Phi = f_\varsigma(\alpha) \) then considering \( d\Phi \) and \( i_\alpha \Phi \). First consider the case \( \Phi = f_\varsigma(\alpha) \). The pullback \( \pi_Y^\dagger \Phi \) requires the following commutative diagram
\[
\begin{array}{ccc}
\pi_Y^\dagger \mathcal{N} & \xrightarrow{\hat{\pi}_Y} & \mathcal{N} \\
\downarrow j & & \downarrow f \\
M_X \times M_Y & \xrightarrow{\pi_Y} & M_Y
\end{array}
\]

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with maps \( \hat{\pi}_Y(x, z) = z \), \( \hat{f}(x, z) = (x, f(z)) \). The pullback manifold is

\[
\pi_Y^*\mathcal{N} = \{ ((x, y), z) \in (\mathcal{M}_X \times \mathcal{M}_Y) \times \mathcal{N} \mid \pi_Y(x, y) = f(z) \}
\]

\[
= \{ (x, f(z)) \mid x \in \mathcal{M}_X, z \in \mathcal{N} \}
\]

\[
= \mathcal{M}_X \times \mathcal{N}.
\]

Using (4.4), the pullback of \( \Phi \) is

\[
\pi_Y^*\Phi = \pi_Y^* f_\varsigma (\alpha) = \hat{f}_\varsigma \hat{\pi}_Y^*(\alpha). \tag{5.7}
\]

Extending the above diagram to include the wedge product \( \Delta^* 1 \wedge \pi_Y^* \Phi \) we obtain

\[
\begin{array}{ccc}
\mathcal{N} & \xleftarrow{\hat{f}} & \mathcal{M}_X \times \mathcal{N} & \xrightarrow{\hat{\pi}_Y} & \mathcal{N} \\
\Delta & \xleftarrow{f \times f} & \mathcal{M} & \xrightarrow{f} & \mathcal{M} \\
\mathcal{M} & \xrightarrow{\Delta} & \mathcal{M}_X \times \mathcal{M}_Y & \xrightarrow{\pi_Y} & \mathcal{M}_Y \\
\end{array}
\]

\[
\xrightarrow{\pi_X 1_{XY}} \mathcal{M}_X
\]

\[
\xrightarrow{1_{XY} \pi_Y} \mathcal{M}_Y
\]

with maps \( \hat{f}(z) = (f(z), z) \), \( \hat{\Delta}(z) = (f(z)) \). Now

\[
\pi_{X_\varsigma} (\Delta^* 1 \wedge \pi_Y^* \Phi) = \pi_{X_\varsigma} (\Delta^* 1 \wedge \hat{f}_\varsigma \hat{\pi}_Y^*(\alpha))
\]

\[
= \pi_{X_\varsigma} ((f \times f)_\varsigma ((\hat{\Delta}^*)(1) \wedge \hat{f}^* \hat{\pi}_Y^*(\alpha)))
\]

\[
= \pi_{X_\varsigma} ((f \times f)_\varsigma ((\hat{\pi}_Y \circ \hat{f})^*(\alpha)))
\]

\[
= \pi_{X_\varsigma} ((f \times f)_\varsigma (\alpha))
\]

\[
= (\pi_X \circ (f \times f)_\varsigma (\alpha)
\]

\[
= \mathbbm{1}_{XY} f_\varsigma (\alpha)
\]

\[
= \mathbbm{1}_{XY} \Phi,
\]

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where we made the observation \((\pi_Y \circ \tilde{f})^* = \mathbb{1}^*_N\). Now consider \(d\Phi\). Using (3.23)

\[\pi_{X^*}(\Delta_1 \wedge \pi_Y^* d\Phi) = \pi_{X^*}(\Delta_1 \wedge d\pi_Y^* \Phi)\]

\[= (-1)^{\text{deg}(\Delta_1)} \pi_{X^*} d(\Delta_1 \wedge \pi_Y^* \Phi) - (-1)^{\text{deg}(\Delta_1)} \pi_{X^*} (d\Delta_1 \wedge \pi_Y^* \Phi)\]

\[= d\pi_{X^*}(\Delta_1 \wedge \pi_Y^* \Phi)\]

\[= d\mathbb{1}^*_{YX^*}(d\Phi),\]

\(d\Delta_1(1) = 0\) since \(d\Delta_1 = \Delta d(1) = 0\). The final inductive step is when \(i_u \Phi\) where \(u \in \Gamma TM_Y\). Let \(\pi_{Y^*} v = u\) with \(v \in \Gamma T(M_X \times M_Y)\). Since \(v\) is not unique we can set it such that it is tangential to \(\Delta\) i.e. \(\Delta \cdot \hat{w} = v\) where \(\hat{w} \in T M_X\), see Figure 5.1. In local coordinates we have

\[u = u^a \frac{\partial}{\partial y^a}, \quad v = v^b \frac{\partial}{\partial x^b} + v^c \frac{\partial}{\partial y^c}, \quad \Delta \cdot \hat{w} = \hat{w}^d \left( \frac{\partial}{\partial x^d} + \frac{\partial}{\partial y^d} \right), \quad w = w^e \frac{\partial}{\partial x^e}.\]

Therefore the components are related as thus;

\[v^c(y) = \hat{w}^c(x), \quad \hat{w}^d(x) = v^d(y), \quad \text{and} \quad w^e(x) = v^e(x) \Rightarrow w = \mathbb{1}^*_{YX^*} u.\]

Now

\[\pi_{X^*}(\Delta_1 \wedge \pi_Y^* i_u \Phi) = \pi_{X^*}(\Delta_1 \wedge i_v \pi_Y^* \Phi)\]

\[= (-1)^{\text{deg}(\Delta_1)} \pi_{X^*} i_v (\Delta_1 \wedge \pi_Y^* \Phi) - (-1)^{\text{deg}(\Delta_1)} \pi_{X^*} (i_v \Delta_1 \wedge \pi_Y^* \Phi)\]

\[= i_v \pi_{X^*}(\Delta_1 \wedge \pi_Y^* \Phi)\]

\[= i_v(\mathbb{1}^*_{YX^*} \Phi) = \mathbb{1}^*_{YX^*}(i_v \Phi).\]

\[\square\]

**Theorem 5.0.3.** Given a linear operator \(L\) extendible to the product manifold and a \(G\) such that there exists a linear operator \(L_{X^*}\) which satisfies (5.3) and (5.4) then \(\Psi\) given by

\[\Psi = \pi_{X^*}(G \wedge \pi_Y^* \Phi),\]

(5.9)

is a solution to (5.1).
5.1 Properties of the Laplace-Beltrami Operator

In this section we consider Minkowski spacetime $M$ with metric $g$ that has signature $(-,+,+,+)$. Furthermore we will be working in Minkowski coordinates $(x^0,x^1,x^2,x^3)$. We are interested in the case when $\mathcal{L} \equiv \Box$ is the Laplace-Beltrami (LB) operator. The Liénard-Wiechert field $A \in \Gamma \Lambda^1 M_X$ is a solution to the wave equation

$$\Box * A = \mathcal{J},$$

such that $dA = F$ where $F \in \Gamma \Lambda^2 M_X$ is the Faraday 2-form and also $d * A = 0$. The latter condition is the Lorentz gauge. In order to use Green’s method for the LB
5.1. Properties of the Laplace-Beltrami Operator

Figure 5.2: \( \pi_Y \Phi \) and \( G \) have support on submanifolds which intersect transversely in \( M_X \times M_Y \).

operator it is necessary to show it is extendible to the product manifold i.e. Definition 5.0.1.

**Definition 5.1.1.** The Laplace-Beltrami operator is defined as

\[
\Box : \Gamma \Lambda^p(M_X \times M_Y) \to \Gamma \Lambda^p(M_X \times M_Y),
\]

\[
\Box \alpha = (-1)^{p(m-p+1)}s(\ast d \ast d\alpha + (-1)^m d \ast d \ast \alpha), \quad \alpha \in \Gamma \Lambda^p M,
\] (5.11)

where \( s = \det(g) = -1 \) and \( m = \dim(M) = 4 \)

From (5.3) we also require the operator \( \Box_X \), which acts on the \( M_X \) part of forms on \( M_X \times M_Y \). Thus the exterior derivative and Hodge dual are defined as

\[
d_X : \Gamma \Lambda^p(M_X \times M_Y) \to \Gamma \Lambda^{p+1}(M_X \times M_Y),
\]

\[
d_X \alpha = \sum_{J^4, K^4} (\partial_{\alpha}^{(x)} \alpha_{J,K}) dx^a \wedge dx^J \wedge dy^K,
\] (5.12)

\[
\ast_X : \Gamma \Lambda^p(M_X \times M_Y) \to \Gamma \Lambda^{4-p}(M_X \times M_Y),
\]

\[
\ast_X \alpha = \sum_{J^4, K^4} \alpha_{J,K}(\ast dx^J) \wedge dy^K.
\] (5.13)
5.1. Properties of the Laplace-Beltrami Operator

From (D.3) in Appendix D

\[ \Box_X : \Gamma^p(M_X \times M_Y) \to \Gamma^p(M_X \times M_Y), \]

\[ \Box_X \alpha = s^2(-1)^{4-p}\left( g^{ab}L_b^{(x)}L_a^{(x)}\alpha - 2g^{ab}l_b^{(x)}L_a^{(x)}d_X\alpha \right). \] \hspace{1cm} (5.14)

Note that since we are in flat space the metric components \( g^{ab} \) are constant, so the right hand side of (5.14) consists of exterior derivatives and internal contractions. In order to solve (5.10) using (5.9) we first must investigate certain properties of the Laplace-Beltrami operator.

**Lemma 5.1.2.** The Laplace-Beltrami operator is extendible to the product manifold, i.e. it satisfies (5.3),

\[ \Box \circ \pi_X \varsigma = \pi_X \varsigma \circ \Box. \] \hspace{1cm} (5.15)

**Proof.** Let \( \phi \in \Gamma_0 \Lambda M_X, \ f : N \leftrightarrow M_X \times M_Y \) and \( \hat{\Phi} \in \Upsilon^{k,p}(f) \). The Laplace-Beltrami operator is given by (5.11) so we only need to show commutativity with \( d \) and \( \ast \). Firstly

\[ \left[ \phi \big| d\pi_X \hat{\Phi} \right]_{M_X} = (-1)^{\deg(\phi)} \left[ \pi_X^* (d\phi) \big| \hat{\Phi} \right]_{M_X \times M_Y}. \]

On a local coordinate chart

\[ \pi_X^* (d\phi) = \pi_X^* \left( \sum_{J=1}^4 (\partial_a^{(x)} \phi_J)dx^{a-J} \right) \]

\[ = \pi_X^* \left( \sum_{J=1}^4 (\partial_a^{(x)} \phi_J)dx^{a-J} \right) \]

\[ = \sum_{J=1}^4 (\partial_a^{(x)} \phi_{J,K})dx^{a-J} \wedge dy^K. \]

However from (5.12)

\[ d_X(\pi_X^* \phi) = d_X\pi_X^* (\sum_{J=1}^4 \phi_J dx^J) \]

\[ = d_X \left( \sum_{J=1}^4 \phi_{J,K}dx^J \wedge dy^K \right) \]

\[ = \sum_{J=1}^4 (\partial_a^{(x)} \phi_{J,K})dx^{a-J} \wedge dy^K. \]
5.1. Properties of the Laplace-Beltrami Operator

thus

\[
\left[ \pi^*_X d\phi \right]_{M_X \times M_Y} = \left[ d_X \pi^*_X \phi \right]_{M_X \times M_Y} = (-1)^{\deg(\phi)} \left[ \pi^*_X \phi \right]_{M_X \times M_Y} = (-1)^{\deg(\phi)} \left[ \phi \right]_{M_X \times M_Y}.
\]

Similarly for the Hodge dual. From (5.13)

\[
\pi^*_X (\ast \phi) = \pi^*_X \left( \sum_{J^4} \ast dx^J \wedge i^{(x)}_J \phi \right)
\]

\[
= \sum_{J^4 \wedge K^4} (\ast dx^J \wedge \phi_{J,K}) \wedge dy^K
\]

\[
= \sum_{J^4 \wedge K^4} \phi_{J,K} \ast dx^J \wedge dy^K,
\]

whereas

\[
\ast_X (\pi^*_X \phi) = \ast_X \left( \pi^*_X \left( \sum_{J^4} \phi_{J} dx^J \right) \right)
\]

\[
= \ast_X \left( \sum_{J^4 \wedge K^4} \phi_{J,K} dx^J \wedge dy^K \right)
\]

\[
= \sum_{J^4 \wedge K^4} \phi_{J,K} (\ast dx^J) \wedge dy^K.
\]

In order to show \( \Box_X \) satisfies (5.4) it suffices to just consider \( L^{(x)}_a \) and \( i^{(x)}_a \). Using (3.23),

\[
L^{(x)}_a (G \wedge \pi^*_Y \Phi) = L^{(x)}_a G \wedge \pi^*_Y \Phi + G \wedge L^{(x)}_a \pi^*_Y \Phi
\]

\[
= L^{(x)}_a G \wedge \pi^*_Y \Phi + G \wedge \pi^*_Y L^{(x)}_a \Phi
\]

\[
= L^{(x)}_a G \wedge \pi^*_Y \Phi.
\]

The Lie derivative commutes with \( \pi^*_Y \) according to (4.6) and (4.7). \( \pi^*_Y L^{(x)}_a \Phi = 0 \) since the components of \( \Phi \) in local coordinates have a \( y \) dependence. The same reasoning applies for \( i^{(x)}_a \) in that \( \Phi \) has no \( dx^a \) terms.
5.2. The Retarded Green’s SMD

In order to find the Liénard-Wiechert field for multipoles we introduce the retarded Green’s SMD. It is necessary to define the domain and codomain of the map \( \iota \) carefully in order for it to be a closed SMD.

**Definition 5.2.1.** The forward lightcone of the point \( z \in N \) is given by the locus

\[
N = \{ z \in M_X | g(z, \bar{z}) = 0, z^0 > 0 \},
\]

with coordinates \((z^1, z^2, z^3)\) such that \( z = (|z|, \bar{z}) \in N \) and \(|z| = \sqrt{(z^1)^2 + (z^2)^2 + (z^3)^2}\).

In order for \( \iota \) to be a closed embedding we have the following definition.

**Definition 5.2.2.**

\[
\iota : N \times M_Y \to (M_X \times M_Y) \setminus \Delta, \quad \iota(z, y) \to (z + y, y), \quad (5.16)
\]

where \((z + y) = (|z| + y^0, z^1 + y^1, z^2 + y^2, z^3 + y^3)\).

**Lemma 5.2.3.** \( \iota \) given by (5.16) is a closed embedding.

**Proof.** Observe that \( \iota \) is a well defined map since if \((x, y) \in \Delta \) and \((x, y) \in \iota(N \times M_Y) \) then \( z + y = y \) i.e. \( z = 0 \) which contradicts \( z \in N \). Hence \( \iota \) is well defined. It is clearly smooth and the rank is maximal i.e. equal to 7. To show it is a closed embedding, we need to show that the complement of \( \iota(N \times M_Y) \) i.e. \( (M_X \times M_Y) \setminus \Delta \setminus \iota(N \times M_Y) \) is open. However

\[
((M_X \times M_Y) \setminus \Delta) \setminus \iota(N \times M_Y)
\]

\[
= \{ (x, y) \in M_X \times M_Y | g(x - y, x - y) \neq 0 \} \cup \{ (x, y) \in M_X \times M_Y | (x - y)^0 < 0 \}
\]

is open as it is the union of two open sets. To see that the second set does not need to be \( \{ (x, y) \in M_X \times M_Y | (x - y)^0 \leq 0 \} \) we note that if \( x^0 = y^0 \) and if \( g(x - y, x - y) = 0 \) then \( x^1 = y^1 \) etc i.e. \( x = y \) which contradicts \( (x, y) \notin \Delta \). \( \square \)
5.2. The Retarded Green’s SMD

Definition 5.2.4. The Green form $G \in \Upsilon^{0,4}(\iota)$ associated with the LB operator is

$$G = \iota_\varsigma(\alpha_N) \quad \text{where} \quad \alpha_N = \frac{dz^{123}}{|z|} \in \Gamma^3 N.$$ \hspace{1cm} (5.17)

Taking the exterior derivative of $G$ we find: $dG = d\iota_\varsigma(\alpha_N) = \iota_\varsigma(d\alpha_N) = 0$. $\alpha_N$ is a top form so $d\alpha_N = 0$ and hence $G$ is closed.

Theorem 5.2.5. The Green form $G$ satisfies

$$\Box X G = \Delta_\varsigma(1).$$ \hspace{1cm} (5.18)

Proof. From (5.14) and using the property that $G$ is closed

$$\Box X G = g^{ab} L_b^{(x)} L_a^{(x)} G - 2g^{ab} \iota_b^{(x)} L_a^{(x)} d_X G = g^{ab} L_b^{(x)} L_a^{(x)} G.$$ 

Let $\phi \in \Gamma_0^4(M_X \times M_Y)$ which in local coordinates is $\phi = \sum_{J^4, K^4} \phi_{J,K} dx^J \wedge dy^K$ with $|J| + |K| = 4$,

$$\left[ \phi | \Box X G \right]_{(M_X \times M_Y) \setminus \Delta} = \left[ g^{ab} L_a^{(x)} L_b^{(x)} \phi \right]_{(M_X \times M_Y) \setminus \Delta} \iota_\varsigma(\alpha_N)$$

$$= \int_{N \times M_Y} t^*(g^{ab} L_a^{(x)} L_b^{(x)} \phi) \wedge \alpha_N$$

$$= \sum_{J^4, K^4} \int_{N \times M_Y} t^*(g^{ab} L_a^{(x)} L_b^{(x)} (\phi_{J,K} dx^J \wedge dy^K)) \wedge \frac{dz^{3}}{|z|}$$

$$= \sum_{J^4, K^4} \int_{N \times M_Y} t^*(g^{ab} \partial_{ab} \phi_{J,K}) t^*(dx^J \wedge dy^K) \wedge \frac{dz^{3}}{|z|}$$

$$= \sum_{J^4, K^4} \int_{N \times M_Y} t^*(g^{ab} \partial_{ab} \phi_{J,K}) ((dx^J + dy^K) \wedge dy^K) \wedge \frac{dz^{3}}{|z|}$$

$$= \int_{\mathbb{R}^3} \int_{\mathbb{R}^4} t^*(g^{ab} \partial_{ab} \phi_{\partial,4}) \frac{1}{|z|} dy^4 \wedge dz^{3}.$$ 

Since $g^{ab} \partial_{ab} \phi_{\partial,4} = 0$ is the wave equation for the components of $\phi$, the above result reduces to standard Green’s function techniques [6, 10].

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Chapter 6

Dipoles and Quadrupoles

6.1 Introduction

Multipoles are a useful approximation to a continuous body of matter. By this method it is possible to represent the system by a finite set of parameters [19, 20]. The first three terms of the approximation go by the name of monopole, dipole and quadrupole. In electrodynamics a monopole is simply a point charge, a dipole is two point charges separated spatially, whereas a quadrupole is two dipoles also separated spatially. The hierarchy continues for higher order multipoles. Point multipoles are constructed by reducing the spatial distance of two monopoles to zero such that the product of the distance between monopoles and the charge of each monopole remains constant. The potentials due to a stationary multipole were first derived by Maxwell over a century ago [21]. Calculations of the radiation fields for a moving dipole are extensively covered in the literature, the most notable are [22, 23, 24, 25, 26, 27]. The source of the fields is given by the current $J$ in Maxwell’s equations and depends on the motion of the particle and also on a scalar quantity $q$ known as the charge. In addition, a particle may also possess an intrinsic dipole moment. These occur in two types: electric and magnetic. Individually, electric and magnetic dipole moments are described by vectors, but together they are described by a bivector. Less known are the fields due
6.2 Closed SMDs Over a Curve

to moving point quadrupoles. Kaufman was the first to write the current as a second order derivative of the Dirac delta function [28] and similar expressions can be found later in Ellis [23, 24]. It has also been shown that quadrupoles also contain “toroidal moments” [29, 30].

In this chapter we express the electromagnetic currents for dipole and quadrupole fields defined on a curve in the language of SMDs. We then investigate the transformation properties and find a peculiar result for quadrupoles. In the first section we do not assume the ambient manifold posses a metric so the definition of dipoles and quadrupoles is metric free [31], and as a consequence this suggest that the curve on which the dipoles reside does not need to be timelike. Furthermore it is possible to model dipoles and quadrupoles on spaces other than spacetime such as phase space or tangent bundles. We then use the wedge product defined in Chapter 3 to calculate the Liénard-Wiechert field in a geometric way.

6.2 Closed SMDs Over a Curve

We now consider SMDs with support on a curve \( C : \mathcal{I} \rightarrow \mathcal{M} \) where \( \mathcal{M} \) is a 4-dimensional manifold and \( \mathcal{I} = \{ \tau \mid \tau_{\text{min}} < \tau < \tau_{\text{max}} \} \subset \mathbb{R} \). At this point it is assumed that \( \mathcal{M} \) does not posses a metric. A current \( \mathcal{J} \) is a source to Maxwell’s equation \( d \cdot F = \mathcal{J} \) where \( F \in \Gamma \Lambda^2 \mathcal{M} \) is the Faraday two form. Therefore \( \mathcal{J} \) is required to be a closed 3-form i.e. \( d\mathcal{J} = 0 \). The left hand side of Maxwell’s equation requires a metric in order to determine the fields however we start by considering a closed 3-form \( \mathcal{J} \) which does not require a metric. In the adapted coordinate system \((\tau, \zeta^1, \zeta^2, \zeta^3)\) the curve is represented by \( C(\tau) = (\tau, \emptyset) \). \( \mathcal{J} \in \Upsilon^{1,3}(C) \), which also contains the zero order terms, can be written as

\[
\mathcal{J} = \sum_{I^h, |I| \leq 1} L_I^{(C)} C_{\varsigma}(\mathcal{J}_I^{I,\emptyset}) + \sum_{I^h, J^h, |I| \leq 1, |J| = 1} i_J^{(C)} L_I^{(C)} C_{\varsigma}(\mathcal{J}_I^{I,J} d\tau). \tag{6.1}
\]
6.2. Closed SMDs Over a Curve

Expanding the summations we have

\[ J = C_\varsigma (J_\phi^\phi,\phi) + \sum_{\mu=1}^{3} L_\mu (J_\phi^\phi,\phi) \sum_{\nu=1}^{3} i_\nu (J_\phi^\phi,\phi) + \sum_{\nu=1}^{3} \frac{3}{\nu} \sum_{\mu=1}^{3} L_\mu (J_\phi^\phi,\phi) d\tau C_\varsigma (J_\phi^\phi,\phi) d\tau. \]

Now setting \( dJ = 0, \)

\[ 0 = dJ = C_\varsigma (dJ_\phi^\phi,\phi) + \sum_{\mu=1}^{3} L_\mu (dJ_\phi^\phi,\phi) + \sum_{\nu=1}^{3} L_\nu (dJ_\phi^\phi,\phi) + \sum_{\mu=1}^{3} \sum_{\nu=1}^{3} L_\mu (dJ_\phi^\phi,\phi) C_\varsigma (J_\phi^\phi,\phi) d\tau. \]

Equating both sides gives the following constraints on the components

\[ \partial_0^{(\tau)} (J_\phi^\phi,\phi) = 0, \quad \partial_\nu^{(\tau)} (J_\phi^\phi,\phi) + J_\phi^\phi,\phi = 0, \quad J_\phi^\phi,\phi + J_\phi^\phi,\phi = 0. \]

Substituting back into (6.1) reduces the 15 components to 6. The first equation says \( J_\phi^\phi,\phi \) is constant on the worldline so we can define it to be equal to the charge \( q \). If \( J \) is written as a sum of monopole and dipole terms \( J = J_M + J_{ED} + J_{MD} \) then from (6.1)

\[ J_M = qC_\varsigma (1), \quad (6.2) \]

\[ J_{ED} = \sum_{\mu=1}^{3} \left( i_0^{(\tau)} L_\mu (J_\phi^\phi,\phi) - i_0^{(\tau)} L_\mu (J_\phi^\phi,\phi) \right) C_\varsigma (J_\phi^\phi,\phi) d\tau, \quad (6.3) \]

\[ J_{MD} = \sum_{\mu<\nu=1}^{3} \left( L_\mu (J_\phi^\phi,\phi) i_\nu (J_\phi^\phi,\phi) - i_\nu (J_\phi^\phi,\phi) L_\mu (J_\phi^\phi,\phi) \right) C_\varsigma (J_\phi^\phi,\phi) d\tau. \quad (6.4) \]

There are 3 electric dipoles given by the components of \( J_{ED} \) and likewise also 3 magnetic dipoles \( J_{MD} \). (6.3) can also be written as

\[ J_{ED} = d_i w C_\varsigma (1), \quad (6.5) \]

where \( w \in \Gamma(C,TM) \) is the electric dipole vector. Even though \( J \) is closed there is no similar constraint on the actual dipole field, at least not by Maxwell’s equations. This means a dipole field can increase from zero, reach a maximum then reduce to zero again. This process conserves charge but not mass (here we are referring to the mass of a particle which has a magnetic or electric dipole, distinct from its electromagnetic mass) so additional constraints, not considered here, may be required.
Lemma 6.2.1. A general dipole consisting of a magnetic and electric part can be written as

\[ \mathcal{J}_{DP} = \frac{1}{2} \sum_{a,b=0}^{3} i_{a}^{(x)} L_{b}^{(x)} C_{c} \left( \gamma_{ab} \right) d\tau \]  

(6.6)

where \( \gamma_{ab} \) is antisymmetric in its indices i.e.

\[ \gamma_{\mu\nu} = -\gamma_{\nu\mu}, \quad \gamma_{\nu\mu} = -\gamma_{\mu\nu}. \]

Proof. Split the summation over \( a \) and \( b \) into four sums with terms \( \gamma_{00}, \gamma_{0\nu}, \gamma_{\mu0} \) and \( \gamma_{\mu\nu} \) where \( \mu, \nu = 1, 2, 3 \). The \( \gamma_{00} \) term is the monopole contribution. If the dipole has zero charge then this vanishes.

\[ \mathcal{J}_{DP} = \mathcal{J}_{MP} + \frac{1}{2} \sum_{\nu=0}^{3} i_{0}^{(x)} L_{\nu}^{(x)} C_{c} \left( \gamma_{0\nu} \right) d\tau + \frac{1}{2} \sum_{\mu=1}^{3} i_{\mu}^{(x)} L_{0}^{(x)} C_{c} \left( \gamma_{\mu0} \right) d\tau \]

\[ + \frac{1}{2} \sum_{\mu=1,\nu=1}^{3} i_{\mu}^{(x)} L_{\nu}^{(x)} C_{c} \left( \gamma_{\mu\nu} \right) d\tau \]

\[ = \mathcal{J}_{MP} + \frac{1}{2} \sum_{\mu=1}^{3} \left( L_{\mu}^{(c)} C_{c} \left( \gamma_{\nu0} \right) + i_{\mu}^{(c)} C_{c} \left( \partial_{0}^{(x)} (\gamma_{0\nu} d\tau) \right) \right) + \frac{1}{2} \sum_{\mu=1,\nu=1}^{3} i_{\mu}^{(c)} L_{\nu}^{(c)} C_{c} \left( \gamma_{\mu\nu} \right) d\tau. \]

Comparing with (6.1) we see \( \gamma_{00} = 2\mathcal{J}_{0}^{\nu\nu} \) and \( \gamma_{00} = -2\mathcal{J}_{0}^{\nu\nu} \) hence \( \gamma_{00} = -\gamma_{00} \). Similarly if we make the identification \( \gamma_{\mu\nu} = 2\mathcal{J}_{0}^{\mu\nu} \) then \( \gamma_{\mu\nu} = -\gamma_{\mu\nu} \). \( \square \)

Lemma 6.2.2. Under a change of coordinates (not necessarily adapted to the world-line) the components \( \gamma_{ab} \) transform tensorially. Thus given another parametrization \( \hat{\tau} \) and a non-adapted coordinate system \( (x^{0}, x^{1}, x^{2}, x^{3}) \)

\[ \mathcal{J}_{DP} = \frac{1}{2} \sum_{a,b=0}^{3} i_{a}^{(x)} L_{b}^{(x)} C_{c} \left( \gamma_{ab} \right) d\tau \]  

where \( \gamma_{ab} = \sum_{c,d=0}^{3} C_{c}(\partial_{c}^{(x)} x^{d})(\partial_{c}^{(x)} x^{d})(\partial_{0}^{(x)} \tau) \gamma_{cd}. \)

(6.7)

Proof. From (B.5) Lemma B.1.2

\[ L_{d}^{(c)} C_{c} \left( \gamma_{(c)} \right) d\tau = \sum_{b=0}^{3} L_{(d)}^{(c)} C_{c} \left( \gamma_{(c)} \right) d\tau = \sum_{b=0}^{3} L_{b}^{(x)} C_{c} \left( \gamma_{(c)} \right) d\tau. \]
Thus

\[ J_{\text{DP}} = \frac{1}{2} \sum_{c,d=0}^{3} \sum_{a,b=0}^{3} i_a^{(x)} \left( \partial_c^{(\xi)} x^a \right) L_b^{(x)} C_\xi \left( C^* \left( \partial_d^{(\xi)} x^b \right) \gamma_{c,d}^{\xi} \right) d\tau \]

\[ = \frac{1}{2} \sum_{c,d=0}^{3} \sum_{a,b=0}^{3} \left( i_a^{(x)} \left( \partial_c^{(\xi)} x^a \right) L_b^{(x)} C_\xi \left( C^* \left( \partial_d^{(\xi)} x^b \right) \gamma_{c,d}^{\xi} \right) \right) d\tau \]

\[ = \frac{1}{2} \sum_{c,d=0}^{3} \sum_{a,b=0}^{3} \left( i_a^{(x)} \left( \partial_c^{(\xi)} x^a \right) L_b^{(x)} C_\xi \left( C^* \left( \partial_d^{(\xi)} x^b \right) \gamma_{c,d}^{\xi} \right) \right) d\tau \]

\[ = \frac{1}{2} \sum_{c,d=0}^{3} \sum_{a,b=0}^{3} \left( i_a^{(x)} \left( \partial_c^{(\xi)} x^a \right) L_b^{(x)} C_\xi \left( C^* \left( \partial_d^{(\xi)} x^b \right) \gamma_{c,d}^{\xi} \right) \right) d\tau \]

\[ = \frac{1}{2} \sum_{c,d=0}^{3} \sum_{a,b=0}^{3} \left( i_a^{(x)} \left( \partial_c^{(\xi)} x^a \right) L_b^{(x)} C_\xi \left( C^* \left( \partial_d^{(\xi)} x^b \right) \gamma_{c,d}^{\xi} \right) \right) d\tau \]

\[ = \frac{1}{2} \sum_{c,d=0}^{3} \sum_{a,b=0}^{3} \left( i_a^{(x)} \left( \partial_c^{(\xi)} x^a \right) L_b^{(x)} C_\xi \left( C^* \left( \partial_d^{(\xi)} x^b \right) \gamma_{c,d}^{\xi} \right) \right) d\tau \]

\[ = \frac{1}{2} \sum_{c,d=0}^{3} \sum_{a,b=0}^{3} \left( i_a^{(x)} \left( \partial_c^{(\xi)} x^a \right) L_b^{(x)} C_\xi \left( C^* \left( \partial_d^{(\xi)} x^b \right) \gamma_{c,d}^{\xi} \right) \right) d\tau \]

The \( C^* \left( \partial_d^{(\xi)} \partial_c^{(\xi)} x^a \right) \gamma_{c,d}^{\xi} \) term vanishes due to the symmetry of \( \gamma_{c,d}^{\xi} \).

In the above lemma the quantity \( \gamma_{c,d}^{ab} \) transforms as a rank (2,0) tensor, also known as a bivector. A general coordinate transformation mixes up the electric and magnetic parts.

**Lemma 6.2.3.** If the coordinate change in Lemma 6.2.2 is between adapted coordinate systems \( (\tau, \xi) \) and \( (\tilde{\tau}, \tilde{\xi}) \) where \( \tilde{\tau} = \tilde{\tau} (\tau, \xi) \) and \( \tilde{\xi} = \tilde{\xi} (\tau, \xi) \) then

\[ \gamma_{c,d}^{\tilde{\tau},\tilde{\xi}} = \sum_{\sigma=1}^{3} \gamma_{c,d}^{\xi,\sigma} \gamma_{\sigma,\sigma}^{\tilde{\tau},\tilde{\xi}} + \sum_{\sigma=1}^{3} \gamma_{c,d}^{\xi,\sigma} \left( \left( \partial^\mu_{\sigma} \tilde{\xi}^\mu \right) \left( \partial^\rho_{\sigma} \tilde{\xi}^\rho \right) \right) \gamma_{\sigma,\sigma}^{\tilde{\tau},\tilde{\xi}}, \]  

(6.8)

\[ \gamma_{c,d}^{\mu,\nu} = \sum_{\sigma=1}^{3} \gamma_{c,d}^{\xi,\sigma} \left( \left( \partial^\mu_{\sigma} \tilde{\xi}^\nu \right) \left( \partial^\nu_{\sigma} \tilde{\xi}^\mu \right) \right) \gamma_{\sigma,\sigma}^{\tilde{\tau},\tilde{\xi}}, \]  

(6.9)

**Proof.** The relations between the adapted coordinate systems are \( \tilde{\tau} = \tilde{\tau} (\tau, \xi) \) and \( \tilde{\xi} = \tilde{\xi} (\tau, \xi) \) with \( \tilde{\xi}^\mu (\tau, \xi) \) then we have \( C^* \left( \partial^\mu_{\sigma} \tilde{\xi}^\nu \right) = 0 \) and \( C^* \left( \partial^\nu_{\sigma} \tilde{\xi}^\mu \right) d\tau = C^* \left( \partial^\nu_{\sigma} \tilde{\xi}^\mu \right) d\tilde{\tau} = d\tilde{\tau} \). Substituting into (6.7) we get the desired result.

From (6.8) and (6.9) it is apparent that if \( J_{MD} = 0 \) and \( J_{ED} \neq 0 \) in one adapted frame then we also have \( J_{MD} = 0 \) and \( J_{ED} \neq 0 \) in any other frame. This means there
6.3 Quadrupoles

Figure 6.1: Diagram that depicts a moving electric dipole (left) and magnetic dipole (right) in non adapted coordinates.

is the concept of a pure intrinsic electric dipole. However, if $\mathcal{J}_{MD} \neq 0$ and $\mathcal{J}_{ED} = 0$ then under a change of adapted coordinates an electric dipole term appears (due to (6.8)) so there isn’t a counterpart pure magnetic dipole. Furthermore if we have a metric then it is possible to define an orthogonal complement to the worldline such that $\dot{z}^\mu = \dot{z}^\mu(\bar{z})$ and consequently $C^\ast(\partial_\sigma^{(C)}\dot{z}^0) = 0$. In this scenario the electric and magnetic dipoles do not mix and one can have a pure intrinsic magnetic dipole.

6.3 Quadrupoles

The same analysis applies for quadrupoles. If $\mathcal{J} \in \Upsilon^{2,3}(C)$ then the current is

$$
\mathcal{J} = C_\varsigma(\mathcal{J}_\emptyset^{\emptyset,\emptyset}) + \sum_{\mu=1}^{3} L_\mu^{(C)} C_\varsigma(\mathcal{J}_\emptyset^{\emptyset^\mu,\emptyset}) + \sum_{\nu=1}^{3} i_\nu^{(C)} C_\varsigma(\mathcal{J}_0^{\emptyset^\nu,\nu} d\tau) + \sum_{\nu,\mu=1}^{3} i_\nu^{(C)} L_\mu^{(C)} C_\varsigma(\mathcal{J}_0^{\mu,\nu} d\tau) + \sum_{\nu \leq \mu = 1}^{3} L_\mu^{(C)} L_\nu^{(C)} C_\varsigma(\mathcal{J}_\emptyset^{\mu^\nu,\emptyset}) + \sum_{\mu \geq \nu = 1}^{3} \sum_{\rho=1}^{3} i_\rho^{(C)} L_\mu^{(C)} L_\nu^{(C)} C_\varsigma(\mathcal{J}_0^{\mu^\nu,\rho} d\tau). \quad (6.10)
$$
6.3. Quadrupoles

Again taking the exterior derivative

\[ 0 = dJ = C_\zeta (\partial_0^{(r)} J^\varphi_\varphi d\tau) + \sum_{\mu=1}^{3} L^{(c)}_\mu C_\zeta (\partial_0^{(r)} J^{\mu\varphi}_\varphi d\tau) + \sum_{\nu,\mu=1}^{3} L^{(c)}_\nu L^{(c)}_\mu C_\zeta (J^{\mu\nu}_0 d\tau) 
+ \sum_{\mu,\nu=1}^{3} L^{(c)}_\mu L^{(c)}_\nu C_\zeta (\partial_0^{(r)} J^{\mu\nu}_\varphi d\tau) + \sum_{\mu=1,\rho=1}^{3} \sum_{\nu}^{3} L^{(c)}_\rho L^{(c)}_\nu L^{(c)}_\mu C_\zeta (J^{\mu\nu\rho}_0 d\tau). \]

Equating each term to zero we obtain the following relation between components

\[ J^\varphi_\varphi = q, \quad \partial_0^{(r)} (J^\mu_\varphi) = J^{\mu\nu}_0 = 0, \]

\[ J^\mu_0 + \partial_0^{(r)} (J^{\mu\varphi}_\varphi) = 0, \quad J^{\nu\mu}_0 + J^{\nu\mu}_0 + \partial_0^{(r)} (J^{\mu\nu\varphi}_\varphi) = 0, \quad \mu < \nu, \]

\[ J^{\mu\nu}_0 = 0, \quad J^{\mu\nu}_0 + J^{\mu\nu}_0 = 0, \quad \mu \neq \nu \quad \text{and} \quad J^{12,3}_0 + J^{31,2}_0 + J^{23,1}_0 = 0. \]

On substituting back into (6.10), in addition to the dipole terms we have the electric quadrupole given by

\[ J_{EQ} = \sum_{\mu,\nu=1}^{3} \left( L^{(c)}_\nu L^{(c)}_\mu C_\zeta (J^{\mu\varphi}_\varphi d\tau) - i^{(c)}_\nu L^{(c)}_\mu C_\zeta (\partial_0^{(r)} J^{\mu\varphi}_\varphi d\tau) \right), \quad (6.11) \]

and the magnetic quadrupole which is

\[ J_{MQ} = \frac{1}{2} \sum_{\mu,\nu=1}^{3} \left( i^{(c)}_\nu L^{(c)}_\mu L^{(c)}_\mu - i^{(c)}_\mu L^{(c)}_\nu L^{(c)}_\nu \right) C_\zeta (J^{\mu\nu}_0 d\tau) 
+ \left( i^{(c)}_3 L^{(c)}_1 L^{(c)}_2 - i^{(c)}_1 L^{(c)}_2 L^{(c)}_3 \right) C_\zeta (J^{123}_0 d\tau) 
+ \left( i^{(c)}_2 L^{(c)}_1 L^{(c)}_3 - i^{(c)}_1 L^{(c)}_2 L^{(c)}_3 \right) C_\zeta (J^{132}_0 d\tau). \quad (6.12) \]

In total there are 14 Quadrupole terms: 6 are electric whilst the remaining 8 are magnetic. The unusual number of magnetic components may be attributed to the toroidal dipoles as mentioned earlier in this chapter. Dipole terms are also present in the above bringing the total number of independent components in (6.10) to 20. This suggests the transformation rules will be more complicated than the dipole case.

**Lemma 6.3.1.** The sum of the quadrupole and dipole terms can be written as

\[ J_{QP} = \frac{1}{4} \sum_{a,b,c=0}^{3} i^{(c)}_a L^{(c)}_{bc} C_\zeta (\gamma^{abc}_0 d\tau), \quad (6.13) \]
6.3. Quadrupoles

where $\gamma^{abc}_{(\zeta)}$ obeys

$$
\gamma^{abc}_{(\zeta)} = \gamma^{acb}_{(\zeta)},
$$

(6.14a)

$$
\gamma^{abc}_{(\zeta)} + \gamma^{cab}_{(\zeta)} + \gamma^{bca}_{(\zeta)} = 0,
$$

(6.14b)

and has the following relation between components

$$
\partial_0^{(r)} (\gamma^{0\mu\nu}_{(\zeta)}) = 2J^{\mu\nu},
$$

$$
\gamma^{0\mu\nu}_{(\zeta)} = -2\gamma^{0\mu\nu}_{(\zeta)} = 4J^{\mu\nu},
$$

and $\partial_0^{(r)} (\gamma^{\mu\nu}_{(\zeta)}) = 2J^{\mu\nu}, \mu < \nu,$

$$
\gamma^{\mu\nu}_{(\zeta)} = 2J^{\mu\nu}, \gamma^{\mu\nu\rho}_{(\zeta)} = 2J^{\mu\nu\rho}, \mu \neq \nu \neq \rho \neq \mu.
$$

Proof. See Appendix D.

Lemma 6.3.2. The coordinate transformation of $\gamma^{abc}_{(\zeta)}$ from the adapted coordinate system $(\tau, \zeta)$ to the non adapted coordinate system $(y^0, y^1, y^2, y^3)$ is non-tensorial.

Proof. The methodology is the same as for the dipole case. From (B.6) and (B.7) in Corollary B.1.3

$$
L^{(\zeta)}_b L^{(\zeta)}_c C_z (\gamma^{abc}_{(\zeta)} d\tau)
$$

$$
= \sum_{e,f=0}^3 L^{(y)}_e L^{(y)}_f C_z \left( C^* \left( (\partial_{b}^{(c)} y^e)(\partial_{c}^{(e)} y^f) \right) \gamma^{abc}_{(\zeta)} d\tau \right)
$$

$$
- L^{(y)}_e C_z \left( C^* \left( \left( \partial_{f}^{(y)} \left( \partial_{b}^{(c)} y^e \right) \left( \partial_{c}^{(e)} y^f \right) \right) \gamma^{abc}_{(\zeta)} d\tau \right) \right)
$$

$$
= \sum_{e,f=0}^3 L^{(y)}_e L^{(y)}_f C_z \left( C^* \left( (\partial_{b}^{(c)} y^e)(\partial_{c}^{(e)} y^f) \right) \gamma^{abc}_{(\zeta)} d\tau \right)
$$

$$
- L^{(y)}_e C_z \left( C^* \left( \left( \partial_{f}^{(y)} \left( \partial_{b}^{(c)} y^e \right) \left( \partial_{c}^{(e)} y^f \right) \right) \gamma^{abc}_{(\zeta)} d\tau \right) \right).
$$
6.3. Quadrupoles

To simplify the notation let \( A_d^d \equiv C^* (\partial_a^{(c)} y^d) \) and \( A_{ba}^d \equiv C^* (\partial_b^{(c)} y^d) \) thus

\[
\frac{1}{4} \sum_{a,b,c=0}^{3} i_a^{(c)} L_b^{(c)} L_c^{(c)} C_\gamma (\gamma_{y}^{abc}) d\tau
\]

\[
= \frac{1}{4} \sum_{a,b,c=0}^{3} \sum_{d,e,f=0}^{3} i_d^{(y)} L_e^{(y)} L_f^{(y)} C_\gamma (A_d^d A_e^e A_f^f \gamma_{y}^{abc}) d\tau
\]

\[
- i_d^{(y)} L_e^{(y)} C_\gamma (\partial_f^{(c)} (A_d^a A_e^b) \gamma_{y}^{abc}) d\tau
\]

\[
- i_d^{(y)} L_e^{(y)} C_\gamma (A_d^a (\partial_e^{(c)} A_e^b) \gamma_{y}^{abc}) d\tau
\]

\[
+ i_d^{(y)} C_\gamma (A_d^a (\partial_e^{(c)} A_e^b) \gamma_{y}^{abc}) d\tau
\]

\[
= \frac{1}{4} \sum_{a,b,c=0}^{3} \left( \sum_{d,e,f=0}^{3} i_d^{(y)} L_e^{(y)} L_f^{(y)} C_\gamma (A_d^d A_e^e A_f^f \gamma_{y}^{abc}) d\tau \right)
\]

\[
- \sum_{d,e=0}^{3} i_d^{(y)} L_e^{(y)} C_\gamma (A_d^d (\partial_e^{(c)} A_e^e) \gamma_{y}^{abc}) d\tau
\]

\[
+ \sum_{d=0}^{3} i_d^{(y)} C_\gamma (A_d^d (\partial_e^{(c)} A_e^e) \gamma_{y}^{abc}) d\tau
\]

\[
= \frac{1}{4} \sum_{a,b,c=0}^{3} \left( \sum_{d,e,f=0}^{3} i_d^{(y)} L_e^{(y)} L_f^{(y)} C_\gamma (A_d^d A_e^e A_f^f \gamma_{y}^{abc}) d\tau + \sum_{d=0}^{3} i_d^{(y)} C_\gamma (A_d^d \gamma_{y}^{abc}) d\tau \right)
\]

\[
- \sum_{d,e=0}^{3} i_d^{(y)} L_e^{(y)} C_\gamma ((A_c^c A_d^d e + A_d^d A_e^e c + A_e^e A_f^f a) \gamma_{y}^{abc}) d\tau
\]

However

\[
A_{cba}^d \gamma_{y}^{abc} = 0 \quad \text{by virtue of} \quad \gamma_{y}^{abc} + \gamma_{y}^{cab} + \gamma_{y}^{bca} = 0,
\]

whereas

\[
A_d^d A_e^e A_f^f \gamma_{y}^{abc} = -A_d^d A_e^e A_f^f \gamma_{y}^{cab} - A_d^d A_e^e A_f^f \gamma_{y}^{bca} = -(A_d^d A_e^e A_f^f c a) \gamma_{y}^{abc} = -2 A_d^d A_e^e A_f^f \gamma_{y}^{abc},
\]

and

\[
A_e^e A_d^d A_b^b \gamma_{y}^{abc} = A_e^e A_d^d A_b^b \gamma_{y}^{acb} = A_d^d A_e^e A_b^b \gamma_{y}^{abc}.
\]
Therefore

\[
(A_b^e A_{ca}^d + A_a^d A_{eb}^c + A_c^e A_{da}^b) \gamma_{(\zeta)}^{abc} = 2(A_b^e A_{ca}^d - A_b^d A_{ca}^e) \gamma_{(\zeta)}^{abc}.
\]

Combining all the above results we obtain

\[
\frac{1}{4} \sum_{a,b,c=0}^{3} i^{(a)} L^{(a)} L^{(c)} C_{\zeta} (\gamma_{(\zeta)}^{abc} d\tau)
\]

\[
= \frac{1}{4} \sum_{a,b,c=0}^{3} \left( \sum_{d,e,f=0}^{3} i^{(d)} L^{(d)} L^{(f)} C_{\zeta} (A_d^e A_b^f A_c^g \gamma_{(\zeta)}^{def} d\tau) - 2 \sum_{d,e=0}^{3} i^{(d)} L^{(d)} C_{\zeta} ((A_b^e A_{ca}^d - A_b^d A_{ca}^e) \gamma_{(\zeta)}^{abc} d\tau) \right).
\]

(6.15)

It is clear that under the change of coordinates a dipole term appears (the second term), however, this contains second order derivatives, thus \( \gamma_{(\zeta)}^{abc} \) does not transform as a tensor.

In its present form (6.15) is not very practical. We rewrite it such that the dipole contribution looks more like a quadrupole term i.e. in accordance with the first term in (6.15). To do this we set the components of the induce dipole as

\[
S_{de} = \frac{1}{4} \sum_{a,b,c=0}^{3} \left( \sum_{d,e,f=0}^{3} i^{(d)} L^{(d)} L^{(f)} C_{\zeta} (A_d^e A_b^f A_c^g \gamma_{(\zeta)}^{def} d\tau) - 2 \sum_{d,e=0}^{3} i^{(d)} L^{(d)} C_{\zeta} ((A_b^e A_{ca}^d - A_b^d A_{ca}^e) \gamma_{(\zeta)}^{abc} d\tau) \right).
\]

(6.16)

which is antisymmetric in its indices, \( S_{de} = -S_{ed} \), and define

\[
P_{de} = \int_{\tau} (A_b^e (\tau') A_{ca}^d (\tau') - A_b^d (\tau') A_{ca}^e (\tau')) \gamma_{(\zeta)}^{abc} (\tau') d\tau',
\]

(6.17)

such that

\[
S_{de} = -\frac{dP_{de}}{d\tau}.
\]

(6.18)

Lemma 6.3.3. \( \gamma_{(\zeta)}^{abc} \) obeys the following transformation rule

\[
\gamma_{(\zeta)}^{def} = A_d^a A_b^e A_c^f \gamma_{(\zeta)}^{abc} + P_{de} \hat{C}^{ef} + P_{df} \hat{C}^{ef},
\]

(6.19)

where \( \hat{C}^{ef} \) are the components of the tangent vector \( C \) to the worldline.
6.3. Quadrupoles

**Proof.** Acting on the test form \( \phi \in \Gamma_0 \Lambda^1 \mathcal{M} \) consider only the second term in (6.15)

\[
\hat{J}_{DP} = \frac{1}{2} \sum_{d,e=0}^{3} i_d^{(y)} L_e^{(y)} C_{\zeta} \left( (A_b^d A_a^e - A_b^e A_a^d) \gamma_{(\zeta)}^{abc} \right) d\tau. \tag{6.20}
\]

Using (6.16) and (6.17) we have

\[
[\phi \rvert \hat{J}_{DP}]_M = \frac{1}{2} \sum_{d,e=0}^{3} \int_{\mathcal{I}} C^\star (L_e^{(y)} i_d^{(y)} \phi) S^{de} d\tau
\]

\[
= -\frac{1}{2} \sum_{d,e=0}^{3} \int_{\mathcal{I}} \left( \frac{\partial \phi_d}{\partial y^e} \right) \left|_C \right. \frac{dP^{de}}{d\tau} d\tau
\]

\[
= \frac{1}{2} \sum_{d,e=0}^{3} \int_{\mathcal{I}} \left( \frac{\partial \phi_d}{\partial y^e} \right) \left|_C \right. P^{de} d\tau
\]

\[
= \frac{1}{2} \sum_{d,e,f=0}^{3} \int_{\mathcal{I}} \hat{C}^{ef} \left( \frac{\partial^2 \phi_d}{\partial y^f \partial y^e} \right) \left|_C \right. P^{de} d\tau
\]

\[
= \frac{1}{4} \sum_{d,e,f=0}^{3} \int_{\mathcal{I}} C^\star (L_f^{(y)} L_e^{(y)} i_d^{(y)} \phi) (P^{de} \hat{C}^{ef} + P^{df} \hat{C}^{ce}) d\tau.
\]

Thus

\[
\hat{J}_{DP} = \frac{1}{4} \sum_{d,e,f=0}^{3} i_d^{(y)} L_e^{(y)} L_f^{(y)} C_{\zeta} \left( (P^{de} \hat{C}^{ef} + P^{df} \hat{C}^{ce}) d\tau \right),
\]

and we have

\[
J_{QP} = \frac{1}{4} \sum_{d,e,f=0}^{3} i_d^{(y)} L_e^{(y)} L_f^{(y)} C_{\zeta} \left( (A_a^d A_b^e A_c^f \gamma_{(\zeta)}^{abc} + P^{de} \hat{C}^{ef} + P^{df} \hat{C}^{ce}) d\tau \right). \tag{6.21}
\]

(6.17) suggests there will be a constant of integration, however, this is not an issue. The components of the induced dipole \( S^{de} \) are equal to the differential of \( P^{de} \) i.e. (6.18), so any constants that do arise are differentiated away, and hence do not affect the overall physical properties of the quadrupole. It is straightforward to show that (6.19) satisfies the symmetry relations (6.14a) and (6.14b)

\[
\gamma_{(y)}^{dfe} = A_a^d A_b^f A_c^e \gamma_{(\zeta)}^{abc} + P^{df} \hat{C}^{ce} + P^{de} \hat{C}^{ef} = \gamma_{(y)}^{def},
\]

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6.4 Event dipoles

Consider a current which has support over a single point \( p \in M \). The embedding map is \( f_p : \{0\} \mapsto M \) such that \( f_p(0) = p \). The closed 3-form \( \mathcal{J}_{\text{event}} \in \Omega^{1,3}(f_p) \) is

\[
\mathcal{J}_{\text{event}} = \sum_{a=0}^{3} \xi_a(\mathcal{J}^a_{\mathcal{Q}}) + \sum_{a=0}^{3} \sum_{b=0}^{3} \xi_a(\mathcal{L}_b) f_p(\mathcal{J}^a_{\mathcal{Q}}).
\]  

(6.22)

Figure 6.2: Diagram showing a moving quadrupole. Under a change of coordinates to a non-adapted system the quadrupole (denoted by the blue ellipses) picks up a dipole term represented by the arrows.

and

\[
\gamma^{def} + \gamma^{edf} + \gamma^{fde} = A^a_b A^b_d \left( \gamma^{abc}_{\zeta} + \gamma^{bca}_{\zeta} + \gamma^{cab}_{\zeta} \right) + (P^{de} + P^{ed}) \dot{C}^f + (P^{df} + P^{fd}) \dot{C}^e + (P^{fe} + P^{ef}) \dot{C}^d = 0.
\]

Since \( \gamma^{abc}_{\zeta} + \gamma^{bca}_{\zeta} + \gamma^{cab}_{\zeta} = 0 \) and \( P^{de} = -P^{ed} \).
6.4. Event dipoles

Again setting \( d\mathcal{J}_{\text{event}} = 0 \) we have

\[
\mathcal{J}_0^{a,a} = 0, \quad \mathcal{J}_0^{b,a} + \mathcal{J}_0^{a,b} = 0.
\]

Substituting back into (6.22)

\[
\mathcal{J}_{\text{event}} = \sum_{a < b = 0}^3 (i_a^{(x)} L_b^{(x)} + i_b^{(x)} L_a^{(x)}) f_{p} (\mathcal{J}_0^{b,a}).
\]

In total there are six independent components \( \mathcal{J}_0^{b,a} \). The three \( \mathcal{J}_0^{0,\mu} \) are electric whilst the remaining three \( \mathcal{J}_0^{\mu,\nu} \) with \( \mu < \nu \) are magnetic. Physically an event dipole can be interpreted as an electron-positron pair that is created and annihilated in an instant. Therefore, an event dipole is the limit of an electron-positron loop shrinking to a point. Since \( \mathcal{J}_{\text{event}} \) is a closed 3-form it is a valid source in Maxwell’s equations and the associated fields will have support on the forward pointing lightcone. A potential use for event dipoles is to approximate the electromagnetic fields of a moving dipole with a discrete sum.
Chapter 7

Liénard-Wiechert Field for
Multipoles

7.1 Introduction

In this chapter we construct the Liénard-Wiechert field as the wedge product of two
submanifold distributions. The SMDs are the current and the Green’s SMD. The
former has support on the worldline whereas the Green’s form is an SMD on the
product manifold with support over the map \( \iota \). One of the key results is the correct
domain and codomain of the map so that it is a closed embedding. We show that we
can write \( G \in \Upsilon^0,4(\iota) \) where

\[
\iota : N \times M_Y \leftrightarrow (M_X \times M_Y) \setminus \Delta
\]

where \( M \) is Minkowski spacetime and \( N \) is the lightcone. Since the vertex is neither
in \( N \) or \( (M_X \times M_Y) \setminus \Delta \), \( \iota \) is a closed embedding as shown earlier. The corresponding
Liénard-Wiechert field associated with this Green’s SMD are only defined on \( M_X \setminus C \).

As the electromagnetic fields are defined on \( M_X \setminus C \) we require a means to “trans-
late” the data from the worldline to \( M_X \setminus C \). This is achieved by parallel transport
along the light cone. It is worth noting there are two cases when transversality breaks
Figure 7.1: (a) When attempting to evaluate the fields on the worldline the tangent space of the light cone is not defined and so the wedge product is also not defined. (b) Whenever the worldline becomes lightlike the tangent space of the worldline and lightcone are not transverse.

down. This occurs when the apex of the lightcone intersects the worldline and also when the curve is tangential (i.e. lightlike) to the lightcone. In these cases the wedge product is not defined, however we have modified the domains and codomains such that this problem does not arise.

## 7.2 Liénard-Wiechert Field in Terms of the Wedge Product

**Definition 7.2.1.** Introduce the operator \( A_{\text{op}} : \Upsilon^k,3(C) \to \Gamma^3(M_X \setminus C) \) such that

\[
A_{\text{op}}[J] = \pi_X(\mathcal{G} \wedge \pi_Y J).
\]  

(7.1)

**Lemma 7.2.2.** Given a closed \( J \in \Upsilon^k,3(C) \) then \( A = \star^{-1} A_{\text{op}}[J] \) satisfies \( d \star A = 0 \) and \( d \star dA = J \).
7.2. Liénard-Wiechert Field in Terms of the Wedge Product

Proof. \( \mathcal{G} \) and \( \mathcal{J} \) are closed so from (7.1) and using the property that \( d \) commutes with pushforwards

\[
d A = d A_{\text{op}}[\mathcal{J}] = d \pi_X (\mathcal{G} \wedge \pi_Y \mathcal{J}) = \pi_X (d \mathcal{G} \wedge \pi_Y \mathcal{J}) - \pi_X (\mathcal{G} \wedge \pi_Y d \mathcal{J}) = 0.
\]

\[\square\]

Lemma 7.2.3. Let \( W \in \Gamma(C, TM_Y) \), \( \widehat{\mathcal{V}}_\mathcal{G} \in \Gamma T(M_X \times M_Y) \) and \( W_G \in \Gamma TM_X \) such that

\[
W = \pi_Y \ast \widehat{\mathcal{V}}_\mathcal{G}|_C \quad \text{and} \quad W_G = \pi_X \ast \widehat{\mathcal{V}}_\mathcal{G}
\]

then

\[
A_{\text{op}}[L W \mathcal{J}] = L W_G A_{\text{op}}[\mathcal{J}] - \pi_X (L \widehat{\mathcal{V}}_\mathcal{G} \mathcal{G} \wedge \pi_Y \mathcal{J}), \quad (7.2a)
\]

\[
A_{\text{op}}[i W \mathcal{J}] = i W_G A_{\text{op}}[\mathcal{J}] - (-1)^{\text{ideg}(\mathcal{G})} \pi_X (i \widehat{\mathcal{V}}_\mathcal{G} \mathcal{G} \wedge \pi_Y \mathcal{J}). \quad (7.2b)
\]

Proof.

\[
A_{\text{op}}[L W \mathcal{J}] = \pi_X (\mathcal{G} \wedge \pi_Y L W \mathcal{J}) = \pi_X (\mathcal{G} \wedge L \widehat{\mathcal{V}}_\mathcal{G} \mathcal{G} \wedge \pi_Y \mathcal{J}), \quad \text{using} \quad W = \pi_Y \ast \widehat{\mathcal{V}}_\mathcal{G}|_C
\]

\[
= \pi_X L \widehat{\mathcal{V}}_\mathcal{G} (\mathcal{G} \wedge \pi_Y \mathcal{J}) - \pi_X (L \widehat{\mathcal{V}}_\mathcal{G} \mathcal{G} \wedge \pi_Y \mathcal{J})
\]

\[
= L W_G \pi_X (\mathcal{G} \wedge \pi_Y \mathcal{J}) - \pi_X (L \widehat{\mathcal{V}}_\mathcal{G} \mathcal{G} \wedge \pi_Y \mathcal{J})
\]

\[
= L W_G A_{\text{op}}[\mathcal{J}] - \pi_X (L \widehat{\mathcal{V}}_\mathcal{G} \mathcal{G} \wedge \pi_Y \mathcal{J}), \quad \text{using} \quad W_G = \pi_X \ast \widehat{\mathcal{V}}_\mathcal{G}.
\]

Similarly for the internal contractions. \[\square\]

Note that given \( W \) neither \( W_G \) or \( \widehat{\mathcal{V}}_\mathcal{G} \) are unique. However, if we now demand \( L \widehat{\mathcal{V}}_\mathcal{G} \mathcal{G} = 0 \) then the following lemmas show it is possible to construct a unique \( W_G \) from \( W \).

Lemma 7.2.4. Given \( \widehat{\mathcal{V}} \in \Gamma T(M_X \times M_Y) \) and \( U \in \Gamma T(N \times M_Y) \) such that \( \iota_*(U) = \widehat{\mathcal{V}} \) and \( L_U (\alpha_N) = 0 \) then \( L \mathcal{G} = 0 \) and \( \widehat{\mathcal{V}} \) has the coordinate representation

\[
\widehat{\mathcal{V}} = U^a(y) \left( \frac{\partial}{\partial x^a} + \frac{\partial}{\partial y^a} \right).
\]

(7.3)

Proof.

\[
L \mathcal{G} = L_{\iota_*(U)} \iota_*(\alpha_N) = \iota_*(L_U (\alpha_N)) = 0.
\]

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7.2. Liénard-Wiechert Field in Terms of the Wedge Product

The pushforward of the basis vectors on $N \times M_Y$ are

$$\iota^* \left( \frac{\partial}{\partial z^\mu} \right) = \frac{\partial x^a}{\partial z^\mu} \frac{\partial}{\partial x^a} + \frac{\partial y^a}{\partial z^\mu} \frac{\partial}{\partial y^a} = \frac{z_\mu}{|z|} \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^a}, \quad \iota^* \left( \frac{\partial}{\partial y^\mu} \right) = \frac{\partial}{\partial y^\mu} + \frac{\partial}{\partial x^a}.$$  

Thus the pushforward of the vector field $U \in \Gamma T(N \times M_Y)$ is:

$$\iota^* U = \iota^* \left( U^\mu(z) \frac{\partial}{\partial z^\mu} + U^b(y) \frac{\partial}{\partial y^b} \right) = \left( \bar{U}^a + U^a(y) \right) \frac{\partial}{\partial x^a} + U^b(y) \frac{\partial}{\partial y^b},$$

where

$$\bar{U}^0 = \frac{U^\mu(z) z_\mu}{|z|}, \quad \bar{U}^\mu = U^\mu(z).$$

Now

$$0 = L_U(\alpha_N) = L_U \left( \frac{dz^{123}}{|z|} \right) = d \iota^* \left( \frac{dz^{123}}{|z|} \right) = d \left( \frac{U^1(z)}{|z|} dz^{23} - \frac{U^2(z)}{|z|} dz^{13} + \frac{U^3(z)}{|z|} dz^{12} \right) = \left( \frac{1}{|z|} dU^1(z) \wedge dz^{23} - \frac{1}{|z|^3} dz^{123} - \frac{1}{|z|} dU^2(z) \wedge dz^{13} - \frac{1}{|z|^3} dU^3(z) \wedge dz^{12} - \frac{1}{|z|^3} dU^3(z) \wedge dz^{123} \right) = \left( \frac{1}{|z|} \frac{\partial U^\mu(z)}{\partial z^\mu} - \frac{U^\mu(z) z_\mu}{|z|^3} \right) dz^{123}.$$

We obtain the following differential equation for $U^\mu(z)$:

$$\frac{\partial U^\mu(z)}{\partial z^\mu} = \frac{U^\mu(z) z_\mu}{|z|^2},$$  \hspace{1cm} (7.4)

which has the solution

$$U^\mu(z) = A^\mu e^{\frac{z_\mu}{|z|}}, \quad \text{such that} \quad \frac{\partial A^\mu}{\partial z^\mu} = 0, \quad \mu = 1, 2, 3. \hspace{1cm} (7.5)$$
7.2. Liénard-Wiechert Field in Terms of the Wedge Product

As we require $U^\mu_{(z)}$ to remain finite as $\|z\| \to \infty$ the obvious choice is $A^\mu = 0$ resulting in $U^\mu_{(z)} = 0$. Therefore

$$\tilde{V} = \iota_x(U) = U^a_{(y)} \left( \frac{\partial}{\partial x^a} + \frac{\partial}{\partial y^a} \right).$$

(7.6)

\[ \square \]

**Definition 7.2.5.** Introduce the linear operator $T_{\text{ret}} : \Gamma(C, TM_Y) \to \Gamma TM_X$ that translates vectors along retarded coordinates. Acting on $W$

$$(T_{\text{ret}} W)|_x = W_{\text{ret}} = W^a(\tau_R | x) \partial_a^{(x)}|_x.$$  

(7.7)

In this notation the tangent vector to the wordline is

$$V|_x = (T_{\text{ret}} \dot{C})|_x = \dot{C}^a(\tau_R | x) \partial_a^{(x)}|_x.$$  

(7.8)

The subscript indicates a vector at a point $x \in M_X$ where $x$ is the apex of the backward lightcone. See Figure 7.2.

**Theorem 7.2.6.** Given $W \in \Gamma(C, TM_Y)$ then

$$A_{\text{op}}[L_W J] = L_{W_{\text{ret}}} A_{\text{op}}[J].$$

(7.9)

**Proof.** In local coordinates the vectors $\tilde{V}$ and $W$ are

$$\tilde{V} = \tilde{V}^a_{(x)}(\bar{x}, \tau) \frac{\partial}{\partial x^a}|_{(x, C(\tau))} + \tilde{V}^b_{(y)}(\bar{y}, \tau) \frac{\partial}{\partial y^b}|_{(x, C(\tau))}, \quad W = W^a(\tau) \frac{\partial}{\partial y^a}|_{C(\tau)}.$$  

The pushforward of $\tilde{V}$ with respect to $\pi_Y$ is

$$\pi_{Y*}(\tilde{V}|_{(x, C(\tau))}) = W|_{C(\tau)} \quad \Rightarrow \quad \tilde{V}^b_{(y)}(\bar{y}, \tau) = W^b(\tau)$$

so

$$\tilde{V} = \tilde{V}^a_{(x)}(\bar{x}, \tau) \frac{\partial}{\partial x^a}|_{(x, C(\tau))} + W^b(\tau) \frac{\partial}{\partial y^b}|_{(x, C(\tau))}. \quad \quad \text{(7.10)}$$

Equating (7.6) with (7.10)

$$U^a_{(y)} = \tilde{V}^a_{(x)} \quad \text{and} \quad U^a_{(y)} = W^a \quad \Rightarrow \quad \tilde{V}^a_{(x)} = W^a.$$  

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7.2. Liénard-Wiechert Field in Terms of the Wedge Product

Figure 7.2: Geometry for calculating the Liénard-Wiechert field. The backward light cone at $x$ intersects the world line at point $C(\tau_R|_x)$. To obtain the vector fields on $M$, vectors at the retarded point are parallel transported by $T_{\text{ret}}$ along the light cone to point $x$.

Therefore

$$\hat{V} = W^a(\tau_R|_x) \left( \frac{\partial}{\partial x^a} + \frac{\partial}{\partial y^a} \right) |_{(z+C(\tau),C(\tau))}.$$  \hfill (7.11)

Finally the pushforward with respect to $\pi_X$ gives

$$W|_{z+C(\tau)} = \pi_X_* \left( (\hat{V}|_{z+C(\tau),C(\tau)}) \right) = W^a(\tau_R|_x) \frac{\partial}{\partial x^a |_{z+C(\tau)}},$$  \hfill (7.12)

which is just $W_{\text{ret}}$. Therefore from (7.2a) we obtain the desired result. 

\textbf{Lemma 7.2.7.} The commutative diagram for the wedge product of the Green form with the current distribution is given by

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\[ \begin{array}{cccc}
N \times \mathcal{I} & \xrightarrow{\iota_0} & \pi_Y^* \mathcal{I} & \xrightarrow{\hat{\pi}_Y} \mathcal{I} \\
\downarrow C_0 & & \downarrow \hat{C} & \downarrow C \\
N \times M_Y & \xrightarrow{\iota} & (M_X \times M_Y) \setminus \Delta & \xrightarrow{\pi_Y} M_Y \\
\pi_X & & & \\
M_X & & & \\
\end{array} \]

with the maps
\[ \begin{align*}
\iota_0(\tilde{z}, \tau) &= (z + C(\tau), \tau), & \hat{\pi}_Y(\tilde{x}, \tau) &= \tau, \\
C_0(\tilde{z}, \tau) &= (z, C(\tau)), & f(\tilde{z}, \tau) &= (z + C(\tau), C(\tau)), & \hat{C}(\tilde{x}, \tau) &= (\tilde{x}, C(\tau)), \\
h(\tilde{z}, \tau) &= (z + C(\tau)).
\end{align*} \]

**Proof.** The pullback diagram of the worldline is given by

\[ \begin{array}{cccc}
\pi_Y^* \mathcal{I} & \xrightarrow{\hat{\pi}_Y} & \mathcal{I} \\
\downarrow \hat{C} & & \downarrow C \\
(M_X \times M_Y) \setminus \Delta & \xrightarrow{\pi_Y} M_Y \\
\end{array} \] (7.13)

with the set
\[ \begin{align*}
\pi_Y^* \mathcal{I} &= ((M_X \times M_Y) \setminus \Delta) \times_{M_Y} \mathcal{I} \\
&= \{ ((x, y), \tau) \mid \pi_Y(x, y) = C(\tau), \ x \neq y \} \\
&= \{ ((x, y), \tau) \mid y = C(\tau), \ x \neq y \} \\
&= \{ (x, \tau) \mid x \neq C(\tau) \} \\
&= (M_X \times \mathcal{I}) \setminus \{ (C(\tau), \tau) \mid \tau \in \mathcal{I} \}.
\end{align*} \]

Now we embed the light cone (support of \( G \)) in \((M_X \times M_Y) \setminus \Delta\) and construct the wedge product. For this we extend the above diagram (7.13)

\[ \begin{array}{cccc}
\iota^*(\pi_Y^* \mathcal{I}) & \xrightarrow{\iota_0} & \pi_Y^* \mathcal{I} & \xrightarrow{\hat{\pi}_Y} \mathcal{I} \\
\downarrow C_0 & & \downarrow \hat{C} & \downarrow C \\
N \times M_Y & \xrightarrow{\iota} & (M_X \times M_Y) \setminus \Delta & \xrightarrow{\pi_Y} M_Y \\
\end{array} \]
7.2. Liénard-Wiechert Field in Terms of the Wedge Product

The induced pullback manifold can be written

\[ \iota^*(\pi_Y^* \mathcal{I}) = (N \times M_Y) \times \pi_Y^* \mathcal{I} \]

\[ = \{ (\mathbb{R}, y, (x, \tau)) \mid \mathbb{R}(\mathbb{R}, y) = \hat{C}(x, \tau), \mathbb{R} \neq C(\tau) \} \]

\[ = \{ (\mathbb{R}, y, (x, \tau)) \mid \mathbb{R} + y = x, y = C(\tau), \mathbb{R} \neq C(\tau) \} \]

\[ = \{ (\mathbb{R}, y, (x, \tau)) \mid \mathbb{R} + C(\tau) = x, \mathbb{R} \neq 0 \} \]

\[ = N \times I, \]

and the resulting diagram is

\[ \begin{array}{c}
N \times I \xrightarrow{\iota_0} \pi_Y^* \mathcal{I} \xrightarrow{\hat{\pi}_Y} I \\
\downarrow C_0 \quad \downarrow \hat{C} \quad \downarrow C \\
N \times M_Y \xrightarrow{\iota} (M_X \times M_Y) \setminus \Delta \xrightarrow{\pi_Y} M_Y
\end{array} \]

Now define the map

\[ h : N \times I \to M_X, \quad h(\mathbb{R}, \tau) = (\mathbb{R} + C(\tau)). \]

As it stands \( h \) is not a bijection since it is not surjective on the set \( M_X \). However, if we restrict the codomain in \( M_X \) to the set

\[ E^+(C) = \{ \mathbb{R} + C(\tau) \mid \mathbb{R} \in N, \tau \in I \} \subset M_X, \tag{7.14} \]

then the maps

\[ H : N \times I \to E^+, \quad R : E^+ \to N \times I, \tag{7.15} \]

are bijective and furthermore diffeomorphisms. In terms of coordinates

\[ R(x) = (x - C(\tau_R|_x), \tau_R|_x). \tag{7.16} \]

Here \( \tau_R|_x \) is the retarded time; the point of intersection between the worldline and the backward light cone centred at \( x \in M_X \). The resulting commutative diagram that incorporates all the geometric structure required for the Liénard-Wiechert field is
7.2. Liénard-Wiechert Field in Terms of the Wedge Product

A general dipole has a current $\mathcal{J} = \sum_{a,b=0}^{3} i_a^{(y)} L^{(y)} C_\varsigma(\gamma_{(y)}^{ab} d\tau)$. In order to calculate the Liénard-Wiechert field we require the following lemma.

**Lemma 7.2.8.** Let $\Phi = C_\varsigma(\gamma_{(y)}^{ab} (\tau) d\tau)$ where $\gamma : \mathcal{I} \to \mathcal{I}$,

$$A_{\text{op}}[C_\varsigma(\gamma_{(y)}^{ab} (\tau) d\tau)] = R^*(C_0^* (\alpha)) \wedge \gamma_{(y)}^{ab} (\tau_R) d\tau_R.$$  \hspace{1cm} (7.17)

**Proof.**

$$A_{\text{op}}[\Phi] = A_{\text{op}}[C_\varsigma(\gamma_{(y)}^{ab} (\tau) d\tau)]$$

$$= \pi_X(\varsigma(\gamma_{(y)}^{ab} (\tau) d\tau))$$

$$= \pi_X(\gamma_{(y)}^{ab} (\tau_R) d\tau)_R$$

$$= \gamma_{(y)}^{ab} (\tau_R) d\tau_R.$$  \hspace{1cm} (7.18)

Here we have used the property that $H$ is a diffeomorphism so from (2.41)

$H_\varsigma = R^*$.

**Definition 7.2.9.** Define the null vector $Z$ which has components

$$Z^a = x^a - C^a(\tau_R|x), \quad dZ^a = dx^a - V^a|x d\tau_R.$$  \hspace{1cm} (7.18)
A derivation of the one form $d\tau_R$ is given in Appendix E. Using the above we can write (7.17) in terms of the vectors $Z$ and $V$.

**Lemma 7.2.10.**

\[ R^*(C^*_0 \alpha) = \frac{\star \tilde{V}}{g(V, Z)}, \quad (7.19) \]

where $\tilde{V}$ is the dual of the vector $V$ with respect to the metric $g$.

**Proof.** From (5.17) and using

\[ d\tau_R = \frac{\tilde{Z}}{g(V, Z)}, \]

we have

\[
R^*(C^*_0 \alpha) \\
= R^* \left( C^*_0 \left( \frac{dz^3}{|z^3|} \right) \right) = R^* \left( \frac{dz^3}{|z^3|} \right) = \frac{dZ^3}{Z_0} \\
= \frac{1}{Z_0} \left( (dx^1 - V^1 d\tau_R) \wedge (dx^2 - V^2 d\tau_R) \wedge (dx^3 - V^3 d\tau_R) \right) \\
= \frac{1}{Z_0} \left( dx^{123} - V^1 dx^{23} \wedge d\tau_R + V^2 dx^{13} \wedge d\tau_R - V^3 dx^{12} \wedge d\tau_R \right) \\
= \frac{1}{g(V, Z)Z_0} \left( (g(V, Z) - V^1 Z_1 - V^2 Z_2 - V^3 Z_3) dx^{123} - V^1 Z_0 dx^{023} + V^2 Z_0 dx^{013} - V^3 Z_0 dx^{012} \right) \\
= \frac{1}{g(V, Z)Z_0} \left( - V^0 Z_0 (\star dx^0) + V^1 Z_0 \star dx^1 + V^2 Z_0 \star dx^2 + V^3 Z_0 \star dx^3 \right) \\
= \frac{\star \tilde{V}}{g(V, Z)}. \]

\[ \square \]

### 7.3 Fields for Monopole, Dipole and Quadrupole

#### 7.3.1 Monopole

The monopole current is $J_M = qC_1(1)$ so from (7.19)

\[ A_M = \star^{-1} R^*(C^*_0 \alpha) = \frac{\tilde{V}}{g(V, Z)}. \quad (7.20) \]
7.3. Fields for Monopole, Dipole and Quadrupole

7.3.2 Dipoles

The dipole current on $M_{7.3}$ in the form (6.6) is

$$\mathcal{J}_{DP} = \frac{1}{2} \sum_{a,b=0}^{3} i_{a}^{(y)} L_{b}^{(y)} C_{c} (\gamma_{(y)}^{ab} d\tau).$$  \hfill (7.21)

**Lemma 7.3.1.** The Liénard-Wiechert field for a moving dipole is

$$A_{DP} = \frac{1}{2} \sum_{a,b=0}^{3} *^{-1} L_{b}^{(x)} i_{a}^{(x)} \left( \frac{\gamma_{(x)}^{ab}(\tau_{R})}{g(V, Z)} * 1 \right).$$  \hfill (7.22)

**Proof.** From (7.2a) and (7.2b)

$$A_{op}[\mathcal{J}_{DP}] = A_{op}[\frac{1}{2} \sum_{a,b=0}^{3} i_{a}^{(y)} L_{b}^{(y)} C_{c}(\gamma_{(y)}^{ab} d\tau)] = \frac{1}{2} \sum_{a,b=0}^{3} i_{a}^{(z)} L_{b}^{(z)} A_{op}[C_{c}(\gamma_{(y)}^{ab} d\tau)].$$  \hfill (7.23)

Using (7.17) and (7.9) we obtain the result. \hfill \Box

Expanding (7.22) in terms of the vectors $V$ and $Z$

$$A_{DP} = -\frac{1}{2} \sum_{a,b=0}^{3} *^{-1} L_{b}^{(x)} i_{a}^{(x)} \left( \frac{\gamma_{(x)}^{ab}(\tau_{R})}{g(V, Z)} * 1 \right)$$

$$= \frac{1}{2} \sum_{a,b=0}^{3} \left( \frac{1}{g(V, Z)} L_{b}^{(x)} \gamma_{(x)}^{ab}(\tau_{R}) - \frac{1}{(g(V, Z))^{2}} \gamma_{(x)}^{ab} L_{b}^{(x)}(g(V, Z)) \right) *^{-1} (i_{a}^{(z)} * 1)$$

$$= \frac{1}{2} \sum_{a,b=0}^{3} \left( \frac{\partial \gamma_{(x)}^{ab}}{g(V, Z)} \frac{\partial \tau_{R}}{\partial x_{a}} - \frac{\tilde{V}_{b}}{(g(V, Z))^{2}} \gamma_{(x)}^{ab}(\tau_{R}) - \frac{1}{(g(V, Z))^{3}} \tilde{Z}_{b} \gamma_{(x)}^{ab}(\tau_{R}) \right) dx^{a}$$

It is clear that the electric and magnetic fields both have the same form and fall off as inverse cube with respect to distance.

7.3.3 Quadrupoles

Calculating the Liénard-Wiechert field for a quadrupole is very similar to the dipole case so we just state the result

$$A_{QP} = A_{op}[\mathcal{J}_{QP}] = \sum_{a,b,c=0}^{3} *^{-1} i_{a}^{(z)} L_{b}^{(x)} L_{c}^{(z)} \left( \frac{\gamma_{(z)}^{abc}(\tau_{R})}{g(V, Z)} * 1 \right).$$  \hfill (7.24)

The same procedure carried out for event dipoles will give a new SMD on the forward lightcone.
Chapter 8

Conclusion and Discussion

This thesis has introduced a new way to determine the product of transverse SMDs in a coordinate free manner. In Chapter 2 we constructed the set of SMDs over a closed embedding using de Rham’s pushforward map. We then defined a new type of product on the intersection of transverse submanifolds. The wedge product was shown to be well defined and can be expressed succinctly in terms of four axioms. At a technical level the definition of the wedge product contains some redundancy. One may ask could one provide a more succinct definition of the wedge product, for example for associativity it is possible to replace one of the distributions with a form. However in its present definition it provided the basis for a new perspective on solutions to linear differential equations via Green’s method. It was shown that the Green’s distribution could be expressed as an SMD on the product manifold \((M_X \times M_Y)\setminus \Delta\). By restricting the codomains of the maps it was possible to make the various subspaces embeddings and hence we could use the wedge product of Chapter 3.

In Chapter 6 we investigated the first three multipoles in terms of SMDs. It was shown that quadrupoles have more complicated transformation rules than suggested by the literature. They involve second order derivatives and an integral. These coordinate transformations are vital since it may be useful to prescribe quadrupoles in terms of adapted coordinates whilst the corresponding fields are derived in terms of...
Cartesian coordinates. This goes to show how unusual and cumbersome it is transforming between coordinates and why a coordinate free approach is advantageous. In Chapter 7 the Liénard-Wiechert field for a moving monopole, dipole and quadrupole was calculated using the wedge product.

There is a lot of natural extensions to this work. One may ask what the coordinate transformations are associated with multipoles over worldsheets. We could extend the ideas to different linear operators with different Green’s functions. For example one could find the fields for moving multipoles in Bopp-Podolsky electromagnetism [32] and also in frequency dependent media.
Bibliography


Bibliography


Appendix A

A.1 Regular Pullback

The definition of the pushforward (Definition 2.4.1) requires the regular pullback of differential forms. This short section gives some important properties of the pullback. We call it regular to distinguish it from the distributional pullback of Chapter 4.

**Definition A.1.1.** Let \( f : \mathcal{N} \to \mathcal{M} \) be a smooth map between manifolds. The regular pullback with respect to \( f \) is the linear map defined as

\[
f^* : \Gamma \Lambda^p \mathcal{M} \to \Gamma \Lambda^p \mathcal{N},
\]

\[
\alpha \mapsto f^* \alpha,
\]

such that for \( \alpha, \beta \in \Gamma \Lambda^p \mathcal{M}, \)

\[
f^*(\alpha + \beta) = f^*(\alpha) + f^*(\beta),
\]

\[
f^*(\alpha \wedge \beta) = f^*(\alpha) \wedge f^*(\beta).
\]

The regular pullback satisfies the following.

(i) The pullback commutes with the exterior derivative

\[
d \circ f^* = f^* \circ d.
\]

(ii) If \( v \in \Gamma T \mathcal{M} \) is transverse to \( u \in \Gamma T \mathcal{N} \) with respect to \( f \) i.e. \( f_* u = v \), then

\[
f^* \circ i_v = i_u \circ f^*.
\]
A.2 Properties of Regular Distributions

(iii) If \( g : \mathcal{M} \to \mathcal{P} \) is a smooth map between manifolds then

\[
(g \circ f)^* = f^* \circ g^*.
\]  

(A.6)

A.2 Properties of Regular Distributions

In this section we give some important properties of regular distributional forms. A smooth form \( \alpha \) leads to a distribution \( \alpha^D \) as follows:

\[
\left[ \phi \, | \, \alpha^D \right] = \int_{\mathcal{M}} \phi \wedge \alpha.
\]  

(A.7)

Lemma A.2.1. For a manifold \( \mathcal{M} \) without boundary, \( \partial \mathcal{M} = \emptyset \), the exterior derivative of a regular \( p \)-form distribution satisfies

\[
(d \alpha)^D = d(\alpha)^D.
\]  

(A.8)

Proof. Using Stokes’ theorem

\[
\left[ \phi \, | \, (d \alpha)^D \right]_{\mathcal{M}} = \int_{\mathcal{M}} \phi \wedge d\alpha
\]

\[
= (-1)^{\deg(\phi)} \int_{\mathcal{M}} d(\phi \wedge \alpha) - (-1)^{\deg(\phi)} \int_{\mathcal{M}} d\phi \wedge \alpha
\]

\[
= (-1)^{\deg(\phi)} \int_{\partial \mathcal{M}} \phi \wedge \alpha - (-1)^{\deg(\phi)} \int_{\mathcal{M}} d\phi \wedge \alpha
\]

\[
= -(-1)^{\deg(\phi)} \left[ d\phi \, | \, \alpha^D \right]_{\mathcal{M}}
\]

\[
= \left[ \phi \, | \, d(\alpha^D) \right]_{\mathcal{M}}.
\]

\[ \square \]

Lemma A.2.2. For a regular distribution

\[
(i_v \alpha)^D = i_v(\alpha^D), \quad v \in \Gamma T \mathcal{M}.
\]  

(A.9)
A.2. Properties of Regular Distributions

Proof.

\[
\left[ \phi \big| (i_\nu \alpha)^D \right]_\mathcal{M} = \int_\mathcal{M} \phi \wedge i_\nu \alpha \\
= (-1)^{\deg(\phi)} \int_\mathcal{M} i_\nu (\phi \wedge \alpha) - (-1)^{\deg(\phi)} \int_\mathcal{M} i_\nu \phi \wedge \alpha \\
= -(-1)^{\deg(\phi)} [i_\nu \phi | \alpha^D]_\mathcal{M} \\
= \left[ \phi \big| i_\nu (\alpha^D) \right]_\mathcal{M}.
\]

On the second line the first term vanishes since \( \deg(\phi \wedge \alpha) = \dim(\mathcal{M}) + 1 \).

The Lie derivative follows naturally from the previous two results,

\[
\left[ \phi \big| L_\nu (\alpha^D) \right]_\mathcal{M} = \left[ \phi \big| d i_\nu (\alpha^D) + i_\nu d(\alpha^D) \right]_\mathcal{M} = \left[ \phi \big| (d i_\nu \alpha)^D + i_\nu (d \alpha)^D \right]_\mathcal{M} \\
= \left[ \phi \big| (d i_\nu \alpha)^D + (i_\nu (d \alpha)^D \right]_\mathcal{M} = \left[ \phi \big| (L_\nu \alpha)^D \right]_\mathcal{M}.
\]

Lemma A.2.3. If \( \mathcal{M} \) has a metric then the Hodge dual satisfies

\[
*(\alpha^D) = (\ast \alpha)^D. \tag{A.10}
\]

Proof. Let \( \phi \in \Gamma_0 \Lambda^p \mathcal{M} \),

\[
\left[ \phi \big| * (\alpha^D) \right]_\mathcal{M} = (-1)^{p(m-p)} \left[ * \phi \big| \alpha^D \right]_\mathcal{M} \\
= (-1)^{p(m-p)} \int_\mathcal{M} * \phi \wedge \alpha \\
= (-1)^{p(m-p)} \int_\mathcal{M} \alpha \wedge * \phi \\
= \int_\mathcal{M} \phi \wedge * \alpha \\
= \left[ \phi \big| (\ast \alpha)^D \right]_\mathcal{M}.
\]

On the third line we have used the star pivot property of the Hodge dual, that is if \( \omega, \beta \in \Gamma \Lambda^p \mathcal{M} \) then \( \omega \wedge * \beta = \beta \wedge * \omega \). \( \square \)

Lemma A.2.4. Suppose \( \mathcal{M} \) is compact and let \( i : \partial \mathcal{M} \to \mathcal{M} \) be the inclusion map.

Lemma A.2.1 now becomes

\[
(d \alpha)^D - d(\alpha^D) = (-1)^{\dim(\mathcal{M}) - \deg(\alpha)} i_\nu (i^* \alpha^D). \tag{A.11}
\]
A.2. Properties of Regular Distributions

Proof. We have

\[ \left[ \phi \right| (\alpha) - d(\alpha) ]_\mathcal{M} = \int_{\mathcal{M}} d(\phi \wedge \alpha). \] (A.12)

Using Stoke’s theorem and noting that \( i^* (\phi \wedge \alpha) \) is the associated top form on \( \partial \mathcal{M} \)

\[ \left[ \phi \right| (\alpha) - d(\alpha) ]_\mathcal{M} = (-1)^{\deg(\phi)} \int_{\partial \mathcal{M}} i^* (\phi \wedge \alpha) \]
\[ = (-1)^{\deg(\phi)} \int_{\partial \mathcal{M}} i^* \phi \wedge i^* \alpha \]
\[ = (-1)^{\deg(\phi)} [i^* \phi \wedge i^* \alpha]_{\partial \mathcal{M}} \]
\[ = (-1)^{\deg(\phi)} [\phi \wedge i^* \alpha]_{\partial \mathcal{M}}. \]

However \( \deg(\phi) = \dim(\partial \mathcal{M}) - \deg(\alpha) \) and we obtain the desired result

\[ (\alpha) - d(\alpha) = (-1)^{\dim(\mathcal{M}) - \deg(\alpha)} i^*_\wedge (i^* \alpha). \] (A.13)

\[ \square \]

Lemma A.2.5. The support of a regular distribution \( \alpha \) coincides with the support of \( \alpha \) as a form, that is

\[ \text{supp}(\alpha) = \text{supp}(\alpha). \] (A.14)

Proof. If \( \text{supp}(\alpha) \cap \text{supp}(\phi) = \emptyset \) then

\[ \left[ \phi \right| \alpha ]_\mathcal{M} = \int_{\mathcal{M}} \phi \wedge \alpha = 0, \]

therefore \( \text{supp}(\alpha) \subset \text{supp}(\alpha) \). Now suppose \( \text{supp}(\alpha) = \emptyset \), this suggests there is an open set \( U \subset \text{supp}(\alpha) \setminus \text{supp}(\alpha) \). It is possible to construct a test form \( \psi \) such that \( \text{supp}(\psi) \subset U \). Now we have

\[ \int \psi \wedge \alpha \neq 0, \]

so \( \text{supp}(\alpha) \cap U \neq \emptyset \) and hence \( \text{supp}(\alpha) \cap (\text{supp}(\alpha) \setminus \text{supp}(\alpha)) \neq \emptyset \), which is a contradiction. Therefore \( \text{supp}(\alpha) = \text{supp}(\alpha) \). \( \square \)
A.3. SMD Results

This part of the Appendix contains proofs for Section 2.5.

Lemma A.3.1. Let \( \phi \in \Gamma_0 \Lambda^{m-p} \mathcal{M} \), \( J \in \text{Snc}(r) \) and \( \Phi \in \Upsilon^{k,p+|J|(f)} \) then

\[
\left[ i_J^{(z)} \phi \big| \Phi \right]_{\mathcal{M}} = (-1)^{s(m-p)+s} \left[ i_J^{(z)} \phi \big| i_J^{(z)} \Phi \right]_{\mathcal{M}},
\]

where \( |J| = s \).

\[
\text{Proof.}
\]

\[
\left[ i_J^{(z)} \phi \big| \Phi \right]_{\mathcal{M}} = \left[ i_{J_s}^{(z)} \cdots i_{J_1}^{(z)} \phi \big| \Phi \right]_{\mathcal{M}} = (-1)^{m-p-s} \left[ i_{J_s}^{(z)} \cdots i_{J_1}^{(z)} \phi \big| i_{J_s}^{(z)} \Phi \right]_{\mathcal{M}}
\]

\[
= (-1)^{2(m-p-s)} \left[ i_{J_s}^{(z)} \cdots i_{J_1}^{(z)} \phi \big| i_{J_s}^{(z)} \Phi \right]_{\mathcal{M}} = (-1)^{s(m-p-s)} \left[ \phi \big| i_J^{(z)} \Phi \right]_{\mathcal{M}}
\]

\[
= (-1)^{s(m-p)-s} \left[ \phi \big| i_J^{(z)} \Phi \right]_{\mathcal{M}} = (-1)^{s(m-p)-s} \left[ \phi \big| i_J^{(z)} \Phi \right]_{\mathcal{M}}.
\]

\[
\square
\]

Proof of Theorem 2.5.5. Clearly (2.62) is a finite number of applications of (2.42)-(2.46). Furthermore there are no more than \( k \) derivatives since \( |I| \leq k \). Hence for \( \Psi \) given by (2.62) then \( \Psi \in \Upsilon^{k,p}(f) \).

Let \( \hat{\Upsilon}^{k,p}(f) \) be the set of SMDs which can be written as (2.62). It is clearly true that \( f_{\alpha}(\alpha) \in \hat{\Upsilon}^{0,p}(f) \). Given \( \Psi \in \hat{\Upsilon}^{k,p}(f) \) it is necessary to show that the application of (2.43)-(2.46) on \( \Psi \) still remains in \( \hat{\Upsilon}^{j,q}(f) \) for the appropriate \( j \) and \( q \). Addition (2.43) follows since we take sums. In the following we can neglect the overall sign of (2.62) as it a choice of convention in order to make (2.64) more usable during the wedge product proofs.

Given \( \beta \in \Gamma \Lambda^q \mathcal{M} \) then we can set \( \beta = \sum_{J^+ \Lambda^m : \text{dim}|J'| = q} \beta_{J',K'} dy^{K'} \wedge dz^{J'} \). Thus using
A.3. SMD Results

lemma (B.1.1) in Appendix B we have

\[
(-1)^{(\deg \phi)(\deg \beta)} \left[ \phi \mid \beta \wedge \Psi \right] = \left[ \beta \wedge \phi \mid \Psi \right] = \sum_{\text{Rng}(I,J,K)} \sum_{J',K'} \sum_{I,J,K} \int_{\mathcal{N}} \Psi^{I,J}_K \, dy^K \wedge f^\star(\iota^{(z)}_I L^{(z)}_I (\beta \wedge \phi))
\]

\[
= \sum_{\text{Rng}(I,J,K)} \sum_{J',K'} (-1)^{|J||K'|} \sum_{I,J,K} \int_{\mathcal{N}} \Psi^{I,J}_K \, dy^K \wedge dy^{K'} \wedge f^\star(\iota^{(z)}_I L^{(z)}_I (\beta \wedge \phi))
\]

\[
= \sum_{\text{Rng}(I,J,K)} \sum_{J',K'} (-1)^{|J||K'|} \sum_{I,J,K} \int_{\mathcal{N}} \Psi^{I,J}_K \, dy^{K+K'} \wedge f^\star(\iota^{(z)}_I L^{(z)}_I (\beta \wedge \phi))
\]

\[
= \sum_{\text{Rng}(I,J,K)} \sum_{J',K'} \sum_{I,J,K} \int_{\mathcal{N}} \Psi^{I,J}_K \, dy^{K+K'} \wedge f^\star(\iota^{(z)}_I L^{(z)}_I (\beta \wedge \phi))
\]

Thus \( \beta \wedge \Psi \in \hat{T}^{k,p,q}(f) \).

Given \( v \in \Gamma TM_C \) then \( v = \sum_a (v^{(y)}_a \partial^{(y)}_a + v^{(z)}_a \partial^{(z)}_a) \). On the basis vectors we have

\[
\left[ \phi \mid \iota^{(y)}_a \right] = (-1)^{\deg \phi - 1} \left[ \iota^{(y)}_a \phi \mid \Psi \right] = (-1)^{\deg \phi - 1} \sum_{\text{Rng}(I,J,K)} \int_{\mathcal{N}} \Psi^{I,J}_K \, dy^K \wedge f^\star(\iota^{(z)}_I L^{(z)}_I \iota^{(y)}_a \phi)
\]

\[
= (-1)^n \sum_{\text{Rng}(I,J,K)} \int_{\mathcal{N}} \Psi^{I,J}_K \, \iota^{(y)}_a (dy^K) \wedge f^\star(\iota^{(z)}_I L^{(z)}_I \iota^{(y)}_a \phi)
\]

\[
= (-1)^n \sum_{\text{Rng}(I,J,K)} \epsilon^{K\wedge a}_{(K\wedge a)} \int_{\mathcal{N}} \Psi^{I,J}_K \, dy^{K\wedge a} \wedge f^\star(\iota^{(z)}_I L^{(z)}_I \phi)
\]
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since \( \deg \phi = m - p \) and \(|K| - |J| = p - r\). Also

\[
\left[ \phi \mid i_a^{(z)} \Psi \right] = (-1)^{\deg \phi - 1} \left[ i_a^{(z)} \phi \mid \Psi \right] = (-1)^{\deg \phi - 1} \sum_{\operatorname{Rng}(I,J,K)} \Psi_{K}^{I,J} dy^{K} \wedge f^{*}(i_j^{(z)}L_i^{(z)}i_a^{(z)} \phi)
\]

\[
= (-1)^{\deg \phi - 1} \sum_{\operatorname{Rng}(I,J,K)} \Psi_{K}^{I,J} dy^{K} \wedge f^{*}(i_j^{(z)}L_i^{(z)} \phi)
\]

Thus \( i_a^{(y)} \Psi \in \hat{\Upsilon}^{k,p-1}(f) \) and \( i_a^{(z)} \Psi \in \hat{\Upsilon}^{k,p-1}(f) \). Hence \( i_a = \sum_a (v_a^{(y)}i_a^{(y)} + v_a^{(z)}i_a^{(z)}) \). Since \( v_a^{(y)} \Psi \in \hat{\Upsilon}^{k,p}(f) \) then \( i_a \Psi \in \hat{\Upsilon}^{k,p-1}(f) \).

For the Lie derivatives (2.46) on the basis vectors we have

\[
\left[ \phi \mid L_a^{(y)} \Psi \right] = -\left[ L_a^{(y)} \phi \mid \Psi \right] = -\sum_{\operatorname{Rng}(I,J,K)} \Psi_{K}^{I,J} dy^{K} \wedge f^{*}(i_j^{(z)}L_i^{(z)}L_a^{(y)} \phi)
\]

\[
= \sum_{\operatorname{Rng}(I,J,K)} \int_{N} (\partial_a^{(y)} \Psi_{K}^{I,J}) dy^{K} \wedge f^{*}(i_j^{(z)}L_i^{(z)} \phi)
\]

and

\[
\left[ \phi \mid L_a^{(z)} \Psi \right] = -\left[ L_a^{(z)} \phi \mid \Psi \right] = -\sum_{\operatorname{Rng}(I,J,K)} \Psi_{K}^{I,J} dy^{K} \wedge f^{*}(i_j^{(z)}L_i^{(z)}L_a^{(z)} \phi)
\]

\[
= -\sum_{\operatorname{Rng}(I,J,K)} \int_{N} \Psi_{K}^{I,J} dy^{K} \wedge f^{*}(i_j^{(z)}L_i^{(z)} \phi).
\]

Now since \( d\Psi = \sum_a dy^a \wedge L_a^{(y)} \Psi + \sum_a dz^a \wedge L_a^{(z)} \Psi \) then \( d\Psi \in \hat{\Upsilon}^{k+1,p+1}(f) \).

As stated applying Cartan’s identity for the Lie derivative implies \( L_v \Psi \in \hat{\Upsilon}^{k+1,p}(f) \).

Thus we have shown every (2.43)-(2.46) on a \( \Psi \in \hat{\Upsilon}^{k,p}(f) \) produces a new SMD in \( \hat{\Upsilon}^{k',p'}(f) \). Hence \( \Upsilon^{k,p}(f) = \hat{\Upsilon}^{k,p}(f) \). \( \square \)
Appendix B

B.1 Wedge Product Results

The following lemma is required for the proof of Theorem 3.5.3.

Lemma B.1.1. Given two forms $\alpha, \beta \in \Gamma \Lambda M$ and indices $I, I' \in \text{Rnc}(r)$ and $J, J' \in \text{Snc}(t)$,

(i) $L_I \alpha \wedge \beta = \sum_{R^r \succeq |R| \succeq |I|} (-1)^{|R|} L_{R \setminus R}(\alpha \wedge L_R \beta)$, \hspace{1cm} (B.1)

(ii) $\alpha \wedge L_{I'} \beta = \sum_{R^r \succeq |R| \succeq |I'|} (-1)^{|R'|} L_{I' \setminus R}(\alpha \wedge L_R \beta)$, \hspace{1cm} (B.2)

(iii) $i_J \alpha \wedge \beta = \sum_{S^t \succeq |S| \succeq |J|} \epsilon_{J \setminus S}^{(J \setminus S) \wedge S} (-1)^{|S|(\deg(\alpha) - |J|)} i_{J \setminus S}(\alpha \wedge i_{S'} \beta)$, \hspace{1cm} (B.3)

(iv) $\alpha \wedge i_{J'} \beta = \sum_{S^t \succeq |S'| \succeq |J'|} \epsilon_{J' \setminus S'}^{(J' \setminus S') \wedge S'} (-1)^{|J'|((\deg(\alpha) - |S'|)} i_{J' \setminus S'}(i_{S'} \alpha \wedge \beta)$, \hspace{1cm} (B.4)

Proof. Via induction. The approach for showing (B.2) and (B.3) are equivalent so we only show the latter, that being slightly more involved. When $|J| = 1$
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\[ \sum_{S \in [J]} \epsilon_{J \setminus S} (\omega) (S) S \left( \begin{array}{c} J \setminus S \setminus 1 \end{array} \right) i_{J \setminus S} \left( \alpha \wedge i_{S} \beta \right) \]

\[ = \sum_{S \in [J]} \epsilon_{J \setminus S} (\omega) (S) S \left( \begin{array}{c} J \setminus S \setminus 1 \end{array} \right) i_{J \setminus S} \left( \alpha \wedge i_{S} \beta \right) \]

\[ + \sum_{S \in [J]} \epsilon_{J \setminus S} (\omega) (S) S \left( \begin{array}{c} J \setminus S \setminus 1 \end{array} \right) i_{J \setminus S} \left( \alpha \wedge i_{S} \beta \right) \]

\[ = \epsilon_{J} \left( \begin{array}{c} J \setminus 1 \end{array} \right) \alpha \wedge i_{1} \beta + \epsilon_{J} \left( \begin{array}{c} J \setminus 1 \end{array} \right) \beta \alpha \]

\[ = i_{J} \left( \begin{array}{c} J \setminus 1 \end{array} \right) \alpha \wedge \beta \]

Assume true for \(|J| = n\). Concatenating \( J \) with an extra element we have

\[ i_{J \setminus a} \alpha \wedge \beta = i_{a} i_{J} \alpha \wedge \beta = i_{a} \left( i_{J} \alpha \wedge \beta \right) + \left( -1 \right) ^{\deg (\alpha) - |J| + 1} i_{J} \alpha \wedge i_{a} \beta \]

\[ = \sum_{S \in [J \setminus a]} \epsilon_{J \setminus a} (\omega) (S) S \left( \begin{array}{c} J \setminus a \setminus 1 \end{array} \right) i_{J \setminus a} \left( \alpha \wedge i_{S} \beta \right) \]

\[ + \left( -1 \right) ^{\deg (\alpha) - |J| - 1} \sum_{S \in [J \setminus a]} \epsilon_{J \setminus a} (\omega) (S) S \left( \begin{array}{c} J \setminus a \setminus 1 \end{array} \right) i_{J \setminus a} \left( \alpha \wedge i_{S} \beta \right) \]

\[ = \sum_{S \in [J \setminus a]} \epsilon_{J \setminus a} (\omega) (S) S \left( \begin{array}{c} J \setminus a \setminus 1 \end{array} \right) i_{J \setminus a} \left( \alpha \wedge i_{S} \beta \right) \]

\[ + \sum_{S \in [J \setminus a]} \epsilon_{J \setminus a} (\omega) (S) S \left( \begin{array}{c} J \setminus a \setminus 1 \end{array} \right) i_{J \setminus a} \left( \alpha \wedge i_{S} \beta \right) \]

\[ = \sum_{S \in [J \setminus a]} \epsilon_{J \setminus a} (\omega) (S) S \left( \begin{array}{c} J \setminus a \setminus 1 \end{array} \right) i_{J \setminus a} \left( \alpha \wedge i_{S} \beta \right) \]

Thus (B.3) must hold true for all values of \( n \). (B.4) now follows via graded
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commutativity

\[ \alpha \wedge i_{J'} \beta = ( -1 )^{\deg(\alpha) \deg(i_{J'} \beta)} i_{J'} \beta \wedge \alpha \]

\[ = ( -1 )^{\deg(\alpha) (\deg(\beta) - |J'|)} \sum_{S^{|J'|} \subseteq S' \subseteq |J'|} \epsilon_{J' \setminus S}^{(J' \setminus S') \cdot S'} ( -1 )^{S' (\deg(\beta) - |J'|)} i_{J' \setminus S} ( \beta \wedge i_{S'} \alpha ) \]

\[ = \sum_{S^{|J'|} \subseteq S' \subseteq |J'|} \epsilon_{J' \setminus S}^{(J' \setminus S') \cdot S'} ( -1 )^{\deg(\beta) (\deg(\alpha) + |S'|) - |J'| (\deg(\alpha) + |S'|)} i_{J' \setminus S} ( \beta \wedge i_{S'} \alpha ) \]

\[ = \sum_{S^{|J'|} \subseteq S' \subseteq |J'|} \epsilon_{J' \setminus S}^{S' \cdot (J' \setminus S')} ( -1 )^{J' (\deg(\alpha) - |S'|)} i_{J' \setminus S} ( i_{S'} \alpha \wedge \beta ) . \]

\[ \Box \]

The next lemma is required for the proof of Lemma 6.2.2.

**Lemma B.1.2.** Given the embedding \( f : N \hookrightarrow M \) with adapted coordinate system \((y^1, \ldots, y^r, z^1, \ldots, z^n)\) such that \( f(y^1, \ldots, y^r) = (y^1, \ldots, y^r, 0, \ldots, 0)\) then the Lie derivative with respect to the transverse vector field \( w \in \Gamma(f, TM) \) satisfies the following

\[ \mathcal{L}_w f_\epsilon(h) = \mathcal{L}_a^z f_\epsilon(f^* (w^a) \alpha) - i_\alpha^z f_\epsilon(f^* (dw^a) \wedge \alpha), \quad \alpha \in \Lambda^p N. \quad (B.5) \]

**Proof.** The transverse vector field \( w \) in adapted coordinates is written as

\[ w = w^a(y) \frac{\partial}{\partial z^a} \bigg|_{(y,0)}. \]
B.1. Wedge Product Results

Using the Leibniz property of the Lie derivative and (2.40) we have

\[ L_w^a \delta^a_{\beta} f_\zeta(\alpha) = w^a L_z^a f_\zeta(\alpha) + dw^a \wedge i^a_{z} f_\zeta(\alpha) \]

\[ = w^a L_z^a f_\zeta(\alpha) + i^a_{z} dw^a \wedge f_\zeta(\alpha) - i^a_{z} f_\zeta(f^*(dw^a \wedge \alpha)) \]

\[ = L_a^z (w^a f_\zeta(\alpha)) - i^a_{z} f_\zeta(f^*(dw^a \wedge \alpha)) \]

\[ = L_a^z (f^*(w^a)\alpha)) - i^a_{z} f_\zeta(f^*(dw^a \wedge \alpha)). \]

\[ \square \]

The coordinate transformation of \( \gamma^{abc}_{(z)} \) in Lemma 6.3.2 uses the following.

**Corollary B.1.3.** If \( \alpha \) is a top form and \( u, v, w \in \Gamma(f, TM) \) are transverse fields then

\[ L_v L_w f_\zeta(\alpha) = L_a^z L_b^z f_\zeta(f^*(v^a w^b)\alpha) - L_a^z f_\zeta(f^*(w^b \frac{\partial v^a}{\partial z^b})\alpha), \]  

\[ (B.6) \]

\[ i_u L_v L_w f_\zeta(\alpha) = i_u L_v L_w f_\zeta(f^*(v^a w^b w^c)\alpha) - i_u L_v f_\zeta(f^*(w^c \frac{\partial}{\partial z^c} (v^a w^b))\alpha) \]

\[ - i_u L_w f_\zeta(f^*(v^b \frac{\partial}{\partial z^b} (w^a w^c))\alpha) + i_u f_\zeta(f^*(w^c \frac{\partial}{\partial z^c} (v^b \frac{\partial w^a}{\partial z^b}))\alpha). \]  

\[ (B.7) \]
Appendix C

C.1 Pullback of SMDs Proofs

This Appendix contains the proofs of the pullback in Chapter 4.

Lemma C.1.1. The pullback, defined locally by (4.9)-(4.10) implies (4.4)-(4.8).

Proof. The pullback is clearly linear across ‘+’, i.e. (4.5).

Setting

\[ f_\varsigma(\alpha) = \Psi = \sum_{K \uparrow t, H \uparrow n-t} f_\varsigma(\Psi_{K,H}^{\otimes} dy^K \wedge dx^H), \]

then (4.10) implies

\[ a^\varsigma(f_\varsigma(\alpha)) = a^\varsigma(\Psi) = \sum_{K \uparrow t, |K| - |J|=p-r} \hat{a}_\varsigma(\Psi_{K,J}^{\otimes} dy^K) = \hat{a}_\varsigma(\hat{f}^\varsigma(\alpha)), \]

hence (4.4).
C.1. Pullback of SMDs Proofs

Consider one term in (4.9) then

\[
\begin{align*}
&d\left(\alpha^{c}\left(i^{(z)}_{j} L^{(z)} f_{c}\left(\Psi^{I,J}_{K,H} dy^{K} \wedge dx^{H}\right)\right)\right) = d\left(i^{(z)}_{j} L^{(z)} \hat{a}_{c}\left(\Psi^{I,J}_{K,H} dy^{K}\right)\right) \\
&= \sum_{\mu=1}^{|J|} (-1)^{\mu-1} i^{(z)}_{J,J_{\mu}} L^{(z)} f_{c}\left(\Psi^{I,J}_{K,H} dy^{K} \wedge dx^{H}\right) \\
&= \sum_{\mu=1}^{|J|} (-1)^{\mu-1} i^{(z)}_{J,J_{\mu}} a^{c}\left(L^{(z)} f_{c}\left(\Psi^{I,J}_{K,H} dy^{K} \wedge dx^{H}\right)\right) \\
&\quad + (-1)^{\mu} a^{c}\left(i^{(z)}_{j} L^{(z)} f_{c}\left(\Psi^{I,J}_{K,H} dy^{K} \wedge dx^{H}\right)\right) \\
&= a^{c}\left(\sum_{\mu=1}^{|J|} (-1)^{\mu-1} i^{(z)}_{J,J_{\mu}} L^{(z)} f_{c}\left(\Psi^{I,J}_{K,H} dy^{K} \wedge dx^{H}\right) \\
&\quad + (-1)^{\mu} i^{(z)}_{j} L^{(z)} f_{c}\left(\Psi^{I,J}_{K,H} dy^{K} \wedge dx^{H}\right)\right) \\
&= a^{c}\left(d\left(i^{(z)}_{j} L^{(z)} f_{c}\left(\Psi^{I,J}_{K,H} dy^{K} \wedge dx^{H}\right)\right)\right),
\end{align*}
\]

hence summing as in (4.9) then \(d\alpha^{c}(\Psi) = a^{c}(d\Psi)\), i.e. (4.6).

To show (4.7) we take \(v \in \Gamma T A\) and \(u \in \Gamma T M\) with

\[
v = \sum_{a} \left( v^{a}_{(w)} \partial^{(w)}_{a} + v^{a}_{(y)} \partial^{(y)}_{a} + v^{a}_{(z)} \partial^{(z)}_{a} \right) \quad \text{and} \quad u = \sum_{a} \left( u^{a}_{(x)} \partial^{(x)}_{a} + u^{a}_{(y)} \partial^{(y)}_{a} + u^{a}_{(z)} \partial^{(z)}_{a} \right),
\]

where \(v^{a}_{(w)} = v^{a}_{(w)}(w, y, z)\) and \(u^{a}_{(x)} = u^{a}_{(x)}(y, x, z)\). Since \(a_{*}(v_{|(w,y,z)}) = u_{|(y,0,0)}\) we have

\[
u^{a}_{(x)}|_{(y,0,0)} = 0, \quad u^{a}_{(y)}|_{(y,0,0)} = v^{a}_{(y)}|_{(w,y,z)} \quad \text{and} \quad u^{a}_{(z)}|_{(y,0,0)} = v^{a}_{(z)}|_{(w,y,z)}.
\]

Consider first \(\Psi = L^{(z)} f_{c}\beta\) and each term of \(u\) in turn. For the \(x\) components,

\[
\begin{align*}
\alpha^{c}\left(i^{(x)}_{f_{c}} L^{(z)} f_{c}\beta\right) &= \alpha^{c}\left(u^{a}_{(x)} i^{(x)}_{f_{c}} L^{(z)} f_{c}\beta\right) = \alpha^{c}\left(L^{(z)} f_{c}\left(u^{a}_{(x)} i^{(x)}_{f_{c}} f_{c}\beta\right)\right) - \alpha^{c}\left(\left(\partial^{(z)} f_{c}\beta\right) i^{(x)}_{f_{c}}\right) \\
&= \alpha^{c}\left(L^{(z)} f_{c}\left(u^{a}_{(x)}|_{x=0} i^{(x)}_{f_{c}}(\beta)\right)\right) - \alpha^{c}\left(f_{c}\left(\left(\partial^{(z)} u^{a}\right)|_{x=0} i^{(x)}_{f_{c}}(\beta)\right)\right) = 0.
\end{align*}
\]
C.1. Pullback of SMDs Proofs

For the $y$ components,

$$a^c(i_{u^{(y)}_a})L^{(z)}_b f_c\beta = a^c(u^{(y)}_a i^{(z)}_a L^{(z)}_b f_c\beta)$$

$$= a^c(L^{(z)}_b(u^{(y)}_a i^{(z)}_a f_c\beta)) - a^c((\partial^{(z)}_b u^{(y)}_a) i^{(z)}_a f_c\beta)$$

$$= a^c(L^{(z)}_b f_c(u^{(y)}_a i^{(z)}_a)) - a^c(f_c((\partial^{(z)}_b u^{(y)}_a) i^{(z)}_a \beta))$$

$$= a^c(L^{(z)}_b f_c(v^{(y)}_a i^{(z)}_a \beta)) - a^c((\partial^{(z)}_b v^{(y)}_a) i^{(z)}_a \beta)$$

$$= v^{(y)}_a i^{(z)}_a(L^{(z)}_b \hat{a}_c(\beta)) = v^{(y)}_a i^{(z)}_a (a^c(L^{(z)}_b f_c\beta)).$$

And for the $z$ components,

$$a^c(i_{u^{(z)}_a})L^{(z)}_b f_c\beta = a^c(u^{(z)}_a i^{(z)}_a L^{(z)}_b f_c\beta)$$

$$= a^c(L^{(z)}_b(u^{(z)}_a i^{(z)}_a f_c\beta)) - a^c((\partial^{(z)}_b u^{(z)}_a) i^{(z)}_a f_c\beta)$$

$$= a^c(i^{(z)}_a L^{(z)}_b f_c(u^{(z)}_a)) - a^c(i^{(z)}_a f_c((\partial^{(z)}_b u^{(z)}_a) \beta))$$

$$= a^c(i^{(z)}_a L^{(z)}_b f_c(v^{(z)}_a)) - a^c((\partial^{(z)}_b v^{(z)}_a) i^{(z)}_a \beta)$$

$$= i^{(z)}_a L^{(z)}_b \hat{a}_c(v^{(z)}_a \beta) - i^{(z)}_a \hat{a}_c((\partial^{(z)}_b v^{(z)}_a) \beta)$$

$$= i^{(z)}_a L^{(z)}_b(v^{(z)}_a \hat{a}_c(\beta)) - (\partial^{(z)}_b v^{(z)}_a) i^{(z)}_a \hat{a}_c(\beta)$$

$$= v^{(z)}_a i^{(z)}_a(L^{(z)}_b \hat{a}_c(\beta)) = v^{(z)}_a i^{(z)}_a (a^c(L^{(z)}_b f_c\beta)).$$

Hence $a^c(i_u L^{(z)}_b f_c\beta) = i_v(a^c L^{(z)}_b f_c\beta)$.

Now consider $\Psi = L^{(z)}_i f_c\beta$. Passing the components $u^{(y)}_a$ through the Lie derivatives will produce higher and higher derivatives of $u^{(y)}_a$ with respect to $z$. These will be replaced by the derivatives of $v^{(y)}_a$ with respect to $z$ and then recombine. Hence $a^c(i_u L^{(z)}_i f_c\beta) = i_v(a^c L^{(z)}_i f_c\beta)$ and since $i_u$ simply graded-commutes with $i^{(z)}_j$ then

$$a^c(i_u i^{(z)}_j L^{(z)}_b f_c\beta) = i_v(a^c i^{(z)}_j L^{(z)}_b f_c\beta),$$

hence (4.7).
C.1. Pullback of SMDs Proofs

**Theorem C.1.2.** The pullback is well defined.

*Proof.* Assume that the pullback was defined with respect to one particular adapted coordinates system \((w, x, y, z)\). Writing \(\Psi\) as \((4.9)\) gives \(\alpha^*\Psi\) uniquely as \((4.10)\). Therefore we must show that \(\alpha^*\) is independent of the coordinate system.

Lemma C.1.1 implies (4.4)-(4.8). Given another coordinate system \((\hat{w}, \hat{x}, \hat{y}, \hat{z})\) then lemma (4.0.2) implies (4.9) (4.10) but with the unhatted coordinates replaced by hatted coordinates. Hence the pullback is invariant under change of coordinates and is therefore well defined. \(\square\)

**Lemma C.1.3.** The pushforward with respect to a regular surjection may be given in terms of an integral.

\[
a_\alpha(\lambda dz^J \wedge dy^K \wedge dw^z) = \mathbb{1}_\alpha \left( \int_{a^{-1}(y, z)} \lambda \left|_{(w, y, z)} \right. dw^z \right) dz^J \wedge dy^K. \tag{C.1}
\]

*Proof.* Since \(\phi\) contains only \(dz^a\) and \(dy^a\) factors,

\[
\left[ \phi \right| a_\alpha(\lambda dz^J \wedge dy^K \wedge dw^z)] = \int_A a^* \phi \wedge (\lambda dz^J \wedge dy^K \wedge dw^z)
\]

\[
= \int_A \lambda a^* \phi \wedge dz^J \wedge dy^K \wedge dw^z
\]

\[
= \int_M a^* \phi \wedge \left( \int_{a^{-1}(y, z)} \lambda \left|_{(w, y, z)} \right. dw^z \right) dz^J \wedge dy^K
\]

\[
= \left[ \phi \right| \left( \int_{a^{-1}(y, z)} \lambda \left|_{(w, y, z)} \right. dw^z \right) dz^J \wedge dy^K].
\]

A result important for Chapter 5.

**Lemma C.1.4.** The pullback of an SMD with respect to regular surjection may be defined in terms of its action of test forms.

\[
\left[ \phi \right| a^*(\Psi)] = (-1)^{s(r-p)}[a^*\phi | \Psi]. \tag{C.2}
\]

*Proof.* Let \(F(\Psi) \in \Upsilon^{k,p}(a)\) be given by \(\left[ \phi \right| F(\Psi)] = (-1)^{s(r-p)}[a_\alpha \phi | \Psi].\) Then when \(\Psi = f_\alpha(\alpha)\) in order for \(\left[ a_\alpha \phi | \Psi \right]\) to be non zero we can let

\[
\phi = \sum_{J^r, |J|=r} \phi_J(w, y, z) dy^{2\lambda(J)} \wedge dw^z \quad \text{and} \quad \alpha = \sum_{K^r, |K|=p-r} \alpha_K(y, z) dy^K.
\]
C.1. Pullback of SMDs Proofs

Thus

\[
[\phi | F(f_c \alpha)]_A = (-1)^s(\sigma - p) [a_c \phi | f_c \alpha] = (-1)^s(\sigma - p) \int_M f^* (a_c \phi) \wedge \alpha
\]

\[
= (-1)^s(\sigma - p) \sum_{J^r, K^r} \int_N f^* \left( \phi \left( \int_{\alpha^{-1}(y,z)} \phi_J \left( w, y, z \right) dw_z \right) dy_2^J \wedge \alpha_K dy^K \right)
\]

\[
= (-1)^s(\sigma - p) \sum_{J^r, K^r} \int_N f^* \left( \phi_J \left( \int_{\alpha^{-1}(y,z)} \phi_J \left( w, y, z \right) dw_z \right) \alpha_J dy_2^J \right)
\]

\[
= (-1)^s(\sigma - p) \sum_{J^r, K^r} \int_N f^* \left( \phi_J \int_{\alpha^{-1}(y,z)} \phi_J \left( w, y, z \right) dw_z \right) \alpha_J \alpha_K dy_2^J \wedge dy^K
\]

\[
= \sum_{J^r, K^r} \int_{a^* N} \hat{\alpha}^* \left( \phi_J, f^* \left( \alpha_K \right) \right) dy_2^J \wedge dw_z \wedge dy^K
\]

\[
= \sum_{J^r, K^r} \int_{a^* N} \hat{\alpha}^* \left( \phi_J \right) \hat{f}^* \left( \alpha_K \right) dy_2^J \wedge dw_z \wedge dy^K
\]

\[
= \int_{a^* N} \hat{\alpha}^* \left( \phi \right) \hat{f}^* \alpha = \left[ \phi | \hat{\alpha}^* \left( \hat{f}^* \alpha \right) \right]_A
\]

\[
= \left[ \phi | a_c \alpha \right]_A
\]

hence \( F(f_c \alpha) = a_c(f_c \alpha) \). In addition

\[
[\phi | dF(\Psi)]_A = (-1)^{\deg \phi + 1} [d\phi | F(\Psi)]_A = (-1)^{\deg \phi + 1} (-1)^{s(r - p)} [a_c(d\phi) | \Psi]
\]

\[
= (-1)^{\deg \phi + 1} (-1)^{s(r - p)} [da_c(\phi) | \Psi] = (-1)^{s(r - p)} [a_c(\phi) | d\Psi]
\]

\[
= \left[ \phi | F(d\Psi) \right]_A
\]
and

\[
[\phi | i_v F(\Psi)]_A = (-1)^{\deg \phi + 1} [i_v \phi | F(\Psi)]_A = (-1)^{\deg \phi + 1} (-1)^s(r-p) [a_c(i_v \phi) | \Psi] \\
= (-1)^{\deg \phi + 1} (-1)^s(r-p) [i_u a_c(\phi) | \Psi] = (-1)^s(r-p) [a_c(\phi) | i_u \Psi] \\
= [\phi | F(i_u \Psi)]_A,
\]

and $^{i+}$-linearity is trivial. Since $F$ agrees with $a^c$ for (4.4)-(4.8) then $F = a^c$. \qed
Appendix D

D.1 Laplace-Beltrami Operator

The Laplace-Beltrami operator is also defined using the codifferential \( \delta = *^{-1}d * \eta \) such that

\[
\Box = -(\delta d + d\delta)
\]  

(D.1)

where \( \eta : \Lambda^p M \to \Lambda^p M \) is defined as \( \eta(\alpha) = (-1)^{\deg(\alpha)}\alpha \). In this Appendix we are using Minkowski coordinates \((x^0, x^1, x^2, x^3)\) on flat spacetime \(M\).

Lemma D.1.1.

\[
\Box \alpha = (-1)^{p(m-p+1)}s(\star d \star d\alpha + (-1)^m d \star d \star \alpha)
\]  

(D.2)

where \( \alpha \in \Lambda^p M \), \( m = \dim(M) \) and \( s = \det(g) \).
Proof.

\[ \Box \alpha = -\left( \star^{-1} d \star \eta(d\alpha) + d \star^{-1} d \star \eta(\alpha) \right) \]

\[ = -\left( (-1)^{p+1} \star^{-1} d \star d\alpha + (-1)^p d \star^{-1} d \star \alpha \right) \]

\[ = (-1)^p \left( \star^{-1} d \star d\alpha - d \star^{-1} d \star \alpha \right) \]

\[ = (-1)^p s \left( (-1)^{(m-p)p} \star d \star d\alpha - (-1)^{(m-p+1)(p+1)} d \star d \star \alpha \right) \]

\[ = (-1)^{p(m-p+1)} s \left( \star d \star d\alpha + (-1)^m d \star d \star \alpha \right). \]

\[ \Box \alpha \]

The next lemma shows that in flat spacetime the LB operator consists of exterior derivatives and internal contractions (Lie derivatives can be expressed in terms of exterior derivatives and internal contractions due to Cartan’s identity).

**Lemma D.1.2.** Let \( \alpha \in \Gamma \Lambda^p(M_X \times M_Y) \),

\[ \Box_X \alpha = s^2 (-1)^{m-p} \left( g^{ab} L_b^{(x)} L_a^{(x)} \alpha - 2 g^{ab} t_b^{(x)} L_a^{(x)} d_X \alpha \right). \] (D.3)

**Proof.** From (D.2)

\[ \Box_X \alpha = (-1)^{p(m-p+1)} s \left( \star_X d_X \star_X d_X \alpha + (-1)^m d_X \star_X d_X \star_X \alpha \right). \] (D.4)

The Hodge dual \( \star_X \) can be written as

\[ \star_X \alpha = \sum_{J \in \mathcal{J}^m, |J|=p} \star d x^J \wedge \iota^{(x)}_J \alpha. \] (D.5)
Taking the exterior derivative

\[ d_X \star_X \alpha = d_X \left( \sum_{J^m, |J| = p} \star dx^J \wedge i_j^{(x)} \alpha \right) \]

\[ = \sum_{J^m, |J| = p} \star dx^J \wedge (dx i_j^{(x)} \alpha) \]

\[ = \sum_{J^m, |J| = p} \star dx^J \wedge (dx^a \wedge L^{(x)}_a i_j^{(x)} \alpha). \]

However \( \star_X \star_X \alpha = (-1)^{(n-p)} s \alpha \) so

\[ \star_X d \star_X \alpha = \sum_{J^m, |J| = p} \star (\star dx^J \wedge dx^a) \wedge L^{(x)}_a i_j^{(x)} \alpha \]

\[ = \sum_{J^m, |J| = p} i_{dx^J} \star dx^J \wedge L^{(x)}_a i_j^{(x)} \alpha \]

\[ = \sum_{J^m, |J| = p} s(-1)^{(m-p)} g^{ab} i_b^{(x)} dx^J \wedge L^{(x)}_a i_j^{(x)} \alpha \]

\[ = s(-1)^{(m-p)} g^{ab} i_b^{(x)} L^{(x)}_a \alpha. \quad (D.6) \]

Thus

\[ d_X \star_X d_X \star_X \alpha = s(-1)^{(m-p)} \left( g^{ab} L^{(x)}_b L^{(x)}_a \alpha - g^{ab} i_b^{(x)} L^{(x)}_a d_X \alpha \right). \quad (D.7) \]

and

\[ \star_X d_X \star_X d_X \alpha = s(-1)^{(p+1)(m-p-1)} g^{ab} i_b^{(x)} L^{(x)}_a d_X \alpha. \quad (D.8) \]

Therefore

\[ \Box_X \alpha = (-1)^{(m-p+1)} s^2 \left( (-1)^{(p+1)(m-p-1)} g^{ab} i_b^{(x)} L^{(x)}_a d_X \alpha \right. \]

\[ + (-1)^{(m-p)} (-1)^m (g^{ab} L^{(x)}_b L^{(x)}_a \alpha - g^{ab} i_b^{(x)} L^{(x)}_a d_X \alpha) \]

\[ = (-1)^{(m-p+1)} s^2 \left( (-1)^{(m-p)} (-1)^m g^{ab} L^{(x)}_b L^{(x)}_a \alpha \right. \]

\[ - 2(-1)^{(m-p)} (-1)^m g^{ab} i_b^{(x)} L^{(x)}_a d_X \alpha \]

\[ = s^2(-1)^{(m-p)} \left( g^{ab} L^{(x)}_b L^{(x)}_a \alpha - 2 g^{ab} i_b^{(x)} L^{(x)}_a d_X \alpha \right). \]
Lemma D.1.3. The LB operator commutes with the Hodge dual,

\[ \square \ast = \ast \square. \]  

Proof. As \( \ast^{-1} \alpha = \ast \alpha(-1)^{p(m-p)} s \) and since \( \square \) preserves the degree of \( \alpha \) it is more convenient working with \( \ast^{-1} \) thus

\[ \square \ast^{-1} \alpha = -\left( \ast^{-1} d \ast \eta(d \ast^{-1} \alpha) + d \ast^{-1} d \ast \eta(\ast^{-1} \alpha) \right) \]

\[ = -\left( \ast^{-1} d \ast d \ast^{-1} \alpha(-1)^{n-p+1} + d \ast^{-1} d \alpha(-1)^{n-p} \right) \]

\[ = (-1)^{m-p}\left( \ast^{-1} d \ast d \ast^{-1} \alpha - d \ast^{-1} d \alpha \right) \]

\[ = (-1)^{m-p} \ast^{-1}\left( d \ast d \ast^{-1} \alpha - \ast d \ast^{-1} d \alpha \right) \]

\[ = (-1)^{m-p} \ast^{-1}\left( d \ast d \ast \alpha(-1)^{p(n-p)} s - \ast d \ast d \alpha(-1)^{(p+1)(n-p-1)} s \right) \]

\[ = (-1)^{(m-p)(p+1)} s \ast^{-1}\left( d \ast d \ast \alpha + (-1)^{n} \ast d \ast d \alpha \right) \]

\[ = (-1)^{(m-p)(p+1)} s \ast^{-1}\left( d \ast^{-1} d \ast \alpha(-1)^{(n-p)(p-1)}(-1)^{p-1} s \right. \]

\[ \quad \left. + (-1)^{n} \ast^{-1} d \ast d \alpha(-1)^{(n-p)p} s \right) \]

\[ = \ast^{-1}\left( d \ast^{-1} d \ast \alpha(-1)^{p-1} + (-1)^{p} \ast^{-1} d \ast d \alpha \right) \]

\[ = \ast^{-1}\left( - d \ast^{-1} d \ast \eta(\alpha) - \ast^{-1} d \ast \eta(d \alpha) \right) \]

\[ = \ast^{-1} \square \alpha. \]
Lemma D.1.4. Let $\Psi \in \Upsilon^{k,p}(f)$ then the codifferential satisfies

$$[\phi|\delta \Psi]_M = s^2(-1)^p[\delta \phi|\Psi]_M. \quad (D.10)$$

Proof.

$$[\phi|\delta \Psi]_M = [\phi|^{-1}d*\Psi]_M$$

$$= (-1)^{(m-p+1)(p-1)}s[\phi|d*\Psi]_M$$

$$= s[\phi|d*\Psi]_M$$

$$= s(-1)^p[\phi|d*\Psi]_M$$

$$= s(-1)^p(-1)^{p(m-p)}[\phi|d*\Psi]_M$$

$$= s^2(-1)^p[\phi|d*\Psi]_M$$

$$= s^2(-1)^p[\delta \phi|\Psi]_M.$$ 

Lemma D.1.5. Let $\Psi \in \Upsilon^{k,p}(f)$ then the LB operator satisfies

$$[\phi|\Box \Psi]_M = s^2(-1)^m[\Box \phi|\Psi]_M. \quad (D.11)$$

Proof.

$$[\phi|\Box \Psi]_M = -[\phi|\delta d\Psi + d\delta \Psi]_M$$

$$= -s^2(-1)^p[\delta \phi|d\Psi]_M + (-1)^{m-p}[d\Phi|\delta \Psi]_M$$

$$= -s^2(-1)^p(-1)^{m-p}[d\delta \phi|\Psi]_M + s^2(-1)^{m-p}(-1)^p[\delta d\phi|\Psi]_M$$

$$= s^2(-1)^m[d\delta \phi + \delta d\phi|\Psi]_M$$

$$= s^2(-1)^m[\Box \phi|\Psi]_M.$$
Appendix E

E.1 Dipoles and Quadrupoles

In this Appendix we show that the electric dipole can be written an alternative way as to what was presented in the main text and we also provide the proof of Lemma 6.3.1.

Lemma E.1.1.

\[ \mathcal{J}_{\text{ED}} = d_i w C_\varsigma(1). \] (E.1)

Proof. In adapted coordinates \( w = w^\mu \partial_\mu^{(c)} \). Let \( w^\mu = \mathcal{J}_\phi^{\mu,\varnothing} \) then

\[
\mathcal{J}_{\text{ED}} = \sum_{\mu=1}^{3} \left( i_0^{(r)} L_\mu^{(c)} - i_\mu^{(c)} L_0^{(r)} \right) C_\varsigma(w^\mu d\tau)
\]

\[ = \sum_{\mu=1}^{3} \left( L_\mu^{(c)} C_\varsigma(w^\mu) - i_\mu^{(c)} C_\varsigma(\partial_0^{(r)}(w^\mu)d\tau) \right)
\]

\[ = \sum_{\mu=1}^{3} \left( L_\mu^{(c)} (w^\mu(C_\varsigma(1)) - i_\mu^{(c)} C_\varsigma(dw^\mu)) \right)
\]

\[ = \sum_{\mu=1}^{3} \left( ((L_\mu^{(c)} w^\mu) C_\varsigma(1) + w^\mu L_\mu^{(c)} C_\varsigma(1) - i_\mu^{(c)} (dw^\mu \wedge C_\varsigma(1)) \right)
\]

\[ = \sum_{\mu=1}^{3} \left( ((L_\mu^{(c)} w^\mu) C_\varsigma(1) + w^\mu L_\mu^{(c)} C_\varsigma(1) - (i_\mu^{(c)} dw^\mu) C_\varsigma(1) + dw^\mu \wedge i_\mu^{(c)} C_\varsigma(1) \right)
\]

\[ = \sum_{\mu=1}^{3} \left( (i_\mu^{(c)} dw^\mu) C_\varsigma(1) + w^\mu L_\mu^{(c)} C_\varsigma(1) - (i_\mu^{(c)} dw^\mu) C_\varsigma(1) + dw^\mu \wedge i_\mu^{(c)} C_\varsigma(1) \right)
\]

\[ = L_w C_\varsigma(1) = d_i w C_\varsigma(1). \]
E.1. Dipoles and Quadrupoles

Proof of 6.3.1. Split the sum as follows

\[
3 \sum_{a,b,c=0} i_a^{(\zeta)} L_{bc}^{(\zeta)} C_\zeta (\gamma_{\zeta}^{abc}) d\tau = i_0^{(\zeta)} L_{00}^{(\zeta)} C_\zeta (\gamma_{\zeta}^{000}) d\tau
\]

\[
+ \sum_{\mu=1}^3 i_0^{(\zeta)} L_{\mu 0}^{(\zeta)} C_\zeta (\gamma_{\zeta}^{00\mu}) d\tau + \sum_{\mu=1}^3 i_0^{(\zeta)} L_{\mu 0}^{(\zeta)} C_\zeta (\gamma_{\zeta}^{0\mu 0}) d\tau + \sum_{\mu=1}^3 i_0^{(\zeta)} L_{\mu 0}^{(\zeta)} C_\zeta (\gamma_{\zeta}^{\mu 0 0}) d\tau
\]

\[
+ \sum_{\mu,\nu,\rho=1}^3 i_0^{(\zeta)} L_{\mu \nu \rho}^{(\zeta)} C_\zeta (\gamma_{\zeta}^{\mu \nu \rho}) d\tau.
\]

The monopole term vanishes by virtue of (6.14b) and therefore does not mix with the quadrupole or dipole terms. The next group of terms that include \(\gamma_{\zeta}^{00\mu}\) is the electric dipole,

\[
4J_{\text{EDP}} = \sum_{\mu=1}^3 i_0^{(\zeta)} L_{\mu 0}^{(\zeta)} C_\zeta (\gamma_{\zeta}^{00\mu}) d\tau + \sum_{\mu=1}^3 i_0^{(\zeta)} L_{\mu 0}^{(\zeta)} C_\zeta (\gamma_{\zeta}^{0\mu 0}) d\tau + \sum_{\mu=1}^3 i_0^{(\zeta)} L_{\mu 0}^{(\zeta)} C_\zeta (\gamma_{\zeta}^{\mu 0 0}) d\tau
\]

\[
= \sum_{\mu=1}^3 \left( L_{\mu}^{(\zeta)} C_\zeta (\partial_0^{(\tau)} (\gamma_{\zeta}^{00\mu})) + L_{\mu}^{(\zeta)} C_\zeta (\partial_0^{(\tau)} (\gamma_{\zeta}^{0\mu 0})) + i_0^{(\zeta)} C_\zeta (\partial_0^{(\tau)} \partial_0^{(\tau)} (\gamma_{\zeta}^{\mu 0 0}) d\tau) \right)
\]

\[
= \sum_{\mu=1}^3 \left( 2\mu L_{\mu}^{(\zeta)} C_\zeta (\partial_0^{(\tau)} (\gamma_{\zeta}^{00\mu})) + i_0^{(\zeta)} C_\zeta (\partial_0^{(\tau)} \partial_0^{(\tau)} (\gamma_{\zeta}^{\mu 0 0}) d\tau) \right).
\]

Thus from (6.14b)

\[
\gamma_{\zeta}^{\mu 0 0} + 2\gamma_{\zeta}^{00\mu} = 0 \quad \Rightarrow \quad \gamma_{\zeta}^{\mu 0 0} = -2\gamma_{\zeta}^{00\mu}
\]

and substituting back into the above

\[
J_{\text{EDP}} = \frac{1}{2} \sum_{\mu=1}^3 \left( L_{\mu}^{(\zeta)} C_\zeta (\partial_0^{(\tau)} (\gamma_{\zeta}^{00\mu})) - i_0^{(\zeta)} C_\zeta (\partial_0^{(\tau)} \partial_0^{(\tau)} (\gamma_{\zeta}^{00\mu}) d\tau) \right).
\]

Comparing with (6.3)

\[
\partial_0^{(\tau)} (\gamma_{\zeta}^{00\mu}) = 2J_{\phi}^{\mu,\phi}.
\]

The next group of terms with \(\gamma_{\zeta}^{0\mu \nu}\), are the electric quadrupole and magnetic dipole
E.1. Dipoles and Quadrupoles

contributions

\[ 4(J_{\text{EDP}} + J_{EdP}) \]
\[ = \sum_{\mu,\nu=1}^{3} i_{0}^{(\zeta)} L^{(\zeta)}_{\mu\nu} C_{\gamma(\zeta)}(\gamma_{0\mu} \gamma_{0\nu} \text{d}\tau) + \sum_{\mu,\nu=1}^{3} i_{\mu}^{(\zeta)} L^{(\zeta)}_{\mu\nu} C_{\gamma(\zeta)}(\gamma_{\mu0} \gamma_{\nu0} \text{d}\tau) \]
\[ + \sum_{\mu,\nu=1}^{3} i_{\mu}^{(\zeta)} L^{(\zeta)}_{\mu\nu} C_{\gamma(\zeta)}(\gamma_{\mu0} \gamma_{0\nu} \text{d}\tau) \]
\[ = \sum_{\mu,\nu=1}^{3} L^{(\zeta)}_{\mu\nu} C_{\gamma(\zeta)}(\gamma_{0\mu}) + \sum_{\mu,\nu=1}^{3} i_{\mu}^{(\zeta)} L^{(\zeta)}_{\mu\nu} C_{\gamma(\zeta)}(\gamma_{0\nu} \gamma^{(\tau)}) \text{d}\tau) \]
\[ + \sum_{\mu,\nu=1}^{3} i_{\mu}^{(\zeta)} L^{(\zeta)}_{\mu\nu} C_{\gamma(\zeta)}(\gamma_{\mu0} \gamma^{(\tau)}) \text{d}\tau) \]
\[ = \sum_{\mu,\nu=1}^{3} L^{(\zeta)}_{\mu\nu} C_{\gamma(\zeta)}(\gamma_{0\mu}) + 2 \sum_{\mu,\nu=1}^{3} i_{\mu}^{(\zeta)} L^{(\zeta)}_{\mu\nu} C_{\gamma(\zeta)}(\gamma_{0\nu} \gamma^{(\tau)}) \text{d}\tau) \]

Further splitting for the cases where \( \mu = \nu \) and \( \mu \neq \nu \) gives

\[ = 2 \sum_{\mu<\nu=1}^{3} L^{(\zeta)}_{\mu\nu} C_{\gamma(\zeta)}(\gamma_{0\mu}) + \sum_{\mu=1}^{3} L^{(\zeta)}_{\mu\mu} C_{\gamma(\zeta)}(\gamma_{0\mu}) + 2 \sum_{\mu=1}^{3} i_{\mu}^{(\zeta)} L^{(\zeta)}_{\mu\mu} C_{\gamma(\zeta)}(\gamma_{0\mu} \gamma^{(\tau)}) \text{d}\tau) \]
\[ + 2 \sum_{\mu<\nu=1}^{3} i_{\mu}^{(\zeta)} L^{(\zeta)}_{\mu\nu} C_{\gamma(\zeta)}(\gamma_{0\nu} \gamma^{(\tau)}) \text{d}\tau) \]
\[ = 2 \sum_{\mu<\nu=1}^{3} L^{(\zeta)}_{\mu\nu} C_{\gamma(\zeta)}(\gamma_{0\mu}) + \sum_{\mu=1}^{3} L^{(\zeta)}_{\mu\mu} C_{\gamma(\zeta)}(\gamma_{0\mu}) + 2 \sum_{\mu=1}^{3} i_{\mu}^{(\zeta)} L^{(\zeta)}_{\mu\mu} C_{\gamma(\zeta)}(\gamma_{0\mu} \gamma^{(\tau)}) \text{d}\tau) \]
\[ + 2 \sum_{\mu<\nu=1}^{3} i_{\mu}^{(\zeta)} L^{(\zeta)}_{\mu\nu} C_{\gamma(\zeta)}(\gamma_{0\nu} \gamma^{(\tau)}) \text{d}\tau) + 2 \sum_{\mu<\nu=1}^{3} i_{\nu}^{(\zeta)} L^{(\zeta)}_{\nu\mu} C_{\gamma(\zeta)}(\gamma_{\nu0} \gamma^{(\tau)}) \text{d}\tau) \]

When \( \mu = \nu \),

\[ 0 = \gamma^{0\mu}_{(\zeta)} + \gamma^{0\mu}_{(\zeta)} + \gamma^{\mu0}_{(\zeta)} = 2\gamma^{0\mu}_{(\zeta)} + \gamma^{0\mu}_{(\zeta)} \quad \Rightarrow \quad 2\gamma^{0\mu}_{(\zeta)} = -\gamma^{0\mu}_{(\zeta)}, \]

whereas for \( \mu < \nu \),

\[ \gamma^{\mu0}_{(\zeta)} = \gamma^{\mu0}_{(\zeta)} \quad \text{and} \quad \gamma^{\nu0}_{(\zeta)} + \gamma^{\mu0}_{(\zeta)} + \gamma^{0\nu}_{(\zeta)} = 0 \quad \Rightarrow \quad \gamma^{\nu0}_{(\zeta)} = -\gamma^{\mu0}_{(\zeta)} - \gamma^{0\nu}_{(\zeta)}. \]
E.1. Dipoles and Quadrupoles

Substituting into the above we get

\[
J_{\text{EQP}} = \frac{1}{4} \sum_{\mu=1}^{3} \left( L^{(c)}_{\mu\nu} C_{c} (\gamma^{0\mu\nu}_{(c)}) - i_{\mu}^{(c)} L^{(c)}_{\mu\nu} C_{c} (\partial_{0}^{(\tau)} (\gamma^{0\mu\nu}_{(c)}) d\tau) \right) \\
+ \frac{1}{2} \sum_{\mu<\nu=1}^{3} \left( L^{(c)}_{\mu\nu} C_{c} (\gamma^{0\mu\nu}_{(c)}) - i_{\mu}^{(c)} L^{(c)}_{\mu\nu} C_{c} (\partial_{0}^{(\tau)} (\gamma^{0\mu\nu}_{(c)}) d\tau) \right).
\]

\[
J_{\text{MDP}} = \frac{1}{2} \sum_{\mu<\nu=1}^{3} \left( i_{\mu}^{(c)} L^{(c)}_{\mu\nu} - i_{\nu}^{(c)} L^{(c)}_{\mu\nu} \right) C_{c} (\partial_{0}^{(\tau)} (\gamma^{\mu\nu}_{(c)}) d\tau).
\]

Therefore comparing with (6.11) and (6.4)

\[
\gamma^{0\mu\nu}_{(c)} = 4J_{\phi}^{\mu..\phi}, \quad \gamma^{0\mu\nu}_{(c)} = 2J_{\phi}^{\mu..\phi} \quad \text{and} \quad \partial_{0}^{(\tau)} (\gamma^{\mu\nu}_{(c)}) = 2J_{0}^{\mu..\nu}, \quad \mu < \nu.
\]

The final term with \(\gamma^{\mu\nu\rho}_{(c)}\) is the magnetic quadrupole

\[
4J_{\text{MDP}} = \sum_{\mu=1, \nu=1, \rho=1}^{3} i_{\mu}^{(c)} L^{(c)}_{\nu\rho} C_{c} (\gamma^{\mu\nu\rho}_{(c)}) d\tau + \sum_{\mu=1, \nu=1, \rho=1}^{3} i_{\mu}^{(c)} L^{(c)}_{\nu\rho} C_{c} (\gamma^{\mu\nu\rho}_{(c)}) d\tau \\
+ \sum_{\nu=\mu=1, \rho=1}^{3} i_{\mu}^{(c)} L^{(c)}_{\nu\rho} C_{c} (\gamma^{\mu\nu\rho}_{(c)}) d\tau + \sum_{\mu=1, \nu=1, \rho=1}^{3} i_{\mu}^{(c)} L^{(c)}_{\nu\rho} C_{c} (\gamma^{\mu\nu\rho}_{(c)}) d\tau \\
+ \sum_{\mu=1, \nu=1, \rho=1}^{3} i_{\mu}^{(c)} L^{(c)}_{\nu\rho} C_{c} (\gamma^{\mu\nu\rho}_{(c)}) d\tau
\]

\[
= \sum_{\mu=1, \nu=1, \rho=1}^{3} i_{\mu}^{(c)} L^{(c)}_{\mu\nu\rho} C_{c} (\gamma^{\mu\nu\rho}_{(c)}) d\tau + \sum_{\mu=1, \nu=1, \rho=1}^{3} i_{\mu}^{(c)} L^{(c)}_{\mu\nu\rho} C_{c} (\gamma^{\mu\nu\rho}_{(c)}) d\tau + \sum_{\mu=1, \nu=1, \rho=1}^{3} i_{\mu}^{(c)} L^{(c)}_{\mu\nu\rho} C_{c} (\gamma^{\mu\nu\rho}_{(c)}) d\tau \\
+ \sum_{\mu=1, \nu=1, \rho=1}^{3} i_{\mu}^{(c)} L^{(c)}_{\mu\nu\rho} C_{c} (\gamma^{\mu\nu\rho}_{(c)}) d\tau + \sum_{\mu=1, \nu=1, \rho=1}^{3} i_{\mu}^{(c)} L^{(c)}_{\mu\nu\rho} C_{c} (\gamma^{\mu\nu\rho}_{(c)}) d\tau
\]

\[
= 2 \sum_{\mu=1, \nu=1, \rho=1}^{3} i_{\mu}^{(c)} L^{(c)}_{\mu\nu\rho} C_{c} (\gamma^{\mu\nu\rho}_{(c)}) d\tau + \sum_{\mu=1, \nu=1, \rho=1}^{3} i_{\mu}^{(c)} L^{(c)}_{\mu\nu\rho} C_{c} (\gamma^{\mu\nu\rho}_{(c)}) d\tau + \sum_{\mu=1}^{3} i_{\mu}^{(c)} L^{(c)}_{\mu\nu\rho} C_{c} (\gamma^{\mu\nu\rho}_{(c)}) d\tau
\]

\[
+ \sum_{\mu=1, \nu=1, \rho=1}^{3} i_{\mu}^{(c)} L^{(c)}_{\mu\nu\rho} C_{c} (\gamma^{\mu\nu\rho}_{(c)}) d\tau.
\]

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E.1. Dipoles and Quadrupoles

However

\[
\gamma_{(\zeta)}^{\mu\mu} = 0,
\]

\[
0 = \gamma_{(\zeta)}^{\mu\mu} + \gamma_{(\zeta)}^{\mu\nu} + \gamma_{(\zeta)}^{\nu\mu} = 2\gamma_{(\zeta)}^{\mu\nu} + \gamma_{(\zeta)}^{\nu\mu},
\]

\[
\Rightarrow 2\gamma_{(\zeta)}^{\mu\nu} = -\gamma_{(\zeta)}^{\nu\mu},
\]

\[
\gamma_{(\zeta)}^{\mu\nu} + \gamma_{(\zeta)}^{\nu\mu} + \gamma_{(\zeta)}^{\mu\mu} = 0, \quad \mu \neq \nu \neq \rho.
\]

\[
\mathcal{J}_{MQP} = \frac{1}{4} \sum_{\mu \neq \nu = 1}^{3} \left( i_{\nu}^{(\zeta)} L_{\mu \mu}^{(\zeta)} - i_{\mu}^{(\zeta)} L_{\nu \nu}^{(\zeta)} \right) C_{\xi} \left( \gamma_{(\zeta)}^{\nu \mu} d\tau \right) + \frac{1}{2} i_{1}^{(\zeta)} L_{23}^{(\zeta)} C_{\xi} \left( \gamma_{(\zeta)}^{123} d\tau \right)
\]

\[
+ \frac{1}{2} i_{2}^{(\zeta)} L_{13}^{(\zeta)} C_{\xi} \left( \gamma_{(\zeta)}^{213} d\tau \right) + \frac{1}{2} i_{3}^{(\zeta)} L_{12}^{(\zeta)} C_{\xi} \left( \gamma_{(\zeta)}^{312} d\tau \right).
\]

\[
= \frac{1}{4} \sum_{\mu \neq \nu = 1}^{3} \left( i_{\nu}^{(\zeta)} L_{\mu \mu}^{(\zeta)} - i_{\mu}^{(\zeta)} L_{\nu \nu}^{(\zeta)} \right) C_{\xi} \left( \gamma_{(\zeta)}^{\nu \mu} d\tau \right) + \frac{1}{2} i_{1}^{(\zeta)} L_{12}^{(\zeta)} - i_{2}^{(\zeta)} L_{23}^{(\zeta)} C_{\xi} \left( \gamma_{(\zeta)}^{123} d\tau \right)
\]

\[
+ \frac{1}{2} \left( i_{2}^{(\zeta)} L_{13}^{(\zeta)} - i_{1}^{(\zeta)} L_{23}^{(\zeta)} \right) C_{\xi} \left( \gamma_{(\zeta)}^{213} d\tau \right).
\]

So

\[
\gamma_{(\zeta)}^{\nu \mu} = 2\mathcal{J}_{0}^{\mu,\nu}, \quad \gamma_{(\zeta)}^{\mu\mu} = 2\mathcal{J}_{0}^{\nu,\mu}, \quad \mu \neq \nu \neq \rho.
\]

Lemma E.1.2. The retarded one form is

\[
d\tau_{R} = \frac{\bar{Z}}{g(V, Z)}. \tag{E.2}
\]

Proof. Z is null so \(g(Z, Z) = 0\) thus

\[
0 = d(g(Z, Z))
\]

\[
= d(g_{ab}Z^{a}Z^{b})
\]

\[
= g_{ab}(dZ^{a})Z^{b} + g_{ab}Z^{a}(dZ^{b})
\]

\[
= g_{ab}(dx^{a} - V^{a}d\tau_{R})Z^{b} + g_{ab}Z^{a}(dx^{b} - V^{b}d\tau_{R})
\]

\[
= Z_{a}dx^{a} - \left( g_{ab}V^{a}Z^{b} \right)d\tau_{R} + Z_{b}dx^{b} - \left( g_{ab}Z^{a}V^{b} \right)d\tau_{R}
\]

\[
= 2\bar{Z} - 2g(V, Z)d\tau_{R}.
\]

Rearranging for \(d\tau_{R}\) gives the result.